Percent knowledge: Effective teaching for learning, relearning and unlearning

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KEY WORDS

Percent knowledge, percent instruction, percent problem solving; percent as a proportion; proportional reasoning; multiplicative structure of percent increase; metacognition; metacognitive training; mathematics diagnosis and intervention; diagnostic-prescriptive mathematics instruction; misconceptions; errors and knowledge; attention, remembering, forgetting; proactive inhibition.

ABSTRACT

This thesis is concerned with percent knowledge and instruction. It explores the relationship between instruction, learning and unlearning in actual classrooms for the purpose of developing instruction to facilitate Year 8 students' access to percent knowledge for solving common percent problems. Research conducted in this study occurred in response to suggestions in the literature that percent is a difficult topic to teach and learn; that no best method for percent instruction has been developed; and, of a more disturbing nature, that many students in senior high school, can not perform two-step percent problems.

In this study, a series of teaching experiments was conducted. A teaching program was implemented, consisting of a proportional method for percent problem solving, and metacognitive training. Implementation of the teaching program was guided by a model of diagnostic-prescriptive instruction which states that prior knowledge must be taken into account in any teaching sequence, and errors, misconceptions and naive conceptions must be dealt with to promote forward development of knowledge. The influence of instruction was monitored through analysis of pre-, post- and delayed posttest results; researcher-generated field notes; observations; students' diaries and artefacts; ad hoc interviews with students and observers. Results of the study indicated that the teaching program developed was extremely effective in promoting students' percent problem solving proficiency; that the metacognitive training component of the program appeared to enhance the development of students' principled-conceptual percent knowledge; and that application of "unteaching" strategies were more effective than good "reteaching strategies" in overcoming inappropriate prior knowledge. This study gave rise to the development of a model of percent instruction, a model of percent knowledge, and a model of diagnostic-prescriptive mathematics teaching.

The teaching experiments in this study were conducted in actual classrooms and therefore in authentic school environments. The students who participated in the study were from intact classes, and the teaching program was implemented during students' normally timetabled mathematics classes. The teaching program spanned the typical allocated time for instruction (2 weeks approximately) in the topic of percent with Year 8 students. Within these constraints, the teaching program presented to students in this study resulted in students operating proficiently on all three types of percent problems, including those involving increase and decrease. Trialling of the teaching program in this naturalistic manner underscored the viability, transplantability, and relevance of the teaching program to the mathematics classroom.

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Statement of original authorship

"The work contained in this thesis has not been previously submitted for a degree or diploma at any other higher education institution. To the best of my knowledge and belief, the thesis contains no material previously published or written by another person except where due reference is made.

Dated:....16.4.99..............................."

CHAPTER 1
INTRODUCTION

CHAPTER OVERVIEW

In this introductory chapter, an overview of the study is presented. This chapter includes a summary of the issues pertaining to the topic under investigation, the context of the study, and also the aims, significance and design of the research carried out. There are four sections in this chapter. In sections 1.1 and 1.2, the background and the context are described. In section 1.3, the aims, significance and design are presented. This chapter concludes (section 1.4) with a summary of the structure of the chapters within the report.

1.1 Background

1.1.1 Overview

This section of the introductory chapter provides a background to the study, and overviews issues in teaching and learning the mathematical topic of percent. The background begins with a description, in section 1.1.2, of the historical roots of percent, delineating the social necessity of percent, and the relative complexity of percent used in society today. This description also suggests that, historically, percent was regarded as a proportion. The proportional nature of percent is addressed in section 1.1.3. In section 1.1.4, the difficulty of defining percent knowing is discussed, and definitions of mathematical knowing are presented. Research on students' understanding of percent is summarised in section 1.1.5. In section 1.1.6, the learning of mathematics, and issues in mathematics instruction are overviewed. A summary of key points in this section of the chapter is presented in section 1.1.7.

1.1.2 The nature of percent

The application of percent in the real world cannot be denied. Percent discounts, profits, losses, savings, increases, are an integral part of our society, as attested to on billboards, in newspapers, in advertisements, in shops. There can be no question as to the social necessity of a knowledge and understanding of percent and therefore its place in the mathematics curriculum. However, percent is often misused or misunderstood when applied in the real world (Watson, 1994), and is a difficult topic to learn and teach (Cole & Weissenfluh, 1974; Parker & Leinhardt, 1995; Smart, 1980; Wendt, 1959).

The notion of percent has its roots in the marketplace; the application of percent-like concepts for interest and tax calculations can be traced to 300BC India (Parker & Leinhardt, 1995). According to Parker and Leinhardt's (1995) historical review, percent was recognised as a statement of simple proportion with a comparative base of 100 in 200BC China, and *The Rule of Three* (a computational procedure for percent calculations - see section 1.1.3 below), was utilised. Since that time, percent calculations have been carried out for various commercial purposes, for example, to determine interest, tax, profit, currency exchange. Procedures employed for such calculations have included the Rule of Three and similar derivations (The Rule of Five), with common (to match base 12) and decimal fraction calculations utilised with the advent of base-12 imperial and base-10 decimal currency respectively (Parker & Leinhardt, 1995). Symbolic recording of percent has evolved to a concise form. According to Parker and Leinhardt (1995), the definitionof percent, meaning *for every hundred*, was shortened to the word *perceto* in India in 1481, followed by the appearance of various symbols for percent in Italy around 1650. The percent symbol used today (%) replaced the words *per cent*, as meaning *for every hundred*, with, for example, expressions such as £6 per cent written as 6%.

According to Parker and Leinhardt (1995), percent is an elusive, concise concept with multiple meanings, as it can be all of the following: (a) a *number*, in that a percent can be written in an equivalent fraction or decimal form; (b) a *comparison* in the part-whole fraction sense (e.g., if a candidate receives 35% of the votes, this percent is the subset of people who voted for this candidate compared to the total number of votes cast); (c) a *ratio* comparison, where the comparison is between two distinct sets (e.g., there are 400% more boys than girls); (d) a *statistic* when data is reduced to manageable form for interpretation (e.g., a state's employment rate of 8.5% is compared to the national average of 10%); and (e) a *function* when amounts are calculated according to a stated percent (e.g., interest rates, discounts, etc). The link between these many dimensions of percent, according to Parker and Leinhardt, is that of proportionality. As they stated:

> The common thread woven through all these descriptions is that percent is an alternative language used to describe proportional relationships - a language that is unique, concise and provides a privileged notation system. (p. 444)

1.1.3 Percent proportionality and the Rule of Three

As previously stated, percent as a proportion comparison with a base of 100 can be traced to the year 200BC. The Rule of Three was utilised as a computational procedure embodying the notion of percent as a statement of simple proportion. Parker & Leinhardt (1995) described the Rule of Three as a procedural method for solving proportion equations, where three numbers in the proportion equation are given and the object is to find the fourth. In India, in AD499, the Rule of Three was described in the following way: "Multiply the fruit by the desire and divide by the measure. The result is the fruit of the desire", and in AD628: "Requisition multiplied by Fruit and divided by Argument, is the Produce" (Parker & Leinhardt, 1995, p. 430).

The Rule of Three can be seen to draw on principles of proportion which still hold today. Touriniaire and Pulos (1985) stated that, "For mathematicians, a proportion is a statement of equality of two ratios, i.e. ${}^{a}/_{b} = {}^{c}/_{d}$" (p. 181). And, Post, Behr and Lesh (1988) stated that, "The standard algorithm for proportionality [is] ${}^{a}/_{b} = {}^{c}/_{x}$, where a, b, c are given and [we] need to find x" (p. 81). The Rule of Three procedure can be seen to follow the cross-multiply method for solving proportion equations defined by Robinson (1981) as the following, "If you multiply the two numbers across from one another and divide by the other number, the correct answer is obtained" (p. 6). The Rule of Three, therefore is the typical proportion equation familiar to mathematicians today.

The Rule of Three procedure encapsulates the meaning of percent as a proportion. However, as Parker and Leinhardt (1995) suggested, the true meaning of percent has become "entangled in the mesh of conversion rules for changing decimals to fractions, fractions to decimals, improper fractions to mixed numbers, and mixed numbers to improper fractions" (p. 434), where the emphasis is on fast, efficient calculation skills rather than meaning. Thus, percent knowledge must comprise more than successful performance of percent calculations. Of percent knowledge, Parker and Leinhardt (1993) stated:

> The black-and-white picture of what students should know about percent that is seen in texts and tests differs from the full-colour version which reflects real appearances of percent in the everyday world. More than conversions, computations, and applications, knowing percent both in school and out means understanding its multiple and often embedded meanings and its relational character. (p. 47)

Parker and Leinhardt (1995) also stated that, "Percent is fundamentally a language of privileged proportion which simplifies and condenses descriptions of multiplicative comparisons" (p. 472).

From Parker and Leinhardt's analysis of percent, it appears that, due to its multiplicity of meaning, there is a need to clearly enunciate the types of knowledge which constitute percent knowledge, and thus serve to inform instruction.

1.1.4 Knowing percent

In the past, basic mathematical knowledge was viewed as a student's proficiency in arithmetical calculation (Putnam, Lampert, & Peterson, 1990), and thus traditionally, the teaching of computational skills was dominant in mathematics instruction (Lampert, 1986). However, computational proficiency is not necessarily a guarantee of mathematical understanding (Leinhardt, 1988).

Resnick (1982) defined mathematical knowledge as comprising *semantic* and *syntactic* knowledge. According to Resnick, syntactic knowledge is correct performance of mathematical procedures (computational proficiency) and semantic knowledge is the understanding of the meaning of those procedures. Similarly, Skemp (1978) described mathematical knowledge as *instrumental*, the knowledge of computational procedures, and *relational*, the understanding of why those procedures work. Anderson (1985) described two components of mathematical knowledge in a similar fashion. *Declarative* mathematical knowledge, according to Anderson, is largely automated algorithmic/computational knowledge, and *procedural* mathematical knowledge is the understanding of those computational procedures which can be applied with meaning to new computational procedures. Taking account of the importance of mathematical knowledge gained prior to formal school instruction, Ginsburg (1977) described mathematical knowledge as *intuitive* and *formal*. According to Ginsburg, *intuitive* mathematical knowledge is constructed by children through problem solving in their own environment, while *formal* knowledge is the result of school instruction, and is seen to be often unconnected to intuitive, logical mathematical structures.

A further definition of mathematical knowledge also acknowledges intuitive mathematical knowledge and suggests that mathematical knowledge is comprised of the categories of *intuitive*, *concrete*, *computational* and *principled/conceptual* knowledge (Lampert, 1986; Leinhardt, 1988). From this perspective, *intuitive* mathematical knowledge is "everyday" real world application knowledge which is normally acquired before formal instruction, *concrete* knowledge is knowledge associated with representation by appropriate concrete materials during instruction, *computational* knowledge is knowledge of and the ability to apply numerical procedures for computation, and *principled/conceptual* knowledge is underlying knowledge of the computational procedures that constrains or justifies those procedures (Leinhardt, 1988).

From the "knowledge types" which constitute mathematical knowledge, it can be seen that, computational knowledge is an essential component of mathematical knowledge, but is not the only component. As suggested by Putnam, Lampert and Peterson (1990), mathematical skill/computational knowledge can develop in relative isolation to conceptual knowledge, but to "know mathematics" is where all mathematical knowledge is linked; where the knower has developed various internalised representations of related mathematical ideas, and easily moves between each representation. It appears that, together, these knowledge types will enable successful performance in problem situations. However, it is generally accepted that performance in mathematical problem situations is influenced by metacognition

(Flavell, 1976; Garofalo & Lester, 1985; Silver, 1982). Therefore, for mathematical knowledge to be accessed, metacognitive knowledge must also be promoted.

