國立臺灣大學電機資訊學院資訊工程學系
碩士論文
Department of Computer Science and Information Engineering
College of Electrical Engineering and Computer Science
National Taiwan University
Master Thesis

具計數量詞雙變量邏輯可滿足性之新演算法
New Algorithms for the Satisfiability of Two-Variable Logic with Counting Quantifiers

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中華民國 111 年 8 月
August, 2022

doi:10.6342/NTU202203098

doi:10.6342/NTU202203098

國立臺灣大學碩士學位論文
口試委員會審定書

具計數量詞雙變量邏輯可滿足性之新演算法

New Algorithms for the Satisfiability of Two-Variable Logic with Counting Quantifiers

本論文係呂佳軒君（學號 R09922064）在國立臺灣大學資訊工程學系完成之碩士學位論文，於民國 111 年 8 月 5 日承下列考試委員審查通過及口試及格，特此證明

口試委員：

（指導教授）

系 主 任

doi:10.6342/NTU202203098

doi:10.6342/NTU202203098

誌謝

首先我得感謝指導教授陳偉松老師，在我升大四暑假接受我冒昧的專題研究請求。在近三年光陰中，不僅是研究上的追求，以及知識上、書寫上的打磨，乃至於應對的方式，都令我獲益良多。這份論文的許多概念與結果有賴於每個星期討論所得。

關於這篇論文的產出，我亦需感謝學位考試口試委員王柏堯老師及江介宏老師，感謝他們撥冗閱讀此論文，並給予寶貴的修改意見。

我也要感謝廷衛學長指導我一些實驗室事務，以及其經驗提供了我很好的參考。

謝謝 6rainee 的偉倫、子賢、耿宇、瑋麟以及晨皓，十分慶幸在新的環境還能遇見同為從物理系轉行的人們。我也要對芸彤提供關於最佳化演算法的建議表示感激，這些知識大幅縮短了我摸索那些不熟悉的理論所需要的時間。

最後要感謝我的家人，感謝他們對於我的關懷與支持，特別是對不停轉換領域的我的包容，以及對於我所選擇的道路從不加以干涉。

十分感謝！

doi:10.6342/NTU202203098

Acknowledgements

First of all, I am glad to thank my advisor Tony Tan. I have learned a lot from him since I joined the lab in my senior of college, including knowledge in the area, research methods, writing skills, and mindsets. All of them are very helpful to me. Many of the concepts and results in this thesis rely on weekly discussions.

Next, I would like to thank the committee members of my oral defense, Bow-Yaw Wang and Jie-Hong Roland Jiang, for taking the time to read my thesis and for their valuable comments.

Finally, I would like to thank my family for their care and support, especially for always accepting my choice.

Thank you all!

doi:10.6342/NTU202203098

doi:10.6342/NTU202203098

摘要

雙變量邏輯（FO^2）為使用至多兩個變數之一階邏輯子類，其可滿足性（satisfiability）問題已知為可判定（decidable），更確切地說，該問題複雜度為非確定性指數時間完備（NEXPTIME-complete）。其上界通常透過指數大小模型（ESM）性質導出，該性質陳述如下，若一雙變量邏輯句子為可滿足，則該句子具有一包含至多其長度指數數量元素之模型。指數大小模型性質亦意味著雙變量邏輯之可滿足性及有限可滿足性（finite satisfiability）問題相等。

具計數量詞雙變量邏輯（C^2）為雙變量邏輯在計數量詞下之擴展。計數量詞之語法為 $\exists^{=k}x\ \varphi(x)$，其語意為於模型中恰有 k 個元素滿足 $\varphi(x)$。具計數量詞雙變量邏輯於知識表示與推理（KRR）領域中有著重要應用。儘管其可滿足性及有限可滿足性問題之複雜度依舊為非確定性指數時間完備，指數大小模型性質在具計數量詞雙變量邏輯上不再成立。一般來說，其複雜度上界乃通過非確定性指數時間規約（non-deterministic exponential-time reduction）至整數線性規劃（ILP）問題導出，因此，其相應之演算法相當繁雜，並且，因規約中包含指數數量之非確定性猜測，此演算法通常難以實現。於此之外，自具計數量詞雙變量邏輯中提取具有效率實現之子類亦相當困難。具計數量詞雙變量邏輯之有效率演算法仍為一未解研究問題。

於此論文中，我們重新審視具計數量詞雙變量邏輯。我們引入並證明組態指數上界（CEB）性質，此性質可視為指數大小模型性質之於雙變量邏輯之擴展。我們演示如何通過組態指數上界性質導出具計數量詞雙變量邏輯之可滿足性及有限可滿足性問題之非確定

doi:10.6342/NTU202203098

指數時間演算法。該性質亦可用於提取具計數量詞雙變量邏輯之強可滿足（strongly satisfiable）語義子類，我們證明具計數量詞雙變量邏輯之強可滿足性及有限強可滿足性問題之複雜度亦為非確定指數時間完備，其複雜度上界乃通過確定性指數時間規約（deterministic exponential-time reduction）至布林可滿足性（Boolean satisfiability）問題導出，此種規約得以導出更佳且有效率之演算法實現。

我們所提出針對具計數量詞雙變量邏輯之強可滿足性及有限強可滿足性問題之演算法亦可應用於具計數量詞雙變量邏輯之守衛子類（guarded fragment）之可滿足性及有限可滿足性問題。我們提出之演算法用於該問題之複雜度為確定性指數時間，此結果與該問題複雜度已知為確定性指數時間完備（EXPTIME-complete）相匹配。由此可知，具計數量詞雙變量邏輯之強可滿足子類乃介於具計數量詞雙變量邏輯之守衛子類與具計數量詞雙變量邏輯之間。

我們亦提出自具計數量詞雙變量邏輯之強可滿足性問題至雙變量邏輯之可滿足性問題之確定性多項式時間規約（deterministic polynomial-time reduction），以及自具計數量詞雙變量邏輯之守衛子類之可滿足性問題至雙變量邏輯之守衛子類之可滿足性問題之確定性多項式時間規約。

關鍵字： 可滿足性；雙變量邏輯；計數量詞；組態指數上界；強可滿足子類

doi:10.6342/NTU202203098

Abstract

Two-variable logic (FO^2) is the subclass of first-order logic using at most two variables. It is a well-known fragment of first-order logic whose satisfiability problem is decidable. More precisely, the exact complexity is NEXPTIME-complete. The upper bound is usually established by the so-called *exponential size model* (ESM) property, i.e., if an FO^2 sentence is satisfiable, then it is satisfiable by a model whose size is exponential in its length. Moreover, the ESM property implies that the satisfiability and finite satisfiability problems of FO^2 coincide.

Two-variable logic with counting quantifiers (C^2) is the extension of FO^2 with counting quantifiers of the form $\exists^{=k} x\ \varphi(x)$. The semantics of the counting quantifier is that exactly k elements exist in the model satisfying $\varphi(x)$. The class C^2 is important for many knowledge representation and reasoning (KRR) applications. Though the exact complexity of the satisfiability and finite satisfiability problems of C^2 is also NEXPTIME-complete, the ESM property no longer holds. In general, these two problems do not coincide. Usually, the upper bound is established by the *non-deterministic* exponential-time reduction to the integer linear programming (ILP) problem. Hence, the algorithms are very involved, and due to the exponential amount of non-determinism in the reduction, they are complicated to implement. It is also difficult to extract the subclasses of C^2 for which there are efficient implementations. Obtaining explicit algorithms for C^2 is still an open research problem.

doi:10.6342/NTU202203098

In this thesis, we revisit C^2. We introduce and prove the *configuration exponential bound* (CEB) property for C^2. This property can be viewed as the extension of the ESM property for FO^2. We show that the CEB property can be used to obtain alternative NEXPTIME algorithms for the satisfiability and finite satisfiability problems of C^2. It can also be used to extract a semantic subclass that we call the *strongly satisfiable fragment*. We show that the complexity of the strong satisfiability and finite strong satisfiability problems of C^2 is NEXPTIME-complete. The upper bound is established by the *deterministic* exponential-time reduction to the Boolean satisfiability problem. Such reduction may yield a better and more efficient implementation.

Our algorithms for the strong satisfiability and finite strong satisfiability problems of C^2 can also be used for the satisfiability and finite satisfiability problems of the *guarded fragment* of C^2 (GC^2). The running time of our algorithms for GC^2 is deterministic exponential in the length of the sentence, which matches that the known complexity of GC^2 is EXPTIME-complete. Thus, the class strong satisfiability of C^2 lies in between the satisfiability of GC^2 and C^2.

Finally, we present a deterministic polynomial-time reduction from the strong satisfiability problem of C^2 to the satisfiability problem of FO^2. We also present a deterministic polynomial-time reduction from the satisfiability problem of GC^2 to the satisfiability problem of GF^2.

Keywords: satisfiability; two variable logic; counting quantifier; configuration exponential bound; strongly satisfiable fragment

doi:10.6342/NTU202203098

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Chapter 1

Introduction

Two-variable logic (FO^2) is the subclass of first-order logic using at most two variables. It is a well-known fragment of first-order logic whose satisfiability problem is decidable. More precisely, the exact complexity is $\mathsf{NEXPTIME}$-complete. The upper bound is usually established by the so-called *exponential size model* (ESM) property, i.e., if an FO^2 sentence is satisfiable, then it is satisfiable by a model whose size is exponential in its length. Moreover, the ESM property implies that the satisfiability and finite satisfiability problems of FO^2 coincide.

Two-variable logic with counting quantifiers (C^2) is the extension of FO^2 with counting quantifiers of the form $\exists^{=k}x\ \varphi(x)$. The semantics of the counting quantifier is that exactly k elements exist in the model satisfying $\varphi(x)$. The class C^2 is important for many knowledge representation and reasoning (KRR) applications. Though the exact complexity of the satisfiability and finite satisfiability problems of C^2 is also $\mathsf{NEXPTIME}$-complete, the ESM property no longer holds. In general, these two problems do not coincide. Usually, the upper bound is established by the *non-deterministic* exponential-time reduction to the integer linear programming (ILP) problem. Hence, the algorithms are very involved, and due to the exponential amount of non-determinism in the reduction, they are complicated to implement. It is also difficult to extract the subclasses of C^2 for which there are efficient implementations. Obtaining explicit algorithms for C^2 is still an open research problem.

doi:10.6342/NTU202203098

1.1 Our contribution

In this thesis, we revisit C^2. Our contribution is as follows.

- We present alternative but more transparent algorithms for the satisfiability and finite satisfiability problems of C^2.
- We extract a semantic subclass of C^2 that we call the *strongly satisfiable fragment*, which may yield a better and more efficient implementation.
- Our approach for the strong satisfiability and finite strong satisfiability problems of C^2 also works for the satisfiability and finite satisfiability problems for GC^2 and yields $\mathsf{EXPTIME}$ algorithms.
- We present deterministic polynomial-time reductions from the strong satisfiability problem of C^2 to the satisfiability problem of FO^2 and from the satisfiability problem of GC^2 to the satisfiability problem of GF^2.

The main idea of our approach is relatively standard. We view the model of C^2 sentence φ as a labeled graph $\mathcal{G}$. We call $\mathcal{G}$ the *pseudo-model* of φ. We introduce the notion of *configuration* to encode the *counting condition* of φ in $\mathcal{G}$. For a vertex in $\mathcal{G}$, its configuration records the number of outgoing edges with specific labels. A configuration is valid in φ if and only if it satisfies the counting condition of φ. We show that a C^2 sentence φ is satisfiable if and only if it has a pseudo-model $\mathcal{G}$ satisfying the configuration of each vertex in $\mathcal{G}$ is valid.

Because the ESM property no longer holds for C^2, there exists a satisfiable C^2 sentence φ such that the size of its pseudo-models are unbounded. However, we observe that φ is satisfiable if and only if it has a pseudo-model $\mathcal{G}$ such that $\mathcal{G}$ can be represented by a pseudo-structure $\mathcal{H}$ and a set of configurations $\mathcal{C}$. Furthermore, the size of $\mathcal{H}$ and $\mathcal{C}$ are bounded by the length of φ. We call this the *configuration exponential bound* (CEB) property for C^2. This property can be viewed as the extension of the ESM property for FO^2. We show that the CEB property can be used to obtain alternative $\mathsf{NEXPTIME}$ algorithms for the satisfiability and finite satisfiability problems of C^2.

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The CEB property can also be used to extract a semantic subclass that we call the *strongly satisfiable fragment*. In general, a C^2 sentence is strongly satisfiable if and only if it has a pseudo-model $\mathcal{G}$ such that $\mathcal{G}$ is the union of infinitely many copies of the identical pseudo-structure $\mathcal{H}$. We show that the complexity of the strong satisfiability and finite strong satisfiability problems of C^2 is $\mathsf{NEXPTIME}$-complete. The upper bound is established by the *deterministic* exponential-time reduction to the Boolean satisfiability problem. Such reduction may yield a better and more efficient implementation.

We also consider the *guarded fragment* of C^2, denoted by GC^2. For a GC^2 sentence φ, we observe that φ is satisfiable if and only if it is strongly satisfiable. Hence, our algorithms for the strong satisfiability and finite strong satisfiability problems of C^2 can also be used. Furthermore, the non-determinism guessing in our algorithms vanishes in this case. The running time of our algorithms for GC^2 is deterministic exponential in the length of φ, which matches that the known complexity of GC^2 is $\mathsf{EXPTIME}$-complete. Thus, the class strong satisfiability of C^2 lies in between the satisfiability of GC^2 and C^2.

Finally, we present a deterministic polynomial-time reduction from the strong satisfiability problem of C^2 to the satisfiability problem of FO^2. We also present a deterministic polynomial-time reduction from the satisfiability problem of GC^2 to the satisfiability problem of GF^2.

1.2 Related works

First-order logic (FO) plays a fundamental role in computer science, but its satisfiability problem is undecidable [6, 5, 31, 29]. Researchers try to find the decidable fragments of FO for practical use and understand the boundary between decidability and undecidability. One approach to obtaining such fragments is restricting the number of variables in the formulae. Henkin [15] considered the k-variable fragment of FO, denoted by FO^k, where $k \geq 1$. The class FO^k is the set of all first-order logic formulae using at most k variables. Since the satisfiability problem of the AEA class, i.e., the fragment of FO with the prefix $\forall\exists\forall$, is undecidable [16], the satisfiability problem of FO^3 is also undecidable. Therefore, it is interesting to consider the satisfiability problem of FO^2, denoted by $\mathsf{SAT}(\mathsf{FO}^2)$. It

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is known that $\mathsf{SAT}(\mathsf{FO}^2)$ is decidable [27, 22, 13], and, in fact, the exact complexity is $\mathsf{NEXPTIME}$-complete [13, 20]. The decidability of FO^2 and its various extensions and restrictions was used to show the decidability of reasoning problems for many modal and description logics [2].

Scott [27] first proved the decidability of $\mathsf{SAT}(\mathsf{FO}^2)$ by reducing it to the *Gödel class*. The Gödel class without the equality predicate is decidable but becomes undecidable when extended with the equality predicate [11]. Therefore, Scott's proof only covered the satisfiability problem of FO^2 *without* the equality predicate. Mortimer [22] proved that every satisfiable FO^2 sentence has a *double-exponential* size model and obtained that $\mathsf{SAT}(\mathsf{FO}^2)$ is in $\mathsf{2\text{-}NEXPTIME}$. Grädel, Kolaitis, and Vardi [13] revisited Scott's procedure and reduced FO^2 to the fragment of FO with the prefix $\forall\forall \wedge \forall\exists^*$. They also improved the upper bound of the model size to only *exponential* and hence derived the *exponential size model* (ESM) property. The ESM property implies that $\mathsf{SAT}(\mathsf{FO}^2)$ is in $\mathsf{NEXPTIME}$. It also implies that the satisfiability and finite satisfiability problems of FO^2 coincide. The lower bound of $\mathsf{SAT}(\mathsf{FO}^2)$ is established by the fact that the satisfiability problem of the fragment $\forall\forall \wedge \forall\exists^*$ is $\mathsf{NEXPTIME}$-hard [19, 10].

Recently, Lin, Lu, and Tan [20] revisited the decision procedure of $\mathsf{SAT}(\mathsf{FO}^2)$ by reducing it to a novel graph theoretic problem called *conditional independent set* (CIS). They also presented a deterministic polynomial-time reduction from the satisfiability problem of FO^2 to the satisfiability problem of FO^2 *without the equality predicate* by encoding the *edge representation* of the model.

Since FO^2 lacks the capability for counting, researchers consider its extension with counting quantifiers, denoted by C^2. The class C^2 was first studied by Grädel, Otto, and Rosen [14]. They showed that the satisfiability and finite satisfiability problems of C^2 are decidable, but unlike FO^2, these two problems do not coincide. They also presented a deterministic exponential-time reduction from C^2 to C^2_1, the fragment of C^2 that used only counting quantifier in the form $\exists^{=1}$. Pacholski, Szwast, and Tendera [23] proved that the satisfiability problem of C^2_1 is in $\mathsf{NEXPTIME}$, which implies that the satisfiability problem of C^2 is in $\mathsf{2\text{-}NEXPTIME}$. They also showed that there is a finitely satisfiable C^2 sentence

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whose model size is at least double-exponential in its length. Therefore, the ESM property no longer holds even for the finite satisfiability problem of C^2. Pratt-Hartmann [25] closed the gap by proving that the satisfiability and finite satisfiability problems of C^2 are in NEXPTIME. The complexity matches the lower bound of FO^2. He also showed that every *finitely satisfiable* C^2 sentence has a *double-exponential* size model, though the complexity of the finite satisfiability problem of C^2 is in NEXPTIME.

The guarded fragment of FO, denoted by GF, is another famous fragment of FO that restricts the usage of quantifiers with the guarded atoms. The satisfiability problem of GF is 2-EXPTIME-complete [1]. Its complexity drops to EXPTIME-complete when the number of variables is fixed [12].

The guarded fragment of FO^2 extended with counting quantifiers, denoted by GC^2, was first studied by Kazakov [18]. He showed that the satisfiability problem of GC^2 is in EXPTIME by reducing it to GF^3. The complexity matches the lower bound of GF^2. Pratt-Hartmann [26] proved that the satisfiability and finite satisfiability problems of GC^2 are in EXPTIME.

1.3 Organization

This thesis is organized as follows. In Chapter 2, we present the detailed definitions of FO^2, C^2, and GC^2. In Chapter 3, we define some notions for C^2, including 1-type, 2-type, and configuration. We also define pseudo-structure and pseudo-model, which can be viewed as the alternative representations of the structure and the model of two-variable logic sentences. We state and prove the CEB property for C^2 in Chapter 4 and, from it, obtain alternative NEXPTIME algorithms for the satisfiability and finite satisfiability problems of C^2 in Chapter 5. In Chapter 6, we define the strongly satisfiable fragment of C^2 and present the algorithms for the strong satisfiability and finite strong satisfiability problems of C^2. In Chapter 7, we demonstrate how to apply our algorithms for the satisfiability and finite satisfiability problems of GC^2. In Chapter 8, we present the deterministic polynomial-time reductions from the strong satisfiability problem of C^2 to the satisfiability problem of FO^2 and from the satisfiability problem of GC^2 to the satisfiability problem

doi:10.6342/NTU202203098

of GF^2. Finally, we conclude with Chapter 9.

doi:10.6342/NTU202203098

Chapter 2

Preliminary

The organization of this chapter is as follows. In Section 2.1, we introduce the detailed definition of two-variable logic (FO^2). In Section 2.2, we define the notion of counting quantifier and present the definition of FO^2 extended with counting quantifiers (C^2). We also state the Scott normalization procedure for C^2. In Section 2.3, we introduce the definition of the guarded fragment, and present the Scott normal of GC^2. In Section 2.4, we briefly review some helpful results for the ILP problem. Finally, in Section 2.5, we clarify some notations used in this thesis.

2.1 Two-variable logic

Two-variable logic, denoted by FO^2, is the fragment of first-order logic where each formula uses at most two variables x and y. In this thesis, we define the syntax and semantics of FO^2 directly. We fix a vocabulary τ which consists of n unary predicates $U_1, U_2, \ldots, U_n$ and m binary predicates $\beta_1, \beta_2, \ldots, \beta_m$.

In this thesis, we only consider the relational first-order logic, that is, without function symbols.

Syntax of FO^2. The syntax of two-variable logic formulae over τ is defined inductively as follows.

- $x = y$ is an FO^2 formula.

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- For an unary predicate $U \in \tau$, for a variable $z \in \{x, y\}$, $U(z)$ is an FO^2 formula.

- For a binary predicate $\beta \in \tau$, for variables $z, w \in \{x, y\}$, $\beta(z, w)$ is an FO^2 formula.

- If φ is an FO^2 formula, then so is $\neg\varphi$.

- If φ_1 and φ_2 are FO^2 formulae, then so are $\varphi_1 \wedge \varphi_2$ and $\varphi_1 \vee \varphi_2$.

- For a variable $z \in \{x, y\}$, if φ is an FO^2 formulae, then so are $\forall z\ \varphi$ and $\exists z\ \varphi$.

For simplicity, let $x \neq y$ be the abbreviation of $\neg(x = y)$, $\varphi_1 \rightarrow \varphi_2$ be the abbreviation of $\neg\varphi_1 \vee \varphi_2$, and $\varphi_1 \leftrightarrow \varphi_2$ be the abbreviation of $(\varphi_1 \rightarrow \varphi_2) \wedge (\varphi_2 \rightarrow \varphi_1)$.

Let φ be an FO^2 formula, the free variable of φ, denoted by $\mathsf{free}(\varphi)$, is defined inductively as follows.

- $\mathsf{free}(x = y) := \{x, y\}$.

- For an unary predicate $U \in \tau$, for a variable $z \in \{x, y\}$, $\mathsf{free}(U(z)) := \{z\}$.

- For a binary predicate $\beta \in \tau$, for variables $z, w \in \{x, y\}$, $\mathsf{free}(\beta(z, w)) = \{z, w\}$.

- For an FO^2 formula φ, $\mathsf{free}(\neg\varphi) := \mathsf{free}(\varphi)$.

- For FO^2 formulae φ_1 and φ_2,

$$\mathsf{free}(\varphi_1 \wedge \varphi_2) = \mathsf{free}(\varphi_1 \vee \varphi_2) := \mathsf{free}(\varphi_1) \cup \mathsf{free}(\varphi_2).$$

- For a variable $z \in \{x, y\}$, for an FO^2 formula φ,

$$\mathsf{free}(\forall z\ \varphi) = \mathsf{free}(\exists z\ \varphi) := \mathsf{free}(\varphi) \setminus \{z\}.$$

Formulae without free variables are called *sentences*. We say that a formula is *quantifier-free* if there is no $\forall$ or $\exists$ in it. We say that a formula is *self-loop-free* if there is no $\beta(x, x)$ or $\beta(y, y)$ in it, where β is a binary predicate.

doi:10.6342/NTU202203098

Semantics of FO^2. A *structure* $\mathcal{A}$ over τ is a tuple $(A, U_1^{\mathcal{A}}, \ldots, U_n^{\mathcal{A}}, \beta_1^{\mathcal{A}}, \ldots, \beta_m^{\mathcal{A}})$ where each component is defined as follows.

- A is a set of elements called the *universe* of $\mathcal{A}$.
- For a unary predicate $U \in \tau$, $U^{\mathcal{A}}$ is a subset of A, that is $U^{\mathcal{A}} \subseteq A$.
- For a binary predicate $\beta \in \tau$, $\beta^{\mathcal{A}}$ is a subset of $A \times A$, that is, $\beta^{\mathcal{A}} \subseteq A \times A$.

A structure $\mathcal{A}$ over τ is *self-loop-free* if for every binary predicate $\beta \in \tau$, for every element $a \in A$, $(a, a) \notin \beta^{\mathcal{A}}$. That is, there is no diagonal term is $\beta^{\mathcal{A}}$.

Let $\mathcal{A}$ be a structure over τ. A *valuation* in $\mathcal{A}$ is a mapping from the set of variable $\{x, y\}$ to the universe A. Let val be a valuation in $\mathcal{A}$. For an element $a \in A$, for a variable $w \in \{x, y\}$, the val restricted w by a, denoted by $val[w \mapsto a]$, is defined as follows.

$$val[w \mapsto a](z) := \begin{cases} a & \text{if } z = w \\ val(z) & \text{otherwise} \end{cases}$$

A *model* over τ is a pair $(\mathcal{A}, val)$, where $\mathcal{A}$ is a structure over τ and val is a valuation in $\mathcal{A}$. The model $(\mathcal{A}, val)$ is *self-loop-free* if $\mathcal{A}$ is self-loop-free.

Given an FO^2 formula φ, and a model $(\mathcal{A}, val)$, we define $(\mathcal{A}, val)$ to be *a model of* φ, denoted by $(\mathcal{A}, val) \models \varphi$, inductively as follows.

- If $val(x) = val(y)$, then $(\mathcal{A}, val) \models x = y$.
- For an unary predicate $U \in \tau$, for a variable $z \in \{x, y\}$, if $val(z) \in U^{\mathcal{A}}$, then $(\mathcal{A}, val) \models U(z)$.
- For a binary predicate $\beta \in \tau$, for variables $z, w \in \{x, y\}$, if $(val(z), val(w)) \in \beta^{\mathcal{A}}$, then $(\mathcal{A}, val) \models \beta(z, w)$.
- For an FO^2 formula φ, if $(\mathcal{A}, val) \models \varphi$ is not true, then $(\mathcal{A}, val) \models \neg\varphi$.
- For FO^2 formulae φ_1 and φ_2,
  - if $(\mathcal{A}, val) \models \varphi_1$ and $(\mathcal{A}, val) \models \varphi_2$, then $(\mathcal{A}, val) \models \varphi_1 \wedge \varphi_2$;

doi:10.6342/NTU202203098

  - if $(\mathcal{A}, val) \models \varphi_1$ or $(\mathcal{A}, val) \models \varphi_2$, then $(\mathcal{A}, val) \models \varphi_1 \vee \varphi_2$.

- For a variable $z \in \{x, y\}$, for an FO^2 formula φ,
  - if *there exists* element $a \in A$ satisfying $(\mathcal{A}, val[z \mapsto a]) \models \varphi$, then $(\mathcal{A}, val) \models \exists z\ \varphi$;
  - if *for every* element $a \in A$ satisfying $(\mathcal{A}, val[z \mapsto a]) \models \varphi$, then $(\mathcal{A}, val) \models \forall z\ \varphi$.

Let $(\mathcal{A}, val) \not\models \varphi$ denote that $(\mathcal{A}, val) \models \varphi$ is not true. When φ is a sentence, that is, there is no free variable in φ, we omit the valuation and only write $\mathcal{A} \models \varphi$.

Remark 2.1. *Recall that $\varphi_1 \rightarrow \varphi_2$ is the abbreviation of $\neg\varphi_1 \vee \varphi_2$. $(\mathcal{A}, val) \models \varphi_1 \rightarrow \varphi_2$ holds if and only if when $(\mathcal{A}, val) \models \varphi_1$, then $(\mathcal{A}, val) \models \varphi_2$,*

An FO^2 sentence φ is *satisfiable* if φ has a model $\mathcal{A}$. It is *finitely satisfiable* if the number of elements in $\mathcal{A}$ is finite. Let $\mathsf{SAT}(\mathsf{FO}^2)$ and $\mathsf{FIN\text{-}SAT}(\mathsf{FO}^2)$ be the set of all satisfiable and finitely satisfiable FO^2 sentences, respectively.

2.2 Two-variable logic with counting quantifiers

The counting quantifier is an extension of the existential quantifier. Unlike the existential quantifier which state there exists some elements in the model satisfying the property, the counting quantifier $\exists^{=k}$ can express that "exactly" k elements exists in the model satisfying the property, where k is a natural number. Two-variable logic with counting quantifiers, denoted by C^2, is the extension of FO^2 with counting quantifiers.

Syntax of C^2. The syntax of C^2 formulae over τ is defined inductively. All rules are the same as the syntax of FO^2 with the following additional rules.

- For a natural number k, for a variable $z \in \{x, y\}$, if φ is a C^2 formula, then so are $\exists^{\geq k} z\ \varphi$, $\exists^{=k} z\ \varphi$, and $\exists^{\leq k} z\ \varphi$.

The definition of the free variable are the same as FO^2 with the following additional rules.

- For a natural number k, for a variable $z \in \{x, y\}$, for a C^2 formula φ,

$$\mathsf{free}(\exists^{\geq k} z\ \varphi) = \mathsf{free}(\exists^{=k} z\ \varphi) = \mathsf{free}(\exists^{\leq k} z\ \varphi) := \mathsf{free}(\varphi) \setminus \{z\}.$$

doi:10.6342/NTU202203098

Semantics of C^2. For a C^2 formula φ, for a model $(\mathcal{A}, val)$, $(\mathcal{A}, val) \models \varphi$ is defined the same as FO^2 with the following additional rules.

- For a natural number k, for a variable $z \in \{x, y\}$, for an C^2 formula φ,
  - if *at least* k elements $a \in A$ satisfy $(\mathcal{A}, val[z \mapsto a]) \models \varphi$, then $(\mathcal{A}, val) \models \exists^{\geq k} x\, \varphi$;
  - if *exactly* k elements $a \in A$ satisfy $(\mathcal{A}, val[z \mapsto a]) \models \varphi$, then $(\mathcal{A}, val) \models \exists^{= k} x\, \varphi$;
  - if *at most* k elements $a \in A$ satisfy $(\mathcal{A}, val[z \mapsto a]) \models \varphi$, then $(\mathcal{A}, val) \models \exists^{\leq k} x\, \varphi$.

Let $\mathsf{SAT}(\mathsf{C}^2)$ and $\mathsf{FIN\text{-}SAT}(\mathsf{C}^2)$ be the set of all satisfiable and finitely satisfiable C^2 sentences, respectively.

Equisatisfibility. Here are some equisatisfibility claims about C^2 that will be used in this thesis.

Claim 2.2. *For a variable $z \in \{x, y\}$, for a C^2 formula φ, the formulae $\exists z\, \varphi$ and $\exists^{\geq 1} z\, \varphi$ are equivalent.*

Proof. The formula $\exists z\, \varphi$ is satisfiable if and only if there is a model $(\mathcal{A}, val) \models \exists z\, \varphi$. By definition, there exists an element $a \in A$ satisfying $(\mathcal{A}, val[z \mapsto a]) \models \varphi$. Hence, there is at least one $a \in A$ satisfying $(\mathcal{A}, val[z \mapsto a]) \models \varphi$. Therefore, $(\mathcal{A}, val) \models \exists^{\geq 1} z\, \varphi$. The other direction is straightforward. □

Claim 2.3. *For a natural number k, for a variable $z \in \{x, y\}$, for a C^2 formula φ, the formulae $\exists^{=k} z\, \varphi$ and $\exists^{\geq k} z\, \varphi \wedge \exists^{\leq k} z\, \varphi$ are equivalent.*

Proof. If $\exists^{=k} z\, \varphi$ is satisfiable, then there is a model $(\mathcal{A}, val) \models \exists^{=k} z\, \varphi$. By definition, exactly k elements $a \in A$ satisfy $(\mathcal{A}, val[z \mapsto a]) \models \varphi$. Hence, at least k elements $a \in A$ satisfy $(\mathcal{A}, val[z \mapsto a]) \models \varphi$, which implies that $(\mathcal{A}, val) \models \exists^{\geq k} z\, \varphi$. Similarly, at most k elements $a \in A$ satisfy the property, which implies that $(\mathcal{A}, val) \models \exists^{\leq k} z\, \varphi$. Therefore, $(\mathcal{A}, val) \models \exists^{\geq k} z\, \varphi \wedge \exists^{\leq k} z\, \varphi$.

If $\exists^{\geq k} z\, \varphi \wedge \exists^{\leq k} z\, \varphi$ is satisfiable, by definition, there is a model $(\mathcal{A}, val)$ satisfying the following.

$$(\mathcal{A}, val) \models \exists^{\geq k} z\, \varphi$$

$$(\mathcal{A}, val) \models \exists^{\leq k} z\, \varphi$$

doi:10.6342/NTU202203098

Let $A' \subseteq A$ be the set of elements a satisfying $(\mathcal{A}, val[z \mapsto a]) \models \varphi$. By definition, $|A'| \geq k$ and $|A'| \leq k$. Hence, $|A'| = k$, and this implies that $(\mathcal{A}, val) \models \exists^{=k} z\, \varphi$. □

Claim 2.4. *For a natural number k, for a variable $z \in \{x, y\}$, for a C^2 formula φ, the formulae $\neg\exists^{\leq k} z\, \varphi$ and $\exists^{\geq k+1} z\, \varphi$ are equivalent.*

Proof. If $\neg\exists^{\leq k} z\, \varphi$ is satisfiable, by definition, there is a model $(\mathcal{A}, val) \not\models \exists^{\leq k} z\, \varphi$. By definition, more than k elements $a \in A$ satisfy $(\mathcal{A}, val[z \mapsto a]) \models \varphi$. Hence, at least $k + 1$ elements $a \in A$ satisfy the property, which this implies that $(\mathcal{A}, val) \models \exists^{\geq k+1} z\, \varphi$.

If $\exists^{\geq k+1} z\, \varphi$ is satisfiable, by definition, there is a model $(\mathcal{A}, val) \models \exists^{\geq k+1} z\, \varphi$. By definition, at least $k + 1$ elements $a \in A$ satisfy $(\mathcal{A}, val[z \mapsto a]) \models \varphi$. Hence, more than k elements $a \in A$ satisfy the property, and this implies that $(\mathcal{A}, val) \not\models \exists^{\leq k} z\, \varphi$. Hence, $(\mathcal{A}, val) \models \neg\exists^{\leq k} z\, \varphi$. □

Claim 2.5. *For a natural number k, for a variable $z \in \{x, y\}$, for a C^2 formula φ, the formulae $\neg\exists^{\geq k+1} z\, \varphi$ and $\exists^{\leq k} z\, \varphi$ are equivalent.*

Proof. Note that $\exists^{\leq k} z\, \varphi$ and $\neg\neg\exists^{\leq k} z\, \varphi$ are equivalent. By Claim 2.4, $\exists^{\geq k+1} z\, \varphi$ and $\neg\exists^{\leq k} z\, \varphi$ are equivalent, then so are $\neg\exists^{\geq k+1} z\, \varphi$ and $\neg\neg\exists^{\leq k} z\, \varphi$. Therefore, the formulae $\neg\exists^{\geq k+1} z\, \varphi$ and $\exists^{\leq k} z\, \varphi$ are equivalent. □

Claim 2.6. *For a natural number k, for a variable $z \in \{x, y\}$, for C^2 formulae φ_1 and φ_2, if z is not in $\mathsf{free}(\varphi_2)$, then the formulae $\left(\exists^{\geq k} z\, \varphi_1\right) \vee \varphi_2$ and $\exists^{\geq k} z\ (\varphi_1 \vee \varphi_2)$ are equisatisfiable over the models with size at least k.*

Proof. If $\left(\exists^{\geq k} z\, \varphi_1\right) \vee \varphi_2$ is satisfiable, by definition, there is a model $(\mathcal{A}, val) \models \left(\exists^{\geq k} z\, \varphi_1\right) \vee \varphi_2$. If $(\mathcal{A}, val) \models \exists^{\geq k} z\, \varphi_1$, then let S be the set of elements a satisfying $(\mathcal{A}, val[z \mapsto a]) \models \varphi_1$. By definition, $|S| \geq k$. For every $a \in S$, $(\mathcal{A}, val[z \mapsto a]) \models \varphi_1 \vee \varphi_2$. Therefore, $(\mathcal{A}, val) \models \exists^{\geq k} z\ (\varphi_1 \vee \varphi_2)$. Otherwise, if $(\mathcal{A}, val) \models \varphi_2$, because z is not in $\mathsf{free}(\varphi_2)$, for every $a \in A$, $(\mathcal{A}, val[z \mapsto a]) \models \varphi_2$. Therefore, $(\mathcal{A}, val[z \mapsto a]) \models \varphi_1 \vee \varphi_2$. Because $|A| \geq k$, this implies that $(\mathcal{A}, val) \models \exists^{\geq k} z\ (\varphi_1 \vee \varphi_2)$.

If $\exists^{\geq k} z\ (\varphi_1 \vee \varphi_2)$ is satisfiable, by definition, there is a model $(\mathcal{A}, val) \models \exists^{\geq k} z\ (\varphi_1 \vee \varphi_2)$. If $(\mathcal{A}, val) \models \varphi_2$, then by definition, $(\mathcal{A}, val) \models \left(\exists^{\geq k} z\, \varphi_1\right) \vee \varphi_2$. Otherwise, if $(\mathcal{A}, val) \not\models \varphi_2$,

doi:10.6342/NTU202203098

let S be the set of elements a satisfying $(\mathcal{A}, val[z \mapsto a]) \models \varphi_1 \vee \varphi_2$. By definition, $|S| \geq k$. For every element $a \in S$, because z is not in $\mathsf{free}(\varphi_2)$, $(\mathcal{A}, val[z \mapsto a]) \not\models \varphi_2$. Hence, $(\mathcal{A}, val[z \mapsto a]) \models \varphi_1$. Therefore, $(\mathcal{A}, val) \models \exists^{\geq k} z\ \varphi_1$. This implies that $(\mathcal{A}, val) \models \left(\exists^{\geq k} z\ \varphi_1\right) \vee \varphi_2$. □

Claim 2.7. *For a natural number k, for a variable $z \in \{x, y\}$, for C^2 formulae φ_1 and φ_2, if z is not in $\mathsf{free}(\varphi_2)$, then the formulae $\left(\exists^{\leq k} z\ \varphi_1\right) \vee \varphi_2$ and $\exists^{\leq k} z\ \ (\varphi_1 \wedge \neg\varphi_2)$ are equivalent.*

Proof. If $\left(\exists^{\leq k} z\ \varphi_1\right) \vee \varphi_2$ is satisfiable, by definition, there is a model $(\mathcal{A}, val) \models \left(\exists^{\leq k} z\ \varphi_1\right) \vee \varphi_2$. If $(\mathcal{A}, val) \models \left(\exists^{\leq k} z\ \varphi_1\right)$, then let S be the set of elements a satisfying $(\mathcal{A}, val[z \mapsto a]) \models \varphi_1$. By definition, $|S| \leq k$. Let S' be the set of elements a satisfying $(\mathcal{A}, val[z \mapsto a]) \models \varphi_1 \wedge \neg\varphi_2$. Because $S' \subseteq S$, then $|S'| \leq |S| \leq k$. Therefore, $(\mathcal{A}, val) \models \exists^{\leq k} z\ \ (\varphi_1 \wedge \neg\varphi_2)$. Otherwise, if $(\mathcal{A}, val) \models \varphi_2$, then for every element $a \in A$, because z is not in $\mathsf{free}(\varphi_2)$, $(\mathcal{A}, val[z \mapsto a]) \not\models \neg\varphi_2$. Therefore, $(\mathcal{A}, val[z \mapsto a]) \not\models \varphi_1 \wedge \neg\varphi_2$. This implies that $(\mathcal{A}, val) \models \exists^{\leq k} z\ \ (\varphi_1 \wedge \neg\varphi_2)$.

If $\exists^{\leq k} z\ \ (\varphi_1 \wedge \neg\varphi_2)$ is satisfiable, by definition, there is a model $(\mathcal{A}, val) \models \exists^{\leq k} z\ \ (\varphi_1 \wedge \neg\varphi_2)$. If $(\mathcal{A}, val) \models \varphi_2$, then by definition, $(\mathcal{A}, val) \models \left(\exists^{\leq k} z\ \varphi_1\right) \vee \varphi_2$. Otherwise, if $(\mathcal{A}, val) \not\models \varphi_2$, let S be the set of elements a satisfying $(\mathcal{A}, val[z \mapsto a]) \models \varphi_1 \wedge \neg\varphi_2$. By definition, $|S| \leq k$. Let S' be the set of elements a satisfying $(\mathcal{A}, val[z \mapsto a]) \models \varphi_1$. For every element $a \in A$, because z is not in $\mathsf{free}(\varphi_2)$, $(\mathcal{A}, val[z \mapsto a]) \models \neg\varphi_2$. Therefore, $S' \subseteq S$, then $|S'| \leq |S| \leq k$. Therefore, $(\mathcal{A}, val) \models \exists^{\leq k} z\ \varphi_1$. This implies that $(\mathcal{A}, val) \models \left(\exists^{\leq k} z\ \varphi_1\right) \vee \varphi_2$. □

Claim 2.8. *For a variable $z \in \{x, y\}$, for C^2 formulae φ_1 and φ_2, if z is not in $\mathsf{free}(\varphi_2)$, then the formulae $\varphi_1 \leftrightarrow (\forall z\ \varphi_2)$ and $\forall z\ \ (\varphi_1 \to \varphi_2) \wedge \exists^{\geq 1} z\ \ (\varphi_2 \to \varphi_1)$ are equivalent.*

Proof. Because the formulae $\varphi_1 \leftrightarrow (\forall z\ \varphi_2)$ and $\forall z\ \ (\varphi_1 \to \varphi_2) \wedge \exists z\ \ (\varphi_1 \to \varphi_2)$ are equivalent. By Claim 2.2, $\exists z\ \ (\varphi_1 \to \varphi_2)$ and $\exists^{\geq 1} z\ \ (\varphi_2 \to \varphi_1)$ are equivalent. This implies the result. □

Claim 2.9. *For a natural number k, for a variable $z \in \{x, y\}$, for C^2 formulae φ_1 and φ_2, if z is not in $\mathsf{free}(\varphi_2)$, then the formulae $\varphi_1 \leftrightarrow \left(\exists^{\geq k+1} z\ \varphi_2\right)$ and*

doi:10.6342/NTU202203098

$\exists^{\geq k+1} z\ (\neg\varphi_1 \vee \varphi_2) \wedge \exists^{\leq k} z\ (\neg\varphi_1 \wedge \varphi_2)$ *are equisatisfiable over the models with size at least* k.

Proof. It is sufficient to show the following.

- Recall that $\varphi_1 \rightarrow \left(\exists^{\geq k+1} z\, \varphi_2\right)$ is the abbreviation of $\neg\varphi_1 \vee \left(\exists^{\geq k+1} z\, \varphi_2\right)$. By Claim 2.6, the formulae $\neg\varphi_1 \vee \left(\exists^{\geq k+1} z\, \varphi_2\right)$ and $\exists^{\geq k+1} z\ (\neg\varphi_1 \vee \varphi_2)$ are equisatisfiable over the models with size at least k.

- Recall that $\left(\exists^{\geq k+1} z\, \varphi_2\right) \rightarrow \varphi_1$ is the abbreviation of $\neg\left(\exists^{\geq k+1} z\, \varphi_2\right) \vee \varphi_1$. By Claim 2.5, the formulae $\neg\left(\exists^{\geq k+1} z\, \varphi_2\right)$ and $\exists^{\leq k} z\, \varphi_2$ are equivalent. By Claim 2.7, the formulae $\left(\exists^{\leq k} z\, \varphi_2\right) \vee \varphi_1$ and $\exists^{\leq k} z\ (\neg\varphi_1 \wedge \varphi_2)$ are equivalent.

□

Claim 2.10. *For a natural number* k*, for a variable* $z \in \{x, y\}$*, for* C^2 *formulae* φ_1 *and* φ_2*, if* z *is not in* $\mathsf{free}(\varphi_2)$*, then the formulae* $\varphi_1 \leftrightarrow \left(\exists^{\leq k} z\, \varphi_2\right)$ *and* $\exists^{\leq k} z\ (\varphi_1 \wedge \varphi_2) \wedge \exists^{\geq k+1} z\ (\varphi_1 \vee \varphi_2)$ *are equisatisfiable over the models with size at least* k.

Proof. It is sufficient to show the following.

- Recall that $\varphi_1 \rightarrow \left(\exists^{\leq k} z\, \varphi_2\right)$ is the abbreviation of $\neg\varphi_1 \vee \left(\exists^{\leq k} z\, \varphi_2\right)$. By Claim 2.7, it is the equivalence of $\exists^{\leq k} z\ (\varphi_1 \wedge \varphi_2)$.

- Recall that $\left(\exists^{\leq k} z\, \varphi_2\right) \rightarrow \varphi_1$ is the abbreviation of $\neg\left(\exists^{\leq k} z\, \varphi_2\right) \vee \varphi_1$. By Claim 2.4, the formulae $\neg\left(\exists^{\leq k} z\, \varphi_2\right)$ and $\exists^{\geq k+1} z\, \varphi_2$ are equivalent. By Claim 2.6, the formulae $\left(\exists^{\geq k+1} z\, \varphi_2\right) \vee \varphi_1$ and $\exists^{\geq k+1} z\ (\varphi_1 \vee \varphi_2)$ are equisatisfiable over the models with size at least k.

□

Claim 2.11. *For a natural number* k*, for a variable* $z \in \{x, y\}$*, for a* C^2 *formula* φ *over the vocabulary* τ*, the formulae* $\exists^{\geq k} z\, \varphi$ *and* $\left(\exists^{=k} z\, U(z)\right) \wedge (\forall z\, U(z) \rightarrow \varphi)$ *are equisatisfiable, where* U *is a fresh unary predicate not in* τ*. Moreover,* $\exists^{\geq k} z\, \varphi$ *is satisfiable by a size* m *model if and only if* $\left(\exists^{=k} z\, U(z)\right) \wedge (\forall z\, U(z) \rightarrow \varphi)$ *is satisfiable by a size* m *model.*

doi:10.6342/NTU202203098

Proof. First, we prove that if $\exists^{\geq k} z\, \varphi$ is satisfiable, then $\big(\exists^{=k} z\, U(z)\big) \wedge (\forall z\, U(z) \rightarrow \varphi)$ is satisfiable. Let $(\mathcal{A}, val)$ be a model of $\exists^{\geq k} z\, \varphi$. Let $S \subseteq A$ be the set of elements s satisfying that $(\mathcal{A}, val[z \mapsto s]) \models \varphi$. By definition, $|S| \geq k$. We define another structure $\mathcal{B}$ over $\tau \cup \{U\}$ as follows. The universe of $\mathcal{B}$ is the same as the universe of $\mathcal{A}$, that is, $B := A$. For every predicate $P \in \tau$, $P^{\mathcal{B}} := P^{\mathcal{A}}$. Finally, $U^{\mathcal{B}}$ is a subset of S satisfying $\left|U^{\mathcal{B}}\right| = k$. Now we claim that $(\mathcal{B}, val) \models \big(\exists^{=k} z\, U(z)\big) \wedge (\forall z\, U(z) \rightarrow \varphi)$. It is sufficient to show the following.

- By definition, $(\mathcal{B}, val[z \mapsto b]) \models U(z)$ if and only if $b \in U^{\mathcal{B}}$. Since $\left|U^{\mathcal{B}}\right| = k$, $(\mathcal{B}, val) \models \exists^{=k} z\, U(z)$.

- For every $b \in B$, if $(\mathcal{B}, val[z \mapsto b]) \models U(z)$, then $b \in U^{\mathcal{B}} \subseteq S$. By definition of S, $(\mathcal{B}, val[z \mapsto b]) \models \varphi$. Hence, $(\mathcal{B}, val[z \mapsto b]) \models U(z) \rightarrow \varphi$. This implies that $(\mathcal{B}, val) \models \forall z\, U(z) \rightarrow \varphi$.

Suppose $\big(\exists^{=k} z\, U(z)\big) \wedge (\forall z\, U(z) \rightarrow \varphi)$ is satisfiable. Let $(\mathcal{A}, val)$ be one of its models. By definition, the following holds.

- $(\mathcal{A}, val) \models \exists^{=k} z\, U(z)$.

- $(\mathcal{A}, val) \models \forall z\, U(z) \rightarrow \varphi$.