Garofalo and Lester (1985) suggested that mathematical knowledge is influenced by three metacognitive categories as *person*, *task* and *strategy* knowledge. Garofalo and Lester defined *person metacognitive knowledge* as "one's assessment of one's own capabilities and limitations with respect to mathematics in general, and also with particular topics or tasks" (p. 167). This is the affective domain, and includes such affective variables as motivation, anxiety and perseverance. According to Garofalo and Lester, *task knowledge* is one's beliefs about the nature of the mathematical tasks, and *strategy knowledge* is awareness of strategies for guiding problem solving. Similarly, Prawatt (1989) suggested that access to knowledge is determined by a learner's organisation (structure) and awareness (metacognition) of three factors: *knowledge base* (concepts, principles, rules, facts and procedures); *strategic and metastrategic thinking* (general problem solving heuristics and executive processes, such as planning, monitoring, checking, revising); and *disposition* (habits of mind).

From the above discussion, it appears that, for students to successfully operate in the domain of percent, they need percent knowledge and metacognitive knowledge. In terms of percent knowledge, knowledge of, and skill in, applying computational procedures for percent calculations would be one element of percent knowledge. Percent knowledge would also consist of conceptual understanding of the meaning of percent in its many dimensions together with knowledge of the principles which legitimise percent calculations, as well as metacognitive knowledge to enhance access to such percent knowledge.

1.1.5 Instructional approaches for teaching percent

The literature provides many and varied suggestions for instruction in percent which appear to focus on: (i) developing the concept of percent, and/or (ii) methods for solving percent application problems. For developing the concept of percent a common approach is through linking percentages to fractions and decimals (e.g., Brueckner & Grossnickle, 1953; Hauck, 1954), and building students' mental visualisation of percent through representing percents on 10x10 grids (Bennett & Nelson, 1994; Reys, Suydam & Lindquist, 1992). It has also been suggested that the concept of percent should be developed through exploration of the special language of percent and building students' estimation skills through exploration of patterns of simple percent calculations (e.g., Cooper & Irons, 1987; Glatzer, 1984). The concept of percent can also be promoted through linking percent to ratio (e.g., Brown & Kinney, 1973) and through studying percent expressions as statements of proportion (e.g., Schmalz, 1977).

As instructional approaches for developing the concept of percent are varied, so too are instructional approaches for solving percent application problems. Percent application problems are of three types. Ashlock, Johnson, Wilson and Jones (1983) described the three percent types as the following:

(i) finding a part or percent of a number (e.g., 25% of 20 is Δ);

(ii) finding a part or percent one number is of another (e.g., Δ% of 15 is 5); and

(iii) finding a number when a certain part or percent of that number is known (e.g., 20% of Δ is 6). (p. 297).

In general, percent application problems give two of the unknown elements in the percent statement, and require students to find the third. (Throughout this report, the three types of percent application problems will be referred to as Type I, Type II and Type III percent problems respectively.)

The literature offers various suggestions for assisting students in solving percent application problems, ranging from the use of concrete materials and representations (Bennett & Nelson, 1994; Cooper & Irons, 1987; Dewar, 1984; Haubner, 1992; Weibe, 1986), to the use of mnemonic strategies (e.g., Boling, 1985; McGivney & Nitschke, 1988; Teahan, 1979). The representations and mnemonic strategies can be seen to advocate particular procedures for solving percent problems, incorporating a variety of multiplication, division, decimal, fraction, and proportion procedures. The various procedures appear to be based on the teaching principle that the teacher's role in mathematics instruction is to provide experiences and opportunities for children to develop mathematical concepts and thus provide a meaningful basis for the application of mathematical skills (Reys, Suydam & Lindquist, 1992).

In their analysis of the literature on percent, Parker and Leinhardt (1995) stated that, by 1960, there were five distinct computational procedures for solving percent equations taught in schools. The five procedures can be summarised as follows:

1. *Traditional/cases*: students classify the problem and apply a different procedure for each problem type (Type I - multiply the number by the percent as a decimal; Type II - divide the numbers and translate the decimal answer to a percent; Type III - divide the number by the percent as a decimal);
2. *Percent formula*: "knowns" are substituted in the formula, P=BR (P is percent as a number, B is base number and R is percent as a rate) and the "unknown" is found by algebraic manipulation;
3. *Equation*: "knowns" are categorised as factors or product and substituted in the formula: factor x factor = product. Algebraic manipulation is used to find the unknown;
4. *Proportion*: percent is considered as a common fraction with a denominator of 100 and is equated to a fraction made up from the two other possible numbers

(i.e., $^{a}/_{100} = ^{c}/_{d}$); the unknown is found either by algebraic manipulation or the cross-multiply method.

5. *Unitary*: 1% of the "known" is calculated and then simple arithmetic computations are performed to calculate the required percent (e.g., 11% of 200 is thought of as the product of 1% of 200 and 11).

According to Smart (1980), the selection of a particular computational procedure studied by students in schools is dependent upon the teacher's personal bias towards that particular method. It appears then, that the instruction students receive on percent during their years at school can be as varied as the approaches for developing the concept of percent as well as instructional methods for performing percent calculations.

1.1.6 Error patterns, misconceptions, remediation and constructivism

Many students experience difficulty with the learning of mathematics; the learning of the mathematical topic of percent in particular is no exception (Parker & Leinhardt, 1995). Constructivist theories of learning state that knowledge is actively constructed by learners (Confrey, 1990a), therefore constructivist learning theory builds awareness of the need to provide rich learning environments to assist students construct appropriate mathematical knowledge and achieve mathematical understanding. However, it is well documented that, through the study of mathematics, students develop error patterns and misconceptions, and this influences their academic performance (Ashlock, 1994; Engelhardt, 1977; Wilson, 1976a). The field of diagnosis and remediation is concerned primarily with helping children overcome difficulties they experience in the study of mathematics.

Traditionally, students' mathematical errors and misconceptions were viewed from a negative perspective, taken as indicative of the absence of knowledge/meaning. Constructivist views of errors and misconceptions offer a more positive perspective, suggesting that errors are an individual's current interpretation of a mathematical situation and thus are indicative of the presence of knowledge (Confrey, 1990b). In terms of mathematics diagnosis and remediation, the problem is dealing with knowledge (albeit inappropriate), not a lack of knowledge. A reinterpretation of errors and mathematical misconceptions as knowledge challenges educators to find the means for effectively dealing with erroneous knowledge rather than engaging in a reteaching program on the assumption of an "absence of knowledge" perspective.

Error pattern research has documented students' (and adults') systematic errors and misconceptions in many topics within the mathematics curriculum (e.g., Borassi, 1994; Dubinsky, Dautermann, Leron & Zazkis, 1994; Fong, 1995; Mansfield & Happs, 1992; Rauff, 1994). Such research has made a significant contribution to the field of mathematical research in several ways. First, awareness of common errors/misconceptions has contributed to mathematical pedagogy by heightening awareness of the need for rich learning environments to facilitate construction of appropriate knowledge. Second, research into teachers' errors and misconceptions has informed teacher-education courses of the need to be aware of the influence of teachers' misconceptions upon students' knowledge constructions. Third, acknowledging that erroneous knowledge impedes forward growth of appropriate knowledge has prompted the search for teaching methods that promote sustained and positive conceptual change through removing the blocking action of the erroneous knowledge.

A common theme through programs of intervention has been the provision of experiences to build students' conceptual understanding of particular mathematical topics (e.g., Ashlock, Johnson, Wilson & Jones, 1983; Booker, Irons & Jones, 1980; Wilson, 1976a). Such programs appear to be based on the notion that students' errors and misconceptions can be overcome through meaningful mathematics teaching. Such programs have been met with mixed success, in that there has been a puzzling tendency for students to be able to demonstrate knowledge of mathematics concepts and principles whilst retaining errors/misconceptions (e.g., Connell & Peck, 1993). This phenomenon has led to the development of programs which use reflection on errors/misconceptions as a starting point for intervention (e.g., Ashlock, 1994; Bell, 1986-87; Borassi, 1994; Rauff, 1994). Current trends in intervention methods appear to be methods which focus specifically on errors, bringing them out into the "open" for discussion, analysis, and/or "exposure" to some extent. Error exposure contrasts traditional intervention methods where errors were not overtly acknowledged in remedial programs.

Taking intervention research to a further level is the Conceptual Mediation Program (CMP) (Lyndon, 1995). CMP provides an explanation for the ineffectual nature of "reteaching" programs together with an explanatory theory for the tenacity of errors/misconceptions. The theoretical background of the program states that *proactive inhibition* (PI), an information protection mechanism common to everyone, is responsible for the human tendency to retain naive conceptions, alternative conceptions and error patterns, in light of rational argument. The theoretical basis of CMP states that, when confronted with information which conflicts with knowledge currently held by the learner, PI causes accelerated forgetting of the new, incoming information whilst protecting the old knowledge from change. To overcome errors and misconceptions, CMP states that errors are the starting point for intervention, and that breaking the protective hold of PI over the error (prior knowledge) is the means of effective remediation. CMP offers a strategy for confronting proactive inhibition, thus providing a vehicle for conceptual change, and self-empowerment and control of learning.

1.1.7 Implications

The focus of this thesis is on the development of instruction to promote Year 8 students' percent knowledge and problem solving skills. The key issues pertinent to this study, and which provide a summary of factors guiding the development and implementation of the study, are encapsulated within the following points:

1. An understanding of and the ability to apply percent knowledge is a life skill.
2. Percent is often misapplied in the real world, thus indicating that many adults have a faulty or tenuous knowledge of percent.
3. The meaning of percent as a proportional statement of comparison has become hidden through the development of various quick computational procedures for calculating percentage amounts.
4. The multi-dimensional nature of percent has made percent difficult to define.
5. Definitions of mathematical knowing may contribute to defining percent knowing.
6. Instructional approaches for developing the concept of percent and for assisting percent calculations are many and varied, drawing on decimal/fraction knowledge, ratio knowledge, proportion knowledge, or a combination.
7. Mathematics diagnosis and remediation literature may provide suggestions for helping students access mathematical knowledge in general, and percent knowledge in particular.
8. Analysis of literature on mathematics diagnosis and remediation will provide the conceptual framework for interpreting the contribution of CMP to the study of mathematics.
9. CMP as an integral part of instruction may provide a lens for evaluating teaching and learning in the classroom and thus guide the development of a model of effective teaching.

1.2 Context

1.2.1 Overview

This section of the chapter describes the context of this study. In section 1.2.2, the mathematics syllabus for teaching percent in Queensland schools is described (Queensland is the state of Australia in which the research for this study was conducted). In section 1.2.3, assessment of students' understanding of percent is presented through summaries of research studies conducted in this field. In section 1.2.4, the significance of the study is described, through discussion of exploring percent instruction in the "real world". A summary of key points in this section of the chapter are presented in section 1.2.5.

1.2.2 Queensland syllabus and practice

In Queensland schools, the mathematical curriculum is guided by the Mathematics Syllabus. A state-school teacher in Queensland must follow the syllabus when developing mathematical learning experiences for students.

In Queensland schools, formal study of the mathematical topic of percent begins in Year 6 (Department of Education, Queensland, 1989a). At this stage, children are not expected to perform written computations for solving percent application problems, but to build their knowledge of percent as hundredths and their knowledge of how percent relates to decimals and fractions. Mental calculations for determining percentages of quantities are encouraged through the application of benchmarks of 50% as one half, and 10% as one tenth. In Year 7, the written algorithm for calculating percentages of numbers or quantities is introduced. This algorithm is applied in practical situations, and the use of mental, written and calculator procedures is practised. In Year 8, the study of the mathematical topic of percent includes:

- analysing the relationships between percentage and decimal and common fractions - particularly hundredths;
- representing 100% as the whole and less than 100% as a part of the whole using appropriate verbal, concrete and pictorial forms;
- representing decimal and common fractions as percentages and vice-versa (whole percentages only);
- estimating and calculating using calculators, mental strategies and algorithms in practical situations to find a percentage of a number or quantity;
- estimating and calculating to increase or decrease a quantity by a given percentage; and
- solving and creating problems involving practical applications of percentages including discount and simple interest (Department of Education, Queensland, 1989b).

In Year 9 the study of percent involves consolidation and extension of all work in previous years, in addition to finding a number or quantity or percentage of that quantity; expressing one number or quantity as a percent of another; and simple and compound rates of growth.

Thus the mathematical topic of percent is introduced to children in a spiral manner, where the relationship of decimals, fractions and percents is the beginning point. Concrete/pictorial models are used widely in the beginning study of percent and typically focus on the 10x10 grid. The grid is used to stress the "hundredthsness" of percent, and to simultaneously depict the diagrammatic form of percents, fractions, and decimals. Conversions between the three forms (decimal, fraction, percent) are integral to the study of the mathematical topic of percent, and are used for the calculation of percent application algorithms (Department of Education, Queensland, 1989a). The three types of percent application problems are introduced in a staggered manner, over three years, beginning with Type I problems in Year 7, Type II problems in Year 8, and Type III problems in Year 9.

1.2.3 Assessment of percent knowledge

Parker and Leinhardt (1995) summarised results of research spanning almost 7 decades into students' knowledge and understanding of percent. From their review, various early studies conducted between 1920 to 1950 indicated that students had great difficulty in problem solving involving percent calculations. Research conducted in the 1960's and 1970's focused on comparing various computational approaches for solving percent equations. For example, in 1959, the ratio method was compared to the case method; in 1961, the unitary analysis method was compared to the equation method; in 1965, the case method, the ratio method, and students' self-discovery methods were compared. Results of these comparative studies did not conclusively suggest one method was superior to another. The comparative studies typically tested students' proficiency with percent calculations and percent word problem solving.

Parker and Leinhardt's (1995) analyses of other studies consistently have shown that the majority of students experience difficulty with the study of percent. In the conclusion of their extensive review, Parker and Leinhardt stated percent is a confusing topic in the mathematics curriculum for both students and teachers, and that basically, "percent is hard" (p. 423).

The fourth National Assessment of Educational Performance (NAEP) of mathematics (Kouba, Brown, Carpenter, Lindquist, Silver & Swafford, 1988) provided evidence that students at the seventh grade level had difficulty with percent calculations and appeared to lack understanding of the concepts of percent underlying calculations. It was found that this lack of understanding of percent and the lack of ability to apply percent knowledge in problem solving situations was a trend which continued through to students in the eleventh grade. Percent items on the NAEP test related to students' understanding of the concept of percent, calculations with percent, and problem solving involving percent. Relating to the concept of percent, it was found that, in general, students understood that the sum of the percentage parts must total 100 percent, and that familiar percents, such as 50% and 25% were well related to common fractional equivalents. However, when required to perform calculations with percent, and apply percent knowledge to problem solving situations, students generally performed at a low level. For example, only 32% of Year 7 students and 62% of Year

11 students could calculate 4% of 75, and only 9% of Year 7 students and 37% of Year 11 students could solve a two-step word problem involving simple interest calculations.

A relatively recent study by Lembke and Reys (1994) looked at Years 5, 7, 9 and 11 students' conceptual and computational percent knowledge before and after formal percent instruction. Their study showed a more promising picture of percent knowledge students may possess. Lembke and Reys interviewed high and middle ability students in each of the four year levels and found that students in Years 5 and 7 who had not received formal instruction in percent used a variety of intuitive strategies to solve (simple) percent problems; that older students (Years 9 and 11) utilised a percent formula for calculating percentages, often making simple errors, but that common benchmarks (100% is a whole, 50% is half, and 25% is half of a half of something) were used by students of all year levels, and were used extensively by the students to check the reasonableness of their calculations.

1.2.4 Authentic classroom instruction in percent

Middle to high achievers experience little difficulty with the topic of percent, by utilising a well-developed "sense-of-percent" to interpret and operate on percent situations (Dole, Cooper, Baturo & Canoplia, 1997; Lembke & Reys, 1994). However, as discussed in section 1.2.3, research into students' knowledge and understanding of percent indicates that students have difficulty completing simple percent calculations and applying percent knowledge and procedures to solve problem situations. The most difficult aspects of percent appear to be those involving problem solving and interpreting and calculating any percent greater than 100 (Parker & Leinhardt, 1995). Students' inability to interpret information relating to percent problems has implications for society. The meaning of information presented in percent language is possibly lost on the majority of our citizens. There is a clear need for research into means of improving instruction in the mathematical topic of percent in order to build students' knowledge and understanding of percent.

Literature on teaching and learning percent provides a wide range of suggestions for instruction, but the specific manner in which to proceed in the actual classroom context is still unclear. As Parker and Leinhardt (1995) stated:

> More recent contributions towards improved instruction in percent suggest ways to improve dialogue, to focus students' attention through models and representation, and to ground percent in the real experiences of students. However, thus far there has been a serious lack of empirical research that actually tests and evaluates these claims in real classroom settings. (p. 472)

The classroom world is "rich, complicated and noisy" (Confrey & Harel, 1994, p. xxv). It is acknowledged in the literature that research provides ideas on the types of mathematical knowledge to be promoted through instruction, but ideas for direct translation into the classroom are less certain (Behr, Harel, Post & Lesh, 1992; Confrey & Harel, 1994). Within the classroom context, there is also the need to develop instruction which assists students learn and achieve in school mathematics now, which contrasts suggestions of the nature of instruction students should have received initially. There is also the dilemma facing teachers of choosing between teaching for understanding and teaching for academic achievement. As Cramer, Post and Currier (1992) stated, "There is currently a mismatch between how and what teachers are encouraged to teach and what skills are evaluated. In the real context, there is subtle pressure to promote academic achievement efficiently, rather than promoting true conceptual understanding" (p.173).

1.2.5 Implications

Issues relating to the context of this study, which further impinge upon the development of the research program, are summarised as follows:

1. In the Queensland school situation, percent is taught from a decimal/fraction perspective, with the three types of percent application problems introduced separately in the curriculum over three years.
2. The concept of percent as a proportion is not overtly taught in Queensland schools.
3. Research studies indicate that students' knowledge and understanding of percent is at a low level and does not improve greatly as students progress through secondary school.
4. There is a need to develop efficient instructional strategies for developing percent knowledge which promotes percent interpretation and problem solving skills; instruction which works in real classroom situations.

Analysis of the issues pertaining to teaching and learning percent have generated the following questions:

1. What instructional sequences best promote Year 8 students' percent knowledge holistically to enable them to solve percent application problems and perform percent computations with meaning?
2. What instructional approaches best enable Year 8 students to come to *know* the multidimensional nature of percent?
3. What instructional sequences are most *efficient* (in terms of required preparation, time, resources, cost) and *effective* (in terms of student knowledge growth, and permanence of knowledge growth) for promoting Year 8 student's knowledge of percent?

4. How can the effects of prior erroneous knowledge be catered for through instruction?
5. Can CMP be successfully utilised in the mathematics classroom, and can definitive guidelines be developed for use of CMP within mathematical classroom situations?

1.3 Aims, significance and design

The aims of this study are:

1. To develop a program for effectively teaching percent applications in real classrooms.
2. To draw implications and construct models for percent knowledge, percent instruction, and mathematics teaching in general.

In more detail, effective teaching is that which:

(a) is diagnostic-prescriptive in that it caters for all categories of learners ranging from learners who have experienced limited formal instruction in percent, to learners who have received considerable formal instruction in percent;

(b) is *effective* (in terms of student outcome) in relation to promoting students' (i) intuitive, concrete, computational, and principled-conceptual percent knowledge; (ii) percent application problem solving skill; and (iii) permanence of knowledge over time; and

(c) is *efficient* (in terms of teacher input) of (i) teacher preparation requirements; (ii) time requirements to implement in the real world (i.e. in the school situation); (iii) resources; and (iv) cost.

Specifically, the implications and models will be in terms of:

(a) students' knowledge of percent;

(b) a model for teaching percent problem solving; and

(c) a model for diagnostic-prescriptive teaching of mathematics.

The significance of the study is embedded in the following statements:

1. Percent notions and percent applications are widely used in society, and thus are culturally valued.
2. Understanding of percent notions and calculation of percent problems is an inherently difficult topic for students (and society members).
3. There is lack of clarity of teaching approaches for developing percent knowledge and application skill, and little consensus amongst mathematics educators as to the most effective program for percent teaching.

4. There is a lack of research into approaches to teaching percent where results readily translate into real classroom situations.

The focus of this study is on developing an instructional program applicable to real classroom situations. The study must utilise an appropriate methodology where a teaching program can be developed and trialled in actual classrooms in a way that enables ongoing refinement to the teaching program to be carried out. The teaching experiment (Kantowski, 1978), therefore, is the most suitable research methodology for this study. The teaching experiment design for this study is described in detail in chapter 4 (see section 4.1).

To achieve the aims of the study, the research was conducted in the following sequence:

1. Review of the literature relating to percent, and development of an initial model of percent knowledge to guide instruction, particularly in relation to developing students' percent problem solving skills.
2. Review of the literature relating specifically to diagnostic-prescriptive approaches to mathematics instruction, and development of an initial diagnostic-prescriptive model of instruction for promoting knowledge change and growth, and for dealing with inappropriate prior knowledge.
3. Development, trialling and refinement of an instructional program which is efficient and effective in Year 8 classrooms for meaningful application of problem types through a series of teaching experiments.
4. Generation of hypotheses regarding knowledge learning, relearning and unlearning, through analysis of relationships between instruction and learning inherent in the teaching experiments.
5. Drawing of implications relating percent knowledge and percent teaching.
6. Construction of models for teaching percent problem solving from a proportional perspective, and for teaching mathematics from a diagnostic-prescriptive perspective.

1.4 Organisation of the thesis

The report on the study is structured into 7 chapters. In chapter 1, the background to the problem was presented. Percent is widely used in our society, yet there is evidence to suggest that percent is a concept which is poorly understood by the majority of citizens. Percent is a multi-dimensional topic, and is a difficult topic to learn and teach. There is a need for research to clarify specifically what constitutes percent knowledge, and instructional approaches to best foster students' understanding of percent. The aims of the study were presented in this introductory chapter.

Chapters 2 and 3 are literature chapters. Literature pertinent to this study is from two fields, (i) teaching and learning percent, and (ii) the diagnostic-prescriptive teaching of mathematics. A summary of the literature within these two fields relevant to the study is presented separately in chapters 2 and 3. Within these chapters, a discussion of how these two fields of literature link to the purposes of the study is presented.

In chapter 4, the research design for the study is described. This chapter presents a discussion of various research methodologies for studying teaching and learning, and specifically the influence of instruction upon knowledge change and growth. The subjects, data sources, and the procedures for analysing the data are discussed.

In chapter 5, the results of the study are presented. A sequence of four teaching experiments was conducted in this study. Implementation of instruction and results of each experiment are detailed in this chapter. The description of the four teaching experiments provides a comprehensive view of the development and modification of the instruction in light of implementation in the classroom.

In chapter 6, a discussion of the results is presented. In this chapter, percent knowledge and percent instruction (incorporating metacognition) are analysed in terms of events within the teaching experiments and with respect to the literature.

In chapter 7, conclusions from the study are drawn. The results of this study in relation to the stated aims of the study listed in chapter 1, are discussed. Limitations of the study are outlined, and implications of the results for further research and for instruction are delineated.

The chapter titles of this report are as follows:

Chapter 1 - Introduction

Chapter 2 - Literature: Teaching and learning percent

Chapter 3 - Literature: The diagnostic-prescriptive teaching of mathematics

Chapter 4 - Design

Chapter 5 - Results

Chapter 6 - Discussion

Chapter 7 - Conclusions

CHAPTER SUMMARY

This introductory chapter has presented an overview of issues relating to teaching and learning percent, particularly for Year 8 students. Through this background to the study, it was seen that percent is a difficult topic to teach and learn, a difficulty compounded by the complexity of defining percent itself. In this chapter, the aims and significance of the study were presented, and the design summarised.