The first one implies that $\left|U^{\mathcal{A}}\right| = k$. The second one implies that for every $a \in A$, if $(\mathcal{A}, val[z \mapsto a]) \models U(z)$, then $(\mathcal{A}, val[z \mapsto a]) \models \varphi$. For every $a \in U^{\mathcal{A}}$, since $(\mathcal{A}, val[z \mapsto a]) \models U(z)$, then $(\mathcal{A}, val[z \mapsto a]) \models \varphi$. We define another structure $\mathcal{B}$ over τ as follows. The universe of $\mathcal{B}$ is the same as the universe of $\mathcal{A}$, that is, $B := A$. For every predicate $P \in \tau$, $P^{\mathcal{B}} := P^{\mathcal{A}}$. Now we claim that $(\mathcal{B}, val)$ is a model of $\exists^{\geq k} z\, \varphi$. For every $b \in U^{\mathcal{A}} \subseteq A = B$, $(\mathcal{B}, val[z \mapsto b]) \models \varphi$. Besides, $\left|U^{\mathcal{A}}\right| = k$. Hence, at least k elements $b \in B$ satisfy $(\mathcal{B}, val[z \mapsto b]) \models \varphi$. This implies that $(\mathcal{B}, val) \models \exists^{\geq k} \varphi$. □

Claim 2.12. *For a natural number k, for a variable $z \in \{x, y\}$, for a C^2 formula φ over τ, the formulae $\exists^{\leq k} z\, \varphi$ and $\big(\exists^{=k} z\, U(z)\big) \wedge (\forall z\, \varphi \rightarrow U(z))$ are equisatisfiable over the models with size at least k, where U is a fresh unary predicate not in τ. Moreover, $\exists^{\leq k} z\, \varphi$ is satisfiable by a size $m \geq k$ model if and only if $\big(\exists^{=k} z\, U(z)\big) \wedge (\forall z\, \varphi \rightarrow U(z))$ is satisfiable by a size m model.*

doi:10.6342/NTU202203098

Proof. First, we prove that if $\exists^{\leq k} z\,\varphi$ is satisfiable over the models with size at least k, then $\left(\exists^{=k} z\, U(z)\right) \wedge (\forall z\, \varphi \to U(z))$ is satisfiable. Let $(\mathcal{A}, val)$ be a model of $\exists^{\leq k} z\,\varphi$ satisfying $|A| \geq k$. Let $S \subseteq A$ be the set of elements s satisfying that $(\mathcal{A}, val[z \mapsto s]) \models \varphi$. By definition, $|S| \leq k$. We define another structure $\mathcal{B}$ over $\tau \cup \{U\}$ as follows. The universe of $\mathcal{B}$ is the same as the universe of $\mathcal{A}$, that is, $B := A$. For every predicate $P \in \tau$, $P^{\mathcal{B}} := P^{\mathcal{A}}$. Finally, $U^{\mathcal{B}}$ is a subset of A satisfying $S \subseteq U^{\mathcal{B}} \subseteq A$ and $\left|U^{\mathcal{B}}\right| = k$. Because $|S| \leq k \leq |A|$, we can always find such $U^{\mathcal{B}}$. Now we claim that $(\mathcal{B}, val)$ is a model of $\left(\exists^{=k} z\, U(z)\right) \wedge (\forall z\, \varphi \to U(z))$. It is sufficient to show the following.

- By definition, $(\mathcal{B}, val[z \mapsto b]) \models U(z)$ if and only if $b \in U^{\mathcal{B}}$. Since $\left|U^{\mathcal{B}}\right| = k$, $(\mathcal{B}, val) \models \exists^{=k} z\, U(z)$.

- For every $b \in B$, if $(\mathcal{B}, val[z \mapsto b]) \models \varphi$, then $b \in S \subseteq U^{\mathcal{B}}$. By definition of $U^{\mathcal{B}}$, $(\mathcal{B}, val[z \mapsto b]) \models U(z)$. Hence, $(\mathcal{B}, val[z \mapsto b]) \models \varphi \to U(z)$. This implies that $(\mathcal{B}, val) \models \forall z\, \varphi \to U(z)$.

Suppose $\left(\exists^{=k} z\, U(z)\right) \wedge (\forall z\, \varphi \to U(z))$ is satisfiable. Let $(\mathcal{A}, val)$ be one of its models. By definition, the following holds.

- $(\mathcal{A}, val) \models \exists^{=k} z\, U(z)$.

- $(\mathcal{A}, val) \models \forall z\, \varphi \to U(z)$.

The first one implies that $\left|U^{\mathcal{A}}\right| = k$. The second one implies that for every $a \in A$, if $(\mathcal{A}, val[z \mapsto a]) \models \varphi$, then $(\mathcal{A}, val[z \mapsto a]) \models U(z)$. For every $a \notin U^{\mathcal{A}}$, since $(\mathcal{A}, val[z \mapsto a]) \not\models U(z)$, then $(\mathcal{A}, val[z \mapsto a]) \not\models \varphi$. We define another structure $\mathcal{B}$ over τ as follows. The universe of $\mathcal{B}$ is the same as the universe of $\mathcal{A}$, that is, $B := A$. For every predicate $P \in \tau$, $P^{\mathcal{B}} := P^{\mathcal{A}}$. Now we claim that $(\mathcal{B}, val)$ is a model of $\exists^{\leq k} z\,\varphi$. For every $b \notin U^{\mathcal{A}} \subseteq A = B$, $(\mathcal{B}, val[z \mapsto b]) \not\models \varphi$. Besides, $\left|U^{\mathcal{A}}\right| = k$. Hence, at most k elements $b \in B$ satisfy $(\mathcal{B}, val[z \mapsto b]) \models \varphi$. This implies that $(\mathcal{B}, val) \models \exists^{\leq k}\varphi$. $\square$

doi:10.6342/NTU202203098

Claim 2.13. *For a natural number k, for a C^2 formula $\varphi(x, y)$ over τ, let*

$$
\begin{aligned}
\varphi_1 :=&\quad \forall x\, \exists^{=k+1} y\ \varphi(x, y), \\
\varphi_2 :=&\quad \forall x\, \forall y\ x = y \vee (\varphi(x, x) \rightarrow (\varphi(x, y) \leftrightarrow \beta_1(x, y))) \vee (\neg\varphi(x, x) \rightarrow (\varphi(x, y) \leftrightarrow \beta_2(x, y))) \\
&\wedge\quad \forall x\, \exists^{=k} y\ \beta_1(x, y) \wedge x \neq y \\
&\wedge\quad \forall x\, \exists^{=k+1} y\ \beta_2(x, y) \wedge x \neq y.
\end{aligned}
$$

The sentences φ_1 and φ_2 are equisatisfiable over the models with size at least $k+2$, where β_1 and β_2 are fresh binary predicates not in τ. Moreover, the sentence φ_1 is satisfiable by a size $m \geq k+2$ model if and only if the sentence φ_2 is satisfiable by a size m model.

Proof. First, we prove that if the sentence φ_1 is satisfiable over the models with size at least $k+2$, then the sentence φ_2 is satisfiable. Let $\mathcal{A}$ be a model of the sentence φ_1 satisfying $|A| \geq k+2$. We define another structure $\mathcal{B}$ over $\tau \cup \{\beta_1, \beta_2\}$ as follows. The universe of $\mathcal{B}$ is the same as the universe of $\mathcal{A}$, that is, $B := A$. For every predicate $P \in \tau$, $P^{\mathcal{B}} := P^{\mathcal{A}}$. Let val be an (arbitrary) valuation in $\mathcal{A}$. For every $a \in A$, we define the sets S_a^1 and S_a^2 by the following rules.

Case 1: $(\mathcal{A}, val[x \mapsto a][y \mapsto a]) \models \varphi_1(x, y)$**.** The set S_a^1 is defined as $\{a' \in A \setminus \{a\} \mid (\mathcal{A}, val[x \mapsto a][y \mapsto a']) \models \varphi_1(x, y)\}$. Because $\mathcal{A}$ is a model of φ_1, there are $k+1$ elements a' satisfy $(\mathcal{A}, val[x \mapsto a][y \mapsto a']) \models \varphi_1(x, y)$. Since $a \notin S_a^1$, the size of S_a^1 is k. Let S_a^2 be a size $k+1$ subset of $A \setminus \{a\}$. Because $|A \setminus \{a\}| \geq k+1$, we can always find such subset.

Case 2: $(\mathcal{A}, val[x \mapsto a][y \mapsto a]) \not\models \varphi_1(x, y)$**.** Let S_a^1 be a size k subset of $A \setminus \{a\}$. Because $|A \setminus \{a\}| \geq k+1$, we can always find such subsets. The set S_a^2 is defined as $\{a' \in A \setminus \{a\} \mid (\mathcal{A}, val[x \mapsto a][y \mapsto a']) \models \varphi_1(x, y)\}$. By the similar argument, the size of S_a^2 is $k+1$.

Finally, $\beta_1^{\mathcal{B}} := \bigcup_{a \in \mathcal{A}} \{(a, a') \mid a' \in S_a^1\}$ and $\beta_2^{\mathcal{B}} := \bigcup_{a \in \mathcal{A}} \{(a, a') \mid a' \in S_a^2\}$. Now we claim that $\mathcal{B}$ is a model of the sentence φ_2. It is sufficient to show the following.

- For every $b \in B$, for every $b' \in B \setminus \{b\}$, note that $(B, val[x \mapsto b][y \mapsto b']) \models \beta_1(x, y)$ if and only if $(A, val[x \mapsto b][y \mapsto b']) \models \beta_1(x, y)$. Therefore, by the construction,

doi:10.6342/NTU202203098

it is not difficult to verify that

$$B \models \forall x \forall y \, x = y \vee (\varphi(x,x) \to (\varphi(x,y) \leftrightarrow \beta_1(x,y))) \vee (\neg\varphi(x,x) \to (\varphi(x,y) \leftrightarrow \beta_2(x,y))).$$

- For every $b \in B$, for every $b' \in B \setminus \{b\}$, $(B, val[x \mapsto b][y \mapsto b']) \models \beta_1(x,y)$ if and only if $b' \in S_b^1$. Recall that the size o S_b^1 is k. Hence, $B \models \forall x \, \forall y^{=k} \, \beta_1(x,y) \wedge x \neq y$.
- For every $b \in B$, for every $b' \in B \setminus \{b\}$, $(B, val[x \mapsto b][y \mapsto b']) \models \beta_2(x,y)$ if and only if $b' \in S_b^2$. Recall that the size o S_b^2 is $k+1$. Hence, $B \models \forall x \, \forall y^{=k+1} \, \beta_2(x,y) \wedge x \neq y$.

Then, we prove that if the sentence φ_2 is satisfiable over the models with size at least $k+2$, then the sentence φ_1 is satisfiable. Let $\mathcal{A}$ be a model of the sentence φ_2 satisfying $|A| \geq k+2$. We define another structure $\mathcal{B}$ over τ as follows. The universe of $\mathcal{B}$ is the same as the universe of $\mathcal{A}$, that is, $B := A$. For every predicate $P \in \tau$, $P^{\mathcal{B}} := P^{\mathcal{A}}$. Now we claim that $\mathcal{B}$ is a model of the sentence φ_1. Let val be an (arbitrary) valuation in $\mathcal{B}$. For every $b \in \mathcal{B}$, consider the following two cases.

Case 1: $(\mathcal{B}, val[x \mapsto b][y \mapsto b]) \models \varphi_1(x,y)$**.** For every $b' \in B \setminus \{b\}$, if $(\mathcal{B}, val[x \mapsto b][y \mapsto b']) \models \varphi_1(x,y)$, because $(\mathcal{A}, val[x \mapsto b][y \mapsto b']) \models \varphi(x,x) \to (\varphi(x,y) \leftrightarrow \beta_1(x,y))$, then $(b,b') \in \beta_1^{\mathcal{A}}$. Since $\mathcal{A}$ is also a model of $\forall x \, \exists^{=k} y \, \beta_1(x,y) \wedge x \neq y$, there are exactly k elements b' satisfying the condition. Therefore, together with b itself, $\mathcal{B} \models \forall x \, \exists^{=k+1} y \, \varphi_1(x,y)$.

Case 2: $(\mathcal{B}, val[x \mapsto b][y \mapsto b]) \not\models \varphi_1(x,y)$**.** For every $b' \in B \setminus \{b\}$, if $(\mathcal{B}, val[x \mapsto b][y \mapsto b']) \models \varphi_1(x,y)$, because $(\mathcal{A}, val[x \mapsto b][y \mapsto b']) \models \neg\varphi(x,x) \to (\varphi(x,y) \leftrightarrow \beta_2(x,y))$, then $(b,b') \in \beta_2^{\mathcal{A}}$. Since $\mathcal{A}$ is also a model of $\forall x \, \exists^{=k+1} y \, \beta_2(x,y) \wedge x \neq y$, there are exactly $k+1$ elements b' satisfying the condition. Therefore, $\mathcal{B} \models \forall x \, \exists^{=k+1} y \, \varphi_1(x,y)$. □

Normal form. It is well-known that there is a linear-time transformation from C^2 sentence φ over τ to the following Scott normal form φ' over extended vocabulary τ' [27, 13, 14, 25].

$$\begin{aligned} \varphi' := \quad & \forall x \, \gamma(x) \\ \wedge \quad & \forall x \, \forall y \, x \neq y \to \alpha(x,y) \\ \wedge \quad & \bigwedge_{i \in [m']} \forall x \, \exists^{=k_i} y \, \beta_i(x,y) \; \wedge \; x \neq y, \end{aligned} \tag{2.1}$$

doi:10.6342/NTU202203098

where $\gamma(x)$ and $\alpha(x, y)$ are quantifier-free formulae, β_i are binary predicates, and $m' \leq m$. The sentence φ is satisfiable over the models with size at least $K + 1$ if and only if φ' is satisfiable over the models with size at least $K + 1$, where K is the summation of counting condition in φ. Moreover, the transformation is model size preserving. That is, φ is satisfiable by a model with size $\ell \geq K + 1$, if and only if φ' is satisfiable by a model with size ℓ.

In general, $m \neq m'$. For the sake of simplicity, we only consider the case $m = m'$ in this thesis. It is not difficult to extended the result in the thesis to the general case.

We present the transformation here for completeness.

Step 1: By Claim 2.2 and 2.3, we can first remove the quantifiers $\exists$ and $\exists^{=k}$ in φ and obtain the equivalent C^2 sentence φ_1.

Step 2: We define the C^2 sentence $\varphi_2 := \mathcal{NF}(\varphi_1)$ by the following Scott procedure $\mathcal{NF}$ inductively. The idea is to represent each subsentence in φ_1 by a fresh predicate. It is quite standard to show that φ_2 and φ_1 are equisatisfiable. Moreover, the transformation is model size preserving. That is, φ_2 has a size m model if and only if φ_1 has a size m model.

- If ψ has two free variables, then let P_ψ be a fresh binary predicate.
  $\mathcal{NF}(\psi) := \forall x \, \forall y \ (P_\psi(x, y) \leftrightarrow \xi_1(\psi)) \wedge \xi_2(\psi)$
- If ψ has one free variable, then let P_ψ be a fresh unary predicate.
  $\mathcal{NF}(\psi) := \forall x \ (P_\psi(x) \leftrightarrow \xi_1(\psi)) \wedge \xi_2(\psi)$
- If ψ has no free variable, then let P_ψ be a fresh unary predicate.
  $\mathcal{NF}(\psi) := \forall x \ (P_\psi(x) \leftrightarrow \xi_1(\psi)) \wedge \xi_2(\psi) \wedge \forall x \, \forall y \ (P_\psi(x) \leftrightarrow P_\psi(y))$

In the third case, we simulate the arity zero predicate with the unary predicate. ξ_1 and ξ_2 are defined as follows.

- If ψ is quantifier-free, then $\xi_2(\psi) := \top$.
  - If ψ is in the form $\psi(z, w)$, then $\xi_1(\psi) := \psi(x, y)$.
  - If ψ is in the form $\psi(z)$, then $\xi_1(\psi) := \psi(x)$.

doi:10.6342/NTU202203098

- If ψ is in the form $\neg\psi_1$, then $\xi_2(\psi) := \mathcal{NF}(\psi_1)$.
  - If ψ_1 is in the form $\psi_1(z, w)$, then $\xi_1(\psi) := \neg P_{\psi_1}(x, y)$.
  - If ψ_1 is in the form $\psi_1(z)$, then $\xi_1(\psi) := \neg P_{\psi_1}(x)$.
  - If ψ_1 is in the form ψ_1, then $\xi_1(\psi) := \neg P_{\psi_1}(x)$.
- If ψ is in the form $\psi_1 \circledast \psi_2$, where $\circledast \in \{\wedge, \vee\}$, then $\xi_2(\psi) := \mathcal{NF}(\psi_1) \wedge \mathcal{NF}(\psi_2)$.
  - If ψ is in the form $\psi(z, w) = \psi_1(z, w) \circledast \psi_2(z, w)$, then $\xi_1(\psi) := P_{\psi_1}(x, y) \circledast P_{\psi_2}(x, y)$.
  - If ψ is in the form $\psi(z, w) = \psi_1(z, w) \circledast \psi_2(w)$, then $\xi_1(\psi) := P_{\psi_1}(x, y) \circledast P_{\psi_2}(y)$.
  - If ψ is in the form $\psi(z, w) = \psi_1(z, w) \circledast \psi_2(z)$, then $\xi_1(\psi) := P_{\psi_1}(x, y) \circledast P_{\psi_2}(x)$.
  - If ψ is in the form $\psi(z, w) = \psi_1(z, w) \circledast \psi_2$, then $\xi_1(\psi) := P_{\psi_1}(x, y) \circledast P_{\psi_2}(x)$.
  - If ψ is in the form $\psi(z, w) = \psi_1(z) \circledast \psi_2(z, w)$, then $\xi_1(\psi) := P_{\psi_1}(x) \circledast P_{\psi_2}(x, y)$.
  - If ψ is in the form $\psi(z, w) = \psi_1(z) \circledast \psi_2(w)$, then $\xi_1(\psi) := P_{\psi_1}(x) \circledast P_{\psi_2}(y)$.
  - If ψ is in the form $\psi(z, w) = \psi_1 \circledast \psi_2(z, w)$, then $\xi_1(\psi) := P_{\psi_1}(x) \circledast P_{\psi_2}(x, y)$.
  - If ψ is in the form $\psi(z) = \psi_1(z) \circledast \psi_2(z)$, then $\xi_1(\psi) := P_{\psi_1}(x) \circledast P_{\psi_2}(x)$.
  - If ψ is in the form $\psi(z) = \psi_1(z) \circledast \psi_2$, then $\xi_1(\psi) := P_{\psi_1}(x) \circledast P_{\psi_2}(x)$.
  - If ψ is in the form $\psi(z) = \psi_1 \circledast \psi_2(z)$, then $\xi_1(\psi) := P_{\psi_1}(x) \circledast P_{\psi_2}(x)$.
  - If ψ is in the form $\psi = \psi_1 \circledast \psi_2$, then $\xi_1(\psi) := P_{\psi_1}(x) \circledast P_{\psi_2}(x)$.
- If ψ is in the form $\psi_1 \vee \psi_2$, then $\xi_1(\psi) := P_{\psi_1}(v) \vee P_{\psi_2}(v)$ and $\xi_2(\psi) := \mathcal{NF}(\psi_1) \wedge \mathcal{NF}(\psi_2)$.
- If ψ is in the form $Qz\ \psi_1$, where Q is a quantifier, then $\xi_2(\psi) := \mathcal{NF}(\psi_1)$.
  - If ψ_1 is in the form $\psi_1(z, w)$, then $\xi_1(\psi) := Qy\ P_{\psi_1}(y, x)$.
  - If ψ_1 is in the form $\psi_1(w)$, then $\xi_1(\psi) := Qy\ P_{\psi_1}(x)$.
  - If ψ_1 is in the form $\psi_1(z)$, then $\xi_1(\psi) := Qy\ P_{\psi_1}(y)$.
  - If ψ_1 is in the form ψ_1, then $\xi_1(\psi) := Qy\ P_{\psi_1}(y)$.

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Step 3: By Claim 2.8, 2.9,and 2.10, we can rewrite φ_2 and obtain φ_3 satisfying the following.

- φ_3 is in the form $\forall\forall \wedge (\forall\exists^{\geq})^* \wedge (\forall\exists^{\leq})^*$.
- φ_2 and φ_3 are equisatisfiable over the models with size at least K.

Step 4: By Claim 2.11 and 2.12, we can rewrite φ_3 and obtain φ_4 satisfying the following.

- φ_4 is in the form $\forall\forall \wedge (\forall\exists^{=})^*$.
- φ_3 and φ_4 are equisatisfiable over the models with size at least K.

Step 5: Finally, observe that $\forall x \forall y\ \alpha(x, y)$ and $\forall x\ \alpha(x, x) \wedge \forall x \forall y\ x \neq y \rightarrow \alpha(x, y)$ are equivalent. Hence, by the observation and Claim 2.13, we obtain desired φ' from φ_4.

It is not hard to show the equisatisfibility of φ and φ'. We conclude this transformation with the following lemma.

Lemma 2.14. *The C^2 sentence φ is satisfiable by a model with size $\ell \geq K+1$, if and only if φ' is satisfiable by a model with size ℓ.*

For the sake of simplicity, we further transform the sentence into the self-loop-free normal form. That is, the formulae $\gamma(x)$ and $\alpha(x, y)$ in the equation (2.1) are self-loop-free.

Lemma 2.15. *The C^2 sentence φ is satisfiable by a model with size $\ell \geq K+1$, if and only if φ'' is satisfiable by a self-loop-free model with size ℓ, where φ'' is in self-loop-free normal form.*

Proof. Let φ' be a C^2 sentence in the normal form. We obtain the self-loop-free normal by the following transformation. The idea of the transformation is to represent the self-loop term $\beta(x, x)$ and $\beta(y, y)$ by a fresh unary predicate, where β is a binary predicate in vocabulary of φ'.

doi:10.6342/NTU202203098

For every $i \in [m]$, let P_i be a fresh unary predicate. We define the formulae $\gamma'(x)$ and $\alpha'(x, y)$ as follows.

$$\gamma'(x) := \gamma(x)[P_t(x)/\beta_i(x,x)]_{t=1}^{m}$$
$$\alpha'(x,y) := \alpha(x,y)[P_t(x)/\beta_i(x,x)]_{t=1}^{m}[P_t(y)/\beta_i(y,y)]_{t=1}^{m}$$

Then φ'' is the desired self-loop-free normal form sentence.

$$\begin{aligned} \varphi'' := \quad & \forall x\, \gamma'(x) \\ \wedge \quad & \forall x\, \forall y\; x \neq y \to \alpha'(x,y) \\ \wedge \quad & \bigwedge_{i \in [m]} \forall x\; \exists^{=k_i} y\; \beta_i(x,y) \;\wedge\; x \neq y, \end{aligned}$$

The idea of constructing the self-loop-free model is represent the diagonal term of the binary predicate by the fresh unary predicate. For every model $\mathcal{A} \models \varphi'$, we define the model $\mathcal{B}$ as follows. Let τ be the vocabulary of φ'. For every unary predicate U in the vocabulary of φ', $U^{\mathcal{B}} := U^{\mathcal{A}}$. For every binary predicate β in the vocabulary of φ', $\beta^{\mathcal{B}} := \beta^{\mathcal{A}} \setminus \{(a,a) \mid a \in A\}$. Finally, for every fresh unary predicate P_i, $P_i^{\mathcal{B}} := \left\{a \;\middle|\; (a,a) \in \beta_i^{\mathcal{A}}\right\}$. It is not hard to verify that $\mathcal{B} \models \varphi''$. □

In the rest of the thesis, we focus on the C^2 sentence in the (self-loop-free) normal form.

2.3 Guarded fragment

The guarded fragment of FO^2, denoted by GF^2, is a syntactic subclass of FO^2 by restricting the usage of quantifier with the guarded atoms. The guarded fragment of FO^2 extended with counting quantifier, denoted by GC^2, is the guarded fragment of C^2. In this section, we present the syntax of GC^2 directly.

Syntax of GC^2. The syntax of GC^2 formulae over τ is defined inductively. All rules are the same as the syntax of C^2 except that the rules for quantifier are as follows.

doi:10.6342/NTU202203098

- For a variable $z, w \in \{x, y\}$, for a binary predicate r, if φ is an GC^2 formulae, then so are $\forall w\ g(z,w) \to \varphi$ and $\exists w\ g(z,w) \wedge \varphi$, where r is called the *guarded atom*.

- For a natural number k, for variables $z, w \in \{x, y\}$, for a binary predicate r, if φ is a GC^2 formula, then so are are $\exists^{\geq k} w\ g(z,w) \wedge \varphi$, $\exists^{=k} w\ g(z,w) \wedge \varphi$, and $\exists^{\leq k} w\ g(z,w) \wedge \varphi$, where r is called the *guarded atom*.

Normal form. Similar to C^2, there is a linear-time transform from GC^2 sentence φ over τ to the following normal form φ' over extended vocabulary τ' [27, 18, 26].

$$\begin{aligned} \varphi' := \quad & \forall x\ \gamma(x) \\ \wedge \quad & \forall x\, \forall y\ x \neq y \to \bigwedge_{i \in [\ell]} (r_i(x,y) \to \alpha_i(x,y)) \\ \wedge \quad & \bigwedge_{i \in [m]} \forall x\ \exists^{=k_i} y\ \beta_i(x,y)\ \wedge\ x \neq y, \end{aligned}$$

where $\gamma(x)$ and $\alpha_i(x,y)$ are quantifier-free (and self-loop-free) formulae.

2.4 Integer linear programming technique

In this Section, we review some helpful ILP results. In this thesis, we consider the ILP system which has the solution in the extended natural number $\mathbb{N}_\infty$. We define the arithmetic of the infinity as follows. For every $n \in \mathbb{N} \setminus \{0\}$, $n + \infty = \infty$ and $n \cdot \infty = \infty$. Otherwise, $0 + \infty = \infty$ and $0 \cdot \infty = 0$.

Lemma 2.16. *Let $\mathcal{Q}$ be an ILP system consists of n variables $\{x_i\}_{i \in [n]}$ and m constraints in the following form,*

$$\mathcal{Q} := \bigwedge_{i \in [m]} \left(\sum_{j \in [n]} c_{i,j} x_j + b_i \circledast \sum_{j \in [n]} c'_{i,j} x_j + b'_i \right),$$

where $\circledast \in \{\geq, =\}$, and $c_{i,j}, c_{i,j}, b_i, b'_i \in \mathbb{N}$. Let $\bar{c}$ be the maximum of $c_{i,j}, c_{i,j}, b_i, b'_i$.

1. *If the system $\mathcal{Q}$ has a solution in $\mathbb{N}$, then it has a solution in $\mathbb{N}$ with at most $2(m+1)\left(\log(m+1) + \lceil \log \bar{c} \rceil + 2\right)$ non-zero elements.*

doi:10.6342/NTU202203098

2. *If the system $\mathcal{Q}$ has a solution in $\mathbb{N}_\infty$, then it has a solution in $\mathbb{N}_\infty$ with at most $2(m+1)(\log(m+1)+\lceil\log\bar{c}\rceil+2)$ non-zero elements.*

Proof.

1. We can introduce m fresh variables $\{z_i\}_{i\in[m]}$ and rewrite the system $\mathcal{Q}$ into the equivalent system $\mathcal{Q}'$.

$$\mathcal{Q}' := \bigwedge_{i\in[m]} \left(\sum_{j\in[n]} (c'_{i,j} - c_{i,j})x_j + d_i z_i = (b_i - b'_i) \right)$$

For every $i \in [n]$, if the i-th operator is $=$, then $d_i = 0$. Otherwise, $d_i = 1$. Note that $\left|c'_{i,j} - c_{i,j}\right| \leq \bar{c}$ and $|b_i - b'_i| \leq \bar{c}$. By Corollary 5 in [8], $\mathcal{Q}'$ has a solution in $\mathbb{N}$ if and only if it has a solution in $\mathbb{N}$ with at most $2(m+1)(\log(m+1)+\lceil\log\bar{c}\rceil+2)$ non-zero elements. It is not hard to check that this solution of $\mathcal{Q}'$ in $\mathbb{N}$ is also a solution of $\mathcal{Q}$.

2. Let $\{x_i \mapsto p_i\}_{i\in[n]}$ be a solution of the system $\mathcal{Q}$ in $\mathbb{N}_\infty$. For every $i \in [m]$, if $\sum_{j\in[n]} c_{i,j}x_j + b_i = \sum_{j\in[n]} c'_{i,j}x_j + b'_i = \infty$, we pick one variable whose assignment is infinity of each side and remove this constraint from the system $\mathcal{Q}$. Then, the remained system $\mathcal{Q}'$ has a solution in $\mathbb{N}$. By the previous result, the system $\mathcal{Q}'$ has a solution in $\mathbb{N}$ with at most $2(m'+1)(\log(m'+1)+\lceil\log\bar{c}\rceil+2)$ non-zero elements, where $m' \leq m$ is the number of constraints in the system $\mathcal{Q}'$. Observe that the solution of $\mathcal{Q}'$ together with the picked variable is still a solution of $\mathcal{Q}$. Since we pick at most $2(m-m')$ variables, the number of non-zero elements in the solution is bounded as follows.

$$2(m-m')+2(m'+1)(\log(m'+1)+\lceil\log\bar{c}\rceil+2) \leq 2(m+1)(\log(m+1)+\lceil\log\bar{c}\rceil+2)$$

□

Lemma 2.17. *Let $\mathcal{Q}$ be an ILP system consists of n variables $\{x_i\}_{i\in[n]}$ and m constraints*

doi:10.6342/NTU202203098

in the following form,

$$\mathcal{Q} := \bigwedge_{i\in[m]} \left(\sum_{j\in[n]} c_{i,j}x_j + b_i \circledast \sum_{j\in[n]} c'_{i,j}x_j + b'_i \right),$$

where $\circledast \in \{\geq, =\}$*,* $c_{i,j}, c_{i,j}, b_i, b'_i \in \mathbb{N}$*. Let* $\bar{c}$ *be the maximal of* $c_{i,j}, c_{i,j}, b_i, b'_i$*.*

1. *If the system* $\mathcal{Q}$ *has a solution in* $\mathbb{N}$*, then it has a solution in* $\mathbb{N}$ *such that each elements is bounded by* $(n+m)(m\bar{c})^{2m+1}$*.*

2. *If the system* $\mathcal{Q}$ *has a solution in* $\mathbb{N}_\infty$*, then it has a solution in* $\mathbb{N}_\infty$ *such that each elements is either infinity or bounded by* $(n+m)(m\bar{c})^{2m+1}$*.*

Proof.

1. As the proof of Lemma 2.16, we first rewrite $\mathcal{Q}$ to the equivalent system $\mathcal{Q}'$. Note that since we introduce m fresh variables, there are $n+m$ variables in the system $\mathcal{Q}'$. By Theorem in [24], the system $\mathcal{Q}'$ has a solution in $\mathbb{N}$ if and only if it has a solution in $\mathbb{N}$ such that each elements is bounded by $(n+m)(m\bar{c})^{2m+1}$. It is not hard to check that this solution of the system $\mathcal{Q}'$ in $\mathbb{N}$ is also a solution of the system $\mathcal{Q}$.

2. The proof is similar to Lemma 2.16. Let $\{x_i \mapsto p_i\}_{i\in[n]}$ be a solution of the system $\mathcal{Q}$ in $\mathbb{N}_\infty$. For every $i \in [m]$, if $\sum_{j\in[n]} c_{i,j}x_j + b_i = \sum_{j\in[n]} c'_{i,j}x_j + b'_i = \infty$, we pick one variable whose assignment is infinity of each side and remove this constraint from the system $\mathcal{Q}$. Then, the remained system $\mathcal{Q}'$ has a solution in $\mathbb{N}$. By the previous result, $\mathcal{Q}'$ has a solution in $\mathbb{N}$ with at most $2(m'+1)\,(\log(m'+1) + \lceil \log \bar{c} \rceil + 2)$ non-zero elements, where $m' \leq m$ is the number of constraints in the system $\mathcal{Q}'$. Observe that the solution of $\mathcal{Q}'$ together with the picked variable is still a solution of $\mathcal{Q}$. Besides, each element in this solution is either infinity or bounded by $(n+m)(m\bar{c})^{2m+1}$.

□

doi:10.6342/NTU202203098

2.5 Notation

Finally, we clarify some notations used in this thesis.

Let $\mathbb{N}_\infty$ denote the *extended natural number*, that is, $\mathbb{N}_\infty := \mathbb{N} \cup \{\infty\}$.

For every positive integer n, let $[n]$ be the set of *positive integers no more than* n. Formally, $[n] := \{1, 2, \ldots, n\}$.

The notation $\phi[b/a]$ denotes the *substitution* that substitute all a occurred in ϕ with b. The notation $\phi[b_t/a_t]_{t=\ell_1}^{\ell_2}$ is the abbreviation of the *multiple substitutions*. That is,

$$\phi[b_t/a_t]_{t=\ell_1}^{\ell_2} := \phi[b_{\ell_1}/a_{\ell_1}][b_{\ell_1+1}/a_{\ell_1+1}] \ldots [b_{\ell_2}/a_{\ell_2}].$$

Because we encode the counting conditions in binary, the length of the C^2 sentence φ is polynomial in n, m, and $\lceil \log K \rceil$, where n is the number of unary predicates, m is the number of binary predicates, and K is the summation of counting conditions. Therefore, we say a function $\theta(n, m, K)$ is *polynomial* in the length of φ if it is polynomial in n, m, and $\lceil \log K \rceil$. We say a function $\theta(n, m, K)$ is *exponential* in the length of φ if it is exponential in n, m, and $\lceil \log K \rceil$. We say a function $\theta(n, m, K)$ is *double-exponential* in the length of φ if it is double-exponential in n, m, and $\lceil \log K \rceil$.

doi:10.6342/NTU202203098

Chapter 3

Type, configuration, and pseudo-structure

In this chapter, we fix a vocabulary τ which consists of n unary predicates $U_1, U_2, \ldots, U_n$ and m binary predicates $\beta_1, \beta_2, \ldots, \beta_m$. Let φ be a C^2 sentence over τ in the normal form, that is,

$$\begin{aligned} \varphi := \quad & \forall x\, \gamma(x) \\ \wedge \quad & \forall x\, \forall y\, x \neq y \rightarrow \alpha(x, y) \\ \wedge \quad & \bigwedge_{i \in [m]} \forall x\, \exists^{=k_i} y\, \beta_i(x, y) \ \wedge\ x \neq y, \end{aligned}$$

where $\gamma(x)$ and $\alpha(x, y)$ are quantifier-free formulae. Let $\vec{k} := (k_1, k_2, \ldots, k_m)$ be the m-dimension (row) vector of the counting condition of φ and $K := \sum_{i \in [m]} k_i$ be the summation of all counting conditions.

We define some notions for C^2 in this chapter. In Section 3.1, we introduce *1-type* and *2-type*. In Section 3.2, we introduce *behavior function* and *configuration* and discuss their properties. Finally, in Section 3.3, we define *pseudo-structure* and *pseudo-model* which can be viewed as an alternative representation of the structure and model of two-variable logic.

doi:10.6342/NTU202203098

3.1 Type

Intuitively, the 1-type of an element records all unary predicates that the element satisfied. The 2-type of a pair of elements records all binary predicates that the element satisfied. The 1-type and 2-type are formally defined as follows.

Definition 3.1. *A unary type (1-type) over τ is a maximally consistent set of unary predicates from τ and their negation using only one variable.*

We usually indicate the 1-types with symbols π (possibly indexed). Let Π^τ be the set of all 1-types over the vocabulary τ. Note that $|\Pi^\tau| = 2^n$, where n is the number of unary predicates in τ.

Definition 3.2. *A binary type (2-type) over τ is a maximally consistent set of binary predicates from τ and their negation using "exactly" two variables.*

Note that since we do not consider the self-loop term in the normal form throughout this thesis, there is no $\beta(x, x)$ or $\beta(y, y)$ in φ, where β is a binary predicate in τ. Hence they are not in the 2-types, too. We usually indicate the 2-types with notation η (possibly indexed). Let $\mathcal{K}^\tau$ be the set of all 2-types over the vocabulary τ. Note that $|\mathcal{K}^\tau| = 2^{2m}$, where m is the number of binary predicates in τ.

For every 2-type $\eta \in \mathcal{K}^\tau$, the *dual of* η, denoted by $\widetilde{\eta}$, is the unique 2-type that swaps the two variables in η. The *null type*, denoted by η_{null}, is the 2-type which consists of only negation of the binary predicates, that is,

$$\eta_{\mathsf{null}} := \{\neg\beta_1(x, y), \neg\beta_2(x, y), \ldots, \neg\beta_m(x, y), \neg\beta_1(y, x), \neg\beta_2(y, x), \ldots, \neg\beta_m(y, x)\}.$$

Remark 3.3. *The dual of 2-types is an involution. That is, for every 2-type $\eta \in \mathcal{K}^\tau$, the dual of $\widetilde{\eta}$ is η.*

Remark 3.4. *The dual of η_{null} is itself.*

For a 2-type η, the *forward vector representation of* η, denoted by $\eta^{\triangleright}$, is an m-dimension (row) vector $(b_1, b_2, \ldots, b_m)$ where $b_i = 1$ if and only if $\beta_i(x, y) \in \eta$, otherwise, $b_i = 0$.

doi:10.6342/NTU202203098

The *backward vector representation of* η, denoted by $\eta^{\triangleleft}$, is also an m-dimension (row) vector $(b_1, b_2, \ldots, b_m)$ where $b_i = 1$ if and only if $\beta_i(y, x) \in \eta$, otherwise, $b_i = 0$.

We say a 2-type η is *trivial* if $\eta^{\triangleright} = \vec{0}$. Otherwise, η is *non-trivial*. Let $\mathcal{K}^{\tau,\mathsf{tri}}$ be the set of all trivial 2-types over τ and $\mathcal{K}^{\tau,\mathsf{ntri}}$ be the set of all non-trivial 2-types over τ. Note that $\mathcal{K}^{\tau} = \mathcal{K}^{\tau,\mathsf{tri}} \mathbin{\dot\cup} \mathcal{K}^{\tau,\mathsf{ntri}}$, $\left|\mathcal{K}^{\tau,\mathsf{tri}}\right| = 2^m$, and $\left|\mathcal{K}^{\tau,\mathsf{ntri}}\right| = 2^{2m} - 2^m$.

Remark 3.5. *For every $\eta \in \mathcal{K}^{\tau}$, because the value of each entry of $\eta^{\triangleright}$ and $\eta^{\triangleleft}$ is non-negative (either 0 or 1), the 1-norm of $\eta^{\triangleright}$ and $\eta^{\triangleleft}$ is just the summation of all entries of it. Hence, we would denote the summation of the entries of $\eta^{\triangleright}$ and $\eta^{\triangleleft}$ by the symbol $\|\eta^{\triangleright}\|_1$ and $\|\eta^{\triangleleft}\|_1$, respectively.*

Remark 3.6. *Note that for every 2-type η, η is the null type, if and only if $\|\eta^{\triangleright}\|_1 + \|\eta^{\triangleleft}\|_1 = 0$.*

Example 3.7. *Consider the vocabulary τ_0 consists of two unary predicates $\{U_1, U_2\}$ and three binary predicates $\{\beta_1, \beta_2, \beta_3\}$. There are 4 different 1-types over τ_0.*

$$\Pi^{\tau_0} = \{\{U_1(x), U_2(x)\}, \{U_1(x), \neg U_2(x)\}, \{\neg U_1(x), U_2(x)\}, \{\neg U_1(x), \neg U_2(x)\}\}$$

There are 64 different 2-types over τ_0.

$$\mathcal{K}^{\tau_0} = \left\{\begin{array}{l} \{\beta_1(x,y), \beta_2(x,y), \beta_3(x,y), \beta_1(y,x), \beta_2(y,x), \beta_3(y,x)\} \\ \{\beta_1(x,y), \beta_2(x,y), \beta_3(x,y), \beta_1(y,x), \beta_2(y,x), \neg\beta_3(y,x)\} \\ \quad\vdots \\ \{\neg\beta_1(x,y), \neg\beta_2(x,y), \neg\beta_3(x,y), \neg\beta_1(y,x), \neg\beta_2(y,x), \neg\beta_3(y,x)\} \end{array}\right\}$$

Let η_1 be the 2-type $\{\beta_1(x,y), \beta_2(x,y), \neg\beta_3(x,y), \neg\beta_1(y,x), \neg\beta_2(y,x), \neg\beta_3(y,x)\}$. The forward and backward vector representation of η_1 are $\eta_1^{\triangleright} = (1,1,0)$ and $\eta_1^{\triangleleft} = (0,0,0)$, respectively. Observe that η_1 is non-trivial, but $\widetilde{\eta}_1$ is trivial.

Validation of the type. For a pair of elements in the model of φ, clearly, the possible 1-types of them and possible 2-type between them are restricted by φ. The idea of validation capture the condition of possible 1-types and 2-types.

doi:10.6342/NTU202203098

Definition 3.8.

1. *A 1-type* $\pi \in \Pi^\tau$ *is valid in* φ *if* $\pi(x) \models \gamma(x)$.

2. *A 2-type* $\eta \in \mathcal{K}^\tau$ *is compatible with the pair of 1-types* $(\pi_1, \pi_2) \in \Pi^\varphi \times \Pi^\varphi$ *in* φ *if*

$$\pi_1(x) \wedge \eta(x,y) \wedge \pi_2(y) \models \alpha(x,y)$$
$$\pi_2(x) \wedge \tilde{\eta}(x,y) \wedge \pi_1(y) \models \alpha(x,y).$$

We say that a tuple $\langle \pi_1, \eta, \pi_2 \rangle \in \Pi^\tau \times \mathcal{K}^\tau \times \Pi^\tau$ is *valid* in φ if π_1 and π_2 are valid in φ and η is compatible with (π_1, π_2) in φ. Let Π^φ be the set of all valid 1-types in φ. Let $\mathcal{K}^\varphi_{\langle \pi_1, *, \pi_2 \rangle}$ be the set of all 2-types compatible with (π_1, π_2) in φ, $\mathcal{K}^{\varphi,\mathsf{tri}}_{\langle \pi_1, *, \pi_2 \rangle}$ be the set of all trivial 2-types compatible with (π_1, π_2) in φ, and $\mathcal{K}^{\varphi,\mathsf{ntri}}_{\langle \pi_1, *, \pi_2 \rangle}$ be the set of all non-trivial 2-types compatible with (π_1, π_2) in φ.

We say that valid 1-types π_1 and π_2 are *compatible* in φ if $\mathcal{K}^\varphi_{\langle \pi_1, *, \pi_2 \rangle}$ is nonempty. We say that valid 1-types π_1 and π_2 are *null-compatible* in φ if $\eta_{\mathsf{null}} \in \mathcal{K}^\varphi_{\langle \pi_1, *, \pi_2 \rangle}$. We say that a set of valid 1-types Π is *mutually null-compatible* in φ if for every 1-types π_1 and π_2 in Π, they are null-compatible in φ. We say that a set of valid 1-types Π is *maximal mutually null-compatible* in φ if Π is mutually null-compatible, and, for every valid 1-type $\pi \notin \Pi$, $\Pi \cup \{\pi\}$ is not mutually null-compatible.

Remark 3.9. *The compatible relation of 1-types is symmetric. Indeed, for every* $(\pi_1, \pi_2) \in \Pi_\varphi \times \Pi_\varphi$, $\eta \in \mathcal{K}^\varphi_{\langle \pi_1, *, \pi_2 \rangle}$ *if and only if* $\widetilde{\eta} \in \mathcal{K}^\varphi_{\langle \pi_2, *, \pi_1 \rangle}$.

Remark 3.10. *Recall that the dual of* η_{null} *is itself. Hence, the null-compatible relation is also symmetric.*

Remark 3.11. *Since the null-compatible relation is symmetric, we can interpret it as a undirected graph* $\mathcal{G}$. *The vertices of* $\mathcal{G}$ *are all valid 1-types in* φ. *For every vertices* π_1 *and* π_2, *there is an edge between them if type are not null-compatible. Then, a set of valid 1-types is mutually null-compatible if and only if it is a independent set in* $\mathcal{G}$. *A set of valid 1-types is maximal mutually null-compatible if and only if it is a maximal independent*

doi:10.6342/NTU202203098

set in $\mathcal{G}$. Note that there are at most $O(3^{n/3})$ maximal independent sets in a graph of n vertices [21].

Example 3.12. *Let τ_0 be as in Example 3.7. Consider the following C^2 formulae over τ_0.*

$$\gamma_0(x) := (U_1(x) \wedge \neg U_2(x)) \vee (\neg U_1(x) \wedge U_2(x))$$

$$\alpha_1(x,y) := \beta_1(x,y) \rightarrow (U_1(y) \wedge (U_1(x) \rightarrow (\beta_1(y,x) \wedge \beta_3(x,y) \wedge \beta_3(y,x))))$$

$$\alpha_2(x,y) := \beta_2(x,y) \rightarrow (U_2(y) \wedge \beta_3(y,x) \wedge \neg\beta_2(y,x))$$

$$\alpha_3(x,y) := \beta_3(x,y) \rightarrow (U_2(x) \rightarrow \beta_2(y,x))$$

$$\alpha_0(x,y) := \alpha_1(x,y) \wedge \alpha_2(x,y) \wedge \alpha_2(x,y)$$

Let φ_0 be the following C^2 sentence.

$$\varphi_0 := \forall x\, \gamma_0(x) \wedge \forall x\, \forall y\, x \neq y \rightarrow \alpha_0(x,y) \wedge \bigwedge_{i \in [3]} \forall x\, \exists^{=1} y\, \beta_i(x,y) \wedge x \neq y$$

Let $\pi_1, \pi_2, \eta_1, \eta_2$, and η_3 be the following 1-types and 2-types.

$$\pi_1 := \{U_1(x), \neg U_2(y)\}$$

$$\pi_2 := \{\neg U_1(x), U_2(y)\}$$

$$\eta_1 := \{\beta_1(x,y), \neg\beta_2(x,y), \beta_3(x,y), \beta_1(y,x), \neg\beta_2(y,x), \beta_3(y,x)\}$$

$$\eta_2 := \{\neg\beta_1(x,y), \beta_2(x,y), \neg\beta_3(x,y), \neg\beta_1(y,x), \neg\beta_2(y,x), \beta_3(y,x)\}$$

$$\eta_3 := \{\neg\beta_1(x,y), \neg\beta_2(x,y), \neg\beta_3(x,y), \beta_1(y,x), \neg\beta_2(y,x), \neg\beta_3(y,x)\}$$

It is not difficult to check the following. There are two valid 1-types in φ_0, $\Pi^{\varphi_0} = \{\pi_1, \pi_2\}$. The sets of compatible 2-types with π_1 and π_2 in φ_0 are as follows.

$$\mathcal{K}^{\varphi_0}_{\langle \pi_1, *, \pi_1 \rangle} = \{\eta_{\mathsf{null}}, \eta_1\}$$

$$\mathcal{K}^{\varphi_0}_{\langle \pi_1, *, \pi_2 \rangle} = \{\eta_{\mathsf{null}}, \eta_2, \eta_3\}$$

$$\mathcal{K}^{\varphi_0}_{\langle \pi_2, *, \pi_1 \rangle} = \{\eta_{\mathsf{null}}, \widetilde{\eta}_2, \widetilde{\eta}_3\}$$

$$\mathcal{K}^{\varphi_0}_{\langle \pi_2, *, \pi_2 \rangle} = \{\eta_{\mathsf{null}}, \eta_2, \widetilde{\eta}_2\}$$

doi:10.6342/NTU202203098

Observe that π_1 and π_2 are compatible and null-compatible.

3.2 Configuration

For an element a in the model of φ, the 2-types between a and all other elements are restricted by the counting condition of φ. In this section, we introduce the notions of behavior function and configuration, which capture the counting condition of elements.

Definition 3.13.

1. *A behavior function F over τ is a function that maps a tuple $\langle \eta, \pi \rangle$ to a natural number, where η is a non-trivial 2-type over τ and π is a 1-type over τ. Formally, $F : \mathcal{K}^{\tau,\mathsf{ntri}} \times \Pi^\tau \to \mathbb{N}$.*

2. *A configuration C over τ is a tuple $\langle \pi_s, F \rangle$ where π_s is a 1-type over τ and F is a behavior function over τ.*

For a configuration $C = \langle \pi_s, F \rangle$ over τ, we call π_s the *source type of C*. For a set of 1-types Π, we say C *conforms* to Π if $\pi_s \in \Pi$ and for every $\pi \in \Pi^\tau \setminus \Pi$, for every $\eta \in \mathcal{K}^{\tau,\mathsf{ntri}}$, $F(\eta, \pi) = 0$. Let $\mathcal{C}^\tau$ be the set of all configurations over τ.

For a set of configurations $\mathcal{C}$ over τ, for every $\pi \in \Pi^\tau$, the *π-restricted subset of $\mathcal{C}$*, denoted by $\mathcal{C}\big|_\pi$, is the set of configurations in $\mathcal{C}$ whose source type is π. The *set of behavior functions of the π-restricted subset of $\mathcal{C}$*, denoted by $\mathcal{F}_\pi^{\mathcal{C}}$, is the set of behavior functions F such that $\langle \pi, F \rangle \in \mathcal{C}$. The *set of 1-type in $\mathcal{C}$*, denoted by $\Pi^{\mathcal{C}}$, is the set of 1-type π such that there exists behavior function F such that $\langle \pi, F \rangle \in \mathcal{C}$. The formal definitions are as follows.

$$\mathcal{C}\big|_\pi := \{\langle \pi_s, F \rangle \in \mathcal{C} \mid \pi_s = \pi\}$$

$$\mathcal{F}_\pi^{\mathcal{C}} := \{F \mid \langle \pi, F \rangle \in \mathcal{C}\}$$

$$\Pi^{\mathcal{C}} := \{\pi \mid \text{There exists } F \text{ such that } \langle \pi, F \rangle \in \mathcal{C}\}$$

Remark 3.14. *For a set of configurations $\mathcal{C}$ over τ, $\mathcal{C}$ can be decomposed by the source type of configurations, that is, $\mathcal{C} = \dot{\bigcup}_{\pi \in \Pi^\tau} \mathcal{C}\big|_\pi$.*

doi:10.6342/NTU202203098

Validation of the configuration. For an element a in the model of φ, the idea of validation captures the necessary condition of its configuration.

Definition 3.15. *A configuration $\langle \pi_s, F \rangle$ over τ is valid in φ if it satisfies the following.*

- *The source type π_s is valid in φ.*
- *For every $\eta \in \mathcal{K}^{\tau,\mathsf{ntri}}$, for every $\pi \in \Pi^{\varphi}$, if $\langle \pi_s, \eta, \pi \rangle$ is not valid in φ, then $F(\eta, \pi) = 0$.*
- *The following* counting condition *with respect to φ holds.*

$$\sum_{\eta \in \mathcal{K}^{\tau,\mathsf{ntri}}} \sum_{\pi \in \Pi^{\tau}} F(\eta, \pi) \cdot \eta^{\triangleright} = \vec{k}$$

Let $\mathcal{C}^{\varphi}$ be the set of all valid configurations in φ. The size of $\mathcal{C}^{\varphi}$ is double-exponential in the length of φ.

Lemma 3.16. *Let $\langle \pi_s, F \rangle$ be a configuration over τ. If $\langle \pi_s, F \rangle$ is valid in φ, then $\sum_{\eta \in \mathcal{K}^{\tau,\mathsf{ntri}}} \sum_{\pi \in \Pi^{\tau}} F(\eta, \pi) \leq K$.*

Proof. Recall that for every $\eta \in \mathcal{K}^{\tau,\mathsf{ntri}}$, $\|\eta\|_1 \geq 1$. Because $\langle \pi_s, F \rangle$ is valid, by the counting condition of it, the following inequality holds.

$$\sum_{\eta \in \mathcal{K}^{\tau,\mathsf{ntri}}} \sum_{\pi \in \Pi^{\tau}} F(\eta, \pi) \leq \sum_{\eta \in \mathcal{K}^{\tau,\mathsf{ntri}}} \sum_{\pi \in \Pi^{\tau}} F(\eta, \pi) \cdot \|\eta^{\triangleright}\|_1 = K$$

□

Lemma 3.17. *Let $\langle \pi_s, F \rangle$ be a configuration over τ, If $\langle \pi_s, F \rangle$ is valid in φ, then for every $\eta \in \mathcal{K}^{\tau,\mathsf{ntri}}$, for every $\pi \in \Pi^{\tau}$, $F(\eta, \pi) \leq K$.*

Proof. By definition of the behavior function, for every $\eta \in \mathcal{K}^{\tau,\mathsf{ntri}}$, for every $\pi \in \Pi^{\tau}$, $F(\eta, \pi) \geq 0$. By Lemma 3.16, the following inequality holds.

$$F(\eta, \pi) \leq K - \sum\sum_{\substack{(\eta', \pi') \in \mathcal{K}^{\tau,\mathsf{ntri}} \times \Pi^{\tau} \\ (\eta', \pi') \neq (\eta, \pi)}} F(\eta', \pi') \leq K$$

doi:10.6342/NTU202203098

$\square$

Lemma 3.18. $|\mathcal{C}^\varphi| \le 2^n (K+1)^{2^{n+2m}}$.

Proof. For every behavior function F of valid configuration in φ, Lemma 3.17 shows that the codomain of F is $[K]$. Hence, for every $\pi \in \Pi^\tau$, the size of $\left|\mathcal{F}_\pi^{\mathcal{C}^\varphi}\right|$ is bounded by the number of possible behavior functions in φ.

$$\left|\mathcal{F}_\pi^{\mathcal{C}^\varphi}\right| \le |[K]|^{\left|\Pi^\tau \times \mathcal{K}^{\tau,\mathsf{ntri}}\right|} \le (K+1)^{2^{n+2m}}$$

Recall that we can decompose $\mathcal{C}^\varphi$ by the source type, that is, $\mathcal{C}^\varphi = \dot{\bigcup}_{\pi\in\Pi^\tau} \left\{\langle \pi, F\rangle \;\middle|\; F \in \mathcal{F}_\pi^{\mathcal{C}^\varphi}\right\}$. The size of $\mathcal{C}^\varphi$ can be bounded by this decomposition.

$$|\mathcal{C}^\varphi| = \sum_{\pi\in\Pi^\tau} \left|\left\{\langle \pi, F\rangle \;\middle|\; F \in \mathcal{F}_\pi^{\mathcal{C}^\varphi}\right\}\right| = \sum_{\pi\in\Pi^\tau} \left|\mathcal{F}_\pi^{\mathcal{C}^\varphi}\right| \le 2^n (K+1)^{2^{n+2m}}$$

$\square$

Example 3.19. *Let $\tau_0, \varphi_0, \pi_1, \pi_2, \eta_1, \eta_2$, and η_3 be as in Example 3.12. Let F_1 and F_2 be the behavior functions defined as follows.*

$$F_1(\eta,\pi) := \begin{cases} 1 & \textit{if } (\eta,\pi) = (\eta_1,\pi_1) \\ 1 & \textit{if } (\eta,\pi) = (\eta_2,\pi_2) \\ 0 & \textit{otherwise} \end{cases}$$

$$F_2(\eta,\pi) := \begin{cases} 1 & \textit{if } (\eta,\pi) = (\widetilde{\eta}_3,\pi_1) \\ 1 & \textit{if } (\eta,\pi) = (\widetilde{\eta}_2,\pi_1) \\ 1 & \textit{if } (\eta,\pi) = (\eta_2,\pi_2) \\ 0 & \textit{otherwise} \end{cases} \quad F_3(\eta,\pi) := \begin{cases} 1 & \textit{if } (\eta,\pi) = (\widetilde{\eta}_3,\pi_1) \\ 1 & \textit{if } (\eta,\pi) = (\widetilde{\eta}_2,\pi_2) \\ 1 & \textit{if } (\eta,\pi) = (\eta_2,\pi_2) \\ 0 & \textit{otherwise} \end{cases}$$

We can verify that the counting condition of F_1 is satisfied.

$$\begin{aligned} \sum_{\eta\in\mathcal{K}^{\tau,\mathsf{ntri}}} \sum_{\pi\in\Pi^\tau} F(\eta,\pi)\cdot\eta^{\triangleright} &= F(\eta_1,\pi_1)\cdot\eta_1^{\triangleright} + F(\eta_2,\pi_2)\cdot\eta_2^{\triangleright} \\ &= 1\cdot(1,0,1) + 1\cdot(0,1,0) \\ &= (1,1,1) \end{aligned}$$

Observe that the configurations $\langle\pi_1, F_1\rangle$, $\langle\pi_2, F_2\rangle$, and $\langle\pi_2, F_3\rangle$ are the only valid con-

doi:10.6342/NTU202203098

figurations in φ. Hence $\mathcal{C}^{\varphi} = \{\langle \pi_1, F_1 \rangle, \langle \pi_2, F_2 \rangle, \langle \pi_2, F_3 \rangle\}$.

3.3 Pseudo-structure

It is well-known that the model of FO^2 can be viewed as a labeled graph. In this section, we introduce the concept of pseudo-structure, which can be viewed as an extension of structure for C^2.

Definition 3.20. *A pseudo-structure $\mathcal{G}$ over τ is a complete directed labeled graph $\left\langle V, \mathsf{lab}_{\mathsf{V}}^{\mathcal{G}}, \mathsf{lab}_{\mathsf{E}}^{\mathcal{G}} \right\rangle$, where each component is defined as follows.*

- *V is the set of vertices called universe of $\mathcal{G}$.*
- *$\mathsf{lab}_{\mathsf{V}}^{\mathcal{G}} : V \to \Pi^{\tau}$ is a function that maps a vertex in V to a 1-type over τ.*
- *$\mathsf{lab}_{\mathsf{E}}^{\mathcal{G}} : V \times V \setminus \{(v, v) \mid v \in V\} \to \mathcal{K}^{\tau}$ is a function that maps an edge in A to a 2-type over τ. Besides, $\mathsf{lab}_{\mathsf{E}}^{\mathcal{G}}$ satisfy the following dual constraint. For every $(v_1, v_2) \in V \times V$, $\mathsf{lab}_{\mathsf{E}}^{\mathcal{G}}[v_1, v_2]$ is the dual of $\mathsf{lab}_{\mathsf{E}}^{\mathcal{G}}[v_2, v_1]$.*

We would use square brackets for the labels $\mathsf{lab}_{\mathsf{V}}^{\mathcal{G}}[*]$ and $\mathsf{lab}_{\mathsf{E}}^{\mathcal{G}}[*, *]$. When the arity of the function is clear from the context, we omit the superscripts and write only $\mathsf{lab}^{\mathcal{G}}[*]$ and $\mathsf{lab}^{\mathcal{G}}[*, *]$. When $\mathcal{G}$ is also clear from the context, we omit it and write only $\mathsf{lab}[*]$ and $\mathsf{lab}[*, *]$.

For the rest of this chapter, we fix a pseudo-structure $\mathcal{G} := \left\langle V, \mathsf{lab}_{\mathsf{V}}^{\mathcal{G}}, \mathsf{lab}_{\mathsf{E}}^{\mathcal{G}} \right\rangle$ over τ.

We say an element $v \in V$ *realizes the 1-type* $\pi \in \Pi^{\tau}$ if $\mathsf{lab}[v] = \pi$. We say a pair of elements $(v_1, v_2) \in V \times V$ *realizes the 2-type* $\eta \in \mathcal{K}^{\tau}$ if $\mathsf{lab}[v_1, v_2] = \eta$. We say a pair of elements $(v_1, v_2) \in V \times V$ *realizes the tuple* $\langle \pi_1, \eta, \pi_2 \rangle \in \Pi^{\tau} \times \mathcal{K}^{\tau} \times \Pi^{\tau}$ if v_1 realizes π_1, v_2 realizes π_2 and (v_1, v_2) realizes η.

We also define two auxiliary labels for the behavior function and the configuration.

Definition 3.21.

1. *For every $v \in V$, the realized behavior function of v, denoted by $F^{\mathcal{G}}[v]$, is the*

doi:10.6342/NTU202203098

function defined as follows. For every $\eta \in \mathcal{K}^\tau$, for every $\pi \in \Pi^\tau$,

$$F^{\mathcal{G}}[v](\eta, \pi) := \left|\left\{v' \in V \setminus \{v\} \,\middle|\, \mathsf{lab}^{\mathcal{G}}[v, v'] := \eta \text{ and } \mathsf{lab}^{\mathcal{G}}[v'] = \pi\right\}\right|.$$

2. *For every $v \in V$, the realized configuration of v, denoted by $C^{\mathcal{G}}[v]$, is the tuple $\left\langle \mathsf{lab}^{\mathcal{G}}[v], F^{\mathcal{G}}[v] \right\rangle$.*

When $\mathcal{G}$ is clear from the context, we omit it and write only $F[*]$ and $C[*]$.

For every set of vertices $U \subseteq V$, let Π^U be the *set of realizable 1-types in U*, and $\mathcal{C}^U$ be the *set of realizable configurations in U*. The formal definitions are as follows.

$$\Pi^U := \{\mathsf{lab}[v] \mid v \in U\}$$
$$\mathcal{C}^U := \{\mathcal{C}[v] \mid v \in U\}$$

For every set of vertices $U \subseteq V$, for every $\pi \in \Pi^\tau$, the *realized subset of π in U*, denoted by $U\big|_\pi$, is the set of elements in U whose label is π and the *number of realizations of π in U*, denoted by n^U_π, is the size of $U\big|_\pi$. For every $C \in \mathcal{C}^\tau$, the *realized subset of C in U*, denoted by $U\big|_C$, is the set of elements in U whose realized configuration is C and the *number of realizations of C in U*, denoted by n^U_C, is the size of $U\big|_C$. The formal definitions are as follows.

$$U\big|_\pi := \{v \in U \mid \mathsf{lab}[v] = \pi\}$$
$$U\big|_C := \{v \in U \mid C[v] = C\}$$
$$n^U_\pi := \left|U\big|_\pi\right|$$
$$n^U_C := \left|U\big|_C\right|$$

Remark 3.22. *Observe that for every $v \in V$, the following is the equivalent definition of realized configuration $F[v]$.*

$$F[v](\eta, \pi) := |\{v' \in V \setminus \{v\} \mid \mathsf{lab}[v, v'] = \eta \text{ and } \mathsf{lab}[v'] = \pi\}|$$
$$= \left|\left\{v' \in V\big|_\pi \setminus \{v\} \mid \mathsf{lab}[v, v'] = \eta\right\}\right|$$

doi:10.6342/NTU202203098

Lemma 3.23. *For every $v \in V$, $C[v]$ conforms to Π^V.*

Proof. By definition of realized 1-types, $\mathsf{lab}[v] \in \Pi^V$. For every $\eta \in \mathcal{K}^{\tau,\mathsf{ntri}}$, for every $\pi \in \Pi^\tau \setminus \Pi^V$, by definition of realized behavior function, the following inequality holds.

$$F[v](\eta,\pi) = \left|\left\{v' \in V\big|_\pi \setminus \{v\} \;\middle|\; \mathsf{lab}[v,v'] = \eta\right\}\right| \leq \left|V\big|_\pi\right| = 0$$

□

For every $V_1, V_2 \subseteq V$, for every tuple $\langle \pi_1, \eta, \pi_2 \rangle \in \Pi_\tau \times \mathcal{K}_\tau \times \Pi_\tau$, we define following notions, the *realized source subset of $\langle \pi_1, \eta, \pi_2 \rangle$ between V_1 and V_2*, denoted by $V_1\big|^{\to V_2}_{\langle \pi_1,\eta,\pi_2\rangle}$, the *number of source realizations of $\langle \pi_1, \eta, \pi_2 \rangle$ between V_1 and V_2*, denoted by $n^{V_1(\to V_2)}_{\langle \pi_1,\eta,\pi_2\rangle}$, the *realized target subset of $\langle \pi_1, \eta, \pi_2 \rangle$ between V_1 and V_2*, denoted by $V_2\big|^{V_1\to}_{\langle \pi_1,\eta,\pi_2\rangle}$, the *number of target realizations of $\langle \pi_1, \eta, \pi_2 \rangle$ between V_1 and V_2*, denoted by $n^{(V_1\to)V_2}_{\langle \pi_1,\eta,\pi_2\rangle}$, and the *number of realizations of $\langle \pi_1, \eta, \pi_2 \rangle$ between V_1 and V_2*, denoted by $n^{V_1\to V_2}_{\langle \pi_1,\eta,\pi_2\rangle}$. The formal definitions are as follows.

$$\begin{aligned}
V_1\big|^{\to V_2}_{\langle \pi_1,\eta,\pi_2\rangle} &:= \left\{v_1 \in V_1\big|_{\pi_1} \;\middle|\; \text{There exists } v_2 \in V_2\big|_{\pi_2} \text{ such that } \mathsf{lab}[v_1,v_2] = \eta\right\} \\
V_2\big|^{V_1\to}_{\langle \pi_1,\eta,\pi_2\rangle} &:= \left\{v_2 \in V_2\big|_{\pi_2} \;\middle|\; \text{There exists } v_1 \in V_1\big|_{\pi_1} \text{ such that } \mathsf{lab}[v_1,v_2] = \eta\right\} \\
n^{V_1(\to V_2)}_{\langle \pi_1,\eta,\pi_2\rangle} &:= \left|V_1\big|^{\to V_2}_{\langle \pi_1,\eta,\pi_2\rangle}\right| \\
n^{(V_1\to)V_2}_{\langle \pi_1,\eta,\pi_2\rangle} &:= \left|V_2\big|^{V_1\to}_{\langle \pi_1,\eta,\pi_2\rangle}\right| \\
n^{V_1\to V_2}_{\langle \pi_1,\eta,\pi_2\rangle} &:= \left|\left\{(v_1,v_2) \in V_1\big|_{\pi_1} \times V_2\big|_{\pi_2} \;\middle|\; \mathsf{lab}[v_1,v_2] = \eta\right\}\right|
\end{aligned}$$

When $V_1 = V_2 = V$ and $\mathcal{G}$ is clear from the context, we omit the superscripts and write only n_π, n_C and $n_{\langle \pi_1,\eta,\pi_2\rangle}$ (for $n^{V\to V}_{\langle \pi_1,\eta,\pi_2\rangle}$).

Remark 3.24. *We observe the following. First, the sets V_1 and V_2 are not necessarily disjoint in the definition. Second, the number of realizations is non-negative and may be infinite.*

Example 3.25. *Let $\tau_0, \varphi_0, \pi_1, \pi_2, \eta_1, \eta_2, \eta_3, C_1, C_2,$ and C_3 be as in Example 3.19. Let $V_0 = \left\langle V_0, \mathsf{lab}^{\mathcal{G}_0}_{\mathsf{V}}, \mathsf{lab}^{\mathcal{G}_0}_{\mathsf{E}} \right\rangle$ be a pseudo-structure over τ_0, where each component is defined as follows.*

doi:10.6342/NTU202203098

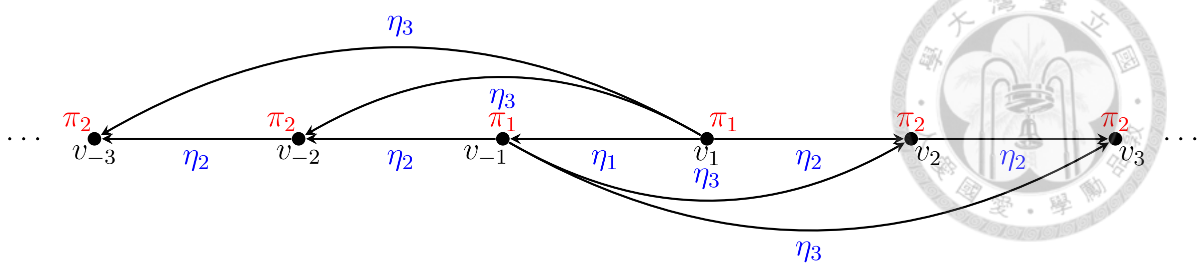

Figure 3.1: The pseudo-structure $\mathcal{G}_0$. The 2-type of remaining edges is η_{null}.

$$V_0 := \{\dots, v_{-2}, v_{-1}, v_1, v_2, \dots\}$$

$$\mathsf{lab}_{\mathsf{V}}^{\mathcal{G}}[v] := \begin{cases} \pi_1 & \textit{if } v = v_1 \textit{ or } v_{-1} \\ \pi_2 & \textit{otherwise} \end{cases}$$

$$\mathsf{lab}_{\mathsf{E}}^{\mathcal{G}}[v, v'] := \begin{cases} \eta_1 & \textit{if } (v, v') = (v_1, v_{-1}) \\ \eta_2 & \textit{if } (v, v') = (v_i, v_{i+1}) \textit{ or } (v, v') = (v_{-i}, v_{-i-1}), \textit{for some } i \geq 1 \\ \eta_3 & \textit{if } (v, v') = (v_{-1}, v_i) \textit{ or } (v, v') = (v_1, v_{-i}), \textit{for some } i \geq 2 \\ \eta_{\mathsf{null}} & \textit{otherwise} \end{cases}$$

Note that due to the constraint of $\mathsf{lab}_{\mathsf{E}}^{\mathcal{G}}$*, once we assign the value of* $\mathsf{lab}_{\mathsf{E}}^{\mathcal{G}}[v, v']$*, the value of* $\mathsf{lab}_{\mathsf{E}}^{\mathcal{G}}[v', v]$ *is fixed automatically. The case "otherwise" is for the edges which are not assigned or fixed. Figure 3.1 shows the pseudo-structure* $\mathcal{G}_0$*. The realized configuration of each element is as follows.*

$$C[v] = \begin{cases} C_1 & v = v_1 \textit{ or } v_{-1} \\ C_2 & v = v_2 \textit{ or } v_{-2} \\ C_3 & \textit{otherwise} \end{cases}$$

Let $V_1 := \{v_i \in V_0 \mid i > 0\}$ *and* $V_2 := \{v_i \in V_0 \mid i < 0\}$*. We observe the following number of realizations.*

$$V_1\big|_{\pi_1} = \{v_1\} \qquad n_{\pi_1}^{V_1} = 1$$

$$V_1\big|_{\langle \pi_1, \eta_3, \pi_2 \rangle}^{\to V_2} = \{v_1\} \qquad n_{\langle \pi_1, \eta_3, \pi_2 \rangle}^{V_1(\to V_2)} = 1$$

$$V_2\big|_{\langle \pi_1, \eta_3, \pi_2 \rangle}^{V_1 \to} = \{v_{-2}, v_{-3}, \dots\} \qquad n_{\langle \pi_1, \eta_3, \pi_2 \rangle}^{(V_1 \to) V_2} = \infty$$

$$n_{\langle \pi_1, \eta_3, \pi_2 \rangle}^{V_1 \to V_2} = \infty$$

doi:10.6342/NTU202203098

Relations on the number of realizations. We are going to explore the relations between the number of realizations of different components in a given pseudo-structure. Recall that we have fixed a pseudo-structure $\mathcal{G}$ in this chapter. In Lemma 3.26 until Lemma 3.28, we fix sets $V_1, V_2, V_3 \subseteq V$ and a tuple $\langle \pi_1, \eta, \pi_2 \rangle \in \Pi^\tau \times \mathcal{K}^\tau \times \Pi^\tau$.

Lemma 3.26.

1. $|V_1| = \sum_{\pi \in \Pi^{V_1}} n_\pi^{V_1}$.

2. $|V_1| = \sum_{C \in \mathcal{C}^{V_1}} n_C^{V_1}$.

3. $n_{\pi_1}^{V_1} = \sum_{C \in \mathcal{C}^{V_1}|_{\pi_1}} n_C^{V_1}$.

4. *If V_1 and V_2 are disjoint, then* $n_{\pi_1}^{V_1 \cup V_2} = n_\pi^{V_1} + n_\pi^{V_2}$.

Proof. Observe that for every $\pi, \pi' \in \Pi^\tau$, if $\pi \neq \pi'$, then $V_1\big|_\pi$ and $V_1\big|_{\pi'}$ are disjoint. For every $C_1, C_2 \in \mathcal{C}^\tau$, if $C_1 \neq C_2$, then $V_1\big|_{C_1}$ and $V_1\big|_{C_2}$ are also disjoint. Therefore, the following set decompositions hold.

1. $V_1 = \dot{\bigcup}_{\pi \in \Pi^{V_1}} V_1\big|_\pi$.

2. $V_1 = \dot{\bigcup}_{C \in \mathcal{C}^{V_1}} V_1\big|_C$.

3. $V_1\big|_{\pi_1} = \dot{\bigcup}_{C \in \mathcal{C}^{V_1}|_{\pi_1}} V_1\big|_C$.

4. $(V_1 \cup V_2)\big|_{\pi_1} = V_1\big|_{\pi_1} \dot{\cup} V_2\big|_{\pi_1}$.

The size of each decomposition implies the lemma. □

Lemma 3.27.

1. $n_{\langle \pi_1, \eta, \pi_2 \rangle}^{V_1(\to V_2)} = n_{\langle \pi_2, \widetilde{\eta}, \pi_1 \rangle}^{(V_2 \to) V_1}$.

2. $n_{\langle \pi_1, \eta, \pi_2 \rangle}^{V_1 \to V_2} = n_{\langle \pi_2, \widetilde{\eta}, \pi_1 \rangle}^{V_2 \to V_1}$.

3. $n_{\langle \pi_1, \eta, \pi_2 \rangle}^{V_1 \to V_2} \leq n_{\langle \pi_1, \eta, \pi_2 \rangle}^{V_1(\to V_2)} \cdot n_{\langle \pi_1, \eta, \pi_2 \rangle}^{(V_1 \to) V_2}$.

4. $n_{\langle \pi_1, \eta, \pi_2 \rangle}^{V_1(\to V_2)} \leq n_{\langle \pi_1, \eta, \pi_2 \rangle}^{V_1 \to V_2}$.

5. $n_{\langle \pi_1, \eta, \pi_2 \rangle}^{(V_1 \to) V_2} \leq n_{\langle \pi_1, \eta, \pi_2 \rangle}^{V_1 \to V_2}$.

doi:10.6342/NTU202203098

6. $n^{V_1(\to V_2)}_{\langle\pi_1,\eta,\pi_2\rangle} \leq n^{V_1}_{\pi_1}$.

7. $n^{(V_1\to)V_2}_{\langle\pi_1,\eta,\pi_2\rangle} \leq n^{V_2}_{\pi_2}$.

8. *If* V_2 *and* V_3 *are disjoint,* $n^{V_1\to(V_2\cup V_3)}_{\langle\pi_1,\eta,\pi_2\rangle} = n^{V_1\to V_2}_{\langle\pi_1,\eta,\pi_2\rangle} + n^{V_1\to V_3}_{\langle\pi_1,\eta,\pi_2\rangle}$.

9. *If* V_1 *and* V_2 *are disjoint,* $n^{(V_1\dot\cup V_2)\to V_3}_{\langle\pi_1,\eta,\pi_2\rangle} = n^{V_1\to V_3}_{\langle\pi_1,\eta,\pi_2\rangle} + n^{V_2\to V_3}_{\langle\pi_1,\eta,\pi_2\rangle}$.

Proof. For (1) and (2), observe that $V_1\big|^{\to V_2}_{\langle\pi_1,\eta,\pi_2\rangle}$ and $V_1\big|^{V_2\to}_{\langle\pi_2,\widetilde{\eta},\pi_1\rangle}$ are the same set. Besides, the elements in the set

$$\left\{(v_1,v_2)\in V_1\big|_{\pi_1}\times V_2\big|_{\pi_2}\;\middle|\;\mathsf{lab}[v_1,v_2]=\eta\right\}$$

are the reverse of the elements in the set

$$\left\{(v_2,v_1)\in V_2\big|_{\pi_2}\times V_1\big|_{\pi_1}\;\middle|\;\mathsf{lab}[v_2,v_1]=\widetilde{\eta}\right\}.$$

Hence, the sizes of two sets are the same. The size of these four sets implies the lemma.

For (3), observe That

$$\left\{(v_1,v_2)\in V_1\big|_{\pi_1}\times V_2\big|_{\pi_2}\;\middle|\;\mathsf{lab}[v_1,v_2]=\eta\right\}\subseteq V_1\big|_{\pi_1}\times V_2\big|_{\pi_2}.$$

Then, $n^{V_1\to V_2}_{\langle\pi_1,\eta,\pi_2\rangle} \leq \left|V_1\big|_{\pi_1}\times V_2\big|_{\pi_2}\right| = n^{V_1(\to V_2)}_{\langle\pi_1,\eta,\pi_2\rangle}\cdot n^{(V_1\to)V_2}_{\langle\pi_1,\eta,\pi_2\rangle}$.

For (4), note that for every $v\in V_1\big|^{\to V_2}_{\langle\pi_1,\eta,\pi_2\rangle}$, there exists v_2 such that

$$(v_1,v_2)\in\left\{(v_1,v_2)\in V_1\big|_{\pi_1}\times V_2\big|_{\pi_2}\;\middle|\;\mathsf{lab}[v_1,v_2]=\eta\right\}.$$

This implies the result.

For (5), by Lemma 3.27.1, 3.27.4, and 3.27.2, the following holds.

$$n^{(V_1\to)V_2}_{\langle\pi_1,\eta,\pi_2\rangle} = n^{V_2(\to V_1)}_{\langle\pi_2,\widetilde{\eta},\pi_1\rangle} \leq n^{V_2\to V_1}_{\langle\pi_2,\widetilde{\eta},\pi_1\rangle} = n^{V_1\to V_2}_{\langle\pi_1,\eta,\pi_2\rangle}$$

For (6), obviously, $V_1\big|^{\to V_2}_{\langle\pi_1,\eta,\pi_2\rangle}\subseteq V_1\big|_{\pi_1}$. Hence the equality holds.

For (7), by Lemma 3.27.1 and 3.27.6, $n^{(V_1\to)V_2}_{\langle\pi_1,\eta,\pi_2\rangle} = n^{V_2(\to V_1)}_{\langle\pi_2,\widetilde{\eta},\pi_1\rangle} \leq n^{V_2}_{\pi_2}$.

doi:10.6342/NTU202203098

For (8), because V_2 and V_3 are disjoint, we can divide the set to two disjoint part.

$$\left\{(v_1, v_2) \in V_1\big|_{\pi_1} \times (V_2 \cup V_3)\big|_{\pi_2} \;\middle|\; \mathsf{lab}[v_1, v_2] = \eta\right\}$$
$$= \left\{(v_1, v_2) \in V_1\big|_{\pi_1} \times V_2\big|_{\pi_2} \;\middle|\; \mathsf{lab}[v_1, v_2] = \eta\right\} \dot\cup \left\{(v_1, v_2) \in V_1\big|_{\pi_1} \times V_3\big|_{\pi_2} \;\middle|\; \mathsf{lab}[v_1, v_2] = \eta\right\}$$

The size of the sets implies the result.

For (9), by Lemma 3.27.1, 3.27.8, and 3.27.2, the following holds.

$$n^{(V_1 \cup V_2) \to V_3}_{\langle \pi_1, \eta, \pi_2 \rangle} = n^{V_3 \to (V_1 \cup V_2)}_{\langle \pi_2, \widetilde{\eta}, \pi_1 \rangle} = n^{V_3 \to V_1}_{\langle \pi_2, \widetilde{\eta}, \pi_1 \rangle} + n^{V_3 \to V_2}_{\langle \pi_2, \widetilde{\eta}, \pi_1 \rangle} = n^{V_1 \to V_3}_{\langle \pi_1, \eta, \pi_2 \rangle} + n^{V_2 \to V_3}_{\langle \pi_1, \eta, \pi_2 \rangle}$$

□

Lemma 3.28.

1. *If* $n_{\langle \pi_1, \eta, \pi_2 \rangle} > 0$*, then* $n_{\pi_1} > 0$.
2. *If* η *is non-trivial,* $n^{V_1 \to V}_{\langle \pi_1, \eta, \pi_2 \rangle} = \sum_{F \in \mathcal{F}^{\mathcal{C}^{V_1}}_{\pi_1}} n^{V_1}_{\langle \pi_1, F \rangle} \cdot F(\eta, \pi_2)$.

Proof. For (1), suppose to the contrary that $n_{\pi_1} = 0$. By Lemma 3.27.3 and 3.27.6.

$$n_{\langle \pi_1, \eta, \pi_2 \rangle} \le n^{V(\to V)}_{\langle \pi_1, \eta, \pi_2 \rangle} \cdot n^{(V \to)V}_{\langle \pi_1, \eta, \pi_2 \rangle} \le n_{\pi_1} \cdot n^{(V \to)V}_{\langle \pi_1, \eta, \pi_2 \rangle} = 0$$

Hence, a contradiction.

For (2), by definition of $n^{V_1 \to V}_{\langle \pi_1, \eta, \pi_2 \rangle}$ and realized behavior functions, the following holds.

$$\begin{aligned}
n^{V_1 \to V}_{\langle \pi_1, \eta, \pi_2 \rangle} &= \left|\left\{(v_1, v_2) \in V_1\big|_{\pi_1} \times V\big|_{\pi_2} \;\middle|\; \mathsf{lab}[v_1, v_2] = \eta\right\}\right| \\
&= \left|\bigcup_{v_1 \in V_1|_{\pi_1}} \left\{(v_1, v_2) \in \{v_1\} \times V\big|_{\pi_2} \;\middle|\; \mathsf{lab}[v_1, v_2] = \eta\right\}\right| \\
&= \sum_{v_1 \in V_1|_{\pi_1}} \left|\left\{v_2 \in V\big|_{\pi_2} \;\middle|\; \mathsf{lab}[v_1, v_2] = \eta\right\}\right| \\
&= \sum_{v_1 \in V_1|_{\pi_1}} F[v_1](\eta, \pi_2) \\
&= \sum_{F \in \mathcal{F}^{\mathcal{C}^{V_1}}_{\pi_1}} n^{V_1}_{\langle \pi_1, F \rangle} \cdot F(\eta, \pi_2)
\end{aligned}$$

doi:10.6342/NTU202203098

$\square$

Pseudo-model of the C^2 sentences. Now we extend the conditions of structure $\mathcal{A}$ satisfying φ to the pseudo-structures. We defined the concept em pseudo-model. The pseudo-model share many properties with the model. In the rest of the thesis, we only consider pseudo-model and their properties.

Lemma 3.29. *A C^2 sentence φ is satisfiable if and only if there is a pseudo-structure $\mathcal{G}$ such that the following holds.*

1. *For every $v \in V$, $\mathsf{lab}[v]$ is valid in φ.*
2. *For every $(v_1, v_2) \in V \times V \setminus \{(v, v) \mid v \in V\}$, the tuple $\langle \mathsf{lab}[v_1], \mathsf{lab}[v_1, v_2], \mathsf{lab}[v_2] \rangle$ is valid in φ.*
3. *For every $v \in V$, $\sum_{v' \in V \setminus \{v\}} \mathsf{lab}^{\triangleright}[v, v'] = \vec{k}$.*

Proof. Let $\mathcal{A} = (A, U_1^{\mathcal{A}}, \ldots, U_n^{\mathcal{A}}, \beta_1^{\mathcal{A}}, \ldots, \beta_m^{\mathcal{A}})$ be a model of φ. We construct the following pseudo-structure $\mathcal{G}$.

- The universe $V := A$.
- For every $v \in V$, $\mathsf{lab}[v] := \{U \mid v \in U^{\mathcal{A}}\} \cup \{\neg U \mid v \notin U^{\mathcal{A}}\}$.
- For every $v_1, v_2 \in V$, $\mathsf{lab}[v_1, v_2]$ is the union of following sets.

$$\begin{aligned}\mathsf{lab}[v_1, v_2] = &\{\beta(x, y) \mid (v_1, v_2) \in \beta^{\mathcal{A}}\} \cup \{\neg\beta(x, y) \mid (v_1, v_2) \notin \beta^{\mathcal{A}}\} \cup \\ &\{\beta(y, x) \mid (v_2, v_1) \in \beta^{\mathcal{A}}\} \cup \{\neg\beta(y, x) \mid (v_2, v_1) \notin \beta^{\mathcal{A}}\}\end{aligned}$$

It is straightforward to check that $\mathcal{G}$ satisfied the conditions. $\square$

Lemma 3.30. *A C^2 sentence φ is satisfiable if and only if there is a pseudo-structure $\mathcal{G}$ such that the conditions (1) and (2) in Lemma 3.29 and the following holds.*

3′. *For every $v \in V$, $C[a]$ is valid in φ.*

doi:10.6342/NTU202203098

Proof. It is sufficient to show that condition (3) in Lemma 3.29 and (3′) are equivalent. For every $v \in V$, clearly, the source type of v is its realized 1-type, that is, $\mathsf{lab}[v]$. By condition (1), the source type is valid in φ. For every $\pi \in \Pi^{\varphi}$, for every $\eta \in \mathcal{K}^{\tau,\mathsf{ntri}}$, if $\langle \pi_s, \eta, \pi \rangle$ is not valid in φ, then there are no edges in V realize the tuple. Hence, $F[v](\eta, \pi) = 0$.

Finally, by definition of realized behavior function, the following holds.

$$\begin{aligned}
\sum_{\eta \in \mathcal{K}^{\tau,\mathsf{ntri}}} \sum_{\pi \in \Pi^{\tau}} F[v](\eta,\pi) \cdot \eta^{\triangleright} &= \sum_{\eta \in \mathcal{K}^{\tau,\mathsf{ntri}}} \sum_{\pi \in \Pi^{\tau}} |\{v' \in V \setminus \{v\} \mid \mathsf{lab}[v,v'] = \eta \text{ and } \mathsf{lab}[v'] = \pi\}| \cdot \eta^{\triangleright} \\
&= \sum_{\eta \in \mathcal{K}^{\tau,\mathsf{ntri}}} |\{v' \in V \setminus \{v\} \mid \mathsf{lab}[v,v'] = \eta\}| \cdot \eta^{\triangleright} \\
&= \sum_{\eta \in \mathcal{K}^{\tau}} |\{v' \in V \setminus \{v\} \mid \mathsf{lab}[v,v'] = \eta\}| \cdot \eta^{\triangleright} \\
&= \sum_{v' \in V \setminus \{v\}} \mathsf{lab}^{\triangleright}[v,v']
\end{aligned}$$

Note that for every $\eta \in \mathcal{K}^{\tau} \setminus \mathcal{K}^{\tau,\mathsf{ntri}} = \mathcal{K}^{\tau,\mathsf{tri}}$, $\eta^{\triangleright} = \vec{0}$. By the counting condition of $C[v]$, $\sum_{\eta \in \mathcal{K}^{\tau,\mathsf{ntri}}} \sum_{\pi \in \Pi^{\tau}} F[v](\eta,\pi) \cdot \eta^{\triangleright} = \vec{k}$. □

Remark 3.31. *In view of Lemma 3.29, to check the satisfiability of a* C^2 *sentence, it suffices to check the existence of a pseudo-structure* $\mathcal{G}$ *that satisfies the conditions.*

In fact, the construction shown in Lemma 3.29 is a bijection between the set of structures over τ *and the set of pseudo-structure over* τ*. Hence we will say a pseudo-structure* $\mathcal{G}$ *is a "pseudo-model" of the* C^2 *sentence* φ*, and* φ *is "satisfied" by* $\mathcal{G}$*. Moreover, it is not hard to verify that this bijection is size preserving. Therefore,* φ *is finitely satisfiable if and only if it has a finite size pseudo-model.*

Example 3.32. *Let* τ_0, φ_0*, and* $\mathcal{G}_0$ *be as in Example 3.25. Condition (1)-(3) in Lemma 3.29 can be verified. Hence,* $\mathcal{G}_0$ *is a pseudo-model of* φ_0*, implying* φ_0 *is satisfiable.*

Intrinsic finiteness. We will show that for every pair of 1-types $(\pi_1, \pi_2) \in \Pi^{\varphi} \times \Pi^{\varphi}$, if they are not null-compatible in φ, then for every pseudo-model of φ, the number of realizations of π_1 and π_2 are bounded. We call this property the *intrinsic finiteness* of φ, which is an important observation to derive the CEB property in the next chapter.

doi:10.6342/NTU202203098

Lemma 3.33. *Let $\mathcal{G}$ be a pseudo-model of φ. For every 1-type $\pi_1, \pi_2 \in \Pi^V$, if they are not null-compatible in φ, then $n_{\pi_1} \leq 2K+1$ or $n_{\pi_2} \leq 2K+1$.*

Proof.

Case 1: $\pi_1 = \pi_2$. Because $\mathcal{G}$ is a pseudo-model of φ, for every $v \in V^{\pi_1}$, by the counting condition, the following holds.

$$\sum_{v' \in V^{\pi_1} \setminus \{v\}} \mathsf{lab}^{\triangleright}[v,v'] + \sum_{v' \in V \setminus V^{\pi_1}} \mathsf{lab}^{\triangleright}[v,v'] = \sum_{v' \in V \setminus \{v\}} \mathsf{lab}^{\triangleright}[v,v'] = \vec{k}$$

Recall that for every 2-type $\eta \in \mathcal{K}^{\tau}$, $\|\eta\|_1 \geq 0$. Summing over all the entries and all elements in V^{π_1}, we have the following.

$$\begin{aligned}\sum_{v \in V^{\pi_1}} \left(\sum_{v' \in V^{\pi_1} \setminus \{v\}} \left\|\mathsf{lab}^{\triangleright}[v,v']\right\|_1 \right) &= \sum_{v \in V^{\pi_1}} \left(K - \sum_{v' \in V \setminus V^{\pi_1}} \left\|\mathsf{lab}^{\triangleright}[v,v']\right\|_1 \right) \\ &\leq \sum_{v \in V^{\pi_1}} K \\ &= n_{\pi_1} K \end{aligned} \tag{3.1}$$

On the other hand, consider the summation of the forward vector representation of all edges between V^{π_1} in $\mathcal{G}$. Because of the constraint of pseudo-structure, for every $v, v' \in V^{\pi_1}$, $\mathsf{lab}^{\triangleright}[v',v] = \mathsf{lab}^{\triangleleft}[v,v']$. Therefore, the following holds.

$$\begin{aligned}\sum_{v \in V^{\pi_1}} \sum_{v' \in V^{\pi_1} \setminus \{v\}} \mathsf{lab}^{\triangleright}[v,v'] &= \frac{1}{2} \sum_{v \in V^{\pi_1}} \sum_{v' \in V^{\pi_1} \setminus \{v\}} \mathsf{lab}^{\triangleright}[v,v'] + \mathsf{lab}^{\triangleright}[v',v] \\ &= \frac{1}{2} \sum_{v \in V^{\pi_1}} \sum_{v' \in V^{\pi_1} \setminus \{v\}} \mathsf{lab}^{\triangleright}[v,v'] + \mathsf{lab}^{\triangleleft}[v,v']\end{aligned}$$

Since π_1 is not null-compatible to itself, for every $v, v' \in V^{\pi_1}$, $\mathsf{lab}^{\triangleright}[v,v']$ is not the null type. Recall that if $\eta \in \mathcal{K}^{\tau}$ is not the null type, $\|\eta^{\triangleright}\|_1 + \|\eta^{\triangleleft}\|_1 \geq 1$. Hence, summing over

doi:10.6342/NTU202203098

all the entries, we have the following.

$$\begin{aligned}\sum_{v\in V^{\pi_1}}\sum_{v'\in V^{\pi_1}\setminus\{v\}}\left\|\mathsf{lab}^{\triangleright}[v,v']\right\|_1 &= \frac{1}{2}\sum_{v\in V^{\pi_1}}\sum_{v'\in V^{\pi_1}\setminus\{v\}}\left\|\mathsf{lab}^{\triangleright}[v,v']\right\|_1+\left\|\mathsf{lab}^{\triangleleft}[v,v']\right\|_1\\ &\geq \frac{1}{2}\sum_{v\in V^{\pi_1}}\sum_{v'\in V^{\pi_1}\setminus\{v\}}1\\ &= \frac{1}{2}n_{\pi_1}(n_{\pi_1}-1)\end{aligned} \tag{3.2}$$

Finally, the equation (3.1) and (3.2) imply that $n_{\pi_1}K \geq \frac{1}{2}n_{\pi_1}(n_{\pi_1}-1)$. Hence, $n_{\pi_1} \leq 2K+1$.

Case 2: $\pi_1 \neq \pi_2$. By the similar strategy for case 1, the following holds.

$$\sum_{v_1\in V^{\pi_1}}\sum_{v_2\in V^{\pi_2}}\left\|\mathsf{lab}^{\triangleright}[v_1,v_2]\right\|_1+\left\|\mathsf{lab}^{\triangleright}[v_2,v_1]\right\|_1 \leq n_{\pi_1}K+n_{\pi_2}K=(n_{\pi_1}+n_{\pi_2})\,K$$

On the other hand, because π_1 and π_2 are not null-compatible, for every $v_1 \in V^{\pi_1}$ and $v_2 \in V^{\pi_2}$, $\left\|\mathsf{lab}^{\triangleright}[v_1,v_2]\right\|_1+\left\|\mathsf{lab}^{\triangleleft}[v_1,v_2]\right\|_1 \geq 1$. Besides, note that $\mathsf{lab}^{\triangleright}[v_2,v_1]=\mathsf{lab}^{\triangleleft}[v_1,v_2]$. Hence, the following holds.

$$\sum_{v_1\in V^{\pi_1}}\sum_{v_2\in V^{\pi_2}}\left\|\mathsf{lab}^{\triangleright}[v_1,v_2]\right\|_1+\left\|\mathsf{lab}^{\triangleleft}[v_1,v_2]\right\|_1 \geq \sum_{v_1\in V^{\pi_1}}\sum_{v_2\in V^{\pi_2}}1=n_{\pi_1}n_{\pi_2}$$

The equation (3.3) and (3.3) imply that $n_{\pi_1}n_{\pi_2} \leq (n_{\pi_1}+n_{\pi_2})\,K$. Suppose to the contrary that both $n_{\pi_1} > 2K+1$ and $n_{\pi_2} > 2K+1$. are strict greater then $2K+1$. We observe that $n_{\pi_1}-K>0$. Therefore, $n_{\pi_2} \leq \frac{K}{1-\frac{K}{n_{\pi_1}}} \leq \frac{K}{1-\frac{K}{2K+1}} \leq 2K$. This contradict that $n_{\pi_2} > 2K+1$. □

doi:10.6342/NTU202203098

doi:10.6342/NTU202203098

Chapter 4

Configurations exponential bound property

In this chapter, we present the main result of this thesis. In Section 4.1, we show that the exponential size bound (ESM) property no longer holds for C^2. In Section 4.2, we state and prove the *configuration exponential bound* (CEB) property for C^2. In Section 4.3, we present the finite version of the CEB property. Finally, we close this chapter with the upper bound of the smallest model in Section 4.4.

4.1 Noncoincidence of $\mathsf{SAT}(\mathsf{C}^2)$ and $\mathsf{FIN\text{-}SAT}(\mathsf{C}^2)$

Unlike FO^2, $\mathsf{SAT}(\mathsf{C}^2)$ and $\mathsf{FIN\text{-}SAT}(\mathsf{C}^2)$ do not coincide. So, in general, it is unlikely that we can design an algorithm for $\mathsf{SAT}(\mathsf{C}^2)$ by guessing and checking the model. In fact, Example 3.32 gives a C^2 sentence that is satisfiable but not finitely satisfiable.