Through description of the aims and significance of the study, it was seen that two fields of research impinge upon the study, namely the mathematical topic of percent, and effective teaching to overcome misconceptions and mathematical learning difficulties. In the next two chapters of the report, issues from within these two fields of research are discussed in detail, with issues in teaching and learning percent presented in chapter 2, and issues in mathematics teaching from a diagnostic-prescriptive approach presented in chapter 3. The focus of these two literature chapters is to identify key factors of effective instruction for teaching and learning the mathematical topic of percent.

CHAPTER 2
LITERATURE

Teaching and learning percent

CHAPTER OVERVIEW

This chapter is concerned with issues in percent teaching and learning. This chapter is divided into four sections. Contained within the first two sections, section 2.1 and section 2.2 respectively, summaries of instructional approaches for: (i) promoting the concept of percent in learners, and (ii) assisting students solve percent application problems, are presented. In section 2.3, other related literature, which provides further suggestions for helping students develop percent knowledge, is discussed. In section 2.4, a model of percent knowledge for guiding instruction is presented.

2.1 Percent concept development

2.1.1 Overview

In this section, instructional approaches for percent concept development are summarised. In section 2.1.2, models, strategies and the language of percent, are described, and are seen to link to fraction/decimal concepts. In section 2.1.3, percent as ratio and the promotion of the ratio concept through percent are described. In section 2.1.4, percent and proportion, and the development of the concept of percent through linking to proportional understanding are described. A summary of the key points contained within this section is presented in section 2.1.5.

2.1.2 Models, strategies, and the language of percent

It is widely suggested that the concept of percent be developed through the use of representations to assist children visualise percent; the most common suggested representation is a 10x10 grid (Bennett & Nelson, 1994; Breuckner & Grossnickle, 1953; Reys, Suydam & Lindquist, 1992). Reys, Suydam & Lindquist (1992) suggested that, as percent is "derived from the Latin work *per centum* which means 'out of every hundred' or 'for every hundred'" (p. 228), percent experiences for children should be those which concentrate on a base of 100, hence the value of a 10x10 pictorial grid as an easily recognisable, visual representation of one whole divided into 100 parts. The key instructional point is to stress that each square within the grid is 1%, and the whole is 100%. Bennett and Nelson (1994) stated that the use of 10x10 grids assists students visualise not only percents to one whole, but also percents greater than 100% and smaller than 1%. For percents greater than 100%, two or more grids are utilised, and for percent less than 1%, one square of the 10x10 grid is further divided. Pictorial examples for representations of 0.5%, 28% and 135% using 10x10 grids are presented in Figure 2.1.

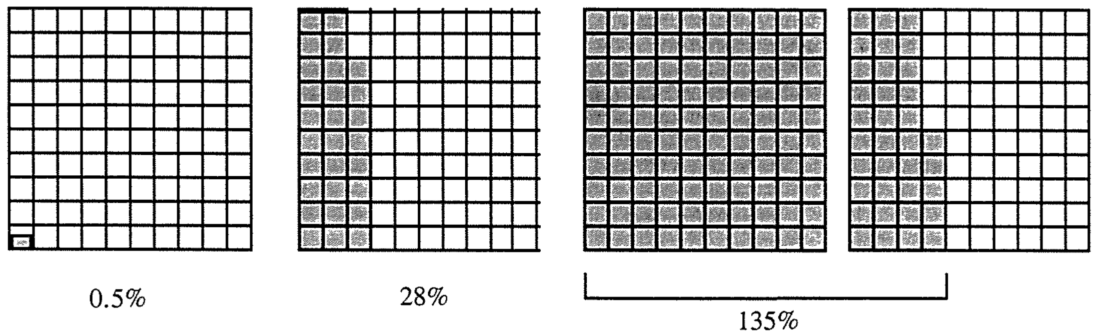


Figure 2.1. Pictorial representation of a percent less than 1%, less that 100%, and greater than 100% and less than 1% on a 10x10 grid.

Reys, Suydam and Lindquist (1992) stated that the time to introduce students to the topic of percent is after they have an understanding of decimals and fractions and have experienced ratios. Similarly, Brueckner and Grossnickle (1953) suggested that children's understanding of percent builds from decimal knowledge, and that the topic of percent should not be introduced until children have a "good background in decimals"(p. 426). Reys et al. (1992) suggested that the 10x10 grid is a useful model for linking percents to fractions and decimals as students can see fraction, decimal, percent equivalence simultaneously.

Another aid for developing the concept of percent is the hundred board, which according to Brueckner and Grossnickle (1953) can be used in teaching to stress the notion that a percent is a part of 100. To begin instruction, Brueckner and Grossnickle suggested that the teacher writes on the board:

1 out of 100
$^{1}/_{100}$
.01

which gives the fraction and decimal form of a verbal statement. The teacher then tells children that another way of writing 1 out of 100 is 1%. Brueckner and Grossnickle also suggested that the pictorial representation in the form of 10x10 grids be used to help children identify percents.

A further model for assisting the visualisation of percent and linking percents to fractions and decimals has been suggested by Hauck (1954). Hauck suggested the use of a "percentage box" which is a concrete model consisting of interconnecting wooden blocks representing 25%, 50%, 75% and so on. The whole block put together represents 100%. On each block the decimal, fraction and percent notation is written. Hauck stated that the percentage box would, "with discussion and question...show a definite relationship between fractions, per cent and decimals"(p. 9).

Cooper and Irons (1987) also suggested that children's understanding of percent should be developed through concrete representations of percents on 10x10 grids, and through linking this to decimal hundredths using appropriate language. The basis for instruction, according to Cooper and Irons, is guided by the triadic interaction of language, symbol and model, as seen in Figure 2.2.

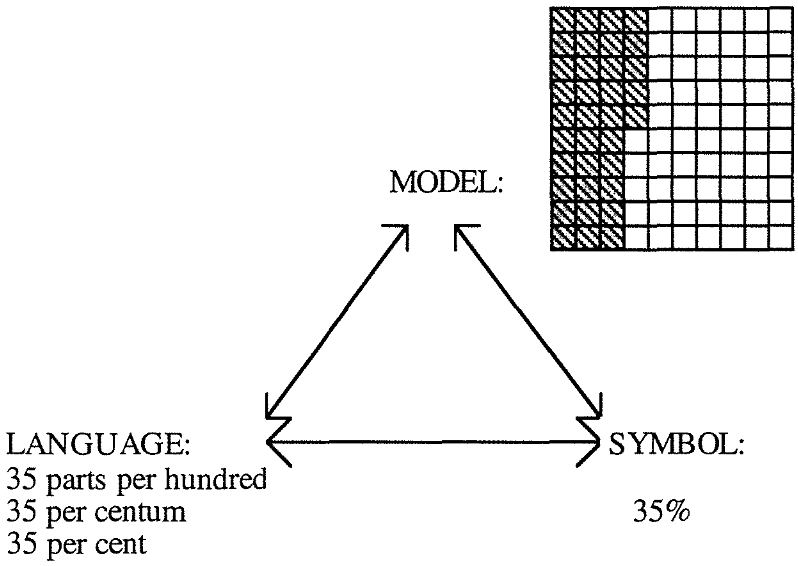


Figure 2.2. Cooper & Irons' (1987) triadic model guiding instruction in percent.

Cooper and Irons (1987) stressed the importance of the need to use appropriate language in all mathematics teaching, and stated that "percent has to be seen as *hundredths*, parts out of one whole, not out of one hundred" (p. 44). They also suggested that the word *centum* be introduced as a mediating step to the word *percent.* And thus, children become familiar with the language sequence (as depicted in Figure 2.2): 35 parts per hundred -> 35 parts per centum -> 35 percent. Cooper and Irons provided the following summary of what they consider to be the three important understandings for percent:

1. that percentages are parts per hundred
2. that percentages are out of one whole; and
3. that percentages are equivalent to hundredths as decimals fractions. (p.45)

To further promote children's understanding of percent, Glatzer (1984) suggested that instruction should capitalise on children's intuitive notions of percent through the teacher asking such questions as:

Can we have 100% class attendance?

Can we have 200% class attendance?

Can a price decrease by 50%?

Can a price decrease by 100%?

Can a price decrease by 200%? and so on.

Instruction should then, according to Glatzer, be based on developing children's awareness of patterns in percent where for example, children look at 1%, 10%, 100% of an amount. Similarly, Brueckner and Grossnickle (1953) suggested that instruction in percent should be continued further using 10x10 grids, where the children identify a percent of 100 as so many for every hundred: for example, 10% of 100 equals 10. This is then built on to finding a percent of multiples of 100: for example, 10% of 200 is 20. Estimation skills are developed through such types of patterning activities (Glatzer, 1984). An example of a practice activity for building mental percent computation, and patterning awareness is the following, as suggested by Glatzer (1984):

Circle the exercises below which can be done mentally:

50% of 86	2% of 800	17% of 10
25% of 88	56% of 89	150% of 40
15% of 60	11% of 50	1.5% of 1 000 (p. 25)

As a guide for teaching, Reys et al. (1992) suggested that further instruction in percent should aim to enable students to: (i) find a certain percent in a given situation (for example, find 25% of an amount), (ii) identify characteristics of that given percent (for example, 25% full means 75% empty), and (iii) compare and contrast that given percent with a range of other percents (e.g., 25% is half 50%, one quarter 100% is five times bigger than 5%, and so on). Thus, continued development of the concept of percent is through building children's mental pictures of common percentages as they relate to fractions (e.g., 25% means $^1/_4$; 50% means $^1/_2$; 10% means $^1/_{10}$, and so on); through children's understanding of the part/whole nature of percent where the whole comprises 100%; and by continually reinforcing how certain percents relate to the whole. The models described for building the concept of percent clearly emphasise the link of percent to fractions and decimals. Equivalence, or viewing various symbolic representations as having the same value (e.g., 25% is the same as $^{25}/_{100}$ which is the same as 0.25) is generally a poorly developed concept in children (Vance, 1992). As the percent literature stresses the development of the concept of percent through linking to the concept of decimals and fractions, it appears that time needs to be spent providing opportunities for children to envisage a percent simultaneously as a decimal and a fraction. According to Glatzer (1984), procedures for solving percent application problems are secondary to developing the concept of percent in children.

2.1.3 The concept of percent as ratio

Brown and Kinney (1973) suggested that the concept of percent should be related to ratio, to thus serve as a vehicle for developing the concept of ratio through the study of percent. Brown and Kinney stated that such definitions of "*percent means hundredths*" and "*percent means x .01*" overly simplify the concept of percent, and are unhelpful in interpreting percent usage in the real world. They suggested that a "percent as fraction" notion would provide little assistance for interpreting such statements as, "There is a budget deficiency of 9 percent" and "Population growth shows no signs of slowing and is running at 1.9 percent" (p. 352). As they stated "Translation of any of these statements into fractional form, according to the rule 'percent means hundredths', is not helpful. It leaves the question, 'fraction of what?'" (p. 352).

According to Brown and Kinney, relating percent to fractions and decimals assumes that there is no new knowledge to be developed in order to understand percent. However, as they stated, children must become aware that percent is a unique and concise language, and because it is used so extensively in our society, children must be able to interpret precent statements (such as in the examples given above).

Brown and Kinney described why percents are ratios rather than numbers (decimals, fractions) as they stated:

> A number is a property of a set, the specific number being determined by how many members are in the set. A ratio on the other hand, expresses the relationship between the number associated with set A and the number associated with reference set B. (p. 354)

Brown and Kinney described three percent problem situations which, as they suggested, need to be interpreted as ratio situations for meaning. The three examples are as follows:

> (1) In the problem, "Find 35 percent of 120", the problem is to find a subset of *n* of the reference set associated with the number 120 whose ratio is as 35 to 100. (2) In the problem, "What percent of 35 is 75?", the problem is to identify the number *n* that has the same ratio to 100 as the set of 75 elements has to the reference set of 35 elements. (3) In the problem "24 is 40 percent of what number?", the problem is to find the reference set *n* so that the ratio between a set of 24 elements and the reference set is as 40 is to 100. (p. 354-355)

It can be seen that the three examples described above are common percent "word problems" which students are exposed to in their study of mathematics. Brown and Kinney (1973) suggested that interpreting percent problems as ratios promotes understanding of ratio as a correspondence between a given set and the reference set and thus promotes ratio understanding.

2.1.4 The concept of percent as proportion

In a similar fashion to Brown and Kinney (1973), Schmalz (1977) stated that for students to understand percent as it is used in our society, they need to be able to interpret the precise language in which percent statements are presented, and this is through conceptualising percent statements as statements of proportion. To develop the concept of percent as proportion, Schmalz suggested a language-based approach where students are instructed on how to interpret percent statements into meaningful language. Schmalz contended that a percent statement is merely a proportion statement as it compares two quantities, and it is this fact that children are not overtly taught.

For this approach to percent instruction, Schmalz suggested using real-world statements as the foci for discussion, and as a means of exploring the semantics of percent statements. For example, Schmalz suggested that a statement such as "Women hold 17% of all management positions" should be expanded into meaningful text, such as "out of every 100 management positions, 17 are held by women" (p. 340). The next instructional step, according to Schmalz, is to write the statement in more mathematical terms, such as:

$$\frac{\text{number of women in management}}{\text{total number of people in management}} = \frac{17}{100}$$

This act of rewriting the real world percent statement into meaningful text, Schmalz labelled as transforming text into a preproportional statement. It is this stage, Schmalz argued, which builds children's understanding of percent; where meaning of percent in the real world provides the basis for problem solution through proportional mathematics in the later stages of percent instruction. Schmalz acknowledged that expressing percent situations into a preproportional statement may take students some time to master, but the dividends pay off during the next phase of instruction when students are required to solve percent problems. As she stated:

> ...a preproportional statement is an immediate step by which any percent statement takes on a manageable form. By helping students to see the comparisons in the problem before starting to work, it eliminates all guesswork. Preproportion statements also make concrete situations more easily understood and thus make problems more easily solved. (p. 343)

2.1.5 Summary of key points

In this section, summaries of instructional approaches for percent concept development were presented. Analysis of the advocated instructional approaches suggests that percent concept development can be promoted through exploring percent as equivalent to fractions and decimals, and in relating percent to 100 as one whole. The part/whole notion of percent relies on prior part/whole knowledge of fractions. Students may have difficulty relating percent to fractions if their fraction knowledge is tenuous. Also, if the part/whole notion of percent is promoted through instruction to the exclusion of other percent notions, students may have a limited knowledge of percent in its other forms. A part/whole concept of percent may provide students with experience in converting percent to common and decimal fractions, but may not assist students to develop understanding of such conversions as applied to percent calculations encountered in the future.

In this section, it was seen that the percent concept can also be developed through promotion of percent in relation to the concept of ratio. This approach suggested inadequacies of the part/whole approach, particularly in the assumption that understanding of percent requires no new knowledge to be developed. A ratio approach suggests that ratio knowledge can be developed through interpreting percent situations as ratios. For this approach, exploration of the language of percent as a statement of ratio was advocated. Such an approach is similar to suggestions that developing percent knowledge should be through proportion. A proportional approach to percent concept development was also presented in this section. Developing the concept of percent both as a ratio and a proportion through analysis of the language of percent may assist understanding of possible future calculations using proportion equations.

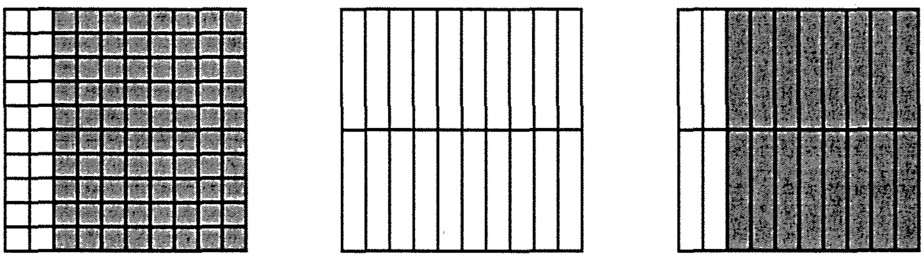
2.2 Solving percent application problems

2.2.1 Overview

In this section, various procedures, models and strategies are described for solving percent application problems. In section 2.2.2, models for representing the solutions to the three types of percent application problems are presented. In section 2.2.3, the application of the 10x10 grid for representing and solving the three types of percent application problems is presented. Building ratio understanding to solve percent application problems is presented in 2.2.4, the use of key words and mnemonics is discussed in section 2.2.5, and proportional representations are presented in section 2.2.6. In section 2.2.7, instructional approaches for solving percent equations proportionally are presented. A summary of the key points contained within this section is presented in section 2.2.8.

2.2.2 Models for representing the solutions to percent equations

Weibe (1986) described a model for solving Type I and II percent application problems. In this model, Weibe suggested the use of a 10x10 transparent grid to represent 100% as consisting of 100 parts. The whole amounts for Type I and II problems are represented by squares congruent to the 10x10 grid, divided into the corresponding number of sections of the whole amount. For example, for the Type I problem, 80% of 20 = Δ, the second grid is divided into 20 equal parts. To solve the problem, the transparent 10x10 grid is shaded to represent 80%, and placed on top of the square divided into 20 parts. The students then simply read off the number of parts out of 20 covered by the 80% shaded on the 10x10 grid (see Figure 2.3).



Percentage transparency shaded to show 80%

Worksheet square divided to show 20

Percentage transparency superimposed on worksheet square (16 rectangles covered by shaded area).

Figure 2.3. Using a percentage transparency to solve the Type I percent application problem: 80% of 20 = Δ (Weibe, 1986).

For Type II problems, for example 10 = Δ% of 25, the whole grid is divided into 25 parts, and 10 of these parts are shaded. The 10x10 grid is placed on top and the corresponding percent is read off (see Figure 2.4).

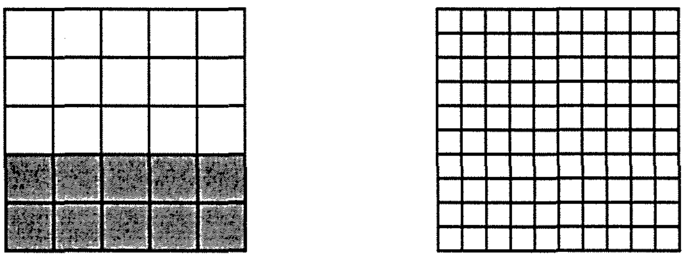


Figure 2.4. Visual model for representing the solution of the Type II percent application problem: 10 = Δ% of 25 (Weibe, 1986).

For Type III problems which, according to Weibe (1986), are difficult using this model, Weibe suggested the use of a piece of elastic divided linearly into 100 equal parts. To solve such Type III problems as 50% of Δ = 44, the elastic is placed against a ruler, with 50% marked on the elastic corresponding to 44 on the ruler. The 100% mark on the elastic is located on the ruler and the percent on the ruler read off as the solution of 88 (see Figure 2.5). Type I and II problems can also be shown with the elastic. Weibe stated that the purpose of the two models is to assist children in building "mental images" and to develop an "understanding of percentage operations" (p. 26).

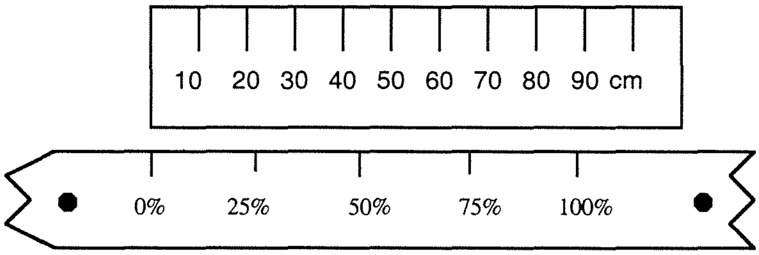


Figure 2.5. Using the percentage elastic to solve the Type III problem: 50% of Δ = 44 (Weibe, 1986).

According to Weibe, these manipulative models are a stepping stone into the symbolic procedures. Their strength, he continues, lies in the fact that they demonstrate the symbolic manipulation for percentage computations, and are also a means through which estimation skills are developed.

2.2.3 Grids (10 x10) for solving percent application problems

Bennett and Nelson (1994) proposed that the 10x10 grid can be used to model percent application problems, thus providing a conceptual model for solving mathematical problems of percent. According to Bennett and Nelson, using the 10x10 grid to "visualise" percent problems is based on prior experience of percent as "parts per hundred", with the understanding that each of the 100 squares of the grid represents 1%. This knowledge is extended to situations where, for example, the 10x10 grid represents 400 people, and students are encouraged to mentally calculate the value of one square within the grid. For example, if the unit square represents 500 rabbits then each square is 5 rabbits; if the unit square represents 395 days, then one square is 3.65 days. Conversely, if provided with information on the value of 1% (one small square of the 10x10 grid) students are expected to be able to determine the value of the unit square. As Bennett and Nelson summarised, "successfully determining the value of one small square (1 percent) is the key to solve percent problems" (p. 21).

Utilisation of the 10x10 grid for representing and solving the three types of percent problems is summarised through the following three examples, as described by

Bennett and Nelson. For the Type I problem, *20% of a business' 240 employees are classified as minorities. How many is this?* the information is represented on the grid as in Figure 2.6. From the grid, the solution can be calculated in a number of ways. Some students may see that each small square (1%) is 2.4, therefore 20 squares would equal 48; or students may interpret 20% as 2 lots of $^1/_{10}$ of the unit square; knowing that $^1/_{10}$ is 24, $^2/_{10}$ would then be simply doubling 24 to give 48.

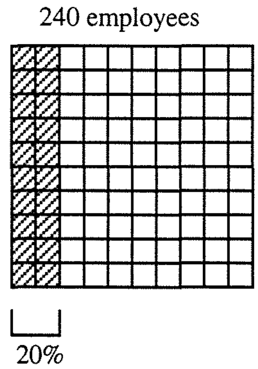


Figure 2.6. Representation of Type I problem using a 10x10 grid (Bennett & Nelson, 1994, p. 22).

For the Type II problem: 25 hectares of land are given to a community, of which 6 hectares must be developed as a playground. What percent of land is for the playground? the information is represented on the 10x10 grid as in Figure 2.7.

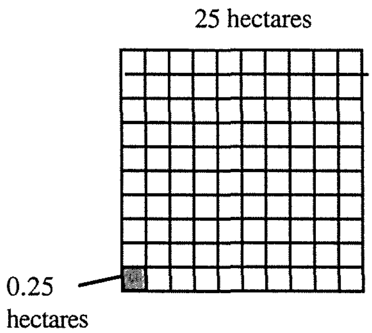


Figure 2.7. Representation of Type II problem using a 10x10 grid (Bennett & Nelson, 1994, p. 22).

From the grid, the unit square represents 25 acres, and one small square (1%) has a value of 0.25 acres. Bennett and Nelson (1994) described the solution process in the following manner:

> Each small square represents 0.25 = $^1/_4$ acres, so 4 small squares represents 1 acre and 24 small squares represents 6 acres. Or since each small square has a value of 0.25 acres, we can determine the number of times 0.25 divides 6 (6 ÷ 25 = 24) to see that 24 squares are needed to represent 6 acres. (p. 22)

For the Type III problem: In a certain county, 57 of the schools have a teacher to student ratio that is greater than the recommendations for accreditation. The 57 schools represent 38 percent of the number of schools in the county. What is the total number of schools in the county? the information is represented on the 10x10 grid as in Figure 2.8.