Lemma 4.1. *Let τ_0, φ_0 be as in Example 3.32. There is no finite pseudo-model satisfying φ_0.*

Proof. The proof is by contradiction. Suppose $\mathcal{H}_0$ is a finite pseudo-model of φ. Recall that we have shown that there are only 3 valid configurations C_1, C_2 and C_3 in φ_0 in Example 3.19. By Lemma 3.28.2, the following equations about the number of realizations

doi:10.6342/NTU202203098

in $\mathcal{H}_0$ hold.

$$
\begin{aligned}
n_{\langle\pi_2,\eta_2,\pi_2\rangle} &= n_{C_2}\cdot F_2(\eta_2,\pi_2) + n_{C_3}\cdot F_3(\eta_2,\pi_2) = n_{C_2}+n_{C_3}\\
n_{\langle\pi_2,\widetilde{\eta_2},\pi_2\rangle} &= n_{C_2}\cdot F_2(\widetilde{\eta_2},\pi_2) + n_{C_3}\cdot F_3(\widetilde{\eta_2},\pi_2) = n_{C_3}\\
n_{\langle\pi_1,\eta_1,\pi_2\rangle} &= n_{C_1}\cdot F_1(\eta_1,\pi_2) = n_{C_1} \qquad (4.1)\\
n_{\langle\pi_2,\widetilde{\eta_1},\pi_1\rangle} &= n_{C_2}\cdot F_2(\widetilde{\eta_1},\pi_1) + n_{C_3}\cdot F_3(\widetilde{\eta_1},\pi_1) = n_{C_2}\\
n_{\langle\pi_2,\widetilde{\eta_3},\pi_1\rangle} &= n_{C_2}\cdot F_2(\widetilde{\eta_3},\pi_1) + n_{C_3}\cdot F_3(\widetilde{\eta_3},\pi_1) = n_{C_2}+n_{C_3}
\end{aligned}
$$

By Lemma 3.27.2, $n_{\langle\pi_2,\eta_2,\pi_2\rangle} = n_{\langle\pi_2,\widetilde{\eta_2},\pi_2\rangle}$ and $n_{\langle\pi_1,\eta_1,\pi_2\rangle} = n_{\langle\pi_2,\widetilde{\eta_1},\pi_1\rangle}$ which implies the following linear system holds.

$$
n_{C_2}+n_{C_3} = n_{\langle\pi_2,\eta_2,\pi_2\rangle} = n_{\langle\pi_2,\widetilde{\eta_2},\pi_2\rangle} = n_{C_3}
$$

$$
n_{C_1} = n_{\langle\pi_1,\eta_1,\pi_2\rangle} = n_{\langle\pi_2,\widetilde{\eta_1},\pi_1\rangle} = n_{C_2}
$$

The above system has finite solution if and only if $n_{C_1} = n_{C_2} = 0$. Note that C_1 is the only valid configuration with source type π_1 in φ_0, by Lemma 3.26.3, $n_{\pi_1} = n_{C_1} = 0$. By the consequence of Lemma 3.28.1 together with equation (4.1), $n_{C_2}+n_{C_3} = n_{\langle\pi_2,\widetilde{\eta_3},\pi_1\rangle} = 0$. Therefore $n_{C_3} = 0$. By Lemma 3.26.2, the size of $\mathcal{H}_0$ is $n_{C_1}+n_{C_2}+n_{C_3} = 0$, which is a contradiction. □

Theorem 4.2. [14, 25] $\mathsf{SAT}(\mathsf{C}^2)$ *and* $\mathsf{FIN\text{-}SAT}(\mathsf{C}^2)$ *do not coincide.*

Proof. Let φ_0 be as in Example 3.32. By Example 3.32, $\varphi_0 \in \mathsf{SAT}(\mathsf{C}^2)$ but by Lemma 4.1, $\varphi_0 \notin \mathsf{FIN\text{-}SAT}(\mathsf{C}^2)$. □

There is, however, an exponential bound on the number of configurations in the pseudo-structures that satisfy C^2 sentences. More precisely, a C^2 sentence φ is satisfiable if and only if it has a pseudo-model $\mathcal{G}$ satisfying that the number of configurations realized in $\mathcal{G}$ is at most exponential in the length of φ. We call this the *configurations exponential bound* (CEB) property for C^2. The rest of this chapter is devoted to proving the CEB property. The purpose is to design the algorithm for $\mathsf{SAT}(\mathsf{C}^2)$ by guessing the set of configurations and checking whether there is a model that realizes it.

doi:10.6342/NTU202203098

4.2 CEB property for $\mathsf{SAT}(\mathsf{C}^2)$

In the rest of this chapter, we fix a vocabulary τ, a C^2 sentence φ over τ in the normal form, and a pseudo-structure $\mathcal{G} = \left\langle V, \mathsf{lab}_{\mathsf{V}}^{\mathcal{G}}, \mathsf{lab}_{\mathsf{E}}^{\mathcal{G}} \right\rangle$ over τ. We need some auxiliary terminology.

Definition 4.3. *A partition* $V_{\mathsf{F}} \dot{\cup} V_{\mathsf{E}} \dot{\cup} V_{\mathsf{S}} = V$ *is a proper partition of* $\mathcal{G}$ *in* φ *if it satisfies the following.*

1. *For every* $v \in V$, $\mathsf{lab}[v] \in \Pi^{\varphi}$.
2. *For every* $v, v' \in V$, *if* $v \neq v'$, *then* $\mathsf{lab}[v, v'] \in \mathcal{K}^{\varphi}_{\langle \mathsf{lab}[v], *, \mathsf{lab}[v'] \rangle}$.
3. *For every* $v \in V_{\mathsf{F}} \cup V_{\mathsf{E}}$, $\sum_{v' \in \mathcal{G} \setminus \{v\}} \mathsf{lab}[v, v'] = \vec{k}$.
4. *For every* $v \in V_{\mathsf{F}}$, *for every* $v' \in V_{\mathsf{S}}$, $\mathsf{lab}^{\triangleright}[v, v'] = \vec{0}$.
5. $\Pi^{V_{\mathsf{F}}}$ *and* $\Pi^{V_{\mathsf{E}} \cup V_{\mathsf{S}}}$ *are disjoint.*
6. $\Pi^{V_{\mathsf{E}} \cup V_{\mathsf{S}}}$ *is mutually null-compatible.*

Remark 4.4. *The proper partition of* $\mathcal{G}$ *is not unique.*

The subscripts F, E, and S are for *finite*, *extended*, and *strong*, respectively. The intuition of proper partition is as follows. For every pseudo-model of φ, by Lemma 3.33, the number of elements in the pseudo-model realized non null-compatible 1-types is bounded. The finite part collects those elements. Hence, the size of the finite part is bounded, and all remaining elements are null-compatible. It is called *strongly satisfiable*, which will be discussed more in Chapter 6. Since the strong part satisfies some good properties, we can represent them with a set of valid configurations instead of the elements themself.

To simplify the conditions of the edges between the finite part and the strong part, we introduce the extended part between them. Therefore, our strategy for the CEB property is as follows. First, we can guess the elements in the pseudo-model and verify conditions directly for the finite part and the extended part. Then, for the strong part, we can guess the set of configurations and check the conditions in the following two definitions.

doi:10.6342/NTU202203098

Definition 4.5. *Let $V_\mathsf{F}\dot\cup V_\mathsf{E}\dot\cup V_\mathsf{S}$ be a proper partition of $\mathcal{G}$ in φ and $\mathcal{C}$ be a set of valid configurations in φ. We say that $\mathcal{G}$, $V_\mathsf{F}\dot\cup V_\mathsf{E}\dot\cup V_\mathsf{S}$ and $\mathcal{C}$ are matching in φ if they satisfy the following.*

1. $\Pi^{\mathcal{C}} \subseteq \Pi^{V_\mathsf{S}}$.
2. *For every $C \in \mathcal{C}$, C conforms Π^V.*
3. *For every $\langle \pi_s, F\rangle \in \mathcal{C}$, for every $\pi \in \Pi^{V_\mathsf{F}}$, for every $\eta \in \mathcal{K}^{\tau,\mathsf{ntri}}$, if $\eta^{\triangleleft} \neq 0$, $F(\eta,\pi) = 0$.*
4. *For every $\langle \pi_s, F\rangle \in \mathcal{C}$, for every $\pi \in \Pi^{V_\mathsf{F}}$, if π_s and π are not null-compatible in φ, then $\sum_{\eta\in\mathcal{K}^{\tau,\mathsf{ntri}}} F(\eta,\pi) = n_\pi$. Otherwise, $\sum_{\eta\in\mathcal{K}^{\tau,\mathsf{ntri}}} F(\eta,\pi) \leq n_\pi$.*

As mentioned above, the intuition of the set of valid configurations is the collection of configurations realized in the strong part of the pseudo-model. However, not every set is the collection of the strong part of some pseudo-model. The notion of matching captures their necessary conditions.

Definition 4.6. *Let $V_\mathsf{F}\dot\cup V_\mathsf{E}\dot\cup V_\mathsf{S}$ be a proper partition of $\mathcal{G}$ in φ and $\mathcal{C}$ be a set of valid configurations in φ. The induced ILP system $\mathcal{Q}_{\mathcal{G},V_\mathsf{F}\dot\cup V_\mathsf{E}\dot\cup V_\mathsf{S},\mathcal{C}}$ is defined as follows.*

The variables in the system are $\{x_C\}_{C\in\mathcal{C}}$. There are four types of constraints in the system. For every $\pi \in \Pi^{V_\mathsf{E}\cup V_\mathsf{S}}$, the following constraint is in the system.

1. $n_\pi^{V_\mathsf{E}} + \sum_{C\in\mathcal{C}|_\pi} x_C \geq 3K+3$.

For every $\pi_1,\pi_2 \in \Pi^{V_\mathsf{E}\cup V_\mathsf{S}}$, for every $\eta \in \mathcal{K}^{\varphi,\mathsf{ntri}}_{\langle\pi_1,*,\pi_2\rangle}$, if $\eta^{\triangleleft} \neq \vec{0}$ and at least one of the following hold,

- $n^{V_\mathsf{E}\to V_\mathsf{S}}_{\langle\pi_1,\eta,\pi_2\rangle} > 0$, *or*
- *there exists $F \in \mathcal{F}^{\mathcal{C}}_{\pi_1}$ such that $F(\eta,\pi_2) > 0$,*

then the following two constraints are in the system.

2. $n^{V_\mathsf{E}\to V_\mathsf{S}}_{\langle\pi_1,\eta,\pi_2\rangle} + \sum_{F\in\mathcal{F}^{\mathcal{C}}_{\pi_1}} x_{\langle\pi_1,F\rangle}\cdot F(\eta,\pi_2) \geq (2K+1)^2$.
3. $n^{V_\mathsf{E}\to V_\mathsf{S}}_{\langle\pi_1,\eta,\pi_2\rangle} + \sum_{F\in\mathcal{F}^{\mathcal{C}}_{\pi_1}} x_{\langle\pi_1,F\rangle}\cdot F(\eta,\pi_2) = n^{V_\mathsf{E}\to V_\mathsf{S}}_{\langle\pi_2,\widetilde{\eta},\pi_1\rangle} + \sum_{F\in\mathcal{F}^{\mathcal{C}}_{\pi_2}} x_{\langle\pi_2,F\rangle}\cdot F(\widetilde{\eta},\pi_1)$.

doi:10.6342/NTU202203098

Finally, if $|V_{\mathsf{F}}| + |V_{\mathsf{E}}| = 0$, then the following constraint is in the system.

4. $\sum_{C \in \mathcal{C}} x_C > 0$.

The induced ILP system captures the sufficient condition of the strong part of the pseudo-model. Recall that, by definition, the null type is valid between every element in the strong part. Hence, unlike the finite part in which we need to assign *all* edges in the pseudo-structure, we only care about the non null type edge for the strong part. In fact, we can set null type to all remaining edges such that the assignment is valid and preserves the counting condition of elements.

Observe that the number of edges increases quadratically in the size of the strong part, but the number of non-trivial edges required increases linearly. Hence, if the strong part is large enough, we can always find a suitable assignment. The first two types of constraints capture this intuition. The third type constraint state that the number of edges with some type and the number of edges with the reverse type should be the same. Finally, the fourth condition prevents the trivial solution.

The following observation and lemma provide the upper bound of the size of the induced ILP system.

Remark 4.7. *Observe that the number of the first type of constraints in the system $\mathcal{Q}_{\mathcal{G},V_{\mathsf{F}}\dot{\cup}V_{\mathsf{E}}\dot{\cup}V_{\mathsf{S}},\mathcal{C}}$ is bounded by $\left|\Pi^{V_{\mathsf{E}}\cup V_{\mathsf{S}}}\right| \leq |\Pi^{\tau}| = 2^n$. The number of the last two types constraints in the system is bounded by the following.*

$$\left|\Pi^{V_{\mathsf{E}}\cup V_{\mathsf{S}}}\right|^2 \left|\mathcal{K}^{\varphi,\mathsf{ntri}}_{\langle \pi_1,*,\pi_2\rangle}\right| \leq |\Pi^{\tau}|^2 |\mathcal{K}^{\tau}| = 2^{2m+2n}$$

Hence, the number of constraints in the system is bounded by $3 \cdot 2^{2m+2n}$.

Lemma 4.8. *Let $V_{\mathsf{F}}\dot{\cup}V_{\mathsf{E}}\dot{\cup}V_{\mathsf{S}}$ be a proper partition of $\mathcal{G}$ in φ and $\mathcal{C}$ be a set of valid configurations in φ. The coefficients in the system $\mathcal{Q}_{\mathcal{G},V_{\mathsf{F}}\dot{\cup}V_{\mathsf{E}}\dot{\cup}V_{\mathsf{S}},\mathcal{C}}$ are non-negative and bounded by* $\max(|V_{\mathsf{E}}|, (2K+1)^2)$.

Proof. For every $\langle \pi_s, F\rangle \in \mathcal{C}$, by definition, it is valid in φ. Therefore, for every $\pi \in \Pi^{\tau}$, for every $\eta \in \mathcal{K}^{\tau,\mathsf{ntri}}$, by Lemma 3.17, $0 \leq F(\pi, \eta) \leq K$.

doi:10.6342/NTU202203098

For every $\pi_1, \pi_2 \in \Pi^{V_{\mathsf{E}} \cup V_{\mathsf{S}}}$, for every $\eta \in \mathcal{K}^{\varphi,\mathsf{ntri}}_{\langle \pi_1, *, \pi_2 \rangle}$, by Lemma 3.27.8 and 3.28.2, the number of realizations is bounded by the behavior functions realized in V_{E}.

$$0 \le n^{V_{\mathsf{E}} \to V_{\mathsf{S}}}_{\langle \pi_1, \eta, \pi_2 \rangle} \le n^{V_{\mathsf{E}} \to V}_{\langle \pi_1, \eta, \pi_2 \rangle} = \sum_{F \in \mathcal{F}^{\mathcal{C}^{V_{\mathsf{E}}}}_{\pi_1}} n^{V_{\mathsf{E}}}_{\langle \pi_1, F \rangle} \cdot F(\eta, \pi_2)$$

Because $V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}$ is a proper partition of $\mathcal{G}$, every realizable configuration in V_{E} is valid. By Lemma 3.17 and 3.26.2, the size of both terms is bounded by the following.

$$n^{V_{\mathsf{E}} \to V_{\mathsf{S}}}_{\langle \pi_1, \eta, \pi_2 \rangle} \le \sum_{F \in \mathcal{F}^{\mathcal{C}^{V_{\mathsf{E}}}}_{\pi_1}} n^{V_{\mathsf{E}}}_{\langle \pi_1, F \rangle} \cdot F(\eta, \pi_2) \le \left(\sum_{F \in \mathcal{F}^{\mathcal{C}^{V_{\mathsf{E}}}}_{\pi_1}} n^{V_{\mathsf{E}}}_{\langle \pi_1, F \rangle} \right) \cdot K \le |V_{\mathsf{E}}| K$$

Clearly, all the coefficients of the system are non-negative. Besides, the coefficients are upper bounded by $\max\left(K, |V_{\mathsf{E}}|K, (2K+1)^2, 3K+3\right) = \max\left(|V_{\mathsf{E}}|K, (2K+1)^2\right)$.

□

Now, we are ready to describe our main result, the configuration exponential bound property for $\mathsf{SAT}(\mathsf{C}^2)$.

Theorem 4.9 (The CEB property for $\mathsf{SAT}(\mathsf{C}^2)$)**.** *The C^2 sentence φ is satisfiable if and only if there exist the following components.*

- *A pseudo-structure $\mathcal{G}$ over τ with a proper partition $V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}$ in φ.*
- *A set of valid configurations $\mathcal{C}$ satisfying $\mathcal{G}$, $V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}$ and $\mathcal{C}$ are matching in φ.*
- *A solution $\{x_C \mapsto p_C\}_{C \in \mathcal{C}}$ of the system $\mathcal{Q}_{\mathcal{G}, V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}, \mathcal{C}}$.*

Furthermore, their sizes are bounded by the length of φ. More precisely,

- *$|V| \le \theta_1(n, m, K)$;*
- *$|\mathcal{C}| \le \theta_2(n, m, K)$;*

doi:10.6342/NTU202203098

- *for every $C \in \mathcal{C}$, $p_C \in [0, \theta_3(n, m, K)] \cup \{\infty\}$, where*

$$\theta_1(n, m, K) := 2^{2m+2n+2}(2K+1)^2$$

$$\theta_2(n, m, K) := 2^{2n+2m+3}\left(4n + 4m + 6 + 2\lceil \log(2K+1) \rceil\right)$$

$$\theta_3(n, m, K) := 2^{2n+2m+4}\left(4n + 4m + 6 + 2\lceil \log(2K+1) \rceil\right)\left(2^{4n+4m+4}(2K+1)^2\right)^{2^{2n+2m+3}}.$$

Proof. By Lemma 4.21 and 4.22. □

Remark 4.10. *Note that $\theta_1(n, m, K)$ and $\theta_2(n, m, K)$ are exponential in the length of φ, and $\theta_3(n, m, K)$ is double-exponential in the length of φ.*

The proof of Theorem 4.9 is rather technical. So we divide it into two parts. Subsection 4.2.1 deals with the "only if" direction, and Subsection 4.2.2 deals with the "if" direction.

4.2.1 Proof of the only if direction

We divide the proof of the only if direction into two stages. In the first stage, we show that the desired components can be obtained by a pseudo-model of φ, but the size of them are unbounded. In the second stage, we show how to bound the size of each component by some techniques from linear algebra.

Stage 1: Standard partition. We first define the *standard partition of the pseudo-model of φ*. The intuitive meaning of this partition is as follows. As mentioned above, the intuition of the finite part is the collection of all elements whose 1-types are not null-compatible. Note that for every pseudo-model of φ, the number of these elements in the pseudo-model is bounded. Therefore, the finite part of the standard partition collects all 1-types which realized by at most $3K + 3$ elements.

Besides, recall that for every 2-type η realized in the strong part, the induced ILP system required that the number of realizations of η is at least $(2K + 1)^2$. Hence, if η doesn't satisfy the condition, we move the elements which realize η into the extended part. We repeat this procedure until the strong part satisfies these conditions.

doi:10.6342/NTU202203098

Definition 4.11. *Let $\mathcal{G}$ be a pseudo-model of φ. The standard partition of $\mathcal{G}$ in φ is a partition $V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}$ where each component is defined as follows.*

- $V_{\mathsf{F}} := \bigcup_{\{\pi \in \Pi^V \mid 0 < n_\pi \leq 3K+3\}} V\big|_{\pi}$

Let V_{F}^C be the complement of V_{F}, that is, $V_{\mathsf{F}}^C := V \setminus V_{\mathsf{F}}$. The set V_{E} and V_{S} are defined inductively. The base case is as follows.

$$V_{\mathsf{E}}^0 := \left\{ v \in V_{\mathsf{F}}^C \;\middle|\; \text{There exists } v' \in V_{\mathsf{F}} \text{ such that } \mathsf{lab}^{\triangleright}[v', v] \neq \vec{0}. \right\}$$

$$V_{\mathsf{S}}^0 := V_{\mathsf{F}}^C \setminus V_{\mathsf{E}}^0$$

For every $i > 0$, let $\langle \pi_1, \eta, \pi_2 \rangle \in \Pi^\tau \times \mathcal{K}^\tau \times \Pi^\tau$ be the tuple with smallest lexicographical order such that $0 < n_{\langle \pi_1, \eta, \pi_2 \rangle}^{V_{\mathsf{S}}^i \to V_{\mathsf{S}}^i} + n_{\langle \pi_1, \eta, \pi_2 \rangle}^{V_{\mathsf{E}}^i \to V_{\mathsf{S}}^i} + n_{\langle \pi_1, \eta, \pi_2 \rangle}^{V_{\mathsf{S}}^i \to V_{\mathsf{E}}^i} \leq (2K+1)^2$. If such tuple exists, then V_{E}^{i+1} and V_{S}^{i+1} are defined as follows.

$$V_{\mathsf{E}}^{i+1} := V_{\mathsf{E}}^i \cup V_{\mathsf{S}}^i\Big|_{\langle \pi_1, \eta, \pi_2 \rangle}^{\to V_{\mathsf{S}}^i} \cup V_{\mathsf{S}}^i\Big|_{\langle \pi_1, \eta, \pi_2 \rangle}^{V_{\mathsf{S}}^i \to} \cup V_{\mathsf{S}}^i\Big|_{\langle \pi_1, \eta, \pi_2 \rangle}^{\to V_{\mathsf{E}}^i} \cup V_{\mathsf{S}}^i\Big|_{\langle \pi_1, \eta, \pi_2 \rangle}^{V_{\mathsf{E}}^i \to}$$

$$V_{\mathsf{S}}^{i+1} := V_{\mathsf{F}}^C \setminus V_{\mathsf{E}}^{i+1}$$

Otherwise, $V_{\mathsf{E}}^{i+1} := V_{\mathsf{E}}^i$ and $V_{\mathsf{S}}^{i+1} := V_{\mathsf{S}}^i$. Finally, V_{F} and V_{S} are defined as follows.

- $V_{\mathsf{F}} := V_{\mathsf{F}}^{2^{2m+2n}}$

- $V_{\mathsf{S}} := V_{\mathsf{S}}^{2^{2m+2n}}$

Remark 4.12. *Note that for every $i \geq 0$, the inductive definition of the standard partition satisfies $V_{\mathsf{E}}^i \cup V_{\mathsf{S}}^i = V_{\mathsf{F}}^C$. Hence, $V_{\mathsf{E}}^i \cup V_{\mathsf{S}}^i$ and V_{F} are disjoint. Besides, since $V_{\mathsf{S}}^i := V_{\mathsf{F}}^C \setminus V_{\mathsf{E}}^i$, V_{E}^i and V_{S}^i are disjoint. This implies that $V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}$ is a partition of V.*

Remark 4.13. *Unlike the proper partition, each pseudo-model has a unique standard partition.*

Lemma 4.14. *Let $\mathcal{G}$ be a pseudo-model of φ. The standard partition of $\mathcal{G}$, $V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}$, satisfies the following.*

1. $|V_{\mathsf{F}}| \leq 2^n(3K+3)$.

doi:10.6342/NTU202203098

2. $|V_{\mathsf{E}}| \leq 2^n(3K+3)K + 2^{2m+2n+1}(2K+1)^2$.

3. *For every* $\pi \in \Pi^\tau$, $n_\pi^{V_{\mathsf{E}} \cup V_{\mathsf{S}}} = 0$ *or* $n_\pi^{V_{\mathsf{E}} \cup V_{\mathsf{S}}} > 3K+3$.

4. *For every* $\langle \pi_1, \eta, \pi_2 \rangle \in \Pi^\tau \times \mathcal{K}^\tau \times \Pi^\tau$, *the value of* $n^{V_{\mathsf{S}} \to V_{\mathsf{S}}}_{\langle \pi_1, \eta, \pi_2 \rangle} + n^{V_{\mathsf{E}} \to V_{\mathsf{S}}}_{\langle \pi_1, \eta, \pi_2 \rangle} + n^{V_{\mathsf{S}} \to V_{\mathsf{E}}}_{\langle \pi_1, \eta, \pi_2 \rangle}$ *is either 0 or more than* $(2K+1)^2$.

Proof. For (1), $|V_{\mathsf{F}}|$ is bounded by the size of each subset.

$$|V_{\mathsf{F}}| = \left| \bigcup_{\{\pi \in \Pi^V \ | \ 0 < n_\pi \leq 3K+3\}} V\big|_\pi \right| \leq \sum_{\{\pi \in \Pi^V \ | \ 0 < n_\pi \leq 3K+3\}} n_\pi \leq \left|\Pi^V\right|(3K+3) \leq 2^n(3K+3)$$

Next, for (2), by definition, an element v is in V_{E}^0 if there exists another elements $v' \in V_{\mathsf{F}}$ such that the 2-type $\mathsf{lab}[v', v]$ is non-trivial. Note that we can decompose V_{E}^0 according to the 2-type of the incoming edges. Therefore, the following equation holds.

$$\begin{aligned}
V_{\mathsf{E}}^0 &= \left\{ v \in V_{\mathsf{F}}^C \ \middle| \ \text{There exists } v' \in V_{\mathsf{F}} \text{ such that } \mathsf{lab}^{\triangleright}[v', v] \neq \vec{0}. \right\} \\
&\subseteq \left\{ v \in V \ \middle| \ \text{There exists } v' \in V_{\mathsf{F}} \text{ such that } \mathsf{lab}^{\triangleright}[v', v] \neq \vec{0}. \right\} \\
&= \bigcup_{\pi \in \Pi^\tau} \left\{ v \in V\big|_\pi \ \middle| \ \text{There exists } v' \in V_{\mathsf{F}} \text{ such that } \mathsf{lab}^{\triangleright}[v', v] \neq \vec{0}. \right\} \\
&\subseteq \bigcup_{v' \in V_{\mathsf{F}}} \bigcup_{\eta \in \mathcal{K}^{\tau,\mathsf{ntri}}} \bigcup_{\pi \in \Pi^\tau} \left\{ v \in V\big|_\pi \ \middle| \ \mathsf{lab}[v', v] = \eta \right\}
\end{aligned}$$

Because $\mathcal{G}$ is a pseudo-model of φ, the size of each subset is bounded by definition of the realized behavior function and Lemma 3.16.

$$\begin{aligned}
\left|V_{\mathsf{E}}^0\right| &\leq \sum_{v' \in V_{\mathsf{F}}} \sum_{\eta \in \mathcal{K}^{\tau,\mathsf{ntri}}} \sum_{\pi \in \Pi^\tau} \left| \left\{ v \in V\big|_\pi \mid \mathsf{lab}[v', v] = \eta \right\} \right| \\
&= \sum_{v' \in V_{\mathsf{F}}} \sum_{\eta \in \mathcal{K}^{\tau,\mathsf{ntri}}} \sum_{\pi \in \Pi^\tau} F[v'](\eta, \pi) \\
&\leq \sum_{v' \in V_{\mathsf{F}}} K \\
&\leq 2^n(3K+3)K
\end{aligned} \tag{4.2}$$

For every $i \geq 0$, if there exists the tuple $\langle \pi_1, \eta, \pi_2 \rangle$ for the construction of V_{E}^{i+1}, the

doi:10.6342/NTU202203098

size of V_{E}^{i+1} is bounded by Lemma 3.27.5.

$$
\begin{aligned}
\left|V_{\mathsf{E}}^{i+1}\right| &\leq \left|V_{\mathsf{E}}^{i}\right| + n_{\langle\pi_1,\eta,\pi_2\rangle}^{V_{\mathsf{S}}^{i}(\to V_{\mathsf{S}}^{i})} + n_{\langle\pi_1,\eta,\pi_2\rangle}^{(V_{\mathsf{S}}^{i}\to)V_{\mathsf{S}}^{i}} + n_{\langle\pi_1,\eta,\pi_2\rangle}^{V_{\mathsf{S}}^{i}(\to V_{\mathsf{E}}^{i})} + n_{\langle\pi_1,\eta,\pi_2\rangle}^{(V_{\mathsf{E}}^{i}\to)V_{\mathsf{S}}^{i}} \\
&\leq \left|V_{\mathsf{E}}^{i}\right| + 2n_{\langle\pi_1,\eta,\pi_2\rangle}^{V_{\mathsf{S}}^{i}\to V_{\mathsf{S}}^{i}} + n_{\langle\pi_1,\eta,\pi_2\rangle}^{V_{\mathsf{S}}^{i}\to V_{\mathsf{E}}^{i}} + n_{\langle\pi_1,\eta,\pi_2\rangle}^{V_{\mathsf{E}}^{i}\to V_{\mathsf{S}}^{i}} \\
&\leq \left|V_{\mathsf{E}}^{i}\right| + 2\left(n_{\langle\pi_1,\eta,\pi_2\rangle}^{V_{\mathsf{S}}^{i}\to V_{\mathsf{S}}^{i}} + n_{\langle\pi_1,\eta,\pi_2\rangle}^{V_{\mathsf{S}}^{i}\to V_{\mathsf{E}}^{i}} + n_{\langle\pi_1,\eta,\pi_2\rangle}^{V_{\mathsf{E}}^{i}\to V_{\mathsf{S}}^{i}}\right) \\
&\leq \left|V_{\mathsf{E}}^{i}\right| + 2(2K+1)^2
\end{aligned}
\tag{4.3}
$$

Otherwise, if such tuple doesn't exist, $\left|V_{\mathsf{E}}^{i+1}\right| = |V_{\mathsf{E}}^{i}|$. The size of V_{E} is bounded by the equation (4.2) and (4.3).

$$
\begin{aligned}
|V_{\mathsf{E}}| &= \left|V_{\mathsf{E}}^{2^{2m+2n}}\right| \\
&\leq \left|V_{\mathsf{E}}^{2^{2m+2n}-1}\right| + 2(2K+1)^2 \\
&\leq \left|V_{\mathsf{E}}^{0}\right| + 2^{2m+2n}\cdot 2(2K+1)^2 \\
&= 2^n(3K+3)K + 2^{2m+2n+1}(2K+1)^2
\end{aligned}
$$

For (3), suppose to the contrary. There exists a 1-type $\pi \in \Pi^\tau$ such that $0 < n_{\pi}^{V_{\mathsf{E}}\cup V_{\mathsf{S}}} \leq 3K+3$, which implies that V_π and $V_{\mathsf{E}} \cup V_{\mathsf{S}}$ are not disjoint. Because $V_\pi \subseteq V^{\mathsf{F}}$ by the construction, this contradicts the fact that V_{F} and $V_{\mathsf{E}} \cup V_{\mathsf{S}}$ are disjoint.

Now, we present the proof of (4), first we view some observations. We consider the sequence $T_0, T_1, T_2, \ldots$ where $T_i \subseteq \Pi^\tau \times \eta^\tau \times \Pi^\tau$ is the number of tuples $\langle\pi_1,\eta,\pi_2\rangle$ satisfying $n_{\langle\pi_1,\eta,\pi_2\rangle}^{V_{\mathsf{S}}^{i}\to V_{\mathsf{S}}^{i}} + n_{\langle\pi_1,\eta,\pi_2\rangle}^{V_{\mathsf{E}}^{i}\to V_{\mathsf{S}}^{i}} + n_{\langle\pi_1,\eta,\pi_2\rangle}^{V_{\mathsf{S}}^{i}\to V_{\mathsf{E}}^{i}} > 0$. Clearly, the size of T_i is bounded by $|\Pi^\tau \times \mathcal{K}^\tau \times \Pi^\tau| = 2^{2m+2n}$. We observe that the sequence T_i satisfies the following.

(O1) $T_0 \supseteq T_1 \supseteq T_2 \ldots$

Recall that $V_{\mathsf{S}}^{i+1} \subseteq V_{\mathsf{S}}^{i}$. For every $\langle\pi_1,\eta,\pi_2\rangle \in \Pi^\tau \times \mathcal{K}^\tau \times \Pi^\tau$, by Lemma 3.27.8 and 3.27.9, the following holds.

$$
n_{\langle\pi_1,\eta,\pi_2\rangle}^{V_{\mathsf{S}}^{i+1}\to V_{\mathsf{S}}^{i+1}} + n_{\langle\pi_1,\eta,\pi_2\rangle}^{V_{\mathsf{E}}^{i+1}\to V_{\mathsf{S}}^{i+1}} + n_{\langle\pi_1,\eta,\pi_2\rangle}^{V_{\mathsf{S}}^{i+1}\to V_{\mathsf{E}}^{i+1}} \leq n_{\langle\pi_1,\eta,\pi_2\rangle}^{V_{\mathsf{S}}^{i}\to V_{\mathsf{S}}^{i}} + n_{\langle\pi_1,\eta,\pi_2\rangle}^{V_{\mathsf{E}}^{i}\to V_{\mathsf{S}}^{i}} + n_{\langle\pi_1,\eta,\pi_2\rangle}^{V_{\mathsf{S}}^{i}\to V_{\mathsf{E}}^{i}}
$$

Hence, if $n_{\langle\pi_1,\eta,\pi_2\rangle}^{V_{\mathsf{S}}^{i}\to V_{\mathsf{S}}^{i}} + n_{\langle\pi_1,\eta,\pi_2\rangle}^{V_{\mathsf{E}}^{i}\to V_{\mathsf{S}}^{i}} + n_{\langle\pi_1,\eta,\pi_2\rangle}^{V_{\mathsf{S}}^{i}\to V_{\mathsf{E}}^{i}} = 0$, then $n_{\langle\pi_1,\eta,\pi_2\rangle}^{V_{\mathsf{S}}^{i}\to V_{\mathsf{S}}^{i+1}} + n_{\langle\pi_1,\eta,\pi_2\rangle}^{V_{\mathsf{E}}^{i}\to V_{\mathsf{S}}^{i+1}} + n_{\langle\pi_1,\eta,\pi_2\rangle}^{V_{\mathsf{S}}^{i+1}\to V_{\mathsf{E}}^{i}} = 0$. Therefore, if $\langle\pi_1,\eta,\pi_2\rangle \in T_{i+1}$, then $\langle\pi_1,\eta,\pi_2\rangle \in T_i$. This implies that $T_{i+1} \subseteq T_i$.

doi:10.6342/NTU202203098

(O2) If there exists the desired tuple for the construction of V_{E}^{i+1}, then $T_{i+1} \subsetneq T_{i-1}$. Otherwise, $T_{i+1} = T_{i-1}$.

Observe that if there exists the desired tuple $\langle \pi_1, \eta, \pi_2 \rangle$ for the construction of V_{E}^{i+1}, then the following equations hold.

$$n_{\langle \pi_1,\eta,\pi_2 \rangle}^{V_{\mathsf{S}}^i \to V_{\mathsf{S}}^i} + n_{\langle \pi_1,\eta,\pi_2 \rangle}^{V_{\mathsf{E}}^i \to V_{\mathsf{S}}^i} + n_{\langle \pi_1,\eta,\pi_2 \rangle}^{V_{\mathsf{S}}^i \to V_{\mathsf{E}}^i} > 0$$
$$n_{\langle \pi_1,\eta,\pi_2 \rangle}^{V_{\mathsf{S}}^{i+1} \to V_{\mathsf{S}}^{i+1}} + n_{\langle \pi_1,\eta,\pi_2 \rangle}^{V_{\mathsf{E}}^{i+1} \to V_{\mathsf{S}}^{i+1}} + n_{\langle \pi_1,\eta,\pi_2 \rangle}^{V_{\mathsf{S}}^{i+1} \to V_{\mathsf{E}}^{i+1}} = 0$$

This implies that $\langle \pi_1, \eta, \pi_2 \rangle \in T_i$, but $\langle \pi_1, \eta, \pi_2 \rangle \notin T_{i+1}$. Hence $T_{i+1} \subsetneq T_i$.

Otherwise, if there exists no desired tuple $\langle \pi_1, \eta, \pi_2 \rangle$ for the construction of V_{E}^{i+1}, Then $V_{\mathsf{E}}^{i+1} = V_{\mathsf{E}}^i$. Hence $T_{i+1} = T_i$.

(O3) If $T^{i+1} = T^i$, then for every $j > i$, $T_j = T_i$.

Observe that if there exists no desired tuple for the construction of V_{E}^{i+1}, then there exists no desired tuple for the construction of V_{E}^j, where $j > i$. Therefore, $V_{\mathsf{S}}^j = V_{\mathsf{S}}^i$, and this implies $T_j = T_i$.

Now we back to the proof of (4), suppose to the contrary that there exists the tuple $\langle \pi_1, \eta, \pi_2 \rangle \in \Pi^\tau \times \mathcal{K}^\tau \times \Pi^\tau$ satisfying the following.

$$0 < n_{\langle \pi_1,\eta,\pi_2 \rangle}^{V_{\mathsf{S}} \to V_{\mathsf{S}}} + n_{\langle \pi_1,\eta,\pi_2 \rangle}^{V_{\mathsf{E}} \to V_{\mathsf{S}}} + n_{\langle \pi_1,\eta,\pi_2 \rangle}^{V_{\mathsf{S}} \to V_{\mathsf{E}}} \leq (2K+1)^2$$

We consider the following two cases.

Case 1: $T_{2^{2m+2n}} = T_{2^{2m+2n}-1}$. By the observation (O2), there exists no desired tuple for the construction of $V_{\mathsf{E}}^{2^{2m+2n}}$, but $\langle \pi_1, \eta, \pi_2 \rangle$ is such tuple. Hence, a contradiction.

Case 2: $T_{2^{2m+2n}} \subsetneq T_{2^{2m+2n}-1}$. If $t_{2^{2m+2n}} \leq t_{2^{2m+2n}-1} - 1$, By the observation (O3), for every $0 \leq i \leq 2^{2m+2n}$, $T_{i-1} \subsetneq T_i$. Therefore, the size of $T_{2^{2m+2n}}$ is bounded.

$$|T_{2^{2m+2n}}| \leq |T_{2^{2m+2n}-1}| - 1 \leq |T_{2^{2m+2n}-2}| - 2 \leq \cdots \leq |T_0| - 2^{2m+2n} \leq 0$$

On the other hand, by the assumption, the tuple $\langle \pi_1, \eta, \pi_2 \rangle$ is in $T_{2^{2m+2n}}$. This implies $t_{2^{2m+2n}} \geq 1$. Hence, also a contradiction. □

doi:10.6342/NTU202203098

Lemma 4.14 provides some useful properties for the standard partition of the pseudo-model. Now we are going to prove that the standard partition is indeed a proper partition of the pseudo-model in φ. Besides, the set of configurations realized in the strong part is matching with the standard partition, and the number of realizations satisfies the induced ILP system. We state this formally in Lemma 4.15, 4.16, and 4.17.

Lemma 4.15. *Let $\mathcal{G}$ be a pseudo-model of φ. The standard partition of $\mathcal{G}$ in φ, $V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}$, is a proper partition of $\mathcal{G}$ in φ.*

Proof. We verify that conditions (1)-(6) in Definition 4.3 hold. To show that conditions (1)-(3) hold, note that $\mathcal{G}$ is a pseudo-model of φ, by definition, for every $v \in V$, $\mathsf{lab}[v_1] \in \Pi^{\varphi}$. For every $v, v' \in V$, if $v_1 \neq v_2$, the 2-type $\mathsf{lab}[v_1, v_2]$ is compatible with $\mathsf{lab}[v_1]$ and $\mathsf{lab}[v_2]$. For every $v \in V_{\mathsf{F}} \cup V_{\mathsf{E}}$, the counting condition holdes, $\sum_{v' \in \mathcal{G} \setminus \{v\}} \mathsf{lab}[v, v'] = \vec{k}$.

Next, for (4), suppose to the contrary that there exists $v_1 \in V_{\mathsf{F}}$ and $v_2 \in V_{\mathsf{S}}$, such that $\mathsf{lab}^{\triangleright}[v_1, v_2] = \vec{0}$. By the construction, $v_2 \in V_{\mathsf{E}}^0 \subseteq V_{\mathsf{E}}$. This contradicts the fact that V_{E} and V_{S} are disjoint.

For (5), suppose to the contrary that there exists $v_1 \in V_{\mathsf{F}}$ and $v_2 \in V_{\mathsf{E}} \cup V_{\mathsf{S}}$ such that $\mathsf{lab}[v_1] = \mathsf{lab}[v_2]$. By the construction, $v_2 \in V^{\mathsf{lab}[v_2]} = V^{\mathsf{lab}[v_1]} \subseteq V_{\mathsf{F}}$. This contradicts that fact that V_{F} and $V_{\mathsf{E}} \cup V_{\mathsf{S}}$ are disjoint.

Finally, for (6), suppose to the contrary that there exists $\pi_1, \pi_2 \in \Pi^{V_{\mathsf{E}} \cup V_{\mathsf{S}}}$ such that π_1 and π_2 are not null-compatible. Because $\mathcal{G}$ is a pseudo-model of φ, by Lemma 3.33, $n_{\pi_1} \leq 2K + 1$ or $n_{\pi_2} \leq 2K + 1$. Therefore, $\pi_1 \in \Pi^{V_{\mathsf{F}}}$ or $\pi_2 \in \Pi^{V_{\mathsf{F}}}$, but $\Pi^{V_{\mathsf{F}}}$ and $\Pi^{V_{\mathsf{E}} \cup V_{\mathsf{S}}}$ are disjoint. Hence, a contradiction. □

Lemma 4.16. *Let $\mathcal{G}$ be a pseudo-model of φ and $V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}$ be the standard partition of $\mathcal{G}$ in φ. $\mathcal{G}$, $V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}$ and $\mathcal{C}^{V_{\mathsf{S}}}$ are matching in φ.*

Proof. We verify that conditions (1)-(4) in Definition 4.5 hold. For (1), for every 1-type $\pi \in \Pi^{V_{\mathsf{S}}}$, by definition, there exists an element $v \in V_{\mathsf{S}}$ such that its 1-type is π. Note that the source type of the configuration $C[v]$ is also π. Then, by definition, $C[v] \in \mathcal{C}^{V_{\mathsf{S}}}$. Hence, $\pi \in \Pi^{\mathcal{C}^{V_{\mathsf{S}}}}$.

Next, for (2), for every configuration $C \in \mathcal{C}^{V_{\mathsf{S}}}$, there exists an element $v \in V_{\mathsf{S}}$ such that $C[v] = C$. By Lemma 3.23, $C[v]$ conforms $\Pi^{V_{\mathsf{S}}} \subseteq \Pi^{V}$.

doi:10.6342/NTU202203098

For (3), for every configuration $\langle \pi_s, F \rangle \in \mathcal{C}^{V_{\mathsf{S}}}$, there exists an element $v \in V_{\mathsf{S}}$ such that $\langle \mathsf{lab}[v], F[v] \rangle = \langle \pi_s, F \rangle$. For every $\pi \in \Pi^{V_{\mathsf{F}}}$, for every $\eta \in \mathcal{K}^{\tau,\mathsf{ntri}}$, if $\eta^{\triangleleft} \neq 0$, then $\widetilde{\eta}$ is non-trivial. Because $V_{\mathsf{F}} \dot{\cup} V_{\mathsf{E}} \dot{\cup} V_{\mathsf{S}}$ is a proper partition and $\widetilde{\eta}$ is non-trivial, $V\big|_{\pi} = V_{\mathsf{F}}\big|_{\pi}$ and $n^{V_{\mathsf{F}} \to V_{\mathsf{S}}}_{\langle \pi, \widetilde{\eta}, \pi_s \rangle} = 0$. By Lemma 3.27.2, the following holds.

$$\begin{aligned} F[v](\eta, \pi) &= \left| \left\{ v' \in V\big|_{\pi} \;\middle|\; \mathsf{lab}^{\mathcal{G}}[v, v'] = \eta \right\} \right| \\ &= \left| \left\{ v' \in V_{\mathsf{F}}\big|_{\pi} \;\middle|\; \mathsf{lab}^{\mathcal{G}}[v, v'] = \eta \right\} \right| \\ &\leq n^{V_{\mathsf{S}} \to V_{\mathsf{F}}}_{\langle \pi_s, \eta, \pi \rangle} = n^{V_{\mathsf{F}} \to V_{\mathsf{S}}}_{\langle \pi, \widetilde{\eta}, \pi_s \rangle} = 0 \end{aligned}$$

Finally, for (4), for every $\langle \pi_s, F \rangle \in \mathcal{C}^{V_{\mathsf{S}}}$, there exists an element $v \in V_{\mathsf{S}}$ such that $\langle \mathsf{lab}[v], F[v] \rangle = \langle \pi_s, F \rangle$. For every $\pi \in \Pi^{V_{\mathsf{F}}}$, consider the following decomposition of $V\big|_{\pi}$.

$$V\big|_{\pi} = \bigcup_{\eta \in \mathcal{K}^{\tau}} \{ v' \in V_{\pi} \mid \mathsf{lab}[v, v'] = \eta \}$$

Clearly, the subsets are mutually disjoint. Note that for every $\eta \in \mathcal{K}^{\tau}$, because of $V_{\mathsf{F}} \dot{\cup} V_{\mathsf{E}} \dot{\cup} V_{\mathsf{S}}$ is a proper subset, if $\eta^{\triangleleft} \neq \vec{0}$, then the size of the subset is 0. Therefore, we can compute the value of n_{π} by the following.

$$\begin{aligned} n_{\pi} &= \sum_{\eta \in \mathcal{K}^{\tau}} |\{ v' \in V_{\pi} \mid \mathsf{lab}[v, v'] = \eta \}| \\ &= \sum_{\eta \in \mathcal{K}^{\tau,\mathsf{ntri}}} |\{ v' \in V_{\pi} \mid \mathsf{lab}[v, v'] = \eta \}| + \sum_{\eta \in \mathcal{K}^{\tau,\mathsf{tri}}} |\{ v' \in V_{\pi} \mid \mathsf{lab}[v, v'] = \eta \}| \\ &= \sum_{\eta \in \mathcal{K}^{\tau,\mathsf{ntri}}} F[v](\eta, \pi) + |\{ v' \in V_{\pi} \mid \mathsf{lab}[v, v'] = \eta_{\mathsf{null}} \}| \end{aligned}$$

If π_s and π are not null-compatible in φ, then $|\{ v' \in V_{\pi} \mid \mathsf{lab}[v, v'] = \eta_{\mathsf{null}} \}| = 0$. Hence,

$$\sum_{\eta \in \mathcal{K}^{\tau,\mathsf{ntri}}} F[v](\eta, \pi) = n^{V_{\mathsf{F}}}_{\pi}.$$

Otherwise, $\sum_{\eta \in \mathcal{K}^{\tau,\mathsf{ntri}}} F[v](\eta, \pi) \leq n^{V_{\mathsf{F}}}_{\pi}$. □

Lemma 4.17. *Let $\mathcal{G}$ be a pseudo-model of φ and $V_{\mathsf{F}} \dot{\cup} V_{\mathsf{E}} \dot{\cup} V_{\mathsf{S}}$ be the standard partition of $\mathcal{G}$ in φ. The system $\mathcal{Q}_{\mathcal{G}, V_{\mathsf{F}} \dot{\cup} V_{\mathsf{E}} \dot{\cup} V_{\mathsf{S}}, \mathcal{C}^{V_{\mathsf{S}}}}$ has a solution in $\mathbb{N}_{\infty}$.*

doi:10.6342/NTU202203098

Proof. We claim that $\left\{x_C \mapsto n_C^{V_\mathsf{S}}\right\}_{C\in\mathcal{C}^{V_\mathsf{S}}}$ is a solution of the system $\mathcal{Q}_{\mathcal{G},V_\mathsf{F}\dot\cup V_\mathsf{E}\dot\cup V_\mathsf{S},\mathcal{C}^{V_\mathsf{S}}}$. We consider four types of constraints in the system.

1. For every $\pi \in \Pi^{V_\mathsf{E}\cup V_\mathsf{S}}$, By Lemma 4.14.3, 3.26.4, and 3.26.3, the following inequality holds.

   $$3K+3 < n_\pi^{V_\mathsf{E}\cup V_\mathsf{S}} = n_\pi^{V_\mathsf{E}} + n_\pi^{V_\mathsf{S}} = n_\pi^{V_\mathsf{E}} + \sum_{C\in\mathcal{C}^{V_\mathsf{S}}\big|_\pi} n_C^{V_\mathsf{S}}$$

   This implies that the first type constraint holds.

   $$n_\pi^{V_\mathsf{E}} + \sum_{C\in\mathcal{C}^{V_\mathsf{S}}\big|_\pi} x_C \geq 3K+3$$

2. For every $\pi_1,\pi_2 \in \Pi^{V_\mathsf{E}\cup V_\mathsf{S}}$, for every $\eta \in \mathcal{K}^{\varphi,\mathsf{ntri}}_{\langle\pi_1,*,\pi_2\rangle}$ satisfying $\eta^{\triangleleft} \neq \vec{0}$, because $V_\mathsf{F}\dot\cup V_\mathsf{E}\dot\cup V_\mathsf{S}$ is a proper partition, $n_{\pi_2}^{V_\mathsf{F}} = 0$. $n^{V_\mathsf{S}\to V_\mathsf{F}}_{\langle\pi_1,\eta,\pi_2\rangle}$ is bounded by Lemma 3.27.3 and 3.27.7.

   $$n^{V_\mathsf{S}\to V_\mathsf{F}}_{\langle\pi_1,\eta,\pi_2\rangle} \leq n^{V_\mathsf{S}(\to V_\mathsf{F})}_{\langle\pi_1,\eta,\pi_2\rangle} \cdot n^{(V_\mathsf{S}\to)V_\mathsf{F}}_{\langle\pi_1,\eta,\pi_2\rangle} \leq n^{V_\mathsf{S}(\to V_\mathsf{F})}_{\langle\pi_1,\eta,\pi_2\rangle} \cdot n^{V_\mathsf{F}}_{\pi_2} = 0$$

   Note that by Lemma 3.27.8 and 3.28.2, we have the following equation.

   $$n^{V_\mathsf{S}\to V_\mathsf{S}}_{\langle\pi_1,\eta,\pi_2\rangle} + n^{V_\mathsf{S}\to V_\mathsf{E}}_{\langle\pi_1,\eta,\pi_2\rangle} = n^{V_\mathsf{S}\to V}_{\langle\pi_1,\eta,\pi_2\rangle} = \sum_{F\in\mathcal{F}^{\mathcal{C}^{V_\mathsf{S}}}_{\pi_1}} x_{\langle\pi_1,F\rangle} \cdot F(\eta,\pi_2)$$

   Consider following two conditions.

   - $n^{V_\mathsf{E}\to V_\mathsf{S}}_{\langle\pi_1,\eta,\pi_2\rangle} > 0$.
   - There exists $v_1 \in V_\mathsf{S}$ such that $\mathsf{lab}[v_1] = \pi_1$ and $F[v_1](\eta,\pi_2) > 0$, which implies $n^{V_\mathsf{S}\to V_\mathsf{S}}_{\langle\pi_1,\eta,\pi_2\rangle} + n^{V_\mathsf{S}\to V_\mathsf{E}}_{\langle\pi_1,\eta,\pi_2\rangle} > 0$.

   By Lemma 4.14.3, both conditions imply the following.

   $$n^{V_\mathsf{S}\to V_\mathsf{S}}_{\langle\pi_1,\eta,\pi_2\rangle} + n^{V_\mathsf{E}\to V_\mathsf{S}}_{\langle\pi_1,\eta,\pi_2\rangle} + n^{V_\mathsf{S}\to V_\mathsf{E}}_{\langle\pi_1,\eta,\pi_2\rangle} > (2K+1)^2$$

doi:10.6342/NTU202203098

Hence the second type constraint holds.

$$n^{V_{\mathsf{E}}\to V_{\mathsf{S}}}_{\langle \pi_1,\eta,\pi_2\rangle} + \sum_{F\in\mathcal{F}^{\mathcal{C}^{V_{\mathsf{S}}}}_{\pi_1}} x_{\langle \pi_1,F\rangle}\cdot F(\eta,\pi_2) \geq (2K+1)^2$$

Lemma 4.18.

3. For every $\pi_1,\pi_2\in\Pi^{V_{\mathsf{E}}\cup V_{\mathsf{S}}}$, for every $\eta\in\mathcal{K}^{\varphi,\mathsf{ntri}}_{\langle \pi_1,*,\pi_2\rangle}$ satisfying $\eta^{\triangleleft}\neq\vec{0}$, by Lemma 3.27.2, the following holds.

$$n^{V_{\mathsf{S}}\to V_{\mathsf{S}}}_{\langle \pi_1,\eta,\pi_2\rangle} + n^{V_{\mathsf{E}}\to V_{\mathsf{S}}}_{\langle \pi_1,\eta,\pi_2\rangle} + n^{V_{\mathsf{S}}\to V_{\mathsf{E}}}_{\langle \pi_1,\eta,\pi_2\rangle} = n^{V_{\mathsf{S}}\to V_{\mathsf{S}}}_{\langle \pi_2,\widetilde{\eta},\pi_1\rangle} + n^{V_{\mathsf{E}}\to V_{\mathsf{S}}}_{\langle \pi_2,\widetilde{\eta},\pi_1\rangle} + n^{V_{\mathsf{S}}\to V_{\mathsf{E}}}_{\langle \pi_2,\widetilde{\eta},\pi_1\rangle}$$

With the similar argument to second type constraint, the following holds.

$$n^{V_{\mathsf{E}}\to V_{\mathsf{S}}}_{\langle \pi_1,\eta,\pi_2\rangle} + \sum_{F\in\mathcal{F}^{\mathcal{C}^{V_{\mathsf{S}}}}_{\pi_1}} n^{V_{\mathsf{S}}}_{\langle \pi_1,F\rangle}\cdot F(\eta,\pi_2) = n^{V_{\mathsf{E}}\to V_{\mathsf{S}}}_{\langle \pi_2,\widetilde{\eta},\pi_1\rangle} + \sum_{F\in\mathcal{F}^{\mathcal{C}^{V_{\mathsf{S}}}}_{\pi_2}} n^{V_{\mathsf{S}}}_{\langle \pi_2,F\rangle}\cdot F(\widetilde{\eta},\pi_1)$$

Hence the third type constraint holds.

$$n^{V_{\mathsf{E}}\to V_{\mathsf{S}}}_{\langle \pi_1,\eta,\pi_2\rangle} + \sum_{F\in\mathcal{F}^{\mathcal{C}^{V_{\mathsf{S}}}}_{\pi_1}} x_{\langle \pi_1,F\rangle}\cdot F(\eta,\pi_2) = n^{V_{\mathsf{E}}\to V_{\mathsf{S}}}_{\langle \pi_2,\widetilde{\eta},\pi_1\rangle} + \sum_{F\in\mathcal{F}^{\mathcal{C}^{V_{\mathsf{S}}}}_{\pi_2}} x_{\langle \pi_2,F\rangle}\cdot F(\widetilde{\eta},\pi_1)$$

4. Note that $|V_{\mathsf{F}}|+|V_{\mathsf{E}}|+\sum_{C\in\mathcal{C}^{V_{\mathsf{S}}}} n_C$ is the size of the pseudo-model. Since the pseudo-model is non-trivial, its size is non-zero. If $|V_{\mathsf{F}}|+|V_{\mathsf{E}}|=0$, then $\sum_{C\in\mathcal{C}^{V_{\mathsf{S}}}} n_C>0$. This implies that the fourth type constraint holds.

$$\sum_{C\in\mathcal{C}^{V_{\mathsf{S}}}} x_C > 0$$

□

Stage 2: Bounding the size of each component. We already constructed a pseudo-structure and a set of valid configurations which satisfied all conditions, but their size is unbounded. The goal of stage 2 is to construct bounded components from them. Our main tools are linear algebra techniques. We construct the desired components with three Lemma 4.18, 4.19, and 4.20.

doi:10.6342/NTU202203098

Let $V_{\mathsf{F}}\dot\cup V_{\mathsf{E}}\dot\cup V_{\mathsf{S}}$ be a proper partition of $\mathcal{G}$ in φ and $\mathcal{C}$ be a set of valid configurations satisfying $\mathcal{G}$, $V_{\mathsf{F}}\dot\cup V_{\mathsf{E}}\dot\cup V_{\mathsf{S}}$, and $\mathcal{C}$ are matching in φ.

There exists a pseudo-structure $\mathcal{H}$ over τ satisfying the following.

- $|U| \leq |V_{\mathsf{F}}| + |V_{\mathsf{E}}| + 2^n K$.
- *$\mathcal{H}$ has a proper partition $U_{\mathsf{F}}\dot\cup U_{\mathsf{E}}\dot\cup U_{\mathsf{S}}$ in φ.*
- *$\mathcal{H}$, $U_{\mathsf{F}}\dot\cup U_{\mathsf{E}}\dot\cup U_{\mathsf{S}}$ and $\mathcal{C}$ are matching in φ.*
- *The system $\mathcal{Q}_{\mathcal{G},V_{\mathsf{F}}\dot\cup V_{\mathsf{E}}\dot\cup V_{\mathsf{S}},\mathcal{C}}$ and $\mathcal{Q}_{\mathcal{H},U_{\mathsf{F}}\dot\cup U_{\mathsf{E}}\dot\cup U_{\mathsf{S}},\mathcal{C}}$ are the same.*

Proof. We construct such pseudo-structure $\mathcal{H}$ directly. Each component is defined as follows.

- Let $U_{\mathsf{F}} := V_{\mathsf{F}}$, $U_{\mathsf{E}} := V_{\mathsf{E}}$, and $U_{\mathsf{S}} := \bigcup_{\pi\in\Pi^{V_{\mathsf{S}}}}\{u_{\pi,1}, u_{\pi,2}, \ldots, u_{\pi,K}\}$ where $u_{\pi,i}$ is a fresh element. The universe $U := U_{\mathsf{F}}\dot\cup U_{\mathsf{E}}\dot\cup U_{\mathsf{S}}$.
- For each $u \in U$, we assign the 1-type of u by the following rules.
  - If $u \in U_{\mathsf{F}} \cup U_{\mathsf{E}}$, $\mathsf{lab}^{\mathcal{H}}_{\mathsf{V}}[u] := \mathsf{lab}^{\mathcal{G}}_{\mathsf{V}}[u]$.
  - If $u = u_{\pi,i}$, $\mathsf{lab}^{\mathcal{H}}_{\mathsf{V}}[u] := \pi$.
- For each $u_1, u_2 \in U$, we assign the 2-type of (u_1, u_2) by the following rules.
  - If $u_1, u_2 \in U_{\mathsf{F}} \cup U_{\mathsf{E}}$, $\mathsf{lab}^{\mathcal{H}}_{\mathsf{E}}[u_1, u_2] := \mathsf{lab}^{\mathcal{G}}_{\mathsf{E}}[u_1, u_2]$.
  - If $u_1, u_2 \in U_{\mathsf{S}}$, $\mathsf{lab}^{\mathcal{H}}_{\mathsf{E}}[u_1, u_2] := \eta_{\mathsf{null}}$.
  - If $u_1 \in U_{\mathsf{F}}$ and $u_2 \in U_{\mathsf{S}}$, $\mathsf{lab}^{\mathcal{H}}_{\mathsf{E}}[u_1, u_2] := \mathsf{lab}^{\mathcal{G}}_{\mathsf{E}}[u_1, v_2]$, where v_2 is an element in V_{S} satisfying $\mathsf{lab}^{\mathcal{G}}[v_2] = \mathsf{lab}^{\mathcal{H}}[u_2]$,
  - Finally, we assign the 2-type of edges between U_{E} and U_{S} by the following procedure. We fix an order of $\mathcal{K}^{\tau,\mathsf{ntri}}$, say $\eta_1, \eta_2, \ldots, \eta_{2^{2m}-2^m}$. For every $u \in U_{\mathsf{E}}$, for every $\pi \in \Pi^{U_{\mathsf{S}}}$, for every $0 \leq i \leq 2^{2m} - 2^m$, let $t^{u,\pi}_i$ be the number of outgoing edges from u to V_{S} whose 2-type is η_j, where $j \leq i$. Formally, $t^{u,\pi}_i$

doi:10.6342/NTU202203098

is defined as follows.

$$t_i^{u,\pi} := \begin{cases} 0 & \text{if } i = 0 \\ \sum_{j\in[i]} F^{\mathcal{G}}[u](\eta_j, \pi) & \text{otherwise} \end{cases}$$

For every $1 \le i \le 2^{2m} - 2^m$, For every $t_{i-1}^{u,\pi} < j \le t_i^{u,\pi}$, let $\mathsf{lab}^{\mathcal{H}}[u, u_{\pi,j}] := \eta_i$. Otherwise, for every $t_{2^{2m}-2^m}^{u,\pi} < j \le K$, $\mathsf{lab}^{\mathcal{H}}[u, u_{\pi,j}] := \eta_{\mathsf{null}}$.

Note that because $V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}$ is a proper of $\mathcal{G}$ in φ, hence,

$$t_{2^{2m}-2^m}^{b,\pi} = \sum_{\eta\in\mathcal{K}^{\tau,\mathsf{ntri}}} F^{\mathcal{G}}[b](\eta, \pi) \le K.$$

This implies that we can always find such elements, and the above procedure is well defined.

Clearly, the size of $\mathcal{H}$ is bounded.

$$|U| = |U_{\mathsf{F}}| + |U_{\mathsf{E}}| + |U_{\mathsf{S}}| = |V_{\mathsf{F}}| + |V_{\mathsf{E}}| + \left|\Pi^{V_{\mathsf{S}}}\right| K \le |V_{\mathsf{F}}| + |V_{\mathsf{E}}| + 2^n K$$

Since $U_{\mathsf{F}} = V_{\mathsf{F}}$, $U_{\mathsf{E}} = V_{\mathsf{E}}$, and $\Pi^{U_{\mathsf{S}}} = \Pi^{V_{\mathsf{S}}}$, it is not difficult to verify that $U_{\mathsf{F}} \dot\cup U_{\mathsf{E}} \dot\cup U_{\mathsf{S}}$ is a proper partition of $\mathcal{H}$ in φ.

Observe that the construction of $\mathcal{H}$ guarantee that for every $\pi_1, \pi_2 \in \Pi^\tau$, for every $\eta \in \mathcal{K}^{\tau,\mathsf{ntri}}$, if $\eta^{\triangleleft} \ne \vec{0}$, the number of realizations of η between V_{E} and V_{S} is preserving. That is $n_{\langle\pi_1,\eta,\pi_2\rangle}^{V_{\mathsf{E}}\to V_{\mathsf{S}}} = n_{\langle\pi_1,\eta,\pi_2\rangle}^{U_{\mathsf{E}}\to U_{\mathsf{S}}}$ and $n_{\langle\pi_1,\eta,\pi_2\rangle}^{V_{\mathsf{S}}\to V_{\mathsf{E}}} = n_{\langle\pi_1,\eta,\pi_2\rangle}^{U_{\mathsf{S}}\to U_{\mathsf{E}}}$. Then it is not difficult to verify that $\mathcal{G}$, $U_{\mathsf{F}} \dot\cup U_{\mathsf{E}} \dot\cup U_{\mathsf{S}}$, and $\mathcal{C}$ are matching in φ, and the system $\mathcal{Q}_{\mathcal{G},V_{\mathsf{F}}\dot\cup V_{\mathsf{E}}\dot\cup V_{\mathsf{S}},\mathcal{C}}$ and $\mathcal{Q}_{\mathcal{H},U_{\mathsf{F}}\dot\cup U_{\mathsf{E}}\dot\cup U_{\mathsf{S}},\mathcal{C}}$ are the same. □

Lemma 4.19. *Let* $V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}$ *be a proper partition of* $\mathcal{G}$ *in* φ *and* $\mathcal{C}$ *be a set of valid configurations satisfying* $\mathcal{G}$, $V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}$, *and* $\mathcal{C}$ *are matching in* φ *and the system* $\mathcal{Q}_{\mathcal{G},V_{\mathsf{F}}\dot\cup V_{\mathsf{E}}\dot\cup V_{\mathsf{S}},\mathcal{C}}$ *has a solution in* $\mathbb{N}_\infty$.

There exists a set of valid configurations in φ, *denoted by* $\mathcal{D}$, *satisfy the following.*

- $|\mathcal{D}| \le 2^{2n+2m+3}\left(2n + 2m + 4 + \lceil \log\left(\max\left(|V_{\mathsf{E}}|, (2K+1)^2\right)\right)\rceil\right)$.

doi:10.6342/NTU202203098

- *$\mathcal{G}$, $V_{\mathsf{F}}\dot{\cup}V_{\mathsf{E}}\dot{\cup}V_{\mathsf{S}}$, and $\mathcal{D}$ are matching in φ.*
- *The system $\mathcal{Q}_{\mathcal{G},V_{\mathsf{F}}\dot{\cup}V_{\mathsf{E}}\dot{\cup}V_{\mathsf{S}},\mathcal{D}}$ has a solution $\{x_C \mapsto p_C\}_{C\in\mathcal{D}}$ in $\mathbb{N}_\infty$.*

Proof. Let d be the number of constraints of the system $\mathcal{Q}_{\mathcal{G},V_{\mathsf{F}}\dot{\cup}V_{\mathsf{E}}\dot{\cup}V_{\mathsf{S}},\mathcal{D}}$. By Remark 4.7, d is bounded by $3\cdot 2^{2m+2n}$. Let $\bar{c}$ be the maximum coefficients in the system $\mathcal{Q}_{\mathcal{G},V_{\mathsf{F}}\dot{\cup}V_{\mathsf{E}}\dot{\cup}V_{\mathsf{S}},\mathcal{D}}$. By Lemma 4.8, $\bar{c}$ is bounded by $\max\left(|V_{\mathsf{E}}|,(2K+1)^2\right)$. By Lemma 2.16, the system has a solution in $\mathbb{N}_\infty$ if and only if it has a solution with at most $2(d+1)(\log(d+1)+\lceil\log\bar{c}\rceil+2)$ non-zero elements. Let $\mathcal{D} := \{C\in\mathcal{C} \mid p_C > 0\}$. Then the size of $\mathcal{D}$ is bounded as follows.

$$
\begin{aligned}
|\mathcal{D}| &\le 2(d+1)(\log(d+1)+\lceil\log\bar{c}\rceil+2)\\
&\le 2^{2n+2m+3}\cdot\left(2n+2m+4+\left\lceil\log\left(\max\left(|V_{\mathsf{E}}|,(2K+1)^2\right)\right)\right\rceil\right)
\end{aligned}
$$

Observe that $\Pi^{\mathcal{D}}\subseteq\Pi^{\mathcal{C}}$, then it is not difficult to verify that $\mathcal{G}$, $V_{\mathsf{F}}\dot{\cup}V_{\mathsf{E}}\dot{\cup}V_{\mathsf{S}}$ and $\mathcal{D}$ are also matching in φ.

Next, we claim that $\{x_C \mapsto p_C\}_{C\in\mathcal{D}}$ is a solution of the system $\mathcal{Q}_{\mathcal{G},V_{\mathsf{F}}\dot{\cup}V_{\mathsf{E}}\dot{\cup}V_{\mathsf{S}},\mathcal{D}}$. The main observation is that for every $\pi_1,\pi_2\in\Pi^\tau$, for every $\eta\in\mathcal{K}^{\tau,\mathsf{ntri}}$, because $\mathcal{D}$ collect all configurations C in $\mathcal{C}$ which has non-zero p_C, the following equations hold.

$$
\sum_{C\in\mathcal{C}|_{\pi_1}} p_C = \sum_{C\in\mathcal{D}|_{\pi_1}} p_C
$$
$$
\sum_{F\in\mathcal{F}^{\mathcal{C}}_{\pi_1}} p_{\langle\pi_1,F\rangle}\cdot F(\eta,\pi_2) = \sum_{F\in\mathcal{F}^{\mathcal{D}}_{\pi_1}} p_{\langle\pi_1,F\rangle}\cdot F(\eta,\pi_2)
$$

We verify four types of constraints in the system as follows.

1. For every $\pi\in\Pi^{V_{\mathsf{E}}\cup V_{\mathsf{S}}}$, by the first type constraint in the system $\mathcal{Q}_{\mathcal{G},V_{\mathsf{F}}\dot{\cup}V_{\mathsf{E}}\dot{\cup}V_{\mathsf{S}},\mathcal{C}}$ and the main observation, the following holds.

$$
n^{V_{\mathsf{E}}}_{\pi} + \sum_{C\in\mathcal{D}|_{\pi}} p_C = n^{V_{\mathsf{E}}}_{\pi} + \sum_{C\in\mathcal{C}|_{\pi}} p_C \ge 3K+3
$$

This implies that the first type constraint in the system $\mathcal{Q}_{\mathcal{G},V_{\mathsf{F}}\dot{\cup}V_{\mathsf{E}}\dot{\cup}V_{\mathsf{S}},\mathcal{C}}$ holds.

$$
n^{V_{\mathsf{E}}}_{\pi} + \sum_{C\in\mathcal{D}|_{\pi}} x_C \ge 3K+3
$$

doi:10.6342/NTU202203098

2. For every $\pi_1, \pi_2 \in \Pi^{V_\mathsf{E} \cup V_\mathsf{S}}$, for every $\eta \in \mathcal{K}^{\varphi,\mathsf{ntri}}_{\langle \pi_1, *, \pi_2 \rangle}$, if $\eta^{\triangleleft} \neq \vec{0}$ and at least one of the following holds,

   - $n^{V_\mathsf{E} \to V_\mathsf{S}}_{\langle \pi_1, \eta, \pi_2 \rangle} > 0$, or
   - there exists $F \in \mathcal{F}^{\mathcal{D}}_{\pi_1} \subseteq \mathcal{F}^{\mathcal{C}}_{\pi_1}$ such that $F(\eta, \pi_2) > 0$,

   by the second type constraint in the system $\mathcal{Q}_{\mathcal{G}, V_\mathsf{F} \dot\cup V_\mathsf{E} \dot\cup V_\mathsf{S}, \mathcal{C}}$ and the main observation, the following equation holds.

$$n^{V_\mathsf{E} \to V_\mathsf{S}}_{\langle \pi_1, \eta, \pi_2 \rangle} + \sum_{F \in \mathcal{F}^{\mathcal{D}}_{\pi_1}} p_{\langle \pi_1, F \rangle} \cdot F(\eta, \pi_2) = n^{V_\mathsf{E} \to V_\mathsf{S}}_{\langle \pi_1, \eta, \pi_2 \rangle} + \sum_{F \in \mathcal{F}^{\mathcal{C}}_{\pi_1}} p_{\langle \pi_1, F \rangle} \cdot F(\eta, \pi_2) \geq (2K+1)^2$$

   This implies that the second type constraints in the system $\mathcal{Q}_{\mathcal{G}, V_\mathsf{F} \dot\cup V_\mathsf{E} \dot\cup V_\mathsf{S}, \mathcal{D}}$ holds.

$$n^{V_\mathsf{E} \to V_\mathsf{S}}_{\langle \pi_1, \eta, \pi_2 \rangle} + \sum_{F \in \mathcal{F}^{\mathcal{D}}_{\pi_1}} x_{\langle \pi_1, F \rangle} \cdot F(\eta, \pi_2) \geq (2K+1)^2$$

3. By the third type constraint in the system $\mathcal{Q}_{\mathcal{G}, V_\mathsf{F} \dot\cup V_\mathsf{E} \dot\cup V_\mathsf{S}, \mathcal{C}}$ and the main observation, the following equation holds.

$$\begin{aligned}
& n^{V_\mathsf{E} \to V_\mathsf{S}}_{\langle \pi_1, \eta, \pi_2 \rangle} + \sum_{F \in \mathcal{F}^{\mathcal{D}}_{\pi_1}} p_{\langle \pi_1, F \rangle} \cdot F(\eta, \pi_2) \\
&= n^{V_\mathsf{E} \to V_\mathsf{S}}_{\langle \pi_1, \eta, \pi_2 \rangle} + \sum_{F \in \mathcal{F}^{\mathcal{C}}_{\pi_1}} p_{\langle \pi_1, F \rangle} \cdot F(\eta, \pi_2) \\
&= n^{V_\mathsf{E} \to V_\mathsf{S}}_{\langle \pi_2, \widetilde{\eta}, \pi_1 \rangle} + \sum_{F \in \mathcal{F}^{\mathcal{C}}_{\pi_2}} p_{\langle \pi_2, F \rangle} \cdot F(\widetilde{\eta}, \pi_1) \\
&= n^{V_\mathsf{E} \to V_\mathsf{S}}_{\langle \pi_2, \widetilde{\eta}, \pi_1 \rangle} + \sum_{F \in \mathcal{F}^{\mathcal{D}}_{\pi_2}} p_{\langle \pi_2, F \rangle} \cdot F(\widetilde{\eta}, \pi_1)
\end{aligned}$$

   These imply that the third type constraint in the system $\mathcal{Q}_{\mathcal{G}, V_\mathsf{F} \dot\cup V_\mathsf{E} \dot\cup V_\mathsf{S}, \mathcal{D}}$ holds.

$$n^{V_\mathsf{E} \to V_\mathsf{S}}_{\langle \pi_1, \eta, \pi_2 \rangle} + \sum_{F \in \mathcal{F}^{\mathcal{D}}_{\pi_1}} x_{\langle \pi_1, F \rangle} \cdot F(\eta, \pi_2) = n^{V_\mathsf{E} \to V_\mathsf{S}}_{\langle \pi_2, \widetilde{\eta}, \pi_1 \rangle} + \sum_{F \in \mathcal{F}^{\mathcal{D}}_{\pi_2}} x_{\langle \pi_2, F \rangle} \cdot F(\widetilde{\eta}, \pi_1)$$

4. By the fourth type constraint in the system $\mathcal{Q}_{\mathcal{G}, V_\mathsf{F} \dot\cup V_\mathsf{E} \dot\cup V_\mathsf{S}, \mathcal{C}}$ and the main observation,

doi:10.6342/NTU202203098

$\sum_{C \in \mathcal{C}^V} n_C = \sum_{C \in \mathcal{D}^V} n_C > 0$ This implies that the fourth type constraint holds.

$$\sum_{C \in \mathcal{D}} x_C > 0$$

$\square$

Lemma 4.20. *Let* $V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}$ *be a proper partition of* $\mathcal{G}$ *in* φ *and* $\mathcal{C}$ *be a set of valid configurations satisfying* $\mathcal{G}$, $V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}$, *and* $\mathcal{C}$ *are matching in* φ *and the system* $\mathcal{Q}_{\mathcal{G}, V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}, \mathcal{C}}$ *has a solution in* $\mathbb{N}_\infty$.

The system $\mathcal{Q}_{\mathcal{G}, V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}, \mathcal{C}}$ *has a solution* $\{x_C \mapsto p_C\}_{C \in \mathcal{C}}$ *such that for every* $C \in \mathcal{C}$, p_C *is either infinity or bounded by* $(3 \cdot 2^{2n+2m} + |\mathcal{C}|) \cdot (3 \cdot 2^{2n+2m} \cdot \max(|V_{\mathsf{E}}|, (2K+1)^2))^{2^{2n+2m+3}}$.

Proof. Let d be the number of constraints of the system $\mathcal{Q}_{\mathcal{G}, V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}, \mathcal{D}}$. By Remark 4.7, d is bounded by $3 \cdot 2^{2m+2n}$. Let $\bar{c}$ be the maximum coefficients in the system $\mathcal{Q}_{\mathcal{G}, V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}, \mathcal{D}}$. By Lemma 4.8, $\bar{c}$ is bounded by $\max(|V_{\mathsf{E}}|, (2K+1)^2)$. There are $|\mathcal{C}|$ variables in the system. By Lemma 2.17, the system has a solution if and only if it has a solution $\{x_C \mapsto p_C\}_{C \in \mathcal{C}}$ in $\mathbb{N}_\infty$ such that for every $C \in \mathcal{C}$, p_C is either ∞ or bounded by $(|\mathcal{C}| + d)(ds)^{(2d+1)}$. Hence, p_C is bounded as follows.

$$\begin{aligned} p_C &\leq (|\mathcal{C}| + d)(ds)^{(2d+1)} \\ &\leq \left(3 \cdot 2^{2n+2m} + |\mathcal{C}|\right)\left(3 \cdot 2^{2n+2m} \cdot \max\left(|V_{\mathsf{E}}|, (2K+1)^2\right)\right)^{2^{2n+2m+3}} \end{aligned}$$

$\square$

Finally, by combining all these lemmas, we can prove the only if direction of the CEB property.

Lemma 4.21. *If the* C^2 *sentence* φ *is satisfiable, then there exists the following components.*

- *A pseudo-structure* $\mathcal{G}$ *over* τ *with a proper partition* $V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}$ *in* φ.
- *A set of valid configurations* $\mathcal{C}$ *satisfying* $\mathcal{G}$, $V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}$ *and* $\mathcal{C}$ *are matching in* φ.
- *A solution* $\{x_C \mapsto p_C\}_{C \in \mathcal{C}}$ *of the system* $\mathcal{Q}_{\mathcal{G}, V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}, \mathcal{C}}$.

doi:10.6342/NTU202203098

Furthermore, their sizes are bounded by the length of φ. More precisely,

- $|V| \leq \theta_1(n, m, K)$*;*
- $|\mathcal{C}| \leq \theta_2(n, m, K)$*;*
- *for every* $C \in \mathcal{C}$*,* $p_C \in [0, \theta_3(n, m, K)] \cup \{\infty\}$*, where* $\theta_1(n, m, K)$*,* $\theta_2(n, m, K)$*, and* $\theta_3(n, m, K)$ *are the same as in Theorem 4.9.*

Proof. Let $\mathcal{H}$ be a pseudo-model of φ and $U_{\mathsf{F}} \dot\cup U_{\mathsf{E}} \dot\cup U_{\mathsf{S}}$ be the standard partition of $\mathcal{H}$ in φ. By Lemma 4.15, $U_{\mathsf{F}} \dot\cup U_{\mathsf{E}} \dot\cup U_{\mathsf{S}}$ is a proper partition of $\mathcal{H}$ in φ. By Lemma 4.16, $\mathcal{H}$,$U_{\mathsf{F}} \dot\cup U_{\mathsf{E}} \dot\cup U_{\mathsf{S}}$, and $\mathcal{C}^{U_{\mathsf{S}}}$ are matching in φ. By Lemma 4.17, the system $\mathcal{Q}_{\mathcal{H}, U_{\mathsf{F}} \dot\cup U_{\mathsf{E}} \dot\cup U_{\mathsf{S}}, \mathcal{C}^{U_{\mathsf{S}}}}$ has a solution in $\mathbb{N}_\infty$.

Because $\mathcal{H}$, $U_{\mathsf{F}} \dot\cup U_{\mathsf{E}} \dot\cup U_{\mathsf{S}}$, and $\mathcal{C}^{U_{\mathsf{S}}}$ are matching in φ, by Lemma 4.18, there is a pseudo-structure $\mathcal{G}$ over τ with a proper partition $V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}$ satisfying $\mathcal{G}$, $V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}$, and $\mathcal{C}^{V_{\mathsf{S}}}$ are matching in φ. Besides, the system $\mathcal{Q}_{\mathcal{H}, U_{\mathsf{F}} \dot\cup U_{\mathsf{E}} \dot\cup U_{\mathsf{S}}, \mathcal{C}^{U_{\mathsf{S}}}}$ and $\mathcal{Q}_{\mathcal{G}, V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}, \mathcal{C}^{V_{\mathsf{S}}}}$ are the same. Hence, the system $\mathcal{Q}_{\mathcal{G}, V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}, \mathcal{C}^{U_{\mathsf{S}}}}$ also has a solution in $\mathbb{N}_\infty$. Note that the size of $\mathcal{G}$ is also bounded by the Lemma 4.18, $|V| \leq |V_{\mathsf{F}}| + |V_{\mathsf{E}}| + 2^n K$. Since $V_{\mathsf{F}} = U_{\mathsf{F}}$ and $V_{\mathsf{E}} = U_{\mathsf{E}}$, together with Lemma 4.14.1 and 4.14.2, the size of $\mathcal{G}$ is bounded as follows.

$$
\begin{aligned}
|V| &\leq |V_{\mathsf{E}}^m| + |V_{\mathsf{F}}^m| + 2^n K \\
&\leq 2^n(3K+3) + \left(2^n(3K+3)K + 2^{2m+2n+1}(2K+1)^2\right) + 2^n K \\
&\leq 2^{2m+2n+2}(2K+1)^2
\end{aligned}
$$

Because $\mathcal{G}$, $V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}$, and $\mathcal{C}^{V_{\mathsf{S}}}$ are matching in φ and the system $\mathcal{Q}_{\mathcal{G}, V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}, \mathcal{C}^{V_{\mathsf{S}}}}$ has a solution in $\mathbb{N}_\infty$, by Lemma 4.19, there is a set of valid configurations $\mathcal{C}$ satisfying $\mathcal{G}$, $V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}$, and $\mathcal{C}$ are matching in φ and the system $\mathcal{Q}_{\mathcal{G}, V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}, \mathcal{C}}$ has a solution in $\mathbb{N}_\infty$. Besides, the size of $\mathcal{C}$ is bounded.

$$
\begin{aligned}
|\mathcal{C}| &\leq 2^{2n+2m+3} \cdot \left(2n + 2m + 4 + \left\lceil \log\left(\max\left(|V_{\mathsf{E}}|, (2K+1)^2\right)\right)\right\rceil\right) \\
&\leq 2^{2n+2m+3} \cdot \left(2n + 2m + 4 + \left\lceil \log\left(2^n(3K+3)K + 2^{2m+2n+1}(2K+1)^2\right)\right\rceil\right) \\
&\leq 2^{2n+2m+3} \cdot \left(2n + 2m + 4 + \left\lceil \log\left(2^{2m+2n+2}(2K+1)^2\right)\right\rceil\right) \\
&\leq 2^{2n+2m+3} \cdot (4n + 4m + 6 + 2\lceil \log(2K+1) \rceil)
\end{aligned}
$$

doi:10.6342/NTU202203098

Finally, by Lemma 4.20, the system $\mathcal{Q}_{\mathcal{G},V_{\mathsf{F}}\dot\cup V_{\mathsf{E}}\dot\cup V_{\mathsf{S}},\mathcal{C}}$ has a solution $\{x_C \mapsto p_C\}_{C\in\mathcal{C}}$ satisfying that for every $C \in \mathcal{C}$, if p_C is finite, then its value is bounded.

$$\begin{aligned}
p_C &\le (3\cdot 2^{2n+2m} + |\mathcal{C}|)\left(3\cdot 2^{2n+2m}\cdot \max\left(|V_{\mathsf{E}}|,(2K+1)^2\right)\right)^{2^{2n+2m+3}} \\
&\le (3\cdot 2^{2n+2m} + |\mathcal{C}|)\left(3\cdot 2^{2n+2m}\cdot \left(2^n(3K+3)K + 2^{2m+2n+1}(2K+1)^2\right)\right)^{2^{2n+2m+3}} \\
&\le (3\cdot 2^{2n+2m} + |\mathcal{C}|)\left(2^{4n+4m+4}(2K+1)^2\right)^{2^{2n+2m+3}} \\
&\le 2^{2n+2m+4}\left(4n+4m+6+2\lceil\log(2K+1)\rceil\right)\left(2^{4n+4m+4}(2K+1)^2\right)^{2^{2n+2m+3}}
\end{aligned}$$

Hence, $\mathcal{G}$, $V_{\mathsf{F}}\dot\cup V_{\mathsf{E}}\dot\cup V_{\mathsf{S}}$, and $\mathcal{C}$ are desired components. □

4.2.2 Proof of the if direction

The proof of the if direction is straightforward. We construct the pseudo-model from such components directly. Recall that the elements in the finite part and the extended part of the proper partition satisfy all conditions for the pseudo-model. Hence, we keep those elements. The main challenge of the construction is to correctly assign the 2-type for the edges in the strong part such that the result pseudo-structure is indeed a pseudo-model. The idea is based on the induced ILP system. Since the number of realizations in the strong part are all large enough, we can always apply some "switching" procedure to find a suitable target element to assign the desired 2-type.

Lemma 4.22 implies the if direction of the CEB property. In fact, it is stronger since it comes with the size of the pseudo-model.

Lemma 4.22. *The C^2 sentence φ is satisfiable, if there exists the following components.*

- *A pseudo-structure $\mathcal{G}$ over τ with a proper partition $V_{\mathsf{F}}\dot\cup V_{\mathsf{E}}\dot\cup V_{\mathsf{S}}$ in φ.*
- *A set of valid configurations $\mathcal{C}$ satisfying $\mathcal{G}$, $V_{\mathsf{F}}\dot\cup V_{\mathsf{E}}\dot\cup V_{\mathsf{S}}$ and $\mathcal{C}$ are matching in φ.*
- *A solution $\{x_C \mapsto p_C\}_{C\in\mathcal{C}}$ of the system $\mathcal{Q}_{\mathcal{G},V_{\mathsf{F}}\dot\cup V_{\mathsf{E}}\dot\cup V_{\mathsf{S}},\mathcal{C}}$.*

Furthermore, φ has a pseudo-model $\mathcal{H}$ with size $|U| = |V_{\mathsf{F}}| + |V_{\mathsf{E}}| + \sum_{C\in\mathcal{C}} p_C$.

Proof. We show the construction of pseudo-model $\mathcal{H}$. For every $C \in \mathcal{C}$, for every $i \in [p_C]$, let $u_{C,i}$ be a fresh element, and U_{S} be the set that collects all those elements.

doi:10.6342/NTU202203098

Formally, $U_{\mathsf{S}} := \bigcup_{C \in \mathcal{C}} \{u_{C,1}, u_{C,2}, \ldots, u_{C,p_C}\}$. By definition, $\Pi^{U_{\mathsf{S}}} = \Pi^{\mathcal{C}}$. Note that because $\mathcal{G}$ and $\mathcal{C}$ are matching, $\Pi^{\mathcal{C}} \subseteq \Pi^{V_{\mathsf{S}}}$. Therefore, $\Pi^{U_{\mathsf{S}}} = \Pi^{\mathcal{C}} \subseteq \Pi^{V_{\mathsf{S}}}$. The intuition of the construction is as follows. We define the *objective configuration* of element u in $\mathcal{H}$. If $u \in V_{\mathsf{F}} \cup V_{\mathsf{E}}$, then its objective configuration is the same as in $\mathcal{G}$. Otherwise, if $u \in U_{\mathsf{S}}$, then $u = u_{C,i}$, and its objective configuration is just C. The following construction guarantee that the configuration of u in $\mathcal{H}$ is exactly its objective configuration. Each component of $\mathcal{H}$ is defined as follows.

- The universe of $\mathcal{H}$ is the union of V_{F}, V_{E}, and U_{S}, that is, $U := V_{\mathsf{F}} \cup V_{\mathsf{E}} \cup U_{\mathsf{S}}$.
- The label of vertices in $\mathcal{H}$ is assigned by the following rules. For every $u \in U$, let $\langle \pi_s, F \rangle$ be the objective configuration of u, and $\mathsf{lab}^{\mathcal{H}}[u] := \pi_s$.
- Finally, the label of edges in $\mathcal{H}$ is assigned with the following procedure. First, for the edges in V_{F} and edges between V_{F} and V_{E}, its label is the same as in $\mathcal{G}$. Second, for the edges between V_{F} and U_{S}, its label is by the matching condition. Finally, for the edges in $V_{\mathsf{E}} \cup U_{\mathsf{S}}$, its label is by the induced ILP system. We divide it into three different cases, the two-direction non-trivial 2-types, the one-direction trivial 2-types, and the null type.

  Step 1: For every $u_1, u_2 \in V_{\mathsf{F}}$ satisfying $u_1 \neq u_2$, its label is the same as in $\mathcal{G}$, that is, $\mathsf{lab}^{\mathcal{H}}[u_1, u_2] := \mathsf{lab}^{\mathcal{G}}[u_1, u_2]$.

  For every $u_1 \in V_{\mathsf{F}}$, for every $u_2 \in V_{\mathsf{E}}$, its label is the same as in $\mathcal{G}$, that is, $\mathsf{lab}^{\mathcal{H}}[u_1, u_2] := \mathsf{lab}^{\mathcal{G}}[u_1, u_2]$.

  Step 2: For every $u \in U_{\mathsf{S}}$, let $\langle \pi_s, F \rangle$ be the objective configuration of u. For every $\pi \in \Pi^{V_{\mathsf{F}}}$, we assign the label of edges between u and the elements in $V_{\mathsf{F}}\big|_{\pi}$, that is, the set of the elements in V_{F} realized 1-type π, with the following procedure. For every $\eta \in \mathcal{K}^{\tau,\mathsf{ntri}}$, pick $F(\eta, \pi)$ unset elements in $V_{\mathsf{F}}\big|_{\pi}$, and assign the edges between u and them with 2-type η. After that, if there are still unset elements in $V_{\mathsf{F}}\big|_{\pi}$, then we assign the edges between u and them with null type.

  Because $\mathcal{G}$, $V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}$ and $\mathcal{C}$ are matching in φ, if π_s and π are not null-compatible in φ, then $\sum_{\eta \in \mathcal{K}^{\tau,\mathsf{ntri}}} F(\eta, \pi) = \left| V_{\mathsf{F}}\big|_{\pi} \right|$. Otherwise, $\sum_{\eta \in \mathcal{K}^{\tau,\mathsf{ntri}}} F(\eta, \pi) \leq \left| V_{\mathsf{F}}\big|_{\pi} \right|$.

doi:10.6342/NTU202203098

Therefore, this guarantee that the existence of unassigned edges in the above procedure. Moreover, by the definition of valid configuration, the 2-type set are compatible with π_s and π.

Step 3: In this step, we assign the label of edges in $V_{\mathsf{E}} \cup U_{\mathsf{S}}$ whose 2-type is two-direction non-trivial. For every $u_1, u_2 \in V_{\mathsf{E}}$ satisfying $u_1 \neq u_2$, if $\mathsf{lab}^{\mathcal{G},\triangleright}[u_1, u_2] \neq \vec{0}$ and $\mathsf{lab}^{\mathcal{G},\triangleleft}[u_1, u_2] \neq \vec{0}$, then $\mathsf{lab}^{\mathcal{H}}[u_1, u_2] := \mathsf{lab}^{\mathcal{G}}[u_1, u_2]$.

For every $\pi_1, \pi_2 \in \Pi^{V_{\mathsf{E}} \cup U_{\mathsf{S}}}$, for every $\eta \in \mathcal{K}^{\varphi,\mathsf{ntri}}_{\langle \pi_1, *, \pi_2 \rangle}$, if $\eta^{\triangleright} \neq \vec{0}$ and $\eta^{\triangleleft} \neq \vec{0}$, then we assign the edges in $V_{\mathsf{E}} \cup U_{\mathsf{S}}$ whose 2-type is η with the following procedure. Because the assignment $\{x_{\mathcal{C}} \mapsto p_{\mathcal{C}}\}_{C \in \mathcal{C}}$ is a solution of the system $\mathcal{Q}_{\mathcal{G}, V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}, \mathcal{C}}$, by the third type constraint in the system, the number of edges which required by the elements' objective configuration are the same. Hence, it is trivial to assign parallel edges in $\mathcal{H}$ with 2-type η satisfying the following. For every element $u \in V_{\mathsf{E}} \cup U_{\mathsf{S}}$, let $\langle \pi_s, F \rangle$ be the objective configuration of u, if $\pi_s = \pi_1$, then the number of outgoing edges from u with type $\langle \pi_1, \eta, \pi_2 \rangle$ is exactly $F(\eta, \pi_2)$.

Note that for every element $u \in V_{\mathsf{E}} \cup U_{\mathsf{S}}$, because the objective configuration of each element is valid in φ, and we only assign the non-trivial edges in $V_{\mathsf{E}} \cup U_{\mathsf{S}}$, there are at most K outgoing edges from u to $V_{\mathsf{E}} \cup U_{\mathsf{S}}$. For every edge (u_1, u_2) in $V_{\mathsf{E}} \cup U_{\mathsf{S}}$, for every element $u \in V_{\mathsf{E}} \cup U_{\mathsf{S}}$, we say u is *related* to (u_1, u_2), if there exists the edge (u_1, u) or (u_2, u). Since there are at most K outgoing edges from u_1 and K outgoing edges from u_2, the number of elements related to (u_1, u_2) is bounded by $2K$. For every edges (u_1, u_2) and (u_3, u_4) in $V_{\mathsf{E}} \cup U_{\mathsf{S}}$, we say they are *related* if u_3 or u_4 is related to (u_1, u_2). The number of edges which related to (u_1, u_2) is bounded. Since there are at most K outgoing edges from each related element, the number of edges related to (u_1, u_2) is bounded by $2K^2$.

To resolve the parallel edges in the above construction, we consider the following "swapping" procedure. Let u_1 and u_2 be elements in $V_{\mathsf{E}} \cup U_{\mathsf{S}}$ satisfying that there are two parallel edges between them with 2-types η and η', respectively. By second type constraint, the number of edges with type $\langle \pi_1, \eta, \pi_2 \rangle$ in $V_{\mathsf{E}} \cup U_{\mathsf{S}}$ is at least $(2K+1)^2$, but the number of edges related to (u_1, u_2) is bounded by $2K^2$. Hence, there exists

doi:10.6342/NTU202203098

a edge (u_3, u_4) satisfying that its type is $\langle \pi_1, \eta, \pi_2 \rangle$, and it is not related to (u_1, u_2). Now, we remove the edges between (u_1, u_2) and (u_3, u_4) and assign new edges (u_1, u_4) and (u_3, u_2) with 2-type η. Clearly, for the elements u_1, u_2, u_3, and u_4, the number of edges connected to it with specific 2-type is preserved. Furthermore, the number of parallel edges in $\mathcal{H}$ decrease. Hence, we can repeat the above procedure until there is no more parallel edges in $\mathcal{H}$.

Step 4: In this step, we assign the label of edges in $V_\mathsf{E} \cup U_\mathsf{S}$ whose 2-type is one-direction trivial. For every $u \in V_\mathsf{E} \cup U_\mathsf{S}$, let $\langle \pi_s, F \rangle$ be its objective configuration, for every $\pi \in \Pi^{V_\mathsf{E} \cup U_\mathsf{S}}$, for every $\eta \in \mathcal{K}^{\tau,\mathsf{ntri}}_{\langle \pi_s, *, \pi \rangle}$ satisfying $\eta^{\lhd} = \vec{0}$, we pick $F(\eta, \pi)$ elements in $V_\mathsf{E} \cup U_\mathsf{S}$ satisfying there are no edges between u and them, and assign the edges with 2-type η. If there is no such element, then we apply the following "swapping" procedure.

Case 1: $\pi_s = \pi$**.** We divide the set $(V_\mathsf{E} \cup U_\mathsf{S})\big|_\pi$ into four parts. Let S_1 be the subset of $(V_\mathsf{E} \cup U_\mathsf{S})\big|_\pi$ such that the 2-types between u and them are non-trivial. Let S_2 be the subset of $(V_\mathsf{E} \cup U_\mathsf{S})\big|_\pi$ such that the 2-types between u and them are trivial. Let S_3 be the subset of $(V_\mathsf{E} \cup U_\mathsf{S})\big|_\pi$ such that the 2-types between u and them are not assigned. Finally, the set of u itself. Clearly, the above four sets are disjoint. Note that the number of outgoing edges from u are bounded by K. Therefore, if $|S_1| + |S_3| \geq K$, then we can always find desired elements from S_3.

Suppose $|S_1| + |S_3| < K$, By the first type constraint, the size of the set S_2 is bounded as follows.

$$|S_2| \geq \left| (V_\mathsf{E} \cup U_\mathsf{S})\big|_{\pi_s} \right| - |S_1| - |S_3| - |\{u\}| \geq (3K+3) - K - 1 \geq 2(K+1)$$

Hence, the number of edge in S_2 is at least $\frac{1}{2}|S_2|\left(|S_2| - 1\right) \geq 2K|S_2|$. On the other hand, the number of assigned edges in S_2 is at most $K|S_2|$. Therefore, there exists a unassigned edge, say (u_1, u_2), in S_3. Let η' be the 2-type of the edge (u, u_1). Now, we remove the edge (u, u_1), and assign the edge (u_2, u_1) with 2-type η'. Because $u_1 \in S_2$, $\eta'^{\rhd} = \vec{0}$, the number of edges connected to u, u_1, and u_2 with specific

doi:10.6342/NTU202203098

non-trivial 2-type is preserved. Furthermore, the size of the set S_3 increase. Hence, we can repeat the above procedure until the set S_3 is large enough.

Case 2: $\pi_s \neq \pi$. The procedure is similar. We divide the set $(V_\mathsf{E} \cup U_\mathsf{S})\big|_\pi$ into three parts. Let S_1 be the subset of $(V_\mathsf{E} \cup U_\mathsf{S})\big|_\pi$ such that the 2-types between u and them are non-trivial. Let S_2 be the subset of $(V_\mathsf{E} \cup U_\mathsf{S})\big|_\pi$ such that the 2-types between u and them are trivial. Let S_3 be the subset of $(V_\mathsf{E} \cup U_\mathsf{S})\big|_\pi$ such that the 2-types between u and them are not assigned. Clearly, the three sets are disjoint. Note that the number of outgoing edges from u are bounded by K. Therefore, if $|S_1| + |S_3| \geq K$, then we can always find desired edges from S_3.

Let U_1 be the subset of $(V_\mathsf{E} \cup U_\mathsf{S})\big|_{\pi_s} \setminus \{u\}$. The number of edge between U_1 and S_2 is at least $|U_1||S_2|$. On the other hand, the number of assigned edges between U_1 and S_2 is at most $K(|U_1| + |S_2|)$. Suppose $|S_1| + |S_3| < K$, By the first type constraint, the size of the set U_1 and S_2 is bounded as follows.

$$|U_1| \geq \left|(V_\mathsf{E} \cup U_\mathsf{S})\big|_{\pi_s}\right| - |\{u\}| \geq (3K+3) - 1 \geq 3K+2$$

$$|S_2| \geq \left|(V_\mathsf{E} \cup U_\mathsf{S})\big|_{\pi}\right| - |S_1| - |S_3| \geq (3K+3) - K \geq 2K+3$$

We observe the following.

$$\frac{|U_1| + |S_2|}{|U_1||S_2|} = \frac{1}{|U_1|} + \frac{1}{|S_2|} \leq \frac{1}{3K+2} + \frac{1}{2K+3} < \frac{1}{K}$$

Therefore, the following holds.

$$|U_1||S_2| > K(|U_1| + |S_2|)$$

The above inequality implies that there exists a unassigned edge, say (u_1, u_2), between U_1 and S_3. Let η' be the 2-type of the edge (u, u_1). We then remove the edge (u, u_1), and assign the edge (u_2, u_1) with 2-type η'. Because $u_1 \in S_2$, $\eta'^{\triangleright} = \vec{0}$, the number of edges connected to u, u_1, and u_2 with specific non-trivial 2-type is preserved. Furthermore, the size of the set S_3 increase. Hence, we can repeat the

doi:10.6342/NTU202203098

above procedure until the set S_3 is large enough.