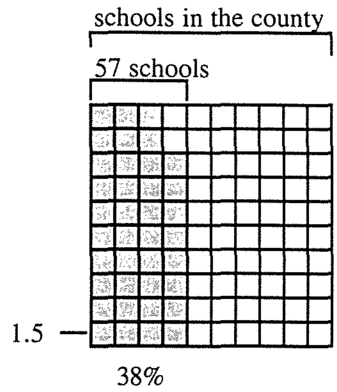


Figure 2.8. Representation of Type III problem using a 10x10 grid (Bennett & Nelson, 1994, p. 22).

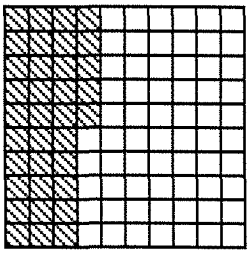
Bennett and Nelson (1994) interpreted the solution to the problem from the 10x10 grid representation in the following manner:

> Since the 38 small shaded squares represent 57 schools, each small shaded square represents 57 ÷ 38 = 1.5 schools. So the unit square represents 100 x 1.5 = 150, which is the number of schools in the county. (p. 22)

Bennett and Nelson also provided examples of how the 10x10 grid can be used to represent problems of percent increase and decrease. The value of this representation, according to Bennett and Nelson, is that it connects the area model used for fractions and decimals, and continually emphasises the basic meaning of percent, as a whole partitioned into 100 parts. Bennett and Nelson also suggested that the 10x10 grid model negates the need to interpret percent problems according to types, as all three types of percent problems can be represented via the model.

2.2.4 Building ratio understanding to solve percent problems

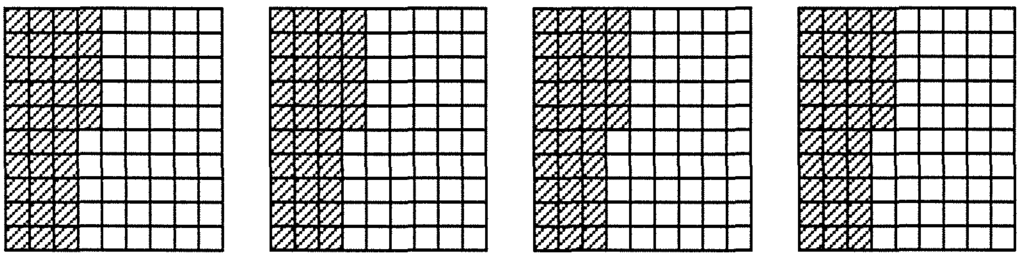
Cooper and Irons (1987) suggested that once children have developed the concept of percent and are familiar with the symbolism for percent, the relationship of percents and ratios should be taught, to thus enable students to solve percent application problems meaningfully through a ratio understanding. They suggested that the ratio/percentage concept is developed through providing a real-world percent context (e.g., money), continuing the use of 10x10 grids as representations for percent, and exploring patterns using a data retrieval table. For example, 35% of $1 would be pictorially represented as seen in Figure 2.9, and students would be asked to shade 35% of the square and questioned: "if the one square is $1, what is the value of the shaded section?" (35c).



Shade 35% of one square.
If one square is $1, what is the value of the shaded section? (35c)

Figure 2.9. Pictorial representation of 35% of $1.

Discussion would focus on the amount shaded and the value of the shaded area. This would be extended to the example of 35% of $4 where children would be encouraged to shade 35% of four squares, each whole square representing $1 (see Figure 2.10).



Shade 35% of 4 squares (each representing $1)
What is the value shaded in each of the squares?

Figure 2.10. Pictorial representation of 35% of $4.

The information determined from such pictorial representations would be recorded on a table (see Table 2.1). The students would be instructed to complete the table to investigate percent patterns. From investigation of pictorial representations and data retrieval tables, children would then be directed to the multiplication method for finding percents of a whole. Cooper and Irons (1987) offered the following guidelines as a means of helping children find a percent of a quantity:

> ...children need to see that per cent means 'so many hundredths out of every one.' This ratio is a part to whole comparison. As a result ratio notation is not usually used with percentage ideas. The fraction ideas of equivalence work just as well. Since decimal fractions are used, it is not even necessary to worry too much about the use of equivalence. It is relatively easy to see that:

1. Per cent means 'so many' hundredths out of every one; e.g., 35% means 0.35 out of every 1.
2. We want to find the percent of an amount; e.g., what is 35% of 27?
3. The amount tells us how many ones; e.g., In 27 there are 27 ones.
4. To find the percentage, we multiply the number of ones by the part that is the per cent; e.g., 35% of 27 is 0.35 x 27. (p. 46)

From the above description it can be seen that children are presented with percentages as ratios in the part-to-whole sense of *so many hundredths for every one*. This instructional sequence appears to be based on a unitary approach where children are encouraged to find 1% of an amount and use this to find the desired percentage part. In the initial instructional phases for solving percent problems, the link of percents to money is made.

Table 2.1

Data Retrieval Table for Finding the Percent of an Amount

Amount earned	Percentage saved	Amount saved
$1	35% of $1	35c
$2	35% of $1	70c
$3	35% of $1	..
$4	35% of $1	..
$5	35% of $1	..
$6	.. ..	..
$7	.. ..	..

Similar to Cooper and Irons' (1987) suggestion of the use of money in percent application problems, Osiecki (1988) also reported on approaching percent calculation through money within the classroom. For ninth-grade students, Osiecki encouraged her students to think of percent in terms of money, for example 29% means 29c for every dollar. When looking at discount, for example a discount of 20% was interpreted as 20c off every dollar. Decimal multiplication was practised to solve percent problems. Osiecki reported much success with teaching percent application problems. In Osiecki's teaching unit, students were encouraged to scour newspaper advertisements to check percent calculations presented once they had mastered percent calculations using decimal multiplication. From this analysis of advertising, the students were asked to create their own percent story problems and compile these as a computer inventory. Of the unit, Osiecki concluded that:

...the students enhanced the concrete level of computation of percents with formal-operational-level creativity, analysis and syntheses of original story problems. They had started to think. (p. 34)

2.2.5 Key words and mnemonics for solving percent problems

To assist in the solving of percent application problems, Teahan (1979) suggested the use of a visual model for translating percent problems into a solvable form. This model is based on incorporating the common verbal cues present in many percentage application problems, being: is, of, percent. The model is a triangular shape divided into three sections. The words is, of and the symbol % are placed in designated sections of the triangle (see Figure 2.11). The positioning of these labels is static.

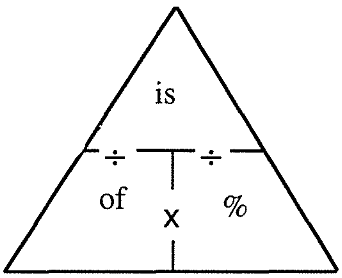
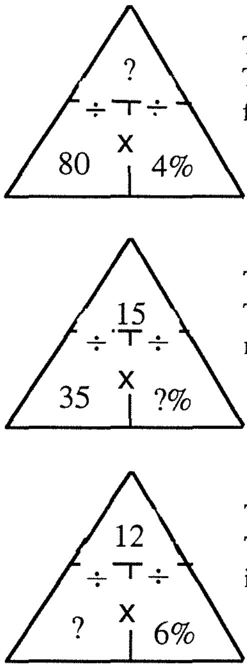


Figure 2.11. Teahan's (1979) model for solving percent application problems.

From Figure 2.11, it can be seen that the procedures for solution have been incorporated into this model with the inclusion of the symbols "x" and "÷". To use this model, students place the numbers given in the problem that precede the three labels (i.e., *is, of, %*) in the corresponding sections of the triangle and perform the stated operation. Examples of how this model can be used to solve the three types of percentage application problems are presented in Figure 2.12.

Similar to Teahan's model, Boling (1985) also suggested the use of a triangle to solve percentage application problems. The difference between the two triangular models is the placement of the labels. Boling's model is presented in Figure 2.13. Boling further delineated prerequisite knowledge necessary for successful application of this model. According to Boling, to utilise this model, students must have (i) the ability to convert percentages to decimals, and (ii) knowledge that fractions can be interpreted as division problems; that is, that the numerator is divided by the denominator. Boling also has cautioned that when this model is utilised, students must check that their solution matches the form required. For example, in Type II problems, the solution is a decimal, and therefore must be converted back to a percent to answer the question. The necessity of such a procedure appears to be a disadvantage of the method.



Type I: What is 4% of 80?
To Solve: The 4% is changed to the decimal form 0.04 and multiplied by 80.

Type II: 15 is what percent of 35?
To Solve: 15 is divided by 35 then multiplied by 100 (15 ÷ 35 x 100)

Type III: 12 is 6% of what number?
To Solve: 6% is changed to 0.06 and divided into 12 (12 ÷ 0.06)

Figure 2.12. Solving the three types of percent application problems using Teahan's (1979) model.

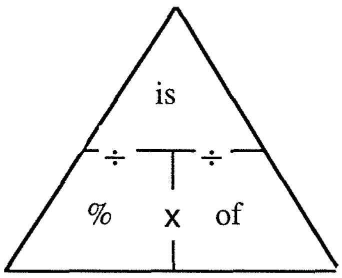


Figure 2.13. Boling's (1985) model for solving percent problems.

McGivney and Nitschke (1988) proposed a mnemonic for helping children translate percentage application problems into a proportion equation. Like Teahan (1979) and Boling (1985), McGivney and Nitschke focused on the key words within most percentage problems: *is, of* and *what.* These words are incorporated into a proportion equation:

$$\frac{is\ \#}{of\ \#} = \frac{what\ \#}{100}$$

Further, to assist children automatise this mnemonic, they suggested that relating *is* and *of* to alphabetical positions ("*i*" comes before "*o*" in the alphabet) helps position the two words in the proportion equation. For percentage application problems not stated in the *is - of* form, McGivney and Nitschke suggested that translation must occur to set up the proportion and then solve the problem.

2.2.6 Proportional representations for solving percent problems

Dewar (1984) suggested a visual approach to setting up proportion equations to solve percentage application problems through the use of a comparison scale diagram. Similar to Weibe's (1986) elastic model for estimating and solving percentage problems, Dewar's comparison scale provides a visual image of a proportion statement. The comparison scale is constructed by drawing a vertical unscaled number line. One side of the number line represents the whole on a linear scale from 0-100%, the other represents the quantity as referred to in the percentage application statement (see Figure 2.14). From Figure 2.14 it can be seen that the whole of the quantity would be positioned on the right hand side corresponding to 100%. By translating the given information from the percentage application problem onto the comparison scale, the elements in the proportion equation are given, and as Dewar (1984) stated, "the correct proportion is right before the students' eyes" (p. 49).

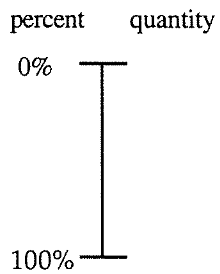


Figure 2.14. Dewar's (1984) comparison scale for solving percent application problems.

Similar to Dewar, Haubner (1992) proposed a more elaborate visual model which simultaneously can assist percentage problem solving through equation or proportion methods. Haubner's model delineates the percentage part from the quantity part of the stated problem, and provides a richer visual image of the stated percent as it relates to the quantity. For example, the Type II problem: *16 is what percent of 40* would be shown as in Figure 2.15, and the proportion equation: $^{x}/_{100} = {}^{16}/_{40}$ would thus be constructed from the model. According to Haubner, this question can also be solved by the equation method where interpretation of the model takes place horizontally, and students construct a typical textbook equation into the form *x% of 40 = 16.* Haubner proposed her model in response to the fact that many difficulties experienced with percentage application problems are due to interpretation problems.

Haubner provided the following example to demonstrate the use of her model for setting up percentage equations:

> The area of North America is about 75% the area of Africa. The area of Africa is about 12 million square miles - About how large is the area of North America? (p. 232)

The given information is translated onto the model as in Figure 2.16.