Step 4: We assign all remaining edges with null type. Note that, by the condition of proper partition, the 1-types realized in $V_{\mathsf{E}} \cup U_{\mathsf{S}}$ are null-compatible. Besides, the null type edges doesn't change the counting conditions of the elements

The size of $\mathcal{H}$ is as follows.

$$
\begin{aligned}
|U| &= |V_{\mathsf{F}}| + |V_{\mathsf{E}}| + |U_{\mathsf{S}}| \\
&= |V_{\mathsf{F}}| + |V_{\mathsf{E}}| + \sum_{C \in \mathcal{C}} |\{u_{C,1}, u_{C,2}, \ldots, u_{C,p_C}\}| \\
&= |V_{\mathsf{F}}| + |V_{\mathsf{E}}| + \sum_{C \in \mathcal{C}} p_C
\end{aligned}
$$

□

4.3 CEB property for $\mathsf{FIN\text{-}SAT}(\mathsf{C}^2)$

The CEB property for $\mathsf{FIN\text{-}SAT}(\mathsf{C}^2)$ is similar to $\mathsf{SAT}(\mathsf{C}^2)$. The only difference is that the solution of the induced ILP system is restricted to finite. In this section, we do not provide the whole proof of the CEB property for $\mathsf{FIN\text{-}SAT}(\mathsf{C}^2)$, but we will point out the main modification in the proof.

Theorem 4.23 (The CEB property for $\mathsf{FIN\text{-}SAT}(\mathsf{C}^2)$)**.** *The C^2 sentence φ is finitely satisfiable if and only if there exist the following components.*

- *A pseudo-structure $\mathcal{G}$ over τ with a proper partition $V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}$ in φ.*
- *A set of valid configurations $\mathcal{C}$ satisfying $\mathcal{G}$, $V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}$ and $\mathcal{C}$ are matching in φ.*
- *A solution $\{x_C \mapsto p_C\}_{C \in \mathcal{C}}$ of the system $\mathcal{Q}_{\mathcal{G}, V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}, \mathcal{C}}$.*

Furthermore, their sizes are bounded by the length of φ. More precisely,

- $|V| \leq \theta_1(n, m, K)$*;*
- $|\mathcal{C}| \leq \theta_2(n, m, K)$*;*

doi:10.6342/NTU202203098

- *for every $C \in \mathcal{C}$, $p_C \in [0, \theta_3(n,m,K)]$, where $\theta_1(n,m,K)$, $\theta_2(n,m,K)$, and $\theta_3(n,m,K)$ are the same as in Theorem 4.9.*

Proof.

(If.) By Lemma 4.25.

(Only if.) It is sufficient to show that the size of the pseudo-model constructed by Lemma 4.22 is finite. Let $\mathcal{H}$ be the such model. Note that the size of $\mathcal{H}$ is $|V_{\mathsf{F}}| + |V_{\mathsf{E}}| + \sum_{C \in \mathcal{C}} p_C$. If for every $C \in \mathcal{C}$, p_C is finite, then the size of $\mathcal{H}$ is also finite. Hence, $\mathcal{H}$ is a finite pseudo-model of φ, and φ is finitely satisfiable. □

Lemma 4.24. *Let $V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}$ be a proper partition of $\mathcal{G}$ in φ and $\mathcal{C}$ be a set of valid configurations satisfying $\mathcal{G}$, $V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}$, and $\mathcal{C}$ are matching in φ and the system $\mathcal{Q}_{\mathcal{G}, V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}, \mathcal{C}}$ has a solution in $\mathbb{N}$.*

There exists a set of valid configurations in φ, denoted by $\mathcal{D}$, satisfy the following.

- $|\mathcal{D}| \leq 2^{2n+2m+3}\left(2n + 2m + 4 + \lceil \log\left(\max\left(|V_{\mathsf{E}}|, (2K+1)^2\right)\right)\rceil\right)$.
- *$\mathcal{G}$, $V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}$, and $\mathcal{D}$ are matching in φ.*
- *The system $\mathcal{Q}_{\mathcal{G}, V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}, \mathcal{D}}$ has a solution $\{x_C \mapsto p_C\}_{C \in \mathcal{D}}$ in $\mathbb{N}$.*

Proof. The proof is similar to Lemma 4.20. The only difference is that the solution of the system $\mathcal{Q}_{\mathcal{G}, V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}, \mathcal{C}}$ is restricted to finite. □

Lemma 4.25. *If the C^2 sentence φ is finitely satisfiable, then there exists the following components.*

- *A pseudo-structure $\mathcal{G}$ over τ with a proper partition $V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}$ in φ.*
- *A set of valid configurations $\mathcal{C}$ satisfying $\mathcal{G}$, $V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}$ and $\mathcal{C}$ are matching in φ.*
- *A solution $\{x_C \mapsto p_C\}_{C \in \mathcal{C}}$ of the system $\mathcal{Q}_{\mathcal{G}, V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}, \mathcal{C}}$.*

Furthermore, the size of them is bounded by the length of φ. More precisely,

- *$|V| \leq \theta_1(n,m,K)$;*
- *$|\mathcal{C}| \leq \theta_2(n,m,K)$;*

doi:10.6342/NTU202203098

- *for every $C \in \mathcal{C}$, $p_C \in [0, \theta_3(n,m,K)]$, where $\theta_1(n,m,K)$, $\theta_2(n,m,K)$, and $\theta_3(n,m,K)$ are the same as in Theorem 4.9.*

Proof. The proof is similar to Lemma 4.21. Here, we only prove that it has a finite assignment. Because φ is finitely satisfiable, we choose a finite pseudo-model for $\mathcal{H}$ in the proof. We have proved that $\left\{x_C \mapsto n_C^{U_\mathsf{S}}\right\}_{C \in \mathcal{C}^{U_\mathsf{S}}}$ is a solution of the system $\mathcal{Q}_{\mathcal{H}, U_\mathsf{F} \dot\cup U_\mathsf{E} \dot\cup U_\mathsf{S}, \mathcal{C}^{U_\mathsf{S}}}$ in Lemma 4.17. Clearly, $n_C^{U_\mathsf{S}} \le |U|$ is finite. Hence, the system has a finite solution.

Finally, by Lemma 4.24 instead of Lemma 4.20, for every $C \in \mathcal{C}$, $p_C \le \theta_3(n,m,K)$.

□

4.4 Bounds on the size of the model

As a direct consequence of the CEB property, we close this chapter by discussing the upper bound of the smallest pseudo-model of φ.

Lemma 4.26. *If the C^2 sentence φ is finitely satisfiable, then it has a pseudo-model whose size is at most $\theta_4(n,m,K)$, where*

$$\theta_4(n,m,K) := 2^{4n+4m+8}\left(4n+4m+6+2\lceil \log(2K+1)\rceil\right)^2 \left(2^{4n+4m+4}(2K+1)^2\right)^{2^{2n+2m+3}}.$$

Proof. Since φ is satisfiable, by Lemma 4.25, there exists the pseudo-structure $\mathcal{G}$ over τ with the proper partition $V_\mathsf{F} \dot\cup V_\mathsf{E} \dot\cup V_\mathsf{S}$ in φ, the set of valid configuration $\mathcal{C}$ in φ and the assignment $\{x_C \mapsto p_C\}_{C \in \mathcal{C}}$ in $\mathbb{N}$, such that their sizes are bounded.

- $|V| \le \theta_1(n,m,K)$;
- $|\mathcal{C}| \le \theta_2(n,m,K)$;
- for every $C \in \mathcal{C}$, $p_C \in [0, \theta_3(n,m,K)]$.

By Lemma 4.22, together with the components $\mathcal{G}$, $V_\mathsf{F} \dot\cup V_\mathsf{E} \dot\cup V_\mathsf{S}$, and $\mathcal{C}$, φ has a pseudo-

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model $\mathcal{H}$ whose size is bounded.

$$
\begin{aligned}
U &\le |V_{\mathsf{F}}| + |V_{\mathsf{E}}| + \sum_{C \in \mathcal{C}} p_C \\
&\le |V| + |\mathcal{C}| \cdot \theta_3(n, m, K) \\
&\le \theta_1(n, m, K) + \theta_2(n, m, K) \cdot \theta_3(n, m, K) \\
&\le 2^{4n+4m+8} \left(4n + 4m + 6 + 2\lceil \log(2K+1) \rceil\right)^2 \left(2^{4n+4m+4}(2K+1)^2\right)^{2^{2n+2m+3}}
\end{aligned}
$$

□

As mentioned before, the ESM property no longer holds for the satisfiability problem of C^2. We have shown that $\mathsf{SAT}(\mathsf{C}^2)$ and $\mathsf{FIN\text{-}SAT}(\mathsf{C}^2)$ do not coincide in Theorem 4.2. Moreover, the ESM property does not hold even for the finite satisfiability problem of C^2. The following *double-exponential size model property* of C^2 is a simple corollary of Lemma 4.26. Though the complexity of $\mathsf{FIN\text{-}SAT}(\mathsf{C}^2)$ is $\mathsf{NEXPTIME}$, still, the guessing and checking algorithm does not match the $\mathsf{NEXPTIME}$ lower bound.

Theorem 4.27. **[25]** *If the C^2 sentence φ is finitely satisfiable, then it has a pseudo-model with size at most double-exponential in its length.*

doi:10.6342/NTU202203098

Chapter 5

Algorithms for the satisfiability of C^2

In this chapter, we present alternative $\mathsf{NEXPTIME}$ algorithms for the satisfiability and finite satisfiability problems of C^2 from the CEB property. In Section 5.1, we discuss the certificate and the verifier for the satisfiability and finite satisfiability problems of C^2. In Section 5.2, we present the algorithm for $\mathsf{SAT}(\mathsf{C}^2)$. Finally, we present the algorithm for $\mathsf{FIN\text{-}SAT}(\mathsf{C}^2)$ in Section 5.3.

In the rest of this chapter, we fix a vocabulary τ and a C^2 sentence φ over τ in the normal form. Let n and m be the number of unary and binary predicates in τ, respectively.

5.1 Certificate for $\mathsf{SAT}(\mathsf{C}^2)$ and $\mathsf{FIN\text{-}SAT}(\mathsf{C}^2)$

Observe that the sizes of each component stated in the CEB property are finite. Furthermore, they can be encoded by only exponential many bits in the length of the C^2 sentence φ. Then, we can obtain the certificate of φ from the CEB property.

Definition 5.1. *The certificate of φ is a tuple $c = \langle \mathcal{G}, V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}, \mathcal{C}, \{x_C \mapsto p_C\}_{C \in \mathcal{C}} \rangle$. Each component satisfies the following.*

1. *$\mathcal{G}$ is a pseudo-structure over τ.*

2. *$V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}$ is a proper partition of $\mathcal{G}$ in φ.*

3. *$\mathcal{C}$ is a set of valid configuration in φ.*

doi:10.6342/NTU202203098

4. *$\mathcal{G}$, $V_{\mathsf{F}}\dot{\cup}V_{\mathsf{E}}\dot{\cup}V_{\mathsf{S}}$ and $\mathcal{C}$ are matching in φ.*

5. *$\{x_C \mapsto p_C\}_{C\in\mathcal{C}}$ is a solution of the system $\mathcal{Q}_{\mathcal{G},V_{\mathsf{F}}\dot{\cup}V_{\mathsf{E}}\dot{\cup}V_{\mathsf{S}},\mathcal{C}}$ in $\mathbb{N}_\infty$.*

Remark 5.2. *The certificate of φ is not unique. In fact, the proof of Lemma 4.21 gives different certificates if we choose different pseudo-models of φ.*

Remark 5.3. *Note that there is no constraint on the sizes of the components in Lemma 4.22. Hence, there may exist a certificate c of φ such that the size of c is unbounded.*

Although the certificate of φ is not unique, and its size is unbounded, the CEB property guarantee that φ has a short certificate. Hence, we can design the algorithm by guessing and checking the certificate instead of the pseudo-model itself.

Lemma 5.4. *The C^2 sentence φ is satisfiable if and only if it has a certificate c such that c can be encoded by exponential bits in the length of φ.*

Proof. By Theorem 4.9, φ is satisfiable if and only if there exists a pseudo-structure $\mathcal{G}$ over τ, a proper partition $V_{\mathsf{F}}\dot{\cup}V_{\mathsf{E}}\dot{\cup}V_{\mathsf{S}}$ of $\mathcal{G}$ in φ, a set of valid configurations $\mathcal{C}$ in φ such that $\mathcal{G}$, $V_{\mathsf{F}}\dot{\cup}V_{\mathsf{E}}\dot{\cup}V_{\mathsf{S}}$ and $\mathcal{C}$ are matching in φ, and a solution $\{x_C \mapsto p_C\}_{C\in\mathcal{C}}$ of the system $\mathcal{Q}_{\mathcal{G},V_{\mathsf{F}}\dot{\cup}V_{\mathsf{E}}\dot{\cup}V_{\mathsf{S}},\mathcal{C}}$. Furthermore, the size of each component is bounded. Hence, these components are a certificate of φ. Then, it is sufficient to show that each of them can be encoded by exponential bits in the length of φ.

For the pseudo-structure $\mathcal{G}$, its size is bounded by $\theta_1(n, m, K)$. Recall that $\theta_1(n, m, K)$ is exponential in the length of φ. We can encode the index of each vertex in binary, which takes only polynomial bit. Besides, the labels of the vertices and edges are 1-types and 2-types which can represent by n and $2m$ bits, respectively.

For the set of valid configuration $\mathcal{C}$, its size is bounded by $\theta_2(n, m, K)$. Recall that $\theta_2(n, m, K)$ is exponential in the length of φ. For each configuration $C \in \mathcal{C}$, C consists of a 1-type and a behavior function. Note that the valid behavior function is a function from $\mathcal{K}^{\tau,\mathsf{ntri}} \times \Pi^\tau \mapsto K$. Then, we can encode this function by $\left|\mathcal{K}^{\tau,\mathsf{ntri}} \times \Pi^\tau\right| \times \lceil \log K \rceil \leq 2^{2n+2m} \lceil \log K \rceil$ bits.

For the assignment $\{x_C \mapsto p_C\}_{C\in\mathcal{C}}$, the number of variables is bounded by $|C|$. For each $C \in \mathcal{C}$, $p_C \in [0, \theta_3(n, m, K)] \cup \{\infty\}$. Though $\theta_3(n, m, K)$ is double-exponential in

doi:10.6342/NTU202203098

the length of φ. It takes only exponential bits to encode the element in $[0, \theta_3(n, m, K)]$ in binary. We can add one extra bit to represent the infinity. Hence, encoding the assignment takes only exponential bits. Therefore, these components are desired certificate. □

Hence, we have the following ALGORITHM-A. The correctness of ALGORITHM-A is by Lemma 5.4. It is not difficult to check that ALGORITHM-A takes deterministic polynomial-time in the length of the input formula φ and the certificate, which implies that ALGORITHM-A is a polynomial-time verifier of $\mathsf{SAT}(\mathsf{C}^2)$.

Theorem 5.5. **[25]** $\mathsf{SAT}(\mathsf{C}^2)$ *is in* $\mathsf{NEXPTIME}$.

Proof. We observe that ALGORITHM-A is a polynomial-time verifier of $\mathsf{SAT}(\mathsf{C}^2)$. Besides, the C^2 sentence φ is satisfiable if and only if it has a exponential length certificate. By definition, $\mathsf{SAT}(\mathsf{C}^2)$ is in $\mathsf{NEXPTIME}$. □

ALGORITHM-A

Input: A C^2 sentence φ and a certificate $c = \langle \mathcal{G}, V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}, \mathcal{C}, \{x_C \mapsto p_C\}_{C \in \mathcal{C}} \rangle$.
Task: Accept if and only if c is a certificate of φ.

Check whether $V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}$ is a proper partition of $\mathcal{G}$ in φ.
if it is not a proper partition **then**
  REJECT
for every configuration C in $\mathcal{C}$ **do**
  Check whether C is valid in φ.
  if it is not valid **then**
    REJECT
Check whether $\mathcal{G}$, $V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}$ and $\mathcal{C}$ are matching in φ.
if they are not matching **then**
  REJECT
Check whether $\{x_C \mapsto p_C\}_{C \in \mathcal{C}}$ is a solution of $\mathcal{Q}_{\mathcal{G}, V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}, \mathcal{C}}$.
if it is not a solution **then**
  REJECT
ACCEPT

5.2 Algorithm for $\mathsf{SAT}(\mathsf{C}^2)$

It is quite standard to construct the non-deterministic algorithm from the verifier by guessing all possible certificates. It is not difficult to check that ALGORITHM-B takes non-deterministic exponential-time in the length of the input sentence φ.

doi:10.6342/NTU202203098

Lemma 5.6. Algorithm-B *decides* $\mathsf{SAT}(\mathsf{C}^2)$.

Proof. Algorithm-B guess all possible certificates c whose size is bounded by Lemma 5.4 and check where c is a certificate of φ by the polynomial-time verifier Algorithm-A. Hence, Algorithm-B decides $\mathsf{SAT}(\mathsf{C}^2)$. □

Remark 5.7. Algorithm-B *makes exponential many guessing from double-exponential many configurations. Hence,* Algorithm-B *is still involved and complicated to implement.*

Algorithm-B

Input: A C^2 sentence φ.
Task: Accept if and only if φ is satisfiable.

Guess a structure $\mathcal{G}$ with size $|\mathcal{G}| \leq \theta_1(n, m, K)$.
Guess a partition $V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}$ of $\mathcal{G}$.
Guess a set of configurations $\mathcal{C}$ such that $|\mathcal{C}| \leq \theta_2(n, m, K)$.
Guess an assignment $\{x_C \mapsto p_C\}_{C \in \mathcal{C}}$ such that for every $C \in \mathcal{C}$,
  $p_C \in [0, \theta_3(n, m, K)] \cup \{\infty\}$.
Run Algorithm-A on φ and $\langle \mathcal{G}, V_{\mathsf{F}} \dot\cup V_{\mathsf{E}} \dot\cup V_{\mathsf{S}}, \mathcal{C}, \{x_C \mapsto p_C\}_{C \in \mathcal{C}} \rangle$.
if it accepts **then**
  ACCEPT
else
  REJECT

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doi:10.6342/NTU202203098

5.3 Algorithm for $\mathsf{FIN\text{-}SAT}(\mathsf{C}^2)$

The CEB property for $\mathsf{SAT}(\mathsf{C}^2)$ and $\mathsf{FIN\text{-}SAT}(\mathsf{C}^2)$ are almost the same. The only difference is that the assignment is restricted to finite for $\mathsf{FIN\text{-}SAT}(\mathsf{C}^2)$. Hence, the arguments for $\mathsf{SAT}(\mathsf{C}^2)$ still hold for $\mathsf{FIN\text{-}SAT}(\mathsf{C}^2)$ with slight modification. We can prove the following finite version results. Besides, we have the following ALGORITHM-C, which decides $\mathsf{FIN\text{-}SAT}(\mathsf{C}^2)$ in non-deterministic exponential-time in the length of the input sentence φ.

Lemma 5.8. ALGORITHM-C *decides* $\mathsf{FIN\text{-}SAT}(\mathsf{C}^2)$.

Theorem 5.9. **[25]** $\mathsf{FIN\text{-}SAT}(\mathsf{C}^2)$ *is in* $\mathsf{NEXPTIME}$.

ALGORITHM-C

Input: A C^2 sentence φ.
Task: Accept if and only if φ is finitely satisfiable.

```
Guess a structure 𝒢 with size |𝒢| ≤ θ_1(n, m, K).
Guess a partition V_F ∪̇ V_E ∪̇ V_S of 𝒢.
Guess a set of configurations 𝒞 such that |𝒞| ≤ θ_2(n, m, K).
Guess an assignment {x_C ↦ p_C}_{C∈𝒞} such that for every C ∈ 𝒞, p_C ∈ [0, θ_3(n, m, K)].
Run ALGORITHM-A on φ and ⟨𝒢, V_F ∪̇ V_E ∪̇ V_S, 𝒞, {x_C ↦ p_C}_{C∈𝒞}⟩.
if it accepts then
    ACCEPT
else
    REJECT
```

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doi:10.6342/NTU202203098

doi:10.6342/NTU202203098

Chapter 6

Strongly satisfiable fragment of C^2

In this chapter, we present the other important result of this thesis. In Section 6.1, we present the detailed definition of the strongly satisfiable fragment of C^2 and its CEB property. In Section 6.2, we present the *graph representation of configurations*, which is a helpful tool to represent the set of configurations with only *exponential* many variables. In Section 6.3, we present the *reduced strong induced ILP system*, which is system whose size is smaller than the size of the induced integer constraint system used in the CEB property. Finally, in Section 6.4, we present the algorithms for the strong satisfiability and finite strong satisfiability problems of C^2.

In the rest of this chapter, we fix a vocabulary τ, a C^2 sentence φ over τ in the normal form, and a pseudo-structure $\mathcal{G}$ over τ.

6.1 Strongly satisfiable fragment of C^2

The strongly satisfiable fragment of C^2 is a semantic fragment of C^2, which captures the idea of the *strong* part in the proper partition of the CEB property. Roughly speaking, a pseudo-model is a strong pseudo-model if the finite part and the extended part of its standard partition are empty. A C^2 sentence is strongly satisfiable if it has a strong pseudo-model. Since the sentence has a strong pseudo-model, we can further simplify the algorithm for deciding the strong satisfiability problem. Though the complexity of this problem is still $\mathsf{NEXPTIME}$-complete, the algorithm for the strong satisfiability problem of C^2

doi:10.6342/NTU202203098

makes less guessing than the algorithm for $\mathsf{SAT}(\mathsf{C}^2)$.

Definition 6.1. *The pseudo-structure $\mathcal{G}$ is a strong pseudo-model of φ, if $\mathcal{G}$ is a pseudo-model of φ and $\Pi^{\mathcal{G}}$ is mutually null-compatible.*

Definition 6.2. *The C^2 sentence φ is strongly satisfiable if φ has a strong pseudo-model. The C^2 sentence φ is finitely strongly satisfiable if φ has a finite size strong pseudo-model.*

Let strong satisfiability $\mathsf{SSAT}(\mathsf{C}^2)$ be the set of all strongly satisfiable C^2 sentences, and finite strong satisfiability $\mathsf{FIN\text{-}SSAT}(\mathsf{C}^2)$ be the set of all finitely strongly satisfiable C^2 sentences. Like the satisfiability problem of C^2, $\mathsf{SSAT}(\mathsf{C}^2)$ and $\mathsf{FIN\text{-}SSAT}(\mathsf{C}^2)$ do not coincide.

Lemma 6.3. $\mathsf{SSAT}(\mathsf{C}^2)$ *and* $\mathsf{FIN\text{-}SSAT}(\mathsf{C}^2)$ *do not coincide.*

Proof. Let φ_0 and $\mathcal{G}_0$ be as in Example 3.32. The sentence φ_0 has a pseudo-model $\mathcal{G}_0$ with $\Pi^{V_0} = \{\pi_1, \pi_2\}$. By Example 3.12, π_1 and π_2 are null-compatible in φ_0. Hence, $\mathcal{G}_0$ is a strong pseudo-model of φ_0 and $\varphi_0 \in \mathsf{SSAT}(\mathsf{C}^2)$.

On the other hand, by Lemma 4.1, since $\varphi_0 \notin \mathsf{FIN\text{-}SAT}(\mathsf{C}^2)$, it has no finite pseudo-model. So it has no finite strong pseudo-model, which implies that $\varphi_0 \notin \mathsf{FIN\text{-}SSAT}(\mathsf{C}^2)$. Hence, the languages $\mathsf{SSAT}(\mathsf{C}^2)$ and $\mathsf{FIN\text{-}SSAT}(\mathsf{C}^2)$ do not coincide. □

An important property for $\mathsf{SSAT}(\mathsf{C}^2)$ is that if a sentence is strongly satisfiable, then it has a arbitrary large strong pseudo-model. Furthermore, it has an infinite size strong pseudo-model.

Lemma 6.4. *If φ is strongly satisfiable, then for every $t \in \mathbb{N}$, φ has a strong pseudo-model $\mathcal{H}$ satisfying the following. For every $\pi \in \Pi^V$, $n_\pi \geq t$, and for every $C \in \mathcal{C}^V$, $n_C \geq t$.*

Proof. Let $\mathcal{G}$ be a strong pseudo-model of φ. We construct another pseudo-model $\mathcal{H}$ of φ by the union of t many copies of $\mathcal{G}$. The edges between the vertex from different copies are assigned null type.

Formally, each component of $\mathcal{H}$ is defined as follows.

- The universe $U := V \times [t]$.

doi:10.6342/NTU202203098

- The label of vertices $\mathsf{lab}^{\mathcal{H}}[(v,i)] := \mathsf{lab}^{\mathcal{G}}[v]$.
- The label of edges $\mathsf{lab}^{\mathcal{H}}[(v_1,i_1),(v_2,i_2)] := \begin{cases} \mathsf{lab}^{\mathcal{G}}[v_1,v_2] & \text{if } i_1 = i_2 \\ \eta_{\mathsf{null}} & \text{otherwise} \end{cases}$.

Here we show that $\mathcal{H}$ is a pseudo-model of φ. For every $(v,i) \in U$, because $\mathcal{G}$ is a pseudo-model of φ, $\mathsf{lab}^{\mathcal{G}}[v]$ is valid in φ. Since $\mathsf{lab}^{\mathcal{H}}[(v,i)] = \mathsf{lab}^{\mathcal{G}}[v]$, $\mathsf{lab}^{\mathcal{H}}[(v,i)]$ is also valid in φ. For every $(v_1,i_1),(v_2,i_2) \in U$, because $\mathcal{G}$ is a pseudo-model of φ, $\left\langle \mathsf{lab}^{\mathcal{G}}[v_1], \mathsf{lab}^{\mathcal{G}}[v_1,v_2], \mathsf{lab}^{\mathcal{G}}[v_2] \right\rangle$ is valid in φ. Because $\mathcal{G}$ is strong, $\left\langle \mathsf{lab}^{\mathcal{G}}[v_1], \eta_{\mathsf{null}}, \mathsf{lab}^{\mathcal{G}}[v_2] \right\rangle$ is also valid in φ. Hence, $\left\langle \mathsf{lab}^{\mathcal{H}}[(v_1,i_1)], \mathsf{lab}^{\mathcal{H}}[(v_1,i_1),(v_2,i_2)], \mathsf{lab}^{\mathcal{H}}[(v_2,i_2)] \right\rangle$ is valid in φ. Finally the counting condition holds for every $(v,i) \in U$.

$$\begin{aligned}
&\sum_{u \in U \setminus \{(v,i)\}} \mathsf{lab}^{\mathcal{H},\triangleright}[(v,i),u] \\
&= \sum_{v' \in V \setminus \{v\}} \mathsf{lab}^{\mathcal{H},\triangleright}[(v,i),(v',i)] + \sum_{i' \in [t] \setminus \{i\}} \sum_{v' \in V} \mathsf{lab}^{\mathcal{H},\triangleright}[(v,i),(v',i')] \\
&= \sum_{v' \in V \setminus \{v\}} \mathsf{lab}^{\mathcal{G},\triangleright}[v,v'] + \sum_{i' \in [t] \setminus \{i\}} \sum_{v' \in V} \eta^{\triangleright}_{\mathsf{null}} \\
&= \vec{k}
\end{aligned}$$

Now we show that $\mathcal{H}$ is strong. Observe that $\Pi^U = \Pi^V$. Because $\mathcal{G}$ is strong, Π^V is mutually null-compatible in φ and so for Π^U.

Finally, observe that $\Pi^U = \Pi^V$ and $\mathcal{C}^U = \mathcal{C}^V$. By definition, if $\pi \in \Pi^V$, then $n^{\mathcal{G}}_{\pi} \geq 1$. Therefore, for every $\pi \in \Pi^U$, $n^{\mathcal{H}}_{\pi} = (t+1) \cdot n^{\mathcal{G}}_{\pi} \geq t$. If $C \in \mathcal{C}^V$, then $n^{\mathcal{G}}_{C} \geq 1$. For every $C \in \mathcal{C}^U$, $n^{\mathcal{H}}_{C} = (t+1) \cdot n^{\mathcal{G}}_{C} \geq t$. These implies that $\mathcal{H}$ is the desired strong pseudo-model of φ. $\square$

Lemma 6.5. *The C^2 sentence φ is strongly satisfiable if and only if φ has a pseudo-model $\mathcal{H}$ satisfying the following. For every $\pi \in \Pi^V$, $n_\pi = \infty$, and for every $C \in \mathcal{C}^V$, $n_C = \infty$.*

Proof.

(If.) Note that for every $C \in \mathcal{C}^U$, $n_C = \infty$ implies that for every $\pi \in \Pi^U$, $n_\pi = \infty$. It is sufficient to show that if $\mathcal{H}$ is a pseudo-model of φ satisfying for every $\pi \in \Pi^U$, $n_\pi = \infty$, then $\mathcal{H}$ is a strong pseudo-model of φ. Suppose the contrary that $\mathcal{H}$ is not strong

doi:10.6342/NTU202203098

and there exists $\pi_1, \pi_2 \in \Pi^U$ such that π_1 and π_2 are not null-compatible. By Lemma 3.33, $n_{\pi_1} \leq 2K+1$ or $n_{\pi_2} \leq 2K+1$. This contradicts that $n_{\pi_1} = \infty$ and $n_{\pi_2} = \infty$. Hence, π_1 and π_2 are null-compatible, and $\mathcal{H}$ is a strong pseudo-model of φ.

(Only if.) The only if direction is the same as the proof of Lemma 6.4. The only difference is that we consider $\mathcal{H}$ the union of infinitely many copies of strong pseudo-model $\mathcal{G}$. □

Remark 6.6. *Lemma 6.5 shows that if a* C^2 *sentence is strongly satisfiable, then it has an infinite size pseudo-model. Since some* C^2 *sentences only have finite size pseudo-models,* $\mathsf{SSAT}(\mathsf{C}^2)$ *and* $\mathsf{SAT}(\mathsf{C}^2)$ *do not coincide.*

The relations of the satisfiability problems are described below.

$$\begin{array}{ccc} \mathsf{SSAT}(\mathsf{C}^2) & \subsetneq & \mathsf{SAT}(\mathsf{C}^2) \\ \cup\!\!\!\!\nmid & & \cup\!\!\!\!\nmid \\ \mathsf{FIN\text{-}SSAT}(\mathsf{C}^2) & \subsetneq & \mathsf{FIN\text{-}SAT}(\mathsf{C}^2) \end{array}$$

Note that $\mathsf{FIN\text{-}SSAT}(\mathsf{C}^2) \neq \mathsf{FIN\text{-}SAT}(\mathsf{C}^2) \cap \mathsf{SSAT}(\mathsf{C}^2)$.

Bounds on the size of the model. Though the strongly satisfiable fragment is a semantic fragment of C^2, we can consider the *strong form* of the sentence which captures the similar fragment.

Definition 6.7. *Let* φ *be a* C^2 *sentence over the vocabulary* τ *in the normal form, that is,*

$$\begin{aligned} \varphi := \quad & \forall x\, \gamma(x) \\ \wedge \quad & \forall x\, \forall y\, x \neq y \rightarrow \alpha(x,y) \\ \wedge \quad & \bigwedge_{i \in [m]} \forall x\, \exists^{=k_i} y\, \beta_i(x,y) \ \wedge\ x \neq y, \end{aligned}$$

where $\gamma(x)$ *and* $\alpha(x,y)$ *are quantifier-free formulae.*

Recall that the notation $\phi[b/a]$ *denotes that substituting all* a *occurred in* ϕ *with* b*. The symbol* $\phi[b_t/a_t]_{t=\ell_1}^{\ell_2}$ *is the abbreviation of multiple substitution. The strong form of* φ*,*

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denoted by φ^S, is the C^2 sentence over τ defined as follows.

$$\alpha'(x,y) := \quad \alpha(x,y)[\bot/\beta_t(x,y)]_{t=1}^{m}[\bot/\beta_t(y,x)]_{t=1}^{m}$$

$$\begin{aligned} \varphi^S := \quad & \forall x\ (\gamma(x) \wedge \alpha'(x,x)) \\ \wedge \quad & \forall x\, \forall y\ x \neq y \rightarrow (\alpha(x,y) \wedge \alpha'(x,y)) \\ \wedge \quad & \bigwedge_{i\in[m]} \forall x\ \exists^{=k_i} y\ \beta_i(x,y)\ \wedge\ x \neq y \end{aligned}$$

Lemma 6.8. *The pseudo-structure $\mathcal{G}$ is a strong pseudo-model of φ if and only if it is a model of φ^S.*

Proof.

(Only if.) For every realizable tuple $\langle \pi_1, \eta, \pi_2\rangle$ in $\mathcal{G}$, because $\mathcal{G}$ is a pseudo-model of φ,

$$\pi_1(x) \wedge \eta(x,y) \wedge \pi_2(y) \models \alpha(x,y). \tag{6.1}$$

Because $\mathcal{G}$ is a strong pseudo-model of φ,

$$\pi_1(x) \wedge \eta_{\mathsf{null}}(x,y) \wedge \pi_2(y) \models \alpha(x,y).$$

This implies that

$$\pi_1(x) \wedge \eta_{\mathsf{null}}(x,y) \wedge \pi_2(y) \models \alpha'(x,y).$$

Moreover, because there are no binary predicate in $\alpha'(x,y)$, the following also holds.

$$\pi_1(x) \wedge \eta(x,y) \wedge \pi_2(y) \models \alpha'(x,y) \tag{6.2}$$

Equations (6.1) and (6.2) imply that

$$\pi_1(x) \wedge \eta(x,y) \wedge \pi_2(y) \models \alpha(x,y) \wedge \alpha'(x,y).$$

Hence, the tuple $\langle \pi_1, \eta, \pi_2\rangle$ is also realizable in φ^S.

For every realizable 1-type π, because $\mathcal{G}$ is a pseudo-model of φ, $\pi(x) \models \gamma(x)$. By

doi:10.6342/NTU202203098

equation (6.2), since there are no binary predicate in $\alpha'(x,x)$, we can omit the 2-type, $\pi(x) \models \alpha'(x,x)$. These implies that $\pi(x) \models \gamma(x) \wedge \alpha'(x,x)$. Hence, the 1-type π is also realizable in φ^S. Therefore, $\mathcal{G}$ is a pseudo-model of φ^S.

(If.) For every realizable tuple $\langle \pi_1, \eta, \pi_2 \rangle$ in $\mathcal{G}$, because $\mathcal{G}$ is a pseudo-model of φ^S,

$$\begin{aligned} \pi_1(x) \wedge \eta(x,y) \wedge \pi_2(y) &\models \alpha(x,y) \\ \pi_1(x) \wedge \eta(x,y) \wedge \pi_2(y) &\models \alpha'(x,y). \end{aligned} \tag{6.3}$$

Hence, the tuple $\langle \pi_1, \eta, \pi_2 \rangle$ is also realizable in φ.

For every realizable tuple $\langle \pi_1, \eta, \pi_2 \rangle$ in $\mathcal{G}$, because $\mathcal{G}$ is a pseudo-model of φ^S,

$$\begin{aligned} \pi(x) &\models \gamma(x) \\ \pi(x) &\models \alpha'(x,x). \end{aligned} \tag{6.4}$$

Hence, the 1-type π is also realizable in φ. Therefore, $\mathcal{G}$ is a pseudo-model of φ.

Now we claim that $\mathcal{G}$ is a strong pseudo-model of φ. For every realizable 1-types π_1 and π_2 in φ. If $\pi_1 \neq \pi_2$, then the equation (6.3) implies that

$$\pi_1(x) \wedge \eta_{\mathsf{null}}(x,y) \wedge \pi_2(y) \models \alpha(x,y).$$

Otherwise, if $\pi_1 = \pi_2$, then the equation (6.4) implies that

$$\pi_1(x) \wedge \eta_{\mathsf{null}}(x,y) \wedge \pi_1(y) \models \alpha(x,y).$$

Hence, $\mathcal{G}$ is a strong pseudo-model of φ. □

The following two lemmas are simple corollaries of Lemma 6.8.

Lemma 6.9. *The* C^2 *sentence* φ *is strongly satisfiable if and only if* φ^S *is satisfiable.*

Lemma 6.10. *The* C^2 *sentence* φ *is finitely strongly satisfiable if and only if* φ^S *is finitely satisfiable.*

Then, we can establish the upper bound of the smallest strong pseudo-model of φ.

doi:10.6342/NTU202203098

In fact, the *double-exponential size model property* still holds for the strongly satisfiable fragment of C^2.

Lemma 6.11. *If the C^2 sentence φ is finitely strongly satisfiable, then it has a strong pseudo-model whose size is at most $\theta_4(n, m, K)$, where $\theta_4(n, m, K)$ is the same as in Lemma 4.26.*

Proof. By Lemma 4.26, φ^S is finitely satisfiable if and only if it has a pseudo-model $\mathcal{G}$ whose size is at most $\theta_4(n, m, K)$. By Lemma 6.8, $\mathcal{G}$ is a strong pseudo-model of φ. Note that the vocabulary of φ and φ^S are the same. Hence, $\mathcal{G}$ is the desire pseudo-model. □

CEB properties for $\mathsf{SSAT}(\mathsf{C}^2)$ and $\mathsf{FIN\text{-}SSAT}(\mathsf{C}^2)$. Now we derive the CEB properties for $\mathsf{SSAT}(\mathsf{C}^2)$ and $\mathsf{FIN\text{-}SSAT}(\mathsf{C}^2)$. They are the simplified version of the CEB property stated in Chapter 4.

Definition 6.12. *Let $\mathcal{C}$ be a set of valid configurations in φ. We say $\mathcal{C}$ is strong self-matching in φ if it satisfies the following.*

1. *$\Pi^{\mathcal{C}}$ is mutually null-compatible in φ.*

2. *For every $C \in \mathcal{C}$, C conforms to $\Pi^{\mathcal{C}}$.*

Definition 6.13. *Let $\mathcal{C}$ be a set of valid configurations in φ. The strong induced ILP system $\mathcal{Q}^S_{\mathcal{C}}$ is defined as follows.*

The variables in the system $\mathcal{Q}^S_{\mathcal{C}}$ are $\{x_C\}_{C\in\mathcal{C}}$. There are three types of constraints in the system. For every $\pi \in \Pi^{\mathcal{C}}$, the following constraint is in the system $\mathcal{Q}^S_{\mathcal{C}}$.

1. $\sum_{C\in\mathcal{C}|_\pi} x_C > 0$.

For every $\pi_1, \pi_2 \in \Pi^{\mathcal{C}}$, for every $\eta \in \mathcal{K}^{\varphi,\mathsf{ntri}}_{\langle\pi_1,*,\pi_2\rangle}$, if $\eta^{\triangleleft} \neq \vec{0}$ and there exists $F \in \mathcal{F}^{\mathcal{C}}_{\pi_1}$ such that $F(\eta, \pi_2) > 0$, then the following constraint is in the system $\mathcal{Q}^S_{\mathcal{C}}$.

2. $\sum_{F\in\mathcal{F}^{\mathcal{C}}_{\pi_1}} x_{\langle\pi_1,F\rangle} \cdot F(\eta, \pi_2) = \sum_{F\in\mathcal{F}^{\mathcal{C}}_{\pi_2}} x_{\langle\pi_2,F\rangle} \cdot F(\widetilde{\eta}, \pi_1)$.

Finally, the following non-trivial constraint is in the system.

3. $\sum_{C\in\mathcal{C}} x_C > 0$.

doi:10.6342/NTU202203098

Theorem 6.14 (The CEB property for $\mathsf{SSAT}(\mathsf{C}^2)$). *The* C^2 *sentence* φ *is strongly satisfiable if and only if there exist the following components.*

- *A set of valid configurations* $\mathcal{C}$ *which is strong self-matching in* φ.
- *A solution* $\{x_C \mapsto p_C\}_{C \in \mathcal{C}}$ *of the system* $\mathcal{Q}^S_{\mathcal{C}}$.

Furthermore, their sizes are bounded by the length of φ. *More precisely,*

- $|\mathcal{C}| \leq \theta_2(n, m, K)$*;*
- *for every* $C \in \mathcal{C}$*,* $p_C \in [0, \theta_3(n, m, K)] \cup \{\infty\}$*, where* $\theta_2(n, m, K)$ *and* $\theta_3(n, m, K)$ *are the same as in Theorem 4.9.*

Proof.

(Only if.) Recall that by Lemma 6.4, φ has a strong pseudo-model $\mathcal{G}$ such that for every $\pi \in \Pi^V$, $n_\pi > 3K + 3$. The rest of the proof is similar to the proof of Lemma 4.21. The main difference is that we choose the specific pseudo-model $\mathcal{G}$. Observe that by the construction in Definition 4.11, the finite part and the extended part of the standard partition of $\mathcal{G}$ are empty. Furthermore, the system $\mathcal{Q}^S_{\mathcal{C}^V}$ is the degenerate case of the system $\mathcal{Q}_{\mathcal{G}, V_\mathsf{F} \dot\cup V_\mathsf{E} \dot\cup V_\mathsf{S}, \mathcal{C}^{V_\mathsf{S}}}$. Hence, the similar proof steps work.

(If.) Let $\mathcal{G}$ be the pseudo-structure over τ defined as follows.

- Let $V_\mathsf{F} := \emptyset$, $V_\mathsf{E} := \emptyset$, and $V_\mathsf{S} := \left\{ v_\pi \mid \pi \in \Pi^{\mathcal{C}} \right\}$ where v_π is a fresh element. The universe $V := V_\mathsf{F} \dot\cup V_\mathsf{E} \dot\cup V_\mathsf{S}$.
- For every $v_\pi \in V$, $\mathsf{lab}[v_\pi] := \pi$.
- For every $v_1, v_2 \in V$, $\mathsf{lab}[v_1, v_2] := \eta_{\mathsf{null}}$.

It is easy to verify that $V_\mathsf{F} \dot\cup V_\mathsf{E} \dot\cup V_\mathsf{S}$ is a proper partition of $\mathcal{G}$ in φ and $\mathcal{G}$, $V_\mathsf{F} \dot\cup V_\mathsf{E} \dot\cup V_\mathsf{S}$, and $\mathcal{C}$ are matching in φ. Note that $\Pi^{V_\mathsf{E} \cup V_\mathsf{S}} = \Pi^{\mathcal{C}}$. For every $\pi_1, \pi_2 \in \Pi^{V_\mathsf{E} \cup V_\mathsf{S}}$, for every $\eta \in \mathcal{K}^{\varphi, \mathsf{ntri}}_{\langle \pi_1, *, \pi_2 \rangle}$, $n^{V_\mathsf{E}}_{\pi_1} = 0$ and $n^{V_\mathsf{E} \to V_\mathsf{S}}_{\langle \pi_1, \eta, \pi_2 \rangle} = 0$.

Now we claim that $\{x_C \mapsto (2K+1)^2 \cdot p_C\}_{C \in \mathcal{C}}$ is a solution of the system $\mathcal{Q}_{\mathcal{G}, V_\mathsf{F} \dot\cup V_\mathsf{E} \dot\cup V_\mathsf{S}, \mathcal{C}}$. For the first type of constraint, for every $\pi \in \Pi^{\mathcal{C}}$, because $\{x_C \mapsto p_C\}_{C \in \mathcal{C}}$ is a solution of $\mathcal{Q}^S_{\mathcal{C}}$, $\sum_{C \in \mathcal{C}|_\pi} p_C \geq 1$. Hence, $\sum_{C \in \mathcal{C}|_\pi} ((2K+1)^2 \cdot p_C) \geq (2K+1)^2 \geq 3K + 1$.

doi:10.6342/NTU202203098

For the other two types of constraints, for every $\pi_1, \pi_2 \in \Pi^{\mathcal{C}}$, for every $\eta \in \mathcal{K}^{\varphi,\mathsf{ntri}}_{\langle \pi_1,*,\pi_2 \rangle}$, if $\eta^{\triangleleft} \neq \vec{0}$ and there exists $F \in \mathcal{F}^{\mathcal{C}}_{\pi_1}$ such that $F(\eta, \pi_2) > 0$, the following holds.

$$\sum_{F \in \mathcal{F}^{\mathcal{C}}_{\pi_1}} p_{\langle \pi_1, F \rangle} \cdot F(\eta, \pi_2) \geq 1$$

$$\sum_{F \in \mathcal{F}^{\mathcal{C}}_{\pi_1}} p_{\langle \pi_1, F \rangle} \cdot F(\eta, \pi_2) = \sum_{F \in \mathcal{F}^{\mathcal{C}}_{\pi_2}} p_{\langle \pi_2, F \rangle} \cdot F(\widetilde{\eta}, \pi_1)$$

Hence, the following holds.

$$\sum_{F \in \mathcal{F}^{\mathcal{C}}_{\pi_1}} ((2K+1)^2 \cdot p_{\langle \pi_1, F \rangle}) \cdot F(\eta, \pi_2) \geq (2K+1)^2$$

$$\sum_{F \in \mathcal{F}^{\mathcal{C}}_{\pi_1}} ((2K+1)^2 \cdot p_{\langle \pi_1, F \rangle}) \cdot F(\eta, \pi_2) = \sum_{F \in \mathcal{F}^{\mathcal{C}}_{\pi_2}} ((2K+1)^2 \cdot p_{\langle \pi_2, F \rangle}) \cdot F(\widetilde{\eta}, \pi_1)$$

Therefore, the assignment $\{x_C \mapsto p_C\}_{C \in (2K+1)^2 \cdot \mathcal{D}}$ is a solution of the system $\mathcal{Q}_{\mathcal{G}, V_{\mathsf{F}} \dot{\cup} V_{\mathsf{E}} \dot{\cup} V_{\mathsf{S}}, \mathcal{C}}$.

Finally, by Lemma 4.22, φ has a pseudo-model $\mathcal{H}$. Furthermore, because $\Pi^U = \Pi^V = \Pi^{\mathcal{C}}$, Π^U is mutually null-compatible, and this implies that $\mathcal{H}$ is a strong pseudo-model of φ. Hence, φ is strongly satisfiable. □

Theorem 6.15 (The CEB property for $\mathsf{FIN\text{-}SAT}(\mathsf{C}^2)$). *The C^2 sentence φ is finitely strongly satisfiable if and only if there exist the following components.*

- *A set of valid configurations $\mathcal{C}$ which is strong self-matching in φ.*
- *A solution $\{x_C \mapsto p_C\}_{C \in \mathcal{C}}$ of the system $\mathcal{Q}^S_{\mathcal{C}}$.*

Furthermore, their sizes are bounded by the length of φ. More precisely,

- *$|\mathcal{C}| \leq \theta_2(n, m, K)$;*
- *for every $C \in \mathcal{C}$, $p_C \in [0, \theta_3(n, m, K)]$, where $\theta_2(n, m, K)$ and $\theta_3(n, m, K)$ are the same as in Theorem 4.9.*

Proof.

The proof is similar to Theorem 6.14. The only difference is that φ has a strong finite pseudo-model $\mathcal{G}$ here. By Lemma 6.4, we choose the union of $(2K+1)^2$ number copies of $\mathcal{G}$. Note that this pseudo-model is still finite. Hence, the similar proof steps work. □

doi:10.6342/NTU202203098

We can obtain the algorithms for $\mathsf{SSAT}(\mathsf{C}^2)$ and $\mathsf{FIN\text{-}SSAT}(\mathsf{C}^2)$ from the CEB property, just like what we do in Chapter 5. However, the algorithms still make exponential many guessing from double-exponential many configurations like the case $\mathsf{SAT}(\mathsf{C}^2)$ and $\mathsf{FIN\text{-}SAT}(\mathsf{C}^2)$, and we got no benefit from the strongly satisfiability. Hence, they are still involved and complicated to implement. To obtain the implementable algorithms for $\mathsf{SSAT}(\mathsf{C}^2)$ and $\mathsf{FIN\text{-}SSAT}(\mathsf{C}^2)$, it is necessary to reduce the number of the non-deterministic in the algorithm.

6.2 Graph representation of configurations

In this section, we fix a set of valid 1-types Π in φ. Let π_s be a 1-type in Π. We say an m-dimension (row) vector $\vec{u}$ *dominated by* an m-dimension (row) vector $\vec{v}$, denoted by $\vec{u} \preceq \vec{v}$, if for every $i \in [m]$, the entry $u_i \le v_i$. Let $\mathcal{C}^{\varphi,\Pi,\pi}$ be the set of all valid configuration with source type π_s which conforms to Π in φ.

The number of valid configurations in φ is double-exponential in the length of φ. We observe that different configurations may share the same components. The graph representation of configurations is a DAG which encodes configurations with only exponential size graph.

Graph representation of configurations.

Definition 6.16. *The graph representation of configurations with source type π_s conforming to Π in φ, denoted by $\mathfrak{G}^{\varphi,\Pi,\pi_s} := \left\langle V^{\varphi,\Pi,\pi_s}, E^{\varphi,\Pi,\pi_s}\right\rangle$, is a directed acyclic graph (DAG) defined as follows.*

- *The set of vertices $\mathfrak{G}^{\varphi,\Pi,\pi_s}$ is the set of m-dimension (row) vectors which dominated by $\vec{k}$, that is, $V^{\varphi,\Pi,\pi_s} := \left\{\vec{v} \in \mathbb{N}^m \;\middle|\; \vec{v} \preceq \vec{k}\right\}$. Recalled that m is the number of binary predicates in τ.*
- *For every $\vec{v} \in V^{\varphi,\Pi,\pi_s}$, for every $\pi \in \Pi$, for every $\eta \in \mathcal{K}^{\varphi,\mathsf{ntri}}_{\langle \pi_s,*,\pi\rangle}$, if $\vec{v} + \eta^{\triangleright} \in V^{\varphi,\Pi,\pi_s}$, then there is a edge from $\vec{v}$ to $\vec{v} + \eta^{\triangleright}$. We denoted this edge by the tuple $\langle \vec{v}, \eta, \pi\rangle$.*

doi:10.6342/NTU202203098

Let $E_{\vec{v},o}^{\varphi,\Pi,\pi_s}$ be the set of outgoing edges from $\vec{v}$, $E_{\vec{v},i}^{\varphi,\Pi,\pi_s}$ be the set of incoming edges to $\vec{v}$, and $E_{\eta,\pi}^{\varphi,\Pi,\pi_s}$ be the set of edges which labeled $\langle *, \eta, \pi\rangle$. The formal definitions are as follows.

$$E_{\vec{v},o}^{\varphi,\Pi,\pi_s} := \left\{\langle \vec{v}_e, \eta_e, \pi_e\rangle \in E^{\varphi,\Pi,\pi} \;\middle|\; \vec{v}_e = \vec{v}\right\}$$

$$E_{\vec{v},i}^{\varphi,\Pi,\pi_s} := \left\{\langle \vec{v}_e, \eta_e, \pi_e\rangle \in E^{\varphi,\Pi,\pi} \;\middle|\; \vec{v}_e + \eta_e^{\triangleright} = \vec{v}\right\}$$

$$E_{\eta,\pi}^{\varphi,\Pi,\pi_s} := \left\{\langle \vec{v}_e, \eta_e, \pi_e\rangle \in E^{\varphi,\Pi,\pi} \;\middle|\; \eta_e = \eta \text{ and } \pi_e = \pi\right\}$$

Definition 6.17. *Let P be a path from $\vec{0}$ to $\vec{k}$ in $\mathfrak{G}^{\varphi,\Pi,\pi_s}$, say $e_1 \to e_2 \to \cdots \to e_\ell$, for every $i \in [\ell]$, $e_i := \langle \vec{v}_i, \eta_i, \pi_i\rangle$. The corresponding configuration of P, denoted by $C[P]$, is defined as follows.*

- *The corresponding behavior function of P, denoted by $F[P]$, is the number of edges in P which labeled with such 2-type and 1-type. Formally, for every $\pi \in \Pi^\tau$, for every $\eta \in \mathcal{K}^{\tau,\mathsf{ntri}}$, $F[P](\eta,\pi) := \left|P \cap E_{\eta,\pi}^{\varphi,\Pi,\pi_s}\right|$.*
- *The corresponding configuration of P, denoted by $C[P]$, is the tuple $\langle \pi_s, F[P]\rangle$.*

Lemma 6.18. *The configuration C with source type π_s is valid in φ and conforms to Π if and only if there is a path P from $\vec{0}$ to $\vec{k}$ in $\mathfrak{G}^{\varphi,\Pi,\pi_s}$ such that $C[P] = C$.*

Proof.

(If.) Let $P :=\ e_1 \to e_2 \to \cdots \to e_\ell$ be a path from $\vec{0}$ to $\vec{k}$ in $\mathfrak{G}^{\varphi,\Pi,\pi_s}$. For every $i \in [\ell]$, $e_i := \langle \vec{v}_i, \eta_i, \pi_i\rangle$.

By definition, the source type of $C[P]$ is π_S. For every $\pi \in \Pi \setminus \Pi^\tau$, there is no edge in $\mathfrak{G}^{\varphi,\Pi,\pi_s}$ labeled with the 1-type π. Hence, by definition, for every $\eta \in \mathcal{K}^{\tau,\mathsf{ntri}}$, $F[P](\eta,\pi) = 0$. Therefore, $C[P]$ conforms to Π.

Because P is a path from $\vec{0}$ to $\vec{k}$, for every $i \in [\ell]$, the following holds.

$$\vec{v}_0 = \vec{0}, \quad \vec{v}_{i+1} = \vec{v}_i + \eta_i^{\triangleright}, \quad \vec{v}_\ell + \eta_\ell^{\triangleright} = \vec{k}$$

By definition of $F[P]$, the following holds.

$$\vec{k} = \vec{v}_\ell + \eta_\ell^{\triangleright} = \vec{v}_0 + \sum_{i\in[\ell]} \eta_i^{\triangleright} = \sum_{\pi\in\Pi^\tau} \sum_{\eta\in\mathcal{K}^{\tau,\mathsf{ntri}}} F[P](\eta,\pi)\cdot \eta^{\triangleright}$$

doi:10.6342/NTU202203098

Hence, $C[P]$ is valid in φ.

(Only if.) We fix an order of elements in the set $\{(\eta, \pi) \in \mathcal{K}^{\tau,\mathsf{ntri}} \times \Pi \mid F(\eta, \pi) > 0\}$, say $(\eta_1, \pi_1), (\eta_2, \pi_2), \ldots, (\eta_\ell, \pi_\ell)$. Because C conforms to Π, for every $i \in [\ell]$, $\pi_i \in \Pi$ and $\eta \in \mathcal{K}^{\varphi,\mathsf{ntri}}_{\langle \pi_s, *, \pi_i \rangle}$. Let $\vec{v}_i = \sum_{1 \leq j < i} F(\eta_j, \pi_j) \cdot \eta_j^{\triangleright}$. Then we define the following path P.

$$
\begin{aligned}
&\langle \vec{v}_1, \eta_1, \pi_1 \rangle \to \langle \vec{v}_1 + \eta_1^{\triangleright}, \eta_1, \pi_1 \rangle \to \cdots \to \langle \vec{v}_1 + F(\eta_1, \pi_1) \cdot \eta_1^{\triangleright}, \eta_1, \pi_1 \rangle \\
\to &\langle \vec{v}_2, \eta_2, \pi_2 \rangle \to \langle \vec{v}_2 + \eta_2^{\triangleright}, \eta_2, \pi_2 \rangle \to \cdots \to \langle \vec{v}_2 + F(\eta_2, \pi_2) \cdot \eta_2^{\triangleright}, \eta_2, \pi_2 \rangle \\
\to &\ldots \\
\to &\langle \vec{v}_\ell, \eta_\ell, \pi_\ell \rangle \to \langle \vec{v}_\ell + \eta_\ell^{\triangleright}, \eta_\ell, \pi_\ell \rangle \to \cdots \to \langle \vec{v}_\ell + F(\eta_\ell, \pi_\ell) \cdot \eta_\ell^{\triangleright}, \eta_\ell, \pi_\ell \rangle
\end{aligned}
$$

Because C is valid in φ, $\vec{v}_\ell + F(\eta_\ell, \pi_\ell) \cdot \eta_\ell^{\triangleright} = \sum_{i \in [\ell]} F(\eta_i, \pi_i) \cdot \eta_i^{\triangleright} = \vec{k}$. Hence, P is a paths from $\vec{0}$ to $\vec{k}$ in $\mathfrak{G}^{\varphi,\Pi,\pi_s}$. It is obvious that $C[P] = C$. □

Remark 6.19. *It is possible that there are two different path P_1 and P_2 from $\vec{0}$ to $\vec{k}$ in $\mathfrak{G}^{\varphi,\Pi,\pi_s}$, but $C[P_1] = C[P_2]$.*

Rewriting the constraints. Now, we define some building blocks helpful in rewriting the induced ILP systems in the graph representation of configurations.

Definition 6.20. *The balance gadget of $\mathfrak{G}^{\varphi,\Pi,\pi_s}$, denoted by $\mathcal{Q}^{bal}_{\mathfrak{G}^{\varphi,\Pi,\pi_s}}$, is the ILP system defined as follows. Its variables are $\{x_e\}_{e \in E^{\varphi,\Pi,\pi_s}}$.*

$$
\mathcal{Q}^{bal}_{\mathfrak{G}^{\varphi.\Pi,\pi_s}} := \bigwedge_{\vec{v} \in V^{\varphi,\Pi,\pi_s} \setminus \{\vec{0}, \vec{k}\}} \sum_{e \in E^{\varphi,\Pi,\pi_s}_{\vec{v},i}} x_e = \sum_{e \in E^{\varphi,\Pi,\pi_s}_{\vec{v},o}} x_e
$$

Definition 6.21. *The standard gadget of $\mathcal{C}^{\varphi,\Pi,\pi_s}$, denoted by $\mathcal{Q}^{std}_{\mathcal{C}^{\varphi,\Pi,\pi_s}}$, is the ILP system with the following set of variables.*

$$
\{x_C \mid C \in \mathcal{C}^{\varphi,\Pi,\pi_s}\} \cup \{y_{\pi_s}\} \cup \left\{ y_{\langle \pi_s, \eta, \pi \rangle} \;\middle|\; \pi \in \Pi \text{ and } \eta \in \mathcal{K}^{\varphi,\mathsf{ntri}}_{\langle \pi_s, *, \pi \rangle} \right\}
$$

doi:10.6342/NTU202203098

The system $\mathcal{Q}^{std}_{\mathcal{C}^{\varphi,\Pi,\pi_s}}$ is defined as follows.

$$\mathcal{Q}^{std}_{\mathcal{C}^{\varphi,\Pi,\pi_s}} := \left(y_{\pi_s} = \sum_{C \in \mathcal{C}} x_C \right) \wedge \bigwedge_{\pi \in \Pi} \bigwedge_{\eta \in \mathcal{K}^{\varphi,\mathsf{ntri}}_{\langle \pi_s,*,\pi \rangle}} \left(y_{\pi_{\langle \pi_s,\eta,\pi \rangle}} = \sum_{\langle \pi_s,F \rangle \in \mathcal{C}} F(\eta,\pi) \cdot x_C \right)$$

Definition 6.22. *The standard gadget of $\mathfrak{G}^{\varphi,\Pi,\pi_s}$, denoted by $\mathcal{Q}^{std}_{\mathfrak{G}^{\varphi,\Pi,\pi_s}}$, is the ILP system with following variables.*

$$\left\{ x_e \;\middle|\; e \in E^{\varphi,\Pi,\pi_s} \right\} \cup \left\{ y_{\pi_s} \right\} \cup \left\{ y_{\langle \pi_s,\eta,\pi \rangle} \;\middle|\; \pi \in \Pi \textit{ and } \eta \in \mathcal{K}^{\varphi,\mathsf{ntri}}_{\langle \pi_s,*,\pi \rangle} \right\}$$

The system $\mathcal{Q}^{std}_{\mathfrak{G}^{\varphi,\Pi,\pi_s}}$ is defined as follows.

$$\mathcal{Q}^{std}_{\mathfrak{G}^{\varphi,\Pi,\pi_s}} := \left(y_{\pi_s} = \sum_{e \in E^{\varphi,\Pi,\pi}_{\vec{0},o}} x_e \right) \wedge \bigwedge_{\pi \in \Pi} \bigwedge_{\eta \in \mathcal{K}^{\varphi,\mathsf{ntri}}_{\langle \pi_s,*,\pi \rangle}} \left(y_{\langle \pi_s,\eta,\pi \rangle} = \sum_{e \in E^{\varphi,\Pi,\pi}_{\eta,\pi}} x_e \right) \wedge \mathcal{Q}^{bal}_{\mathfrak{G}^{\varphi,\Pi,\pi_s}}$$

We have shown that the valid configurations in φ can be encoded by the graph representation. Suppose $\mathcal{Q}$ is an ILP system. If $\mathcal{Q}$ is only about the properties of the valid configurations in φ, then we can rewrite $\mathcal{Q}$ by the graph representation. Since the size of the graph representation is only exponential in the length of φ. We can, in general, compress the size of $\mathcal{Q}$ and obtain a smaller but equisatisfiable system $\mathcal{Q}'$.

Lemma 6.23. *The system $\mathcal{Q}^{std}_{\mathcal{C}^{\varphi,\Pi,\pi}}$ has a solution*

$$\left\{ x_e \mapsto p_e \right\}_{e \in E} \cup \left\{ y_{\pi_s} \mapsto p_{\pi_s} \right\} \cup \left\{ y_{\langle \pi_s,\eta,\pi \rangle} \mapsto p_{\langle \pi_s,\eta,\pi \rangle} \right\}$$

in $\mathbb{N}_\infty$ if and only if the system $\mathcal{Q}^{std}_{\mathfrak{G}^{\varphi,\Pi,\pi}}$ has a solution

$$\left\{ x_e \mapsto q_C \right\}_{C \in \mathcal{C}} \cup \left\{ y_{\pi_s} \mapsto q_{\pi_s} \right\} \cup \left\{ y_{\langle \pi_s,\eta,\pi \rangle} \mapsto q_{\langle \pi_s,\eta,\pi \rangle} \right\}$$

in $\mathbb{N}_\infty$ satisfying the following. For every $\pi \in \Pi$, for every $\eta \in \mathcal{K}^{\varphi,\mathsf{ntri}}_{\langle \pi_s,*,\pi \rangle}$, $p_{\langle \pi_s,\eta,\pi \rangle} = q_{\langle \pi_s,\eta,\pi \rangle}$ and $p_{\pi_s} = q_{\pi_s}$.

Proof. Observe that the variable y_{π_s} and $y_{\langle \pi_s,\eta,\pi \rangle}$ are dummy in the system $\mathcal{Q}^{\mathrm{std}}_{\mathcal{C}^{\varphi,\Pi,\pi}}$ and

doi:10.6342/NTU202203098

$\mathcal{Q}^{\text{std}}_{\mathfrak{G}^{\varphi,\Pi,\pi}}$. Therefore, it is sufficient to only consider the variables $\{x_e \mapsto p_e\}_{e\in E}$ and $\{x_e \mapsto q_C\}_{C\in\mathcal{C}}$ in the proof.

(If.) Let $\{x_e \mapsto p_e\}_{e\in E}$ be a solution of the system $\mathcal{Q}^{\text{std}}_{\mathfrak{G}^{\varphi,\Pi,\pi}}$. Now we view p_e as the flow in the graph $\mathfrak{G}^{\varphi.\Pi,\pi_s}$. For every $\vec{v} \in V \setminus \{\vec{0}, \vec{k}\}$, because this assignment is also a solution of the balance gadget $\mathcal{Q}^{\text{bal}}_{\mathfrak{G}^{\varphi.\Pi,\pi_s}}$, the incoming flow and outgoing flow of the vertex $\vec{v}$ are the same. Therefore, $\vec{v}$ are a "balance" vertex. Besides, the vertices $\vec{0}$ and $\vec{k}$ are the only "source" and "sink" vertex in the graph $\mathfrak{G}^{\varphi.\Pi,\pi_s}$, respectively. Because $\mathfrak{G}^{\varphi.\Pi,\pi_s}$ is a DAG, there is no circle flow in it. Hence, by flow decomposition theorem [9], the flow can be decomposed into a set of path flows. That is, there exists a set of path $\mathcal{P}$ from the source vertex $\vec{0}$ to the sink vertex $\vec{k}$ with path flow q_P such that for every $e \in E$, $\sum_{\{P\in\mathcal{P} \mid e\in P\}} q_P = p_e$.

We claim that the assignment $\left\{x_C \mapsto \sum_{\{P\in\mathcal{P} \mid C[P]=C\}} q_P\right\}_{C\in\mathcal{C}^{\varphi,\Pi,\pi}}$ is a solution of the system $\mathcal{Q}^{\text{std}}_{\mathcal{C}^{\varphi,\Pi,\pi}}$. Observe that $\sum_{C\in\mathcal{C}^{\varphi,\Pi,\pi}} \left(\sum_{\{P\in\mathcal{P} \mid C[P]=C^{\varphi,\Pi,\pi}\}} q_P\right) = \sum_{P\in\mathcal{P}} q_P$ is the total outgoing flow from the source vertex $\vec{0}$. Hence, the following holds.

$$q_{\pi_s} = \sum_{C\in\mathcal{C}^{\varphi,\Pi,\pi}} \left(\sum_{\{P\in\mathcal{P} \mid C[P]=C\}} q_P\right) = \sum_{e\in E^{\varphi,\Pi,\pi_s}_{\vec{0},o}} p_e = p_{\pi_s}$$

For every $\pi \in \Pi$, for every $\eta \in \mathcal{K}^{\varphi,\text{ntri}}_{\langle\pi_s,*,\pi\rangle}$, by definition of $F[P]$, the following holds.

$$\begin{aligned}
q_{\langle\pi_s,\eta,\pi\rangle} &= \sum_{\langle\pi_s,F\rangle\in\mathcal{C}^{\varphi,\Pi,\pi_s}} \left(\sum_{\{P\in\mathcal{P} \mid C[P]=C\}} q_P\right) \cdot F(\eta,\pi) \\
&= \sum_{P\in\mathcal{P}} q_P \cdot F[P](\eta,\pi) \\
&= \sum_{e\in E^{\varphi,\Pi,\pi_s}_{\eta,\pi}} \sum_{\{P\in\mathcal{P} \mid e\in P\}} q_P \\
&= \sum_{e\in E^{\varphi,\Pi,\pi_s}_{\eta,\pi}} p_e \\
&= p_{\langle\pi_s,\eta,\pi\rangle}
\end{aligned}$$

(Only if.) Let $\{x_C \mapsto q_C\}_{C\in\mathcal{C}^{\varphi,\Pi,\pi_s}}$ be a solution of the system $\mathcal{Q}^{\text{std}}_{\mathcal{C}^{\varphi,\Pi,\pi}}$. For every $C \in \mathcal{C}^{\varphi,\Pi,\pi_s}$, by Lemma 6.18, there is a path P_C from $\vec{0}$ to $\vec{k}$ in $\mathfrak{G}^{\varphi.\Pi,\pi_s}$ such that $C[P] = C$.

doi:10.6342/NTU202203098

We claim that the assignment $\left\{x_e \mapsto \sum_{\{C\in\mathcal{C}^{\varphi,\Pi,\pi_s} \mid e\in P_C\}} q_C\right\}_{e\in E}$ is a solution of the system $\mathcal{Q}^{\text{std}}_{\mathfrak{G}^{\varphi,\Pi,\pi}}$. For every $C \in \mathcal{C}^{\varphi,\Pi,\pi_s}$, because start P_C start from $\vec{0}$, there is a unique element in the set $P_C \cap E^{\varphi,\Pi,\pi}_{\vec{0},o}$. Therefor, the following holds.

$$p_{\pi_s} = \sum_{e\in E^{\varphi,\Pi,\pi}_{\vec{0},o}} \left(\sum_{\{C\in\mathcal{C}^{\varphi,\Pi,\pi_s} \mid e\in P_C\}} q_C \right) = \sum_{C\in\mathcal{C}^{\varphi,\Pi,\pi_s}} q_C = q_{\pi_s}$$

For every $\pi \in \Pi$, for every $\eta \in \mathcal{K}^{\varphi,\mathsf{ntri}}_{\langle\pi_s,*,\pi\rangle}$, observe that the following holds.

$$p_{\langle\pi_s,\eta,\pi\rangle} = \sum_{e\in E^{\varphi,\Pi,\pi}_{\eta,\pi}} \left(\sum_{\{C\in\mathcal{C}^{\varphi,\Pi,\pi_s} \mid e\in P_C\}} q_C \right) = \sum_{C\in\mathcal{C}^{\varphi,\Pi,\pi_s}} q_C \cdot F(\eta,\pi) = q_{\langle\pi_s,\eta,\pi\rangle}$$

Finally, for every $\vec{v} \in V \setminus \{\vec{0},\vec{k}\}$, for every $C \in \mathcal{C}^{\varphi,\Pi,\pi_s}$, because P_C is a path from $\vec{0}$ to $\vec{k}$ in $\mathfrak{G}^{\varphi.\Pi,\pi_s}$, if there exists an edge $e \in P_C \cup E^{\varphi,\Pi,\pi}_{\vec{v},i}$, then there also exists the edge $e' \in P_C \cup E^{\varphi,\Pi,\pi}_{\vec{v},o}$. These imply that for every vertex $v \in V \setminus \{\vec{0},\vec{k}\}$, the incoming flow and outgoing flow are the same. Hence this assignment is a solution of the balance gadget. □

Lemma 6.24. *For an ILP system $\mathcal{Q}_0$ with variables*

$$\{y_{\pi_s}\} \cup \left\{ y_{\langle\pi_s,\eta,\pi\rangle} \;\middle|\; \pi \in \Pi \text{ and } \eta \in \mathcal{K}^{\varphi,\mathsf{ntri}}_{\langle\pi_s,*,\pi\rangle} \right\}.$$

Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be the ILP systems defined as follows.

- *The variable of $\mathcal{Q}_1$ is as the system $\mathcal{Q}_{\mathcal{C}^{\varphi,\Pi,\pi}}$ and the system defined as $\mathcal{Q}_1 := \mathcal{Q}^{std}_{\mathcal{C}^{\varphi,\Pi,\pi}} \wedge \mathcal{Q}_0$.*
- *The variable of $\mathcal{Q}_2$ is as the system $\mathcal{Q}_{\mathfrak{G}^{\varphi,\Pi,\pi}}$ and the system defined as $\mathcal{Q}_2 := \mathcal{Q}^{std}_{\mathfrak{G}^{\varphi,\Pi,\pi}} \wedge \mathcal{Q}_0$.*

The system $\mathcal{Q}_1$ has a solution in $\mathbb{N}_\infty$ if and only $\mathcal{Q}_2$ has a solution in $\mathbb{N}_\infty$.

Proof.

(If.) Suppose the system $\mathcal{Q}_2$ is satisfiable. Then, the assignment is a solution of the system $\mathcal{Q}^{\text{std}}_{\mathfrak{G}^{\varphi,\Pi,\pi}}$. By Lemma 6.23, there exists a solution of the system $\mathcal{Q}^{\text{std}}_{\mathcal{C}^{\varphi,\Pi,\pi}}$. Besides, the assignment for the variable y_{π_s} and $y_{\langle\pi_s,\eta,\pi\rangle}$ are the same. Hence, it is also a solution of the system $\mathcal{Q}_0$, therefore, a solution a solution of the system $\mathcal{Q}_1$.

doi:10.6342/NTU202203098

(Only if.) Suppose the system $\mathcal{Q}_1$ is satisfiable. Then, the assignment is a solution of the system $\mathcal{Q}^{\text{std}}_{\mathcal{C}^{\varphi,\Pi,\pi}}$. By Lemma 6.23, there exists a solution of the system $\mathcal{Q}^{\text{std}}_{\mathfrak{G}^{\varphi,\Pi,\pi}}$. Besides, the assignment for the variable y_{π_s} and $y_{\langle \pi_s,\eta,\pi\rangle}$ are the same. Hence, it is also a solution of the system $\mathcal{Q}_0$, therefore, a solution a solution of the system $\mathcal{Q}_2$. $\square$

Rewriting the constraints to boolean formulae. We also define the boolean version building blocks, which can be view as the case that the solution of the ILP system in restricted in $\{0, \infty\}$. We will use them later.

Definition 6.25. *The boolean balance gadget of $\mathfrak{G}^{\varphi,\Pi,\pi_s}$, denoted by $\Psi^{bal}_{\mathfrak{G}^{\varphi,\Pi,\pi_s}}$, is the boolean formula defined as follows. Its variables are $\{b_e\}_{e\in E^{\varphi,\Pi,\pi_s}}$.*

$$\Psi^{bal}_{\mathfrak{G}^{\varphi.\Pi,\pi_s}} := \bigwedge_{\vec{v}\in V^{\varphi,\Pi,\pi_s}\setminus\{\vec{0},\vec{k}\}} \left(\bigvee_{e\in E^{\varphi,\Pi,\pi_s}_{\vec{v},i}} \neg b_e \leftrightarrow \bigvee_{e\in E^{\varphi,\Pi,\pi_s}_{\vec{v},o}} \neg b_e \right)$$

Definition 6.26. *The boolean standard gadget of $\mathfrak{G}^{\varphi,\Pi,\pi_s}$, denoted by $\Psi^{std}_{\mathfrak{G}^{\varphi,\Pi,\pi_s}}$, is the boolean formula with following variables.*

$$\left\{b_e \;\middle|\; e \in E^{\varphi,\Pi,\pi_s}\right\} \cup \{b_{\pi_s}\} \cup \left\{ b_{\langle \pi_s,\eta,\pi\rangle} \;\middle|\; \pi\in\Pi \textit{ and } \eta\in \mathcal{K}^{\varphi,\mathsf{ntri}}_{\langle \pi_s,*,\pi\rangle}\right\}$$

The formula $\Psi^{std}_{\mathfrak{G}^{\varphi,\Pi,\pi_s}}$ is defined as follows.

$$\Psi^{std}_{\mathfrak{G}^{\varphi.\Pi,\pi_s}} := \left(\neg b_{\pi_s} \leftrightarrow \sum_{e\in E^{\varphi,\Pi,\pi}_{\vec{0},o}} \neg b_e\right) \wedge \bigwedge_{\pi\in\,\Pi}\; \bigwedge_{\eta\in\, \mathcal{K}^{\varphi,\mathsf{ntri}}_{\langle \pi_s,*,\pi\rangle}} \left(\neg b_{\langle \pi_s,\eta,\pi\rangle} \leftrightarrow \sum_{e\in E^{\varphi,\Pi,\pi}_{\eta,\pi}} \neg b_e\right) \wedge \Psi^{bal}_{\mathfrak{G}^{\varphi.\Pi,\pi_s}}$$

Remark 6.27. *Observe that for every edge $e \in E^{\varphi,\Pi,\pi_s}$, the variable b_e occur at most twice in the boolean balance gadget $\Psi^{bal}_{\mathfrak{G}^{\varphi,\Pi,\pi_s}}$. Therefore, the length of $\Psi^{bal}_{\mathfrak{G}^{\varphi,\Pi,\pi_s}}$ is bounded by $2\left|E^{\varphi,\Pi,\pi_s}\right| \leq 2\left|V^{\varphi,\Pi,\pi_s}\right||\Pi^\tau||\mathcal{K}^\tau| = 2K^m2^{n+2m}$.*

Therefore, the length of the standard boolean gadget $\Psi^{std}_{\mathfrak{G}^{\varphi,\Pi,\pi_s}}$ is bounded as follows.

$$\Psi^{std}_{\mathfrak{G}^{\varphi,\Pi,\pi_s}} \leq 1 + \left|E^{\varphi,\Pi,\pi_s}\right| + |\Pi^\tau||\mathcal{K}^\tau|\left(1+\left|E^{\varphi,\Pi,\pi_s}\right|\right) + \left|\Psi^{bal}_{\mathfrak{G}^{\varphi,\Pi,\pi_s}}\right| \leq 5K^m2^{2n+4m}$$

doi:10.6342/NTU202203098

6.3 Unified ILP systems

We are going to rewrite the strong induced ILP system by the graph representation of configurations in this section.

Extended strong induced ILP system. We first introduce the *extended strong induced ILP system*. The intuition is as follows. For the strong induced ILP system, we need to guess a set of valid configurations first, which takes exponential many guessing from double-exponential many configurations. Therefore, the algorithms are involved and difficult to implement. However, for the extended strong induced ILP system, we consider the 1-types instead of the configurations. Note that the number of 1-types is only *exponential* in the length of φ.

Definition 6.28. *Let* Π *be a mutually null-compatible set of valid 1-types in* φ*. Let* $\mathcal{C}$ *be the set of valid configurations conformed to* Π*. The extended strong induced ILP system* $\mathcal{Q}_{\Pi}^{ES}$ *is defined as follows.*

The variables in the system $\mathcal{Q}_{\Pi}^{ES}$ *are* $\{x_C\}_{C\in\mathcal{C}}$*. For every* $\pi_1, \pi_2 \in \Pi$*, for every* $\eta \in \mathcal{K}^{\varphi,\mathsf{ntri}}_{\langle \pi_1, *, \pi_2\rangle}$*, if* $\eta^{\triangleleft} = \vec{0}$*, then the following constraint is in the system* $\mathcal{Q}_{\Pi}^{ES}$*.*

1. $\sum_{F\in\mathcal{F}^{\mathcal{C}}_{\pi_1}} x_{\langle \pi_1, F\rangle} \cdot F(\eta, \pi_2) > 0 \implies \sum_{F\in\mathcal{C}\big|_{\pi_2}} x_C > 0.$

Otherwise, if $\eta^{\triangleleft} \neq \vec{0}$*, then the following constraint is in the system* $\mathcal{Q}_{\Pi}^{ES}$*.*

2. $\sum_{F\in\mathcal{F}^{\mathcal{C}}_{\pi_1}} x_{\langle \pi_1, F\rangle} \cdot F(\eta, \pi_2) = \sum_{F\in\mathcal{F}^{\mathcal{C}}_{\pi_2}} x_{\langle \pi_2, F\rangle} \cdot F(\widetilde{\eta}, \pi_1).$

Finally, the non-trivial constraint.

3. $\sum_{C\in\mathcal{C}} x_C > 0.$

Lemma 6.29. *The* C^2 *sentence* φ *is strongly satisfiable if and only if there exists a mutually null-compatible set of valid 1-types* Π *in* φ *such that the system* $\mathcal{Q}_{\Pi}^{ES}$ *has a solution in* $\mathbb{N}_{\infty}$*.*

Proof.

(Only if.) Because φ is strongly satisfiable, by Theorem 6.14, there exists the following components.

doi:10.6342/NTU202203098

- A set of valid configurations $\mathcal{C}$ in φ such that $\mathcal{C}$ is strong self-matching in φ.

- A solution $\{x_C \mapsto p_C\}_{C\in\mathcal{C}}$ of the system $\mathcal{Q}^S_{\mathcal{C}}$ in $\mathbb{N}_\infty$.

Because $\mathcal{C}$ is strong self-matching, for every $C \in \mathcal{C}$, C conforms to $\Pi^{\mathcal{C}}$. Let $\mathcal{D}$ be the set of valid configurations conform to $\Pi^{\mathcal{C}}$. Then, $\mathcal{C} \subseteq \mathcal{D}$. Besides, by definition, $\Pi^{\mathcal{C}}$ is mutually null-compatible.

Now we claim that $\{x_C \mapsto p_C\}_{C\in\mathcal{C}} \cup \{x_C \mapsto 0\}_{C\in\mathcal{D}\setminus\mathcal{C}}$ is a solution of the system $\mathcal{Q}^{ES}_{\Pi^{\mathcal{D}}}$. For every $\pi_1, \pi_2 \in \Pi^{\mathcal{C}}$, for every $\eta \in \mathcal{K}^{\varphi,\mathsf{ntri}}_{\langle\pi_1,*,\pi_2\rangle}$, if $\eta^{\triangleleft} = \vec{0}$. Because the assignment $\{x_C \mapsto p_C\}_{C\in\mathcal{C}}$ is a solution of the system $\mathcal{Q}^S_{\mathcal{C}}$, the following holds.

$$\sum_{C\in\mathcal{D}|_{\pi_2}} p_C = \sum_{C\in\mathcal{D}\setminus\mathcal{C}|_{\pi_2}} p_C + \sum_{C\in\mathcal{C}|_{\pi_2}} p_C = 0 + \sum_{C\in\mathcal{C}|_{\pi_2}} p_C > 0 \tag{6.5}$$

Because the equation (6.5) always holds, the following implication also holds.

$$\sum_{F\in\mathcal{F}^{\mathcal{D}}_{\pi_1}} p_{\langle\pi_1,F\rangle} \cdot F(\eta,\pi_2) > 0 \implies \sum_{C\in\mathcal{D}\big|_{\pi_2}} p_C > 0$$

Therefore, the first type constraints in the system $\mathcal{Q}^{ES}_{\Pi^{\mathcal{C}}}$ holds.

Otherwise, if $\eta^{\triangleleft} \neq \vec{0}$. Because the assignment $\{x_C \mapsto p_C\}_{C\in\mathcal{C}}$ is a solution of the system $\mathcal{Q}^S_{\mathcal{C}}$, the following equation holds.

$$\begin{aligned}
\sum_{F\in\mathcal{F}^{\mathcal{D}}_{\pi_1}} p_{\langle\pi_1,F\rangle} \cdot F(\eta,\pi_2) &= \sum_{F\in\mathcal{F}^{\mathcal{C}}_{\pi_1}} p_{\langle\pi_1,F\rangle} \cdot F(\eta,\pi_2) + \sum_{F\in\mathcal{F}^{\mathcal{D}\setminus\mathcal{C}}_{\pi_1}} p_{\langle\pi_1,F\rangle} \cdot F(\eta,\pi_2) \\
&= \sum_{F\in\mathcal{F}^{\mathcal{C}}_{\pi_1}} p_{\langle\pi_1,F\rangle} \cdot F(\eta,\pi_2) \\
&= \sum_{F\in\mathcal{F}^{\mathcal{C}}_{\pi_2}} p_{\langle\pi_2,F\rangle} \cdot F(\widetilde{\eta},\pi_1) \\
&= \sum_{F\in\mathcal{F}^{\mathcal{C}}_{\pi_2}} p_{\langle\pi_2,F\rangle} \cdot F(\widetilde{\eta},\pi_1) + \sum_{F\in\mathcal{F}^{\mathcal{D}\setminus\mathcal{C}}_{\pi_2}} p_{\langle\pi_2,F\rangle} \cdot F(\widetilde{\eta},\pi_1) \\
&= \sum_{F\in\mathcal{F}^{\mathcal{D}}_{\pi_2}} p_{\langle\pi_2,F\rangle} \cdot F(\widetilde{\eta},\pi_1)
\end{aligned}$$

Therefore, the second type constraints in the system $\mathcal{Q}^{ES}_{\Pi^{\mathcal{C}}}$ holds.

doi:10.6342/NTU202203098

Finally, by the third type constraint in the system $\mathcal{Q}^S_{\mathcal{C}}$.

$$\sum_{C \in \mathcal{D}} x_C = \sum_{C \in \mathcal{D} \setminus \mathcal{C}} x_C + \sum_{C \in \mathcal{C}} x_C = \sum_{C \in \mathcal{C}} x_C > 0$$

Therefore, the third type constraints in the system $\mathcal{Q}^{ES}_{\Pi^{\mathcal{C}}}$ holds. Hence, the assignment $\{x_C \mapsto p_C\}_{C \in \mathcal{C}} \cup \{x_C \mapsto 0\}_{C \in \mathcal{D} \setminus \mathcal{C}}$ is a solution of the system $\mathcal{Q}^{ES}_{\Pi^{\mathcal{C}}}$.

(If.) Let $\mathcal{D}$ be the set of all valid configurations conform to Π in φ and the assignment $\{x_C \mapsto p_C\}_{C \in \mathcal{D}}$ be a solution of the system $\mathcal{Q}^{ES}_{\Pi}$ in $\mathbb{N}_\infty$. Let $\mathcal{C}$ be the set $\{C \in \mathcal{D} \mid p_C > 0\}$.

We first claim that $\mathcal{C}$ is strong self-matching. Clearly, $\Pi^{\mathcal{C}} \subseteq \Pi$ is mutually null-compatible. Suppose to the contrary that there is a configuration $\langle \pi_1, F_1 \rangle \in \mathcal{C}$ such that it does not conform to $\Pi^{\mathcal{C}}$. Let $\pi_2 \in \Pi$ and $\eta \in \mathcal{K}^{\varphi,\mathsf{ntri}}_{\langle \pi_1, *, \pi_2 \rangle}$ satisfy that $F_1(\eta, \pi_2) > 0$ but $\pi_2 \notin \Pi^{\mathcal{C}}$.

- If $\eta^{\triangleleft} = \vec{0}$, because $\sum_{F \in \mathcal{F}^{\mathcal{D}}_{\pi_1}} p_{\langle \pi_1, F \rangle} \cdot F(\eta, \pi_2) \geq p_{\langle \pi_1, F_1 \rangle} \cdot F_1(\eta, \pi_2) > 0$, together with the first type constraint of the system $\mathcal{Q}^{ES}_{\Pi^{\mathcal{C}}}$, then $\sum_{F \in \mathcal{F}^{\mathcal{D}}_{\pi_2}} p_{\langle \pi_2, F \rangle} > 0$. Hence, there exists a configuration $\langle \pi_2, F_2 \rangle \in \mathcal{D}$ such that $p_{\langle \pi_2, F_2 \rangle} > 0$. By definition, $\langle \pi_2, F_2 \rangle \in \mathcal{C}$, but this contradict to the fact that $\pi_2 \notin \Pi^{\mathcal{C}}$.

- If $\eta^{\triangleleft} \neq \vec{0}$, because $\sum_{F \in \mathcal{F}^{\mathcal{D}}_{\pi_1}} p_{\langle \pi_1, F \rangle} \cdot F(\eta, \pi_2) > 0$, together with the second type constraint of the system $\mathcal{Q}^{ES}_{\Pi^{\mathcal{C}}}$, then $\sum_{F \in \mathcal{F}^{\mathcal{D}}_{\pi_2}} p_{\langle \pi_2, F \rangle} \cdot F(\widetilde{\eta}, \pi_1) > 0$. Hence, there is a configuration $\langle \pi_2, F_2 \rangle \in \mathcal{D}$ such that $p_{\langle \pi_2, F_2 \rangle} > 0$. By definition, $\langle \pi_2, F_2 \rangle \in \mathcal{C}$, but this contradict to the fact that $\pi_2 \notin \Pi^{\mathcal{C}}$.

Therefore, for every $C \in \mathcal{C}$, C conforms to $\Pi^{\mathcal{C}}$. Hence, $\mathcal{C}$ is strong self-matching.

We then claim that $\{x_C \mapsto p_C\}_{C \in \mathcal{C}}$ is a solution of the system $\mathcal{Q}^S_{\mathcal{C}}$. For every $\pi \in \Pi^{\mathcal{C}}$, there exists F such that $\langle \pi, F \rangle \in \mathcal{C}$. By definition, $p_{\langle \pi, F \rangle} > 0$, therefore, $\sum_{C \in \mathcal{C}|_\pi} p_C \geq p_{\langle \pi, F \rangle} > 0$. This implies the first type constraint of the system $\mathcal{Q}^S_{\mathcal{C}}$ holds. For every $\pi_1, \pi_2 \in \Pi^{\mathcal{C}}$, for every $\eta \in \mathcal{K}^{\varphi,\mathsf{ntri}}_{\langle \pi_1, *, \pi_2 \rangle}$, by the second type constraint of the system $\mathcal{Q}^{ES}_{\Pi}$, if $\eta^{\triangleleft} \neq \vec{0}$, the following holds.

$$\sum_{F \in \mathcal{F}^{\mathcal{C}}_{\pi_1}} x_{\langle \pi_1, F \rangle} \cdot F(\eta, \pi_2) = \sum_{F \in \mathcal{F}^{\mathcal{D}}_{\pi_1}} x_{\langle \pi_1, F \rangle} \cdot F(\eta, \pi_2) = \sum_{F \in \mathcal{F}^{\mathcal{C}}_{\pi_2}} x_{\langle \pi_2, F \rangle} \cdot F(\widetilde{\eta}, \pi_1) = \sum_{F \in \mathcal{F}^{\mathcal{C}}_{\pi_2}} x_{\langle \pi_2, F \rangle} \cdot F(\widetilde{\eta}, \pi_1)$$

doi:10.6342/NTU202203098

Therefore, the second type constraint of the system $\mathcal{Q}_{\mathcal{C}}^{S}$ holds. Note that $\sum_{C\in\mathcal{C}} x_C = \sum_{C\in\mathcal{D}} x_C > 0$. The third type constraint of the system $\mathcal{Q}_{\mathcal{C}}^{S}$ also holds.

Finally, because $\mathcal{C}$ is strong self-matching and the system $\mathcal{Q}_{\mathcal{C}}^{S}$ has a solution in $\mathbb{N}_\infty$, by Theorem 6.14, φ is strongly satisfiable. □

Lemma 6.30. *For every mutually null-compatible sets of valid 1-types* Π_1 *and* Π_2 *in* φ *satisfying* $\Pi_1 \subset \Pi_2$. *If the system* $\mathcal{Q}_{\Pi_1}^{ES}$ *has a solution in* $\mathbb{N}_\infty$, *then the system* $\mathcal{Q}_{\Pi_2}^{ES}$ *has a solution in* $\mathbb{N}_\infty$.

Proof. Let $\mathcal{V}_1$ be the set of variables of the system $\mathcal{Q}_{\Pi_1}^{ES}$ and $\mathcal{V}_2$ be the set of variables of the system $\mathcal{Q}_{\Pi_2}^{ES}$. Because $\Pi_1 \subseteq \Pi_2$, $\mathcal{V}_1 \subset \mathcal{V}_2$. Let $\{v \mapsto p_v\}_{v\in\mathcal{V}_1}$ be a solution of the system $\mathcal{Q}_{\Pi_1}^{ES}$. Since the system $\mathcal{Q}_{\Pi_2}^{ES}$ is the extension of the system $\mathcal{Q}_{\Pi_1}^{ES}$. It is not difficult to check that the assignment $\{v \mapsto p_v\}_{v\in\mathcal{V}_1} \cup \{v \mapsto 0\}_{v\in\mathcal{V}_2\setminus\mathcal{V}_1}$ is a solution of the system $\mathcal{Q}_{\Pi_2}^{ES}$. □

Note that the number of maximal mutually null-compatible sets of valid 1-types is less than the number of mutually null-compatible sets of valid 1-types. Hence, the following theorem may yield a more efficient algorithm than Lemma 6.29.

Lemma 6.31. *The* C^2 *sentence* φ *is strongly satisfiable if and only if there exists a maximal mutually null-compatible set of valid 1-types* Π *in* φ *such that the system* $\mathcal{Q}_{\Pi}^{ES}$ *has a solution in* $\mathbb{N}_\infty$.

Proof.
(If.) It is a simple corollary of Lemma 6.29.
(Only if.) By Lemma 6.29, if φ is strongly satisfiable, then there exists a mutually null-compatible set of valid 1-types Π in φ such that the system $\mathcal{Q}_{\Pi}^{ES}$ has a solution in $\mathbb{N}_\infty$. Let Π' be a maximal mutually null-compatible set of valid 1-types satisfying $\Pi' \subseteq \Pi$. By Lemma 6.30, if the system $\mathcal{Q}_{\Pi}^{ES}$ has a solution in $\mathbb{N}_\infty$, then the system $\mathcal{Q}_{\Pi'}^{ES}$ has a solution in $\mathbb{N}_\infty$. Hence, Π' is the desired set of 1-types. □

Reduced strong induced ILP system. The algorithm for $\mathsf{SAT}(\mathsf{C}^2)$ from then extended strong induced ILP system takes only exponential many guessing for exponential many

doi:10.6342/NTU202203098

1-types. However the size of the extended strong induced ILP system is unbounded, in fact, its size is double-exponential in the length of φ, since it enumerate all possible configurations.

To compress the size of the ILP system, we will apply the techniques presented in Section 6.2 and obtain the *reduced strong induced ILP system* whose size is only *exponential* in the length of φ.

Definition 6.32. *Let Π be a mutually null-compatible set of valid 1-types in φ. The reduced strong induced ILP system $\mathcal{Q}_{\Pi}^{RS}$ is defined as follows.*

The variables in the system $\mathcal{Q}_{\Pi}^{RS}$ are $\{x_e\}_{e\in\bigcup_{\pi\in\Pi} E^{\varphi,\Pi,\pi}}$. For every $\pi_1,\pi_2\in\Pi$, for every $\eta\in\mathcal{K}^{\varphi,\mathsf{ntri}}_{\langle\pi_1,*,\pi_2\rangle}$, if $\eta^{\triangleleft}=\vec{0}$, then the following constraint is in the system $\mathcal{Q}_{\Pi}^{RS}$.

1. $\sum_{e\in E^{\varphi,\Pi,\pi_1}_{\eta,\pi_2}} x_e>0 \implies \sum_{e\in E^{\varphi,\Pi,\pi_2}_{\vec{0},o}} x_e>0.$

Otherwise, if $\eta^{\triangleleft}\neq\vec{0}$, then the following constraint is in the system $\mathcal{Q}_{\Pi}^{RS}$.

2. $\sum_{e\in E^{\varphi,\Pi,\pi_1}_{\eta,\pi_2}} x_e=\sum_{e\in E^{\varphi,\Pi,\pi_2}_{\tilde{\eta},\pi_1}} x_e.$

For every $\pi\in\Pi$, the following balance gadget is in the system $\mathcal{Q}_{\Pi}^{RS}$.

3. $\mathcal{Q}^{bal}_{\mathcal{G}^{\varphi,\Pi,\pi}}.$

Finally, the non-trivial constraint.

4. $\sum_{e\in\bigcup_{\pi\in\Pi} E^{\varphi,\Pi,\pi}} x_e>0.$

Lemma 6.33. *The C^2 sentence φ is strongly satisfiable if and only if there exists a maximal mutually null-compatible set of valid 1-types Π in φ such that the system $\mathcal{Q}_{\Pi}^{RS}$ has a solution in $\mathbb{N}_\infty$.*

Proof. We fix an order of elements in Π, say $\pi_1,\pi_2,\ldots,\pi_\ell$. For every $i\in[\ell]$, let $\mathcal{C}_i$ be the set of valid configurations with source type π_i conform to Π in φ. For $0\le i\le\ell$, let $\mathcal{Q}_i$ be the ILP system defined as follows. The variables of the system $\mathcal{Q}_i$ are as follows.

$$\bigcup_{1\le j\le i}\left\{x_C \;\middle|\; C\in\mathcal{C}^{\varphi,\Pi,\pi_j}\right\}\cup\bigcup_{i<j\le\ell}\left\{x_e \;\middle|\; e\in E^{\varphi,\Pi,\pi_j}\right\}\cup$$
$$\bigcup_{1\le j\le\ell}\{y_{\pi_j}\}\cup\bigcup_{1\le j\le\ell}\left\{y_{\langle\pi_j,\eta,\pi\rangle}\;\middle|\;\pi\in\Pi \text{ and } \eta\in\mathcal{K}^{\varphi,\mathsf{ntri}}_{\langle\pi_j,*,\pi\rangle}\right\}$$

doi:10.6342/NTU202203098

The system $\mathcal{Q}_i$ is defined as follows.

$$
\begin{aligned}
\mathcal{Q}_i := \quad & \bigwedge_{1\le j\le i} \mathcal{Q}^{\mathsf{std}}_{\mathcal{C}^{\varphi,\Pi,\pi_j}} \wedge \bigwedge_{i<j\le \ell} \mathcal{Q}^{\mathsf{std}}_{\mathcal{G}^{\varphi,\Pi,\pi_j}} \\
\wedge \quad & \bigwedge_{\pi_1\in\Pi}\ \bigwedge_{\pi_2\in\Pi}\ \bigwedge_{\eta\,\in\,\mathcal{K}^{\varphi,\mathsf{ntri}}_{\langle\pi_1,*,\pi_2\rangle}\text{ s.t. }\eta^{\triangleright}=\vec{0}} y_{\langle\pi_1,\eta,\pi_2\rangle} > 0 \implies y_{\pi_2} > 0 \\
\wedge \quad & \bigwedge_{\pi_1\in\Pi}\ \bigwedge_{\pi_2\in\Pi}\ \bigwedge_{\eta\,\in\,\mathcal{K}^{\varphi,\mathsf{ntri}}_{\langle\pi_1,*,\pi_2\rangle}\text{ s.t. }\eta^{\triangleright}\neq\vec{0}} y_{\langle\pi_1,\eta,\pi_2\rangle} = y_{\langle\pi_2,\widetilde{\eta},\pi_1\rangle}
\end{aligned}
$$

For every $i \in [\ell]$, by Lemma 6.24, $\mathcal{Q}_{i-1}$ has a solution in $\mathbb{N}_\infty$ if and only if $\mathcal{Q}_i$ also has a solution in $\mathbb{N}_\infty$. Observe that $\mathcal{Q}_0$ is $\mathcal{Q}^{RS}_{\Pi}$ and $\mathcal{Q}_\ell$ is $\mathcal{Q}^{ES}_{\Pi}$. Hence, $\mathcal{Q}^{RS}_{\Pi}$ has a solution in $\mathbb{N}_\infty$ if and only if $\mathcal{Q}^{ES}_{\Pi}$ has a solution in $\mathbb{N}_\infty$. By Lemma 6.31, $\mathcal{Q}^{ES}_{\Pi}$ has a solution in $\mathbb{N}_\infty$ if and only if φ is strongly satisfiable. □

Reduced strong induced boolean formula. Recall that φ is strongly satisfiable, by Lemma 6.5, if and only if it has an infinite size strong pseudo-model $\mathcal{G}$. Moreover, for every configuration C, if C is realized in $\mathcal{G}$, then the number of realizations of C is infinite. Hence, for the strong satisfiability problem of C^2, it is sufficient to consider whether the reduced strong induced ILP system has a solution in $\{0, \infty\}$. In fact, we can further rewrite the system to the boolean formula and obtain more efficient algorithm.