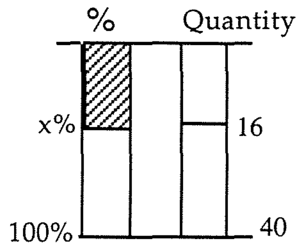


Figure 2.15. Using Haubner's (1992) model for solving the Type II percent application problem: *16 is what percent of 40?*

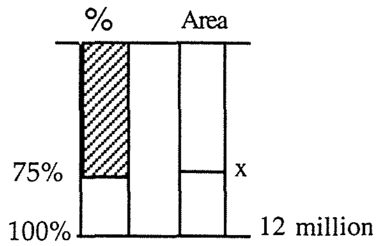


Figure 2.16. Using Haubner's (1992) visual model to translate percent application problems into solvable equations.

To translate into an equation, the students must think about the information given and how this relates to the question asked. Reading from the model, the student thinks "100% of 12 million is 12 million; 75% of 12 million is x.", and thus the equation is developed. Haubner suggested that both proportion and equation approaches to solving percentage application problems need to be familiar to students, and that the equation method promotes use of mental estimation skills.

2.2.7 Approaches for solving percent equations proportionally

Once percent situations have been represented as a proportional equation, the equation must be solved to complete the problem. Examples of strategies for solving percent equations using a proportion equation discussed thus far (e.g., Dewar, 1984; Haubner, 1992; McGivney & Nitschke, 1988) do not describe the procedures involved in solving proportion equations. For solving proportion equations, initial instruction generally limits proportion examples to factors or multiples of the given proportion (e.g., Reys et al., 1992). Simple exercises such as the following would be common examples for children to solve at this stage:

1 : 3	=	4 : __
5 : 7	=	__ : 35
2 : __	=	6 : 8
40 : 30	=	4 : __

However, when instruction in percent begins to relate to proportion and students are required to solve percent application problems by solving a proportion equation, the numbers in such problems rarely are multiples or factors of the other. This is an issue which only infrequently is addressed in the literature. Wendt (1959) and Cole and Weissenfluh (1974) attempted such a task.

Wendt (1959) suggested that as children are taught such formulae as "A = L x W", and "V = $^{1}/_{2}$ b x h", so too can children learn "A is P percent of B" and translate this into the proportion equation: $^{P}/_{100} = {}^{A}/_{60}$. To solve a percent application problem such as, *What number is 30 percent of 60?*, Wendt suggested the following procedure:

> After the proportion equation has been set up, the values are inserted into their appropriate place: $^{80}/_{100} = {}^{A}/_{60}$. To solve this equation requires the A to be left by itself. Both sides of the equation are multiplied by $^{60}/_{1}$, and elaborate cancelling begins:
>
> $$\overset{6}{\cancel{60}}/_{1} \quad x \quad \overset{3}{\cancel{30}}/_{100} \quad = \quad {}^{A}/_{\cancel{60}} \quad x \quad {}^{\cancel{60}}/_{1}$$
>
> $$18 \quad = \quad A$$
>
> to reveal the solution of A equals 18. (p. 227)

Wendt acknowledged that this procedure is algebraic, but prefers to call it "generalised arithmetic." He suggested that generalised arithmetic can be taught to Year 6 children by building on their prior arithmetical knowledge. For example, children can solve 3 x N = 12, and this can be expanded to show that by dividing both sides of the equation by 3, the solution also can be found. Hence, Wendt suggested, that generalised arithmetic should be introduced to children in the upper primary school to give meaning to the arithmetical manipulation of numbers required to solve percentage proportion problems.

Cole and Weissenfluh (1974) suggested that the cross multiply rule be utilised to solve percent proportion problems, but that children need to be shown why this procedure works. For example, to solve the problem: *What is 10% of 20?*, the proportion equation is set up: $^{10}/_{100} = {}^{x}/_{20}$. Upon cross multiplying, the 10 is multiplied by the 20, and the 100 is multiplied by the *x*:

$$10 \times 20 = 100 \times x$$
$$200 = 100 \times x$$

To explain how this is possible, Cole and Weissenfluh suggested that teachers demonstrate how, by multiplying "both equivalent comparisons by the same numbers, $^{100}/_{1}$ and $^{20}/_{1}$" (p. 227), the equivalency is unchanged:

$$^{20}/_{1} \times {}^{100}/_{1} \times {}^{10}/_{100} = {}^{x}/_{20} \times {}^{100}/_{1} \times {}^{20}/_{1}$$

to become $20 \times 10 = x \times 100$ as after the cross multiplication procedure.

To solve for "x", Cole and Weissenfluh suggested converting the product of the cross multiply procedure to the fractional form as follows:

$$^{200}/_{1} = {}^{100x}/_{1}$$
$$^{1}/_{100} \times {}^{200}/_{1} = {}^{100x}/_{1} \times {}^{1}/_{100}$$
$$^{2}/_{1} = {}^{x}/_{1}$$
$$\text{and } 2 = x$$

For Type II and III percentage application problems, Cole and Weissenfluh (1974) presented the following worked examples:

Type II: 2 is what percent of 20?

$$^{x}/_{100} = {}^{2}/_{20}$$
$$200 = 20 \times x$$
$$10 = x$$

Type III: 2 is 10% of what number?

$$^{10}/_{100} = {}^{2}/_{x}$$
$$200 = 10x$$
$$20 = x$$ (p. 228)

In these last two cases, converting products to fractions over 1 are not presented. The examples given by Cole and Weissenfluh use simple numbers, and finding x is merely a mental exercise. However, if the numbers were not so "neat", the step from the cross multiply to solving for x is not fully delineated.

Reys et al. (1992) provided the following approach to solving percent application problems using proportion equations:

Type I

$$10\% \text{ of } 60 = \Delta$$
$$^{10\%}/_{100\%} = {}^{x}/_{60}$$
$$x = 6$$

Type II

$$\Delta\% \text{ of } 20 = 15$$
$$^{x\%}/_{100\%} = {}^{15}/_{20}$$
$$x = 75$$

Type III	50% of Δ	=	\$40
	$^{50\%}/_{100\%}$	=	$^{\$40}/_{x}$
	x	=	\$80 (p. 232)

In the above examples given by Reys et al., the proportion equation and the solution is given. No discussion, however, is presented as how to guide children from the equation to the solution. This appears to be the case in many textbook approaches to solving proportion equations, as Fisher (1988) stated, "The most common textbook strategy for solving a proportion problem is to write an equation in the $^{a}/_{d} = {}^{c}/_{d}$ form with an unknown as one of the four terms, cross-multiply and solve for the unknown" (p. 157).

2.2.8 Summary of key points

In this section, summaries of methods for representing and solving percent application problems were presented. These methods included the use of various models, such as: fraction/percent overlays, elastic strips, 10x10 grids, identification of key words, use of mnemonic strategies, and comparison scales. All methods presented appear to have certain shortcomings, which override, to an extent, the benefits purported. The fraction/percent overlays exemplify the part/whole notion of percent situations but can be used only for Type I and Type II percent situations. Type III percent situations can be adequately modelled using an elastic strip, but this models the proportional nature of percent situations, not the part/whole notion of the overlays. The 10x10 grids require extensive decimal and fraction multiplication and division procedures as well as a high level of computational estimation skill. The grids are time-consuming to construct and also do not appear to lend themselves to percents greater than 100. Using mnemomics and key words appear to be very procedural methods, unrelated to percent concepts and principles, and therefore appear limited in their ability to promote students' principled-conceptual knowledge. Comparison scales appear to be simple to construct and powerful for representing percent situations as statements of proportion, but offer few suggestions for assisting students solve proportion equations meaningfully. In this section, procedures for calculating the proportion meaningfully were described, but such procedures were seen to use relatively simple numbers, and failed to address understanding of the calculations required in proportion equations when the numbers are not multiples or factors of 100.

It has been seen that, through analysis of instructional approaches to percent, percent knowledge and problem solving skills can be developed in multifarious ways. The question is, which ways are best? Throughout their comprehensive analysis of percent literature, Parker and Leinhardt (1995) frequently suggested that the key to percent understanding is in understanding the relationship of percent to proportion. However, there appears to be a paucity of literature which specifically describes the promotion of percent knowledge from a proportional perspective which encompasses teaching of the proportion equation meaningfully. In the next section, related literature for pedagogic insight into percent is reviewed.

2.3 Related issues in percent teaching and learning

2.3.1 Overview

`This section is a summary of the exploration of issues pertaining to percent teaching and learning which may serve to guide instructional planning. There are five parts. In section 2.3.2, various studies which have focused on researching instructional approaches to teaching percent are summarised. In section 2.3.3, the mathematical topic of proportion and the issue of percent as a component of the concept of proportion, is addressed. In section 2.3.4, the simplicity and complexity of the proportion equation applied to percent calculations is described. In section 2.3.5, other instructional strategies which may be useful for guiding percent instruction are presented through discussion of mental models, problem interpretation, and holistic versus sequential instruction. A summary of key points contained within this section is presented in section 2.3.6.

2.3.2 Studies of percent instruction

As described in chapter 1 (section 1.2.3), research studies have indicated that students' understanding of percent is generally poor. Parker and Leinhardt (1995) provided an extensive review of studies of various approaches to percent instruction. In the following paragraphs, summaries of the studies, as described by Parker and Leinhardt, are presented.

Parker and Leinhardt (1995) described a comparative study conducted in 1963 by Tredway and Hollister, where students received instruction which was either drill and practice of the three cases of percent problems, or was more activity based, focusing on meaningful understanding of the three types of percent problems using 10x10 grids and the hundreds board. The two approaches can be seen as quite distinct. For one group, each of the three types of percent problems were introduced to the students on different days, and computational procedures for solving each of the types were practised. For the other group, the three types of percent problems were intermingled throughout the instructional sequence, and concrete referents were used to assist meaning. Of this study, Parker and Leinhardt stated: "This study provides one of the first pieces of evidence for the benefits of teaching percent as an integrated process. It also provides some evidence of the value of using visual representations of percent relationships" (p. 454).

Parker and Leinhardt described another study conducted in 1975 by Mason, where an alternative representation for percent problems was taught to adult students. This method was called the time-function model, and Parker and Leinhardt provided the following example of the time-function model for interpreting and solving percent problems:

Problem: *After an increase of 15%, the cost of an item was $28.75. Find the original cost.*

To solve the problem, Parker and Leinhardt (1995) stated, that "the base is the first quantity to appear in time, that is the original cost. Using the percent relationship, if the original price had been $100, the increase would be $15" (p. 455). The symbolic representation of this information would be:

original	increase	new cost	
$100	$15	$115	*relative data*
x	y	$28.75	*actual data*

From this representation, there are two sets of data visible, relative and actual. From this data, there can be seen three statements of proportion: $^{100}/_{x} = {}^{15}/_{y}$; $^{100}/_{x} = {}^{115}/_{28.75}$; and $^{15}/_{y} = {}^{115}/_{28.75}$. Solution proceeds via cross multiplication. Of this method, Parker and Leinhardt, stated that it "emphasised both the multiplicative and additive relationship found in percent problems, eliminated one known trouble spot (conversions) and addressed another (base identification). Considering the clear evidence of the entanglement that students experience between the notations of decimal and percents, it is significant that this model for percent problems gave meaning to the base of the percent and avoided the necessary conversion between symbol systems" (p. 455).

Parker and Leinhardt reported on a relatively recent comparative study conducted by Maxim in 1982, where students were instructed either in solving percent exercises using the proportion method or the equation methods (see chapter 1, section 1.1.5 for descriptions of the procedures involved in these two solution methods). Results indicated that neither method improved students' performance significantly, with Maxim (1982) concluding that students' percent performance is affected by students' knowledge of fractions, the difficulty of the percent symbol, and the wording of percent problems. From such a conclusion, it appears that, execution of the computational procedure may have assisted students in computing percent problems, but interpretation of percent situations hindered application of the computational procedure. In conclusion to their review of studies on percent instruction, Parker and Leinhardt stated that the results of all studies collectively were inconclusive, and no single "best" approach was apparent.