Definition 6.34. *Let Π be a mutually null-compatible set of valid 1-types in φ. The reduced strong induced boolean formula Ψ^{RS}_{Π} is defined as follows. The variables in the formula*

doi:10.6342/NTU202203098

Ψ_{Π}^{RS} *are* $\{b_e\}_{e \in \bigcup_{\pi\in\Pi} E^{\varphi,\Pi,\pi}}$.

$$
\begin{aligned}
\Psi_{\Pi}^{RS} := \quad & \bigwedge_{\pi_1\in\Pi} \bigwedge_{\pi_2\in\Pi} \bigwedge_{\eta \in \mathcal{K}^{\varphi,\mathsf{ntri}}_{\langle \pi_1,*,\pi_2\rangle} \ s.t.\ \eta^{\triangleright} = \vec{0}} \left(\bigvee_{e\in E^{\varphi,\Pi,\pi_1}_{\eta,\pi_2}} \neg b_e \rightarrow \bigvee_{e\in E^{\varphi,\Pi,\pi_1}_{\vec{0},o}} \neg b_e \right) \\
\wedge \quad & \bigwedge_{\pi_1\in\Pi} \bigwedge_{\pi_2\in\Pi} \bigwedge_{\eta \in \mathcal{K}^{\varphi,\mathsf{ntri}}_{\langle \pi_1,*,\pi_2\rangle} \ s.t.\ \eta^{\triangleright} \neq \vec{0}} \left(\bigvee_{e\in E^{\varphi,\Pi,\pi_1}_{\eta,\pi_2}} \neg b_e \leftrightarrow \bigvee_{e\in E^{\varphi,\Pi,\pi_2}_{\tilde{\eta},\pi_1}} \neg b_e \right) \\
\wedge \quad & \bigwedge_{\pi\in\Pi} \Psi^{bal}_{\mathcal{G}^{\varphi.\Pi,\pi}} \\
\wedge \quad & \bigvee_{\pi_1\in\Pi} \bigvee_{\pi_2\in\Pi} \bigvee_{\eta\in\mathcal{K}^{\varphi,\mathsf{ntri}}_{\langle \pi_1,*,\pi_2\rangle}} \bigvee_{e\in E^{\varphi,\Pi,\pi_1}_{\eta,\pi_2}} \neg b_e
\end{aligned}
$$

Remark 6.35. *The size of the formula* Ψ_{Π}^{RS} *is bounded by the following.*

$$
\left|\Psi_{\Pi}^{RS}\right| \leq 5|\Pi^{\tau}|^2|\mathcal{K}^{\tau}|\left|E^{\varphi,\Pi,\pi_1}\right| + |\Pi^{\tau}|\left|\Psi^{bal}_{\mathcal{G}^{\varphi.\Pi,\pi}}\right| \leq 8K^m 2^{3n+4m}
$$

Note that by the following semantics equivalent rule, we can rewrite Ψ_{Π}^{RS} *into equisatisfiable Horn formula in polynomial-time in its length.*

1. *For ever clause* c_1 *and* c_2, $c_1 \leftrightarrow c_2 \equiv (c_1 \rightarrow c_2) \wedge (c_2 \rightarrow c_1)$.
2. *For ever clause* c_1, c_2, *and* c_3, $(c_1 \vee c_2) \rightarrow c_3 \equiv (c_1 \rightarrow c_3) \wedge (c_2 \rightarrow c_3)$.
3. *For ever clause* c_1 *and literal* ℓ_1, $\ell_1 \rightarrow c_1 \equiv \neg\ell_1 \vee c_1$.

Theorem 6.36. *The* C^2 *sentence* φ *is strongly satisfiable if and only if there exists a maximal mutually null-compatible set of valid 1-types* Π *in* φ *such that the formula* Ψ_{Π}^{RS} *is satisfiable.*

Proof. By Lemma 6.33, it is sufficient to show that for every mutually null-compatible set of valid 1-types Π in φ, Ψ_{Π}^{RS} is satisfiable if and only if the system $\mathcal{Q}_{\Pi}^{RS}$ has a solution in $\mathbb{N}_{\infty}$. The proof is straightforward, we construct the corresponding assignment, and it is not difficult to verify it is a solution.

(If.) Let the assignment $\{x_e \mapsto n_e\}_{e\in\bigcup_{\pi\in\Pi} E^{\varphi,\Pi,\pi}}$ be a solution of the system $\mathcal{Q}_{\Pi}^{RS}$. For

doi:10.6342/NTU202203098

every $e \in \bigcup_{\pi\in\Pi} E^{\varphi,\Pi,\pi}$, the following assignment is a valid assignment of Ψ_{Π}^{RS}.

$$b_e \mapsto \begin{cases} \bot & \text{if } x_e > 0 \\ \top & \text{otherwise} \end{cases}$$

(Only if.) Let the assignment $\{b_e \mapsto v_e\}_{e\in\bigcup_{\pi\in\Pi} E^{\varphi,\Pi,\pi}}$ be a valid assignment of Ψ_{Π}^{RS}. For every $e \in \bigcup_{\pi\in\Pi} E^{\varphi,\Pi,\pi}$, the following assignment is a solution of the system $\mathcal{Q}_{\Pi}^{RS}$.

$$b_e \mapsto \begin{cases} \infty & \text{if } v_e = \bot \\ 0 & \text{if } v_e = \top \end{cases}$$

□

In fact, we can also encode the mutually null-compatible set of valid 1-types and obtain a *deterministic* exponential-time reduction to the Boolean satisfiability problem.

Theorem 6.37. *There exists a deterministic exponential-time reduction from the strong satisfiability problem of* C^2 *to the Boolean satisfiability problem.*

Proof. Recall that the relation of null-compatible can be represented as a graph, and the mutually null-compatible set of valid 1-types can be represented as a independent set by Remark 3.11, and Our approach is quite standard, we encode this graph and the condition of the independent set in the boolean formula and let the variable in the reduced strong induced ILP system be controlled by the independent set.

Let $R(x, y)$ be a relation over the set of valid configuration Π^{φ}. The relation $R(x, y)$ holds if and only if x and y are not null-compatible. Note that it takes exponential-time in the length of φ to compute Π^{φ} and $R(x, y)$.

The variables in the formula Ψ are $\{b_e\}_{e\in\bigcup_{\pi\in\Pi^{\varphi}} E^{\varphi,\Pi^{\varphi},\pi}} \cup \{b_\pi\}_{\pi\in\Pi^{\varphi}}$. The formula Ψ is defined as follows.

$$\Psi := \bigwedge_{\{(\pi_1,\pi_2)\ \mid\ R(\pi_1,\pi_2)\}} (\neg b_{\pi_1} \vee \neg b_{\pi_2}) \wedge \bigwedge_{\pi\in\Pi^{\varphi}} \bigwedge_{e\in E^{\varphi,\Pi^{\varphi},\pi}} (\neg b_\pi \rightarrow b_e) \wedge \Psi_{\Pi^{\varphi}}^{RS}$$

doi:10.6342/NTU202203098

The intuition of the variable b_π is whether the 1-type π realized in the pseudo-model. Then, the first term encode that if two 1-types are not null-compatible, then they can't realize at the same time. The second term encode that if a 1-type is not realized, then there is no edge outgoing from this 1-type. That is, the variable in the system $\Phi^{RS}_{\Pi^\varphi}$ is controlled by the independent set. The correctness of the reduction follows the above intuition. □

Finite strong satisfiability problem of C^2. Finally, we close this section with the finite strong satisfiability problem of C^2. Like Chapter 5, the algorithms are from theorems mentioned above.

Lemma 6.38. *The C^2 sentence φ is finitely strongly satisfiable if and only if there exists a mutually null-compatible set of valid 1-types Π in φ such that the system $\mathcal{Q}^{RS}_\Pi$ has a solution in $\mathbb{N}$.*

Proof. The proof is similar to Lemma 6.33, 6.31, 6.29, 6.24, and 6.23. The only difference is that the solution of the system is restricted to finite. □

However, there are implications in the system $\mathcal{Q}^{RS}_\Pi$, We define the following *finite reduced strong induced ILP system* to obtain efficient algorithm.

Definition 6.39. *Let Π be a mutually null-compatible set of valid 1-types in φ. The finite reduced strong induced ILP system $\mathcal{Q}^{FRS}_\Pi$ is defined as follows.*

The variables in the system $\mathcal{Q}^{FRS}_\Pi$ are $\{x_e\}_{e\in\bigcup_{\pi\in\Pi} E^{\varphi,\Pi,\pi}}$. Let $M := (\theta_4(n,m,K))^2$. For every $\pi_1,\pi_2\in\Pi$, for every $\eta\in\mathcal{K}^{\varphi,\mathsf{ntri}}_{\langle\pi_1,*,\pi_2\rangle}$, if $\eta^{\triangleleft}=\vec{0}$, then the following constraint is in the system $\mathcal{Q}^{FRS}_\Pi$.

1. $M\cdot\sum_{e\in E^{\varphi,\Pi,\pi_2}_{\vec{0},o}} x_e \geq \sum_{e\in E^{\varphi,\Pi,\pi_1}_{\eta,\pi_2}} x_e$.

Otherwise, if $\eta^{\triangleleft}\neq\vec{0}$, then the following constraint is in the system $\mathcal{Q}^{FRS}_\Pi$.

2. $\sum_{e\in E^{\varphi,\Pi,\pi_1}_{\eta,\pi_2}} x_e = \sum_{e\in E^{\varphi,\Pi,\pi_2}_{\tilde{\eta},\pi_1}} x_e$.

For every $\pi\in\Pi$, the following balance gadget is in the system $\mathcal{Q}^{FRS}_\Pi$.

3. $\mathcal{Q}^{bal}_{\mathcal{G}^{\varphi,\Pi,\pi}}$.

doi:10.6342/NTU202203098

Finally, the non-trivial constraint.

4. $\sum_{e \in \bigcup_{\pi \in \Pi} E^{\varphi,\Pi,\pi}} x_e > 0$.

Remark 6.40. *Though the constant M is double-exponential in the length of φ, it can be encoded in binary by only exponential bits. The size of the system $\mathcal{Q}_{\Pi}^{FRS}$ is exponential in the length of φ.*

Note that the only difference between the system $\mathcal{Q}_{\Pi}^{RS}$ and $\mathcal{Q}_{\Pi}^{FRS}$ is the first type constraint.

Theorem 6.41. *The C^2 sentence φ is strongly satisfiable if and only if there exists a maximal mutually null-compatible set of valid 1-types Π in φ such that the system $\mathcal{Q}_{\Pi}^{FRS}$ has a solution in $\mathbb{N}$.*

Proof.
(If.) By Lemma 6.38, it is sufficient to show that if the system $\mathcal{Q}_{\Pi}^{FRS}$ has a solution in $\mathbb{N}$, then the system $\mathcal{Q}_{\Pi}^{RS}$ has a solution in $\mathbb{N}$. In fact, every finite solution of the system $\mathcal{Q}_{\Pi}^{FRS}$ is also the solution of the system $\mathcal{Q}_{\Pi}^{RS}$.

Let the assignment $\{x_e \mapsto n_e\}_{e \in \bigcup_{\pi \in \Pi} E^{\varphi,\Pi,\pi}}$ be a solution of the system $\mathcal{Q}_{\Pi}^{FRS}$. For every $\pi_1, \pi_2 \in \Pi$, for every $\eta \in \mathcal{K}^{\varphi,\mathsf{ntri}}_{\langle \pi_1, *, \pi_2 \rangle}$, if $\eta^{\triangleleft} = \vec{0}$, then $M \cdot \sum_{e \in E^{\varphi,\Pi,\pi_2}_{\vec{0},o}} p_e \geq \sum_{e \in E^{\varphi,\Pi,\pi_1}_{\eta,\pi_2}} p_e$. If $\sum_{e \in E^{\varphi,\Pi,\pi_1}_{\eta,\pi_2}} p_e > 0$, then the following holds.

$$\sum_{e \in E^{\varphi,\Pi,\pi_2}_{\vec{0},o}} p_e \geq \frac{1}{M} \cdot \sum_{e \in E^{\varphi,\Pi,\pi_1}_{\eta,\pi_2}} p_e > 0$$

Therefore, the following first type constraint in the system $\mathcal{Q}_{\Pi}^{RS}$ holds.

$$\sum_{e \in E^{\varphi,\Pi,\pi_1}_{\eta,\pi_2}} p_e > 0 \implies \sum_{e \in E^{\varphi,\Pi,\pi_2}_{\vec{0},o}} p_e > 0$$

(Only if.) By Lemma 6.11, if φ is strongly satisfiable, then it has a pseudo-model $\mathcal{G}$ whose size is at most $\theta_4(n, m, K)$. Recall that the number of realizations in $\mathcal{G}$ is a solution of the system $\mathcal{Q}_{\Pi}^{RS}$. We claim that it is also a solution of the system $\mathcal{Q}_{\Pi}^{FRS}$.

doi:10.6342/NTU202203098

Since the number of realizations in $\mathcal{G}$ is a solution of the system $\mathcal{Q}_{\Pi}^{RS}$, for every $\pi_1, \pi_2 \in \Pi$, for every $\eta \in \mathcal{K}_{\langle \pi_1, *, \pi_2 \rangle}^{\varphi, \mathsf{ntri}}$, if $\eta^{\triangleleft} = \vec{0}$, then the following first type constraint holds.

$$\sum_{e \in E_{\eta, \pi_2}^{\varphi, \Pi, \pi_1}} p_e > 0 \implies \sum_{e \in E_{\vec{0}, o}^{\varphi, \Pi, \pi_2}} p_e > 0$$

Note that $\sum_{e \in E_{\vec{0}, o}^{\varphi, \Pi, \pi_2}} p_e$ is bounded by the number of edges in $\mathcal{G}$, that is, M. Therefore, if $\sum_{e \in E_{\eta, \pi_2}^{\varphi, \Pi, \pi_1}} p_e > 0$, then the following holds.

$$M \cdot \sum_{e \in E_{\vec{0}, o}^{\varphi, \Pi, \pi_2}} p_e \geq M \cdot 1 \geq \sum_{e \in E_{\vec{0}, o}^{\varphi, \Pi, \pi_2}} p_e$$

Otherwise, if $\sum_{e \in E_{\eta, \pi_2}^{\varphi, \Pi, \pi_1}} p_e = 0$, then $M \cdot \sum_{e \in E_{\vec{0}, o}^{\varphi, \Pi, \pi_2}} p_e \geq 0$ always holds. Finally, the following first type constraint in the system $\mathcal{Q}_{\Pi}^{FRS}$ holds.

$$M \cdot \sum_{e \in E_{\vec{0}, o}^{\varphi, \Pi, \pi_2}} p_e \geq \sum_{e \in E_{\eta, \pi_2}^{\varphi, \Pi, \pi_1}} p_e$$

□

6.4 Algorithms for $\mathsf{SSAT}(\mathsf{C}^2)$ and $\mathsf{FIN\text{-}SSAT}(\mathsf{C}^2)$

Finally, we present algorithms for $\mathsf{SSAT}(\mathsf{C}^2)$ and $\mathsf{FIN\text{-}SSAT}(\mathsf{C}^2)$.

Algorithm for $\mathsf{SSAT}(\mathsf{C}^2)$. We obtain ALGORITHM-D from the reduced strong induced boolean formula. Since the length of the formula Ψ_{Π}^{RS} is exponential in the length of φ, it is not difficult to check that ALGORITHM-D takes non-deterministic exponential-time in the length of the input sentence φ.

Lemma 6.42. ALGORITHM-D *decides* $\mathsf{SSAT}(\mathsf{C}^2)$.

Proof. ALGORITHM-D guess a mutually null-compatible set of valid 1-types in φ and check the satisfiability of the system Ψ_{Π}^{RS}. By Theorem 6.36, φ is strongly satisfiable if and only ALGORITHM-D accepts φ. □

doi:10.6342/NTU202203098

Theorem 6.43. $\mathsf{SSAT}(\mathsf{C}^2)$ *is in* $\mathsf{NEXPTIME}$.

Remark 6.44. *Recall that we can rewrite the formula* Ψ_{Π}^{RS} *to Horn formula in polynomial-time in its length. The complexity of the satisfiability problem of Horn formulae is linear in the length of the formula [7]. Hence, there exists a polynomial-time solver for the satisfiability of* Ψ_{Π}^{RS}.

Remark 6.45. ALGORITHM-D *makes exponential many guessing from exponential many 1-types. Hence,* ALGORITHM-D *is more efficient and implementable than the algorithm form the CEB property directly.*

Remark 6.46. *Recall that the maximal mutually null-compatible set of valid 1-types can be represented as a independent set in the null-compatible relations graph. There are some algorithms that enumerate all independent sets in* $O(3^{n/3} \cdot poly(n))$ *[4, 30, 28], which are efficient for implementing* ALGORITHM-D.

ALGORITHM-D

Input: A C^2 sentence φ.
Task: Accept if and only if φ is strongly satisfiable.

Compute Π^{φ}.
Guess a set of 1-types $\Pi \subseteq \Pi^{\varphi}$.
if Π is not maximal mutually null-compatible **then**
    REJECT
if Ψ_{Π}^{RS} is satisfiable **then**
    ACCEPT
else
    REJECT

---

doi:10.6342/NTU202203098

Algorithm for $\mathsf{FIN\text{-}SSAT}(\mathsf{C}^2)$. We obtain ALGORITHM-E from the finite reduced strong induced ILP system. Since the size of the system $\mathcal{Q}_{\Pi}^{FRS}$ is exponential in the length of φ, it is not difficult to check that ALGORITHM-E takes non-deterministic exponential-time in the length of the input sentence φ.

Lemma 6.47. ALGORITHM-E *decides* $\mathsf{FIN\text{-}SSAT}(\mathsf{C}^2)$.

Proof. ALGORITHM-E guess a mutually null-compatible set of valid 1-types in φ and check whether the system $\mathcal{Q}_{\Pi}^{FRS}$ has a solution. By Theorem 6.41, φ is finitely strongly satisfiable if and only ALGORITHM-E accepts φ. □

Theorem 6.48. $\mathsf{FIN\text{-}SSAT}(\mathsf{C}^2)$ *is in* $\mathsf{NEXPTIME}$.

Remark 6.49. *Observe that there is no constant terms in the system $\mathcal{Q}_{\Pi}^{FRS}$. Therefore, if $\mathcal{Q}_{\Pi}^{FRS}$ has a positive rational solution, then it has a solution in $\mathbb{N}$. There is algorithm solve the integer programming problem in polynomial-time [17]. Hence, there exists a polynomial-time solver for the satisfiability of $\mathcal{Q}_{\Pi}^{FRS}$.*

ALGORITHM-E

Input: A C^2 sentence φ.
Task: Accept if and only if φ is finitely strongly satisfiable.

```
Compute Π^φ.
Guess a set of 1-types Π ⊆ Π^φ.
if Π is not maximal mutually null-compatible then
    REJECT
if Q_Π^FRS has a finite solution then
    ACCEPT
else
    REJECT
```


doi:10.6342/NTU202203098

doi:10.6342/NTU202203098

Chapter 7

Guarded fragment of C^2

We consider the guarded fragment of C^2 in this chapter. In Section 7.1, we prove some properties about GC^2. In Section 7.2, we present the algorithms for satisfiability and finite satisfiability problems of GC^2.

In this chapter, we fix a vocabulary τ which consists of n unary predicates $U_1, U_2, \ldots, U_n$ and m binary predicates $\beta_1, \beta_2, \ldots, \beta_m$. Let φ be a GC^2 sentence over τ in the normal form, that is,

$$\begin{aligned} \varphi := \quad & \forall x \ \gamma(x) \\ \wedge \quad & \forall x \, \forall y \ x \neq y \rightarrow \bigwedge_{i \in [\ell]} (r_i(x,y) \rightarrow \alpha_i(x,y)) \\ \wedge \quad & \bigwedge_{i \in [m]} \forall x \ \exists^{=k_i} y \ \beta_i(x,y) \ \wedge \ x \neq y, \end{aligned}$$

where $\gamma(x)$ and $\alpha_i(x,y)$ are quantifier-free formulae, and r_i is a binary predicate called guard-atom. Let $\vec{k} := (k_1, k_2, \ldots, k_m)$ be the m-dimension (row) vector of the counting condition of φ and $K := \sum_{i \in [m]} k_i$ be the summation of all counting conditions.

7.1 Guarded fragment of C^2

Our main observation for GC^2 is that the GC^2 sentence φ is satisfiable if and only if it is strongly satisfiable. Hence, our algorithm for the strong satisfiability and finite strong satisfiability problems of C^2 developed in Chapter 6 can also be used for GC^2.

doi:10.6342/NTU202203098

Lemma 7.1. *For every valid 1-type π_1 and π_2 in φ, π_1 and π_2 are null-compatible.*

Proof. Note that the assignment of the guarded atoms are all negative in the null type. Hence, null type is valid between π_1 and π_2 and this implies that π_1 and π_2 are null-compatible. □

Lemma 7.2. *The set of all valid 1-types Π^φ is mutually null-compatible. Besides, Π^φ is the only maximal mutually null-compatible set of valid 1-types.*

Proof. For every $\pi_1, \pi_2 \in \Pi^\varphi$, by Lemma 7.1, π_1 and π_2 are null-compatible. Hence, Π^φ is mutually null-compatible.

For every set mutually null-compatible of valid 1-types Π, by definition, $\Pi \subseteq \Pi^\varphi$. Then, Π^φ is the only maximal mutually null-compatible set of valid 1-types. □

Satisfiability problem of GC^2.

Lemma 7.3. *The GC^2 sentence φ is satisfiable only if it is strongly satisfiable.*

Proof. We claim that every pseudo-model of φ is a strong pseudo-model. Let $\mathcal{G}$ be a model of φ. By definition, $\Pi^\mathcal{G} \subseteq \Pi^\varphi$. By Lemma 7.2, because Π^φ is mutually null-compatible, $\Pi^\mathcal{G}$ is also mutually null-compatible. Hence, $\mathcal{G}$ is a strong pseudo-model. □

Theorem 7.4. *The GC^2 sentence φ is satisfiable if and only if the formula $\Psi^{RS}_{\Pi^\varphi}$ is satisfiable.*

Proof.

(if.) By Lemma 7.2, Π^φ is maximal mutually null-compatible. Hence, by Theorem 6.36, if the formula $\Psi^{RS}_{\Pi^\varphi}$ is satisfiable, then φ is strongly satisfiable. Then, φ is satisfiable.

(Only if.) By Lemma 7.3, if φ is satisfiable, then it is strongly satisfiable. By Theorem 6.36, φ is strongly satisfiable, then there exists a maximal mutually null-compatible set of valid 1-types Π in φ such that Ψ^{RS}_{Π} is satisfiable. By Lemma 7.2, Π^φ is the only maximal mutually null-compatible set of valid 1-types. Hence, $\Pi = \Pi^\varphi$. □

doi:10.6342/NTU202203098

Finite satisfiability problem of GC^2. We can prove the following properties for finite satisfiability of GC^2 with the similar arguments.

Lemma 7.5. *The GC^2 sentence φ is finitely satisfiable only if it is finitely strongly satisfiable.*

Theorem 7.6. *The GC^2 sentence φ is finitely satisfiable if and only if the system $\mathcal{Q}^{FRS}_{\Pi\varphi}$ is satisfiable.*

7.2 Algorithms for $\mathsf{SAT}(\mathsf{GC}^2)$ and $\mathsf{FIN\text{-}SAT}(\mathsf{GC}^2)$

In this section, we present the algorithm for $\mathsf{SAT}(\mathsf{GC}^2)$ and $\mathsf{FIN\text{-}SAT}(\mathsf{GC}^2)$. We observe that by Lemma 7.2, there is only one maximal mutually null-compatible set of valid 1-types. Hence, the non-deterministic guessing vanish in the algorithms presented in Section 5.3, and we obtain *deterministic* algorithms for $\mathsf{SAT}(\mathsf{GC}^2)$ and $\mathsf{FIN\text{-}SAT}(\mathsf{GC}^2)$.

Algorithm for $\mathsf{SAT}(\mathsf{GC}^2)$. We obtain Algorithm-F from the reduced strong induced boolean formula. Recall that there exists a polynomial-time solver for the formula $\Psi^{RS}_{\Pi\varphi}$ by Remark 6.44. Since the length of the formula $\Psi^{RS}_{\Pi\varphi}$ is exponential in the length of φ, it is not difficult to check that Algorithm-F takes deterministic exponential-time in the length of the input sentence φ.

Lemma 7.7. Algorithm-F *decides* $\mathsf{SAT}(\mathsf{GC}^2)$.

Proof. Algorithm-F compute the set of valid 1-types in φ and check the satisfiability of the system $\Psi^{RS}_{\Pi\varphi}$. By Theorem 7.4, φ is satisfiable if and only Algorithm-F accepts φ. □

Theorem 7.8. **[18, 26]** $\mathsf{SAT}(\mathsf{GC}^2)$ *is in* $\mathsf{EXPTIME}$.

doi:10.6342/NTU202203098

Algorithm-F

Input: A GC^2 sentence φ.
Task: Accept if and only if φ is satisfiable.

```
Compute Π^φ.
if Ψ^{RS}_{Π^φ} is satisfiable then
    ACCEPT
else
    REJECT
```

Algorithm for $\mathsf{FIN\text{-}SAT}(\mathsf{GC}^2)$. We obtain Algorithm-G from the finite reduced strong induced ILP system. Recall that there exists a polynomial-time solver for the system $\mathcal{Q}^{FRS}_{\Pi^\varphi}$ by Remark 6.49. Since the size of the system $\mathcal{Q}^{FRS}_{\Pi^\varphi}$ is exponential in the length of φ, it is not difficult to check that Algorithm-G takes deterministic exponential-time in the length of the input sentence φ.

Lemma 7.9. Algorithm-G *decides* $\mathsf{FIN\text{-}SAT}(\mathsf{GC}^2)$.

Proof. Algorithm-G compute the set of valid 1-types in φ and check whether the system $\mathcal{Q}^{FRS}_{\Pi^\varphi}$ has a solution. By Theorem 7.6, φ is satisfiable if and only Algorithm-G accepts φ. □

Theorem 7.10. **[26]** $\mathsf{FIN\text{-}SAT}(\mathsf{GC}^2)$ *is in* $\mathsf{EXPTIME}$.

Algorithm-G

Input: A GC^2 sentence φ.
Task: Accept if and only if φ is finitely satisfiable.

```
Compute Π^φ.
if Q^{RS}_{Π^φ} has a finite solution then
    ACCEPT
else
    REJECT
```


doi:10.6342/NTU202203098

Chapter 8

Reductions

In this chapter, we present reductions between different fragments of two-variable logic. In Section 8.1, we present a deterministic polynomial-time reduction from $\mathsf{SSAT}(\mathsf{C}^2)$ to $\mathsf{SAT}(\mathsf{FO}^2)$. In Section 8.2, we present a deterministic polynomial-time reduction from $\mathsf{SAT}(\mathsf{GC}^2)$ to $\mathsf{SAT}(\mathsf{GF}^2)$.

We fix a vocabulary τ which consists of n unary predicates $U_1, U_2, \ldots, U_n$ and m binary predicates $\beta_1, \beta_2, \ldots, \beta_m$ in this chapter. Let φ be a C^2 sentence over τ in the normal form, that is,

$$\begin{aligned} \varphi := \quad & \forall x\, \gamma(x) \\ \wedge \quad & \forall x\, \forall y\; x \neq y \rightarrow \alpha(x, y) \\ \wedge \quad & \bigwedge_{i \in [m]} \forall x\, \exists^{=k_i} y\; \beta_i(x, y) \;\wedge\; x \neq y, \end{aligned}$$

where $\gamma(x)$ and $\alpha(x, y)$ are quantifier-free formulae. Let $\vec{k} := (k_1, k_2, \ldots, k_m)$ be the m-dimension (row) vector of the counting condition of φ and $K := \sum_{i \in [m]} k_i$ be the summation of all counting conditions. We also fix a pseudo-structure $\mathcal{G}$ over τ.

8.1 Reduction from $\mathsf{SSAT}(\mathsf{C}^2)$ to $\mathsf{SAT}(\mathsf{FO}^2)$

In this section, we describe a deterministic polynomial-time reduction from $\mathsf{SSAT}(\mathsf{C}^2)$ to $\mathsf{SAT}(\mathsf{FO}^2)$. The intuition of the reduction is to encode the non-trivial edges in the strong

doi:10.6342/NTU202203098

pseudo-model of the original C^2 sentence φ as an element in the pseudo-model of the new FO^2 sentence φ'. Recall that an edge is non-trivial if it realizes at least one binary predicate forwardly. Besides, there is a partial order structure in the pseudo-model of the new sentence φ' such that this structure guarantees the counting conditions of the original sentence φ.

Vocabulary. We first introduce the vocabulary and terminology of the reduction. The vocabulary τ' of the new FO^2 sentence φ' consists of only unary predicates.

$$\tau' = \{S_1, \ldots, S_n, T_1, \ldots, T_n, \beta_1^{\triangleright}, \ldots, \beta_m^{\triangleright}, \beta_1^{\triangleleft}, \ldots, \beta_m^{\triangleleft},$$
$$K_{1,1}, \ldots, K_{1,\lceil \lg k_1 \rceil}, \ldots, K_{m,1}, \ldots, K_{m,\lceil \lg k_m \rceil}, G\}$$

We define the following terms to describe the intuition of the vocabulary τ'.

- An S-type over τ' is a maximally consistent set of unary predicates from $\{S_1, \ldots, S_n\}$ and their negation.
- A T-type over τ' is a maximally consistent set of unary predicates from $\{T_1, \ldots, T_n\}$ and their negation.
- An E-type over τ' is a maximally consistent set of unary predicates from $\{\beta_1^{\triangleright}, \ldots, \beta_m^{\triangleright}, \beta_1^{\triangleleft}, \ldots, \beta_m^{\triangleleft}\}$ and their negation.
- A K-type over τ' is a maximally consistent set of unary predicates from $\{K_{1,1}, \ldots, K_{1,\lceil \lg k_1 \rceil}, \ldots, K_{m,1}, \ldots, K_{m,\lceil \lg k_m \rceil}\}$ and their negation.

We say that a 1-type π over τ' realizes the S/T/E/K-type t if and only if $t \subseteq \pi$. Clearly, for a 1-type π, it realized a *unique* S/T/E/K-type. For the sake of simplicity, for every $i \in m$, we would view $\{K_{i,1}, \ldots, K_{i,\lceil \log(k_i) \rceil}\}$ as the binary encoding of an integer $0 \leq K_i < 2^{\lceil \log(k_i) \rceil}$ and denote a K-type by an m-dimension (row) vector.

The intuition of the S-type and the T-type is the 1-type of the edge's source and target element, respectively. For every $i \in n$, the unary predicate S_i and T_i are for the original unary predicate U_i. The intuition of the E-type is the 2-type of the edge. For every $i \in m$, the unary predicate $\beta_i^{\triangleright}$ and $\beta_i^{\triangleleft}$ are for the original binary predicate $\beta_i(x, y)$ and $\beta_i(y, x)$,

doi:10.6342/NTU202203098

respectively. Finally, the K-type deal with the counting condition of the original sentence. The intuition of the K-type is the number of binary predicates already realized forwardly. We will describe more on the K-type later.

Edge representation. The edge representation of the pseudo-structure $\mathcal{G}$ over the vocabulary τ, denoted by $\mathcal{H}$, is a pseudo-pseudo over the vocabulary τ'. We construct $\mathcal{H}$ with the following rules. We fix an order of elements in V, say $v_1, v_2, \ldots$. The universe of $\mathcal{H}$ is the set of all non-trivial edges in $\mathcal{G}$, that is, $U := \left\{(v_1, v_2) \in V \times V \;\middle|\; v_1 \neq v_2 \text{ and } \mathsf{lab}^{\mathcal{G},\triangleright}[v_1, v_2] \neq \vec{0}\right\}$. Let (v_i, v_j) be an element in U. The S/T/E/K-type of (v_i, v_j) are defined as follows.

- The S-type S is derived from the 1-type of the source element v_i, that is, $S := \mathsf{lab}^{\mathcal{G}}[v_i][S_t/U_t]_{t=1}^{n}$. Recall that the notation $\phi[b/a]$ denotes that substituting all a occurred in ϕ with b. The symbol $\phi[b_t/a_t]_{t=\ell_1}^{\ell_2}$ is the abbreviation of multiple substitution.
- The T-type T is derived from the 1-type of the target element v_j, that is, $T := \mathsf{lab}^{\mathcal{G}}[v_j][T_t/U_t]_{t=1}^{n}$.
- The E-type E is derived from the 2-type of (v_i, v_j), that is, $E := \mathsf{lab}^{\mathcal{G}}[v_i, v_j][\beta_t^{\triangleright}/\beta_t(x, y)]_{t=1}^{m}[\beta_t^{\triangleleft}/\beta_t(y, x)]_{t=1}^{m}$.
- The K-type K is the number of binary predicates which already realized, that is, $K := \sum_{t\in[j]} \mathsf{lab}^{\mathcal{G},\triangleright}[v_i, v_t]$.

Then, the 1-type of (v_i, v_j) is defined as $\mathsf{lab}^{\mathcal{H}}[(v_i, v_j)] := S \cup T \cup E \cup K \cup \{G\}$. Since there is no binary predicate in τ', the 2-type of edge in $\mathcal{H}$ is trivial.

As mentioned above, we would interpret the element in $\mathcal{H}$ as a non-trivial edge in $\mathcal{G}$. The S-type and the T-type encode the 1-type of the source and target element of the edge, respectively. The E-type E represents the 2-type of the edge. For every element $v \in V$, there is a set of elements in $\mathcal{H}$ that encode the non-trivial outgoing edges from v. For every element (v, v') in the set, it has a unique K-type. Moreover, the value of its K-type encode the number of binary predicates already realized. We can view the set as a chain structure by order of element's K-type. Clearly, if $\mathcal{G}$ is a pseudo-model of φ, then the K-type of the last element in the chain should be $\vec{k}$, that is, the counting conditions of the original sentence φ.

doi:10.6342/NTU202203098

Reduction. To construct the FO^2 sentence φ', we first introduce the following auxiliary formulae.

$$
\begin{aligned}
\mathsf{PAIR}_{\mathsf{ST}}(x,y) &:= \bigwedge_{i\in[n]} (S_i(x) \leftrightarrow T_i(y)) \;\wedge\; (T_i(x) \leftrightarrow S_i(y)) \\
\mathsf{PAIR}_{\mathsf{E}}(x,y) &:= \bigwedge_{i\in[m]} (\beta_i^{\triangleright}(x) \leftrightarrow \beta_i^{\triangleleft}(y)) \;\wedge\; (\beta_i^{\triangleleft}(x) \leftrightarrow \beta_i^{\triangleright}(y)) \\
\mathsf{PREV}(x,y) &:= \bigwedge_{i\in[n]} (S_i(x) \leftrightarrow S_i(y)) \;\wedge \\
&\qquad \bigwedge_{i\in[m]} (\beta_i^{\triangleright}(x) \rightarrow K_i(x) = K_i(y)+1) \wedge (\neg\beta_i^{\triangleright}(x) \rightarrow K_i(x) = K_i(y)) \\
\mathsf{ROOT}(x) &:= \bigwedge_{i\in[m]} (\beta_i^{\triangleright}(x) \rightarrow K_i(x) = 1) \wedge (\neg\beta_i^{\triangleright}(x) \rightarrow K_i(x) = 0) \\
\mathsf{LEAF}(x) &:= \bigwedge_{i\in[m]} K_i(x) = k_i
\end{aligned}
$$

Then we define following sentences. Note that the symbol $\phi[b/a]$ denotes that substituting all a occurred in ϕ with b. The symbol $\phi[b_t/a_t]_{t=\ell_1}^{\ell_2}$ is the abbreviation of multiple substitution.

- $\varphi'_1 := \forall x\, \gamma(x)[S_t(x)/U_t(x)]_{t=1}^{n}$
- $\varphi'_2 := \forall x\, \gamma(x)[T_t(x)/U_t(x)]_{t=1}^{n}$
- $\varphi'_3 := \forall x\, \alpha(x,y)[S_t(x)/U_t(x)]_{t=1}^{n}[T_t(x)/U_t(y)]_{t=1}^{n}[\beta_t^{\triangleright}(x)/\beta_t(x,y)]_{t=1}^{m}[\beta_t^{\triangleleft}(x)/\beta_t(y,x)]_{t=1}^{m}$
- $\varphi'_4 := \forall x\, \forall y\, \alpha(x,y)[S_t(x)/U_t(x)]_{t=1}^{n}[S_t(y)/U_t(y)]_{t=1}^{n}[\bot/\beta_t(x,y)]_{t=1}^{m}[\bot/\beta_t(y,x)]_{t=1}^{m}$
- $\varphi'_5 := \forall x \bigvee_{i\in[m]} \beta_i^{\triangleright}(x)$
- $\varphi'_6 := \forall x\, \exists y\, \mathsf{PAIR}_{\mathsf{ST}}(x,y) \wedge \left(\left(\bigvee_{i\in[m]} \beta_i^{\triangleleft}(x)\right) \rightarrow \mathsf{PAIR}_{\mathsf{E}}(x,y)\right)$
- $\varphi'_7 := \forall x\, \mathsf{ROOT}(x) \vee \exists y\, \mathsf{PREV}(x,y)$
- $\varphi'_8 := \forall x\, G(x) \rightarrow \mathsf{LEAF}(x) \vee (\exists y\, \mathsf{PREV}(y,x) \wedge G(y))$
- $\varphi'_9 := \forall x\, G(x)$

doi:10.6342/NTU202203098

Finally, $\varphi' := \bigwedge_{i \in [9]} \varphi'_i$.

We describe the intuition of each sentence here. The sentences φ'_1, φ'_2, and φ'_3 encode the constraints for the S-type, T-type, and E-type, that is, the constraint for the valid 1-type and 2-type in the original sentence φ. The sentence φ'_4 is the condition for non-trivial edges, that is, the E-type should realize at least one binary predicate forwardly. The sentence φ'_5 is the condition for edge pairing. The sentence φ'_6 encodes the necessary condition of the strong pseudo-model, that is, all realizable 1-types are null-compatible. The sentences φ'_7, φ'_8, and φ'_9 describe the partial order structure by the relation PREV and each path from the ROOT element to the LEAF element in this structure represents a valid configuration in $\mathcal{G}$. We will show this property later.

Correctness of the reduction. The correctness of the reduction follows the intuitions described before. We prove two directions by constructing the corresponding pseudo-models directly.

Lemma 8.1. *If the* C^2 *sentence* φ *is strongly satisfiable, then the* FO^2 *sentence* φ' *is satisfiable.*

Proof. Let $\mathcal{G}$ be a strong pseudo-model of φ. We show that the edge representation of $\mathcal{G}$, denoted by $\mathcal{H}$, is a pseudo-model of φ'.

Because of the construction rules of $\mathcal{H}$, for every element $u \in U$, u's S-type, T-type, or E-type are derived from some valid 1-type or 2-type realized in $\mathcal{G}$. Therefore, the sentences φ'_1, φ'_2, and φ'_3, which derived from the conditions of valid 1-types and 2-types of the original sentence φ, are also satisfied by u. Moreover, since $\mathcal{G}$ is a strong pseudo-model, for every 1-types π_1 and π_2 realized in $\mathcal{G}$, π_1 and π_2 are null-compatible. Thus, the sentence φ'_4 is satisfied. Because the universe of $\mathcal{H}$ is the set of all non-trivial edges in $\mathcal{G}$, the original 2-types are non-trivial and so for the E-types. Therefore, the sentence φ'_5 is satisfied.

For every element $(v_1, v_2) \in U$, if $\mathsf{lab}^{\mathcal{G},\lhd}[v_1, v_2] \neq \vec{0}$, then (v_2, v_1) is a non-trivial edges, and this implies that $(v_2, v_1) \in U$. Therefore, the assignment $x \mapsto (v_1, v_2)$ and $y \mapsto (v_2, v_1)$ is a valid assignment for the sentence φ'_6. Otherwise, if $\mathsf{lab}^{\mathcal{G},\lhd}[v_1, v_2] = \vec{0}$,

doi:10.6342/NTU202203098

we choose another element $v_3 \in V$ such that (v_2, v_3) is a non-trivial edge. Thus, (v_2, v_3) is an element in $\mathcal{H}$, and $x \mapsto (v_1, v_2)$ and $y \mapsto (v_2, v_3)$ is a valid assignment for the sentence φ'_6.

Finally, constructing the K-type guarantees the partial order structure in $\mathcal{H}$. For every element $v \in V$, it is represented by a set of outgoing non-trivial edges in $\mathcal{H}$. We assign the K-type of each element in the set satisfying the formulae PREV. Because for every element $u \in U$, $G \in \mathsf{lab}^{\mathcal{H}}[u]$ by the construction of $\mathcal{H}$, these implies that $\mathcal{H}$ satisfies the sentences φ'_7, φ'_8, and φ'_9. □

Lemma 8.2. *If the* FO^2 *sentence* φ' *is satisfiable, then the* C^2 *sentence* φ *is strongly satisfiable.*

Proof. Let $\mathcal{H}$ be a pseudo-model of φ'. For every element $u \in U$, by the sentence φ'_6, either $\mathsf{ROOT}(u)$ or there is another element u' such that $\mathsf{PREV}(u', u)$. We can link the elements in $\mathcal{G}$ with one of its PREV and view $\mathcal{G}$ as a partial order structure, though there might be multiple y satisfying it. By φ'_7, φ'_8, and φ'_9, the elements in the forest are linked with the relation PREV, and the root element satisfies ROOT. Also, the collection of edges of a maximum path in $\mathcal{H}$ from a valid configuration of φ.

We construct a pseudo-structure $\mathcal{G}$ over the vocabulary τ with the following two steps. First, we construct a labeled graph $\mathcal{G}_0$ which consists of parallel edges.

1. For every maximal chain $P := u_1 \to u_2 \to \cdots \to u_\ell$ in the partial order structure. Observe that the S-type of these elements are the same, say S. We add an element v_P in $\mathcal{G}_0$ and assign S to its 1-type.

2. For every maximal chain $P' := u_1 \to u_2 \to \cdots \to u_\ell$ in the partial order structure, for every element u in the chain, by the sentence φ'_6, it has a pairing element, say u'. Let P' be a maximal chain that contains u', and $v_{P'}$ be the corresponding element of P' in $\mathcal{G}_0$. We add an edge between v_P and $v_{P'}$ and set it 2-type to u's E-type.

Then, we resolve these edges in $\mathcal{G}_0$ and obtain the desired pseudo-structure $\mathcal{G}$. The universe of $\mathcal{G}$ consists of infinitely many copies of $\mathcal{G}_0$.

1. For every element in $\mathcal{G}$, its 1-type is the 1-type in $\mathcal{G}_0$.

doi:10.6342/NTU202203098

2. For every pair of elements v_1 and v_2, if there are parallel edges between them, then let v_2' be an element satisfying the following.

    - The 1-type of v_2 and v_2' are the same.
    - There is no edge between (v_1, v_2').

    We can resolve the parallel edges between v_1 and v_2 by replacing one edge between (v_1, v_2) with (v_1, v_2').

3. Finally, we set the null type to the 2-type of all remaining edges.

Here, we show that $\mathcal{G}$ is a strong pseudo-model of φ. For every element v_1 in $\mathcal{G}$, the 1-type of v_1 is the S-type of some element in $\mathcal{H}$. By the sentence φ_1', the 1-type (S-type) satisfies γ. For every edge (v_1, v_2) in $\mathcal{G}$, if the 2-type of (v_1, v_2) is the null type, then by the sentence φ_6', (v_1, v_2) satisfies α. Otherwise, the 2-type of (v_1, v_2) is the E-type of some element in $\mathcal{H}$. By the sentence φ_3', the 2-type (E-type) together with its source's 1-type (S-type) and target's 1-type (T-type) satisfy α. Because the edges of the maximum path in $\mathcal{H}$ form a valid configuration in φ and we assign the non-trivial edges of each element in $\mathcal{G}$ by it during the construction, each element satisfies the counting condition. Hence, $\mathcal{G}$ is the desired strong pseudo-model. $\square$

Theorem 8.3. *The* C^2 *sentence* φ *is strongly satisfiable, if and only if the* FO^2 *sentence* φ' *is satisfiable.*

It is clear that the reduction is deterministic. The size of the vocabulary τ' is polynomial in m, n, and $\lceil \lg k_i \rceil$, that is, the length of the original C^2 sentence φ. Hence, the length of the FO^2 sentence φ' is polynomial in the length of φ.

Remark 8.4. *Notice that the equality symbol in the formulae* PREV*,* ROOT*, and* LEAF *is the abbreviation of the integer arithmetic. Therefore, they should be expanded as some boolean formulae. Therefore, there is no equality in the* FO^2 *sentence* φ'*. Besides, there is no binary predicate in the vocabulary* τ'*.*

doi:10.6342/NTU202203098

8.2 Reduction from $\mathsf{SAT}(\mathsf{GC}^2)$ to $\mathsf{SAT}(\mathsf{GF}^2)$

If φ is a GC^2 sentence, then by Lemma 7.1, every valid 1-types in φ are null-compatible with each other. Therefore, the sentence φ'_4 is unnecessary. Let $\varphi' := \bigwedge_{i=1}^{3} \varphi'_i \wedge \bigwedge_{i=5}^{9} \varphi'_i$. It is not difficult to show that φ is satisfiable if and only if φ'' is satisfiable.

Observe that the sentences φ'' are in the form $\forall^* \wedge \forall\exists$. It is possible to introduce some new binary predicates and rewrite it into guarded form. Hence, it is in the normal form of GF^2. Since the semantics of each sentence are the same, the correctness of the reduction can be proved by the similar arguments.

doi:10.6342/NTU202203098

Chapter 9

Concluding remarks

In this thesis, we revisit C^2. We introduce and prove the *configuration exponential bound* (CEB) property for C^2. This property can be viewed as the extension of the ESM property for FO^2. We present alternative but more transparent algorithms for the satisfiability and finite satisfiability problems of C^2. Then, we extract a semantic subclass of C^2 that we call the strongly satisfiable fragment, which may yield a better and more efficient implementation.

Our approach for the strong satisfiability and finite strong satisfiability problems of C^2 also works for the satisfiability and finite satisfiability problems for GC^2 and yields $\mathsf{EXPTIME}$ algorithms. Finally, we present deterministic polynomial-time reductions from the strong satisfiability problem of C^2 to the satisfiability problem of FO^2 and from the satisfiability problem of GC^2 to the satisfiability problem of GF^2.

There is still much to do in this field. The following are some possible future works.

- The algorithms for C^2 presented in this thesis are still by non-deterministic reducing to the ILP problem. Therefore, it is still an open research problem to obtain explicit and implementable algorithms for C^2.

- In this thesis, we present a semantic subclass of C^2 that we call the strongly satisfiable fragment. We also show that the class strong satisfiability of C^2 lies in between the satisfiability of GC^2 and C^2. Since both GC^2 and C^2 capture various description logics, it is interesting to figure out the relation between the class strong satisfiability

doi:10.6342/NTU202203098

of C^2 and description logics.

- There are only a few known reductions between NEXPTIME-complete problems. In this thesis, we present a deterministic polynomial-time reduction from the strong satisfiability problem of C^2 to the satisfiability problem of FO^2. We note that it may be possible to extended our techniques for the reduction from the satisfiability problem of C^2 to the satisfiability problem of FO^2.

- There are some quantifiers which has stronger expressive power than counting quantifier. Recently, Bednarczyk, Orłowska, Pacanowska, and Tan [3] considered Presburger quantifiers, which can express local numerical properties. They showed that FO^2 extended with Presburger quantifiers is undecidable, but GF^2 extended with the Presburger quantifier is in 3-NEXPTIME. We note that it may be possible to extended our techniques for the satisfiability problem of GC^2 to obtain a better upper bound for GF^2 extended with Presburger quantifiers.

doi:10.6342/NTU202203098

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