The study by Lembke and Reys (1994), briefly described in section 1.2.3, looked at Years 5, 7, 9 and 11 students' conceptual and computational percent knowledge, before and after formal percent instruction. Lembke and Reys interviewed high and middle ability students in each of the four year levels to determine their: (i) knowledge of pictorial representations of percent; (ii) knowledge of relationships between fractions, decimals and percents; (iii) use of strategies including benchmarks; (iv) use of mental computation; and (v) sensibility to reasonableness of the solution. Lembke and Reys' found that students utilised a variety of strategies for solving percent problems, but with formal instruction, such creativity appeared to give way to application of learned procedures. This finding has clear implications for instruction. According to Parker and Leinhardt (1995), instruction should not focus on execution of mechanical procedures, but on providing students with strategies and experiences to enable them to apply their own computational procedures.

The value of allowing students opportunities to develop their own percent computational procedures in the school situation can be seen as a noble suggestion, well grounded in research on children's intuitive percent computational procedures. However, the provision of such experiences may not be viable in all school settings. A study by Allinger (1985) provided an insight into the dilemma facing teachers between providing students with opportunity to develop their own understanding of percent situations, and also providing helping strategies to promote achievement. Contrasting Lembke and Reys (1994) study (described above) on the use of intuitive strategies for solving percent problems by middle and high ability students, Allinger (1985) reported on a study with Year 10 students who had experienced a long history of failure in the study of mathematics. In this study, students were provided with instruction in solving Type I and Type II percent problems using the proportion equation and the cross-multiply technique. Instruction began with the focus on equivalent fractions. The procedure for fraction equivalence was then related to solve Type II percent problems. For example, the problem: *24 is what percent of 80?* was translated to the proportion form of $^{24}/_{80} = {}^{N}/_{100}$. Instruction also included the application of 10x10 grids to develop meaning of percent as part of a whole where the whole is 100%. Students were given opportunities to practise the Type I and Type II problems through constructing a proportion equation, and using a calculator to solve the equation. As Allinger stated, "Our goal of the unit was to build concept understanding and thought processes but some mechanisation was desirable in helping these slow/reluctant learners achieve" (p. 568). Of this study Allinger reported that motivation and general interest in mathematics was promoted with the use of the calculator as a tool in computation. Throughout the instructional sequence, Allinger stated that the concept of percent as a statement of proportion was continually impressed upon the students. Allinger presented this concept as the following:

"*Percent means hundredths, and is a part/whole comparison; therefore 20% of 75 can be computed using a proportion*" (p. 572). According to Allinger, the proportion is thus set up by reminding students that percent means hundredths, and that, in this particular example, 75 is the whole, creating the equation $^{N}/_{75} = {}^{20}/_{100}$. Solution thus proceeds using cross-multiplication. According to Allinger, by drawing students' attention to the components of the problem (i.e., percent means hundredths, the whole relates to hundredths), they were provided with strategies for interpreting and analysing the components of percent problem situations. The use of the calculator was simply to attain a solution.

In relation to students' own intuitive procedures for percent calculations, Parker and Leinhardt (1995) stated:

> There is strong evidence that students can do the mechanical parts of percent problems. For the student who has correctly analysed the referents and their relationships to each other and understands the proportion relationship that exists between the quantities, any number of solution procedures might be accessed to solve the problem. (p. 463)

In this statement, it can be seen that such procedures are dependent upon "correctly analysing the referents" and "understanding the proportional relationships". In Allinger's (1985) study, it can be argued that an attempt has been made to assist students experiencing difficulty with percent (students who have received formal instruction in percent over many years) to provide scaffolding for the analysis of referents and understanding the proportion relationship.

2.3.3 Percent as a proportion

As repeatedly stated by Parker and Leinhardt (1995), the underlying meaning of percent is primarily proportion. It follows that, to understand the proportional nature of percent, a well-developed understanding of proportion is required. It could be argued that building students' understanding of proportion would be the means to promote students' understanding of percent. Developing instructional programs in percent, from a proportional perspective, thus would rely on a knowledge of instructional programs for building proportional understanding.

Analysis of the proportion concept itself has been a major field of research (e.g., Behr, Harel, Post & Lesh, 1992), and the importance of students' developing an understanding of proportion concepts and proportional reasoning skills is becoming widely accepted as a key aspect of the mathematics curriculum. As Lesh, Post and Behr (1988) stated, "Proportional reasoning is the capstone of children's elementary school arithmetic and the cornerstone of all that is to follow" (p. 93-94). However, the development of proportional reasoning is a complex operation, as Post, Behr and Lesh (1988) stated that proportional reasoning:

> ...requires firm grasp of various rational number concepts such as order and equivalence, the relationship between the unit and its parts, the meaning and interpretation of ratio, and issues dealing with division, especially as this relates to dividing smaller numbers by larger ones. A proportional reasoner has the mental flexibility to approach problems from multiple perspectives and at the same time has understandings that are stable enough not to be radically affected by large or "awkward" numbers, or the context within which a problem is posed. (p. 80)

Proportional reasoning, therefore, relies on a well-developed concept of proportion, but building students' understanding of proportion appears to be a complex process as the proportion concept is interrelated with many other concepts. For example, English and Halford (1995) stated that: "Fractions are the building blocks of proportion" (p. 254). Similarly, Behr et al. (1992) stated that "the concept of fraction order and equivalence and proportionality are one component of this very significant and global mathematical concept" (p. 316). Also, Streefland (1985) suggested that "Learning to view something 'in proportion', or 'in proportion with...' precedes the acquisition of the proper concept of ratio" (p. 83). Developing students' understanding of ratio and proportion is difficult because the concepts of multiplication, division, fractions and decimals are the building blocks of proportional reasoning, and students' knowledge of such topics is generally poor (Lo & Watanabe, 1997).

The proportion concept, therefore, is intertwined with many mathematical concepts. This has implications for instruction. The development of a rich concept of rational number, and thus proportional relationships, takes a long time (Streefland, 1985). The proportional nature of various rational number topics must be the focus of instruction as these topics are revisited continually throughout the curriculum, in order to build and link students' proportional understanding (Behr et al., 1992). Building proportional reasoning must be through multiple perspectives (Post et al., 1988).

The literature provides various suggestions for activities and strategies for promoting the proportion concept. The use of ratio tables has been suggested as one means for building students' ratio understanding (English & Halford, 1995; Middleton & Van den Heuvel-Panhuizen, 1994; Robinson, 1981; Streefland, 1985). English and Halford (1995) provided the following example of a ratio table, which assists in the comparison of the number of soup cubes per person:

soup cubes	2	4	6	8
people	4	8	12	16

English and Halford stated, "A table of this nature provides an effective means of organising the problem data and enables children to detect more readily all the relations displayed, both within and between the series...it serves as a permanent record of proportion as an equivalence relation" (p. 254).

A common theme for initial instruction in ratio and proportion, is on providing students with many mathematical experiences and activities which require them to explore and test their own intuitive procedures for solving proportion problems (Behr et al., 1992; Hiebert & Carpenter, 1992; Lo & Watanabe, 1997; Streefland, 1985). The essence of proportional reasoning is on understanding the multiplicative structures inherent in proportion situations (Behr et al., 1992). Children's intuitive strategies for solving proportion problems typically are additive (Hart, 1981). The teacher's role, therefore, is to build on students' intuitive additive strategies and guide them towards building multiplicative structures. Multiplicative structures develop as early as Grade 2 for some children, but are also seen to take time to develop to a level of conceptual stability (often beyond Grade 5) (Clark & Kamii, 1996). Behr et al. (1992) suggested that exploring "change" will help students develop multiplicative understanding. For example, students can be encouraged to discuss the *change* to 4 which will result in 8. From an additive view, 4 can change to 8 by adding 4. From a multiplicative view, 4 can change to 8 by multiplying by 2. The difference between the additive and multiplicative view can be seen by looking at other numbers. The additive rule holds for 13 changing to 17, but not the multiplicative rule. According to Behr et al. (1992), "the ability to represent change (or difference) in both additive and multiplicative terms and to understand their behaviour under transformation is fundamental to understanding fraction and ratio equivalence" (p. 316). Moving students towards formal ratio and proportion principles and procedures is termed by Streefland (1985) as "anticipating ratio", where the teacher capitalises on students' informal intuitive problem solving procedures guiding students to "formulae and algorithmisation" (p. 84). Such an approach was taken in a teaching experiment conducted by Lo and Watanabe (1997) where a Year 5 child was exposed to proportional reasoning tasks to promote intuitive multiplicative reasoning skills and hence develop proportional reasoning.

Research has indicated that students' (and teachers') understanding of proportion is generally poor (e.g., Behr et al., 1992; Fisher, 1988; Hart, 1981; Post, Harel, Behr & Lesh, 1991;). Streefland (1985) stated that: "Ratio is introduced too late to be connected with mathematically related ideas such as equivalence of fractions, scale, percentage" (p. 78). English and Halford (1995) suggested that proportional reasoning is taught in isolation and thus remains unrelated to other topics. Behr et al. (1992) stated, "We believe that the elementary school curriculum is deficient by failing to include the basic concepts and principles relating to multiplicative structures necessary for later learning in intermediate grades"(p. 300). Behr et al. also added, "There is a great deal of agreement that learning rational number concepts remains a serious obstacle in the mathematical development of children...In contrast there is no clear argument about how to facilitate learning of rational number concepts" (p. 300). Developing students' understanding of percent through linking it to the concept of proportion, appears therefore, to be a complex process, as students appear to be unlikely to have a well-developed concept of proportion at the time they are introduced to the percent concept.

2.3.4 The proportion equation and its relation to percent

In linking percent to proportion, Post, Behr and Lesh (1988) stated simply, that "percent is a special type of rate", and that "in percent situations, one rate pair will always be 100" (p. 80). They then provided an example of each of the three types of percent problems as simple statements of proportion, and stated that "all percent-related situations can be solved with the use of proportions and an essentially identical conceptual framework" (p. 80), suggesting that, percent situations are conceptually the same as proportion situations. Indeed, Vernguard (1988; 1983) stated that multiplication and division problems are also proportion situations, solvable using the Rule of Three (see Greer, 1992, for a full description). Post et al. provided examples of how the three types of percent situations can be symbolised as proportion situations as follows:

Type 1 *Jessica scored 85 points on a 115 point test. What percent was this?*

$^{85}/_{115} = {}^{x}/_{100}$

Type II *If Jessica scored 74% on a test with 115 items, how many did she get correct?*

$^{74}/_{100} = {}^{x}/_{115}$

Type III *Jessica had 85 items on a test correct. This was 74%. How many items were on the test?*

$^{74}/_{100} = {}^{85}/_{x}$ or $^{85}/_{74} = {}^{x}/_{100}$

Solving the proportion equations, as suggested by Post and colleagues, is via the "standard" approach which is to "cross-multiply and solve for *x*" (p. 81). If this is the standard method, the question is asked as to how this is taught to students.

Teaching students the standard solution procedure for proportion equations appears to be a controversial issue. There is agreement that the cross-multiply method is a simple procedure, but concern for how it is introduced to students in the classroom. For example, Hart (1981) stated, "Teaching an algorithm such as $^{a}/_{b} = {}^{c}/_{d}$ is of little value unless the child understands the need for it and is capable of using it.

Children who are not at a suitable level to the understanding of $^a/_b = {}^c/_d$ will just forget the formula" (p. 101). Similarly, Parker and Leinhardt (1995) stated:

> ...the proportion method has become one of the most popular methods for solving percent problems. Conceptually, we would agree that a true understanding of the proportionality of percent would uncover the hidden referents. But in actual case, the proportion method has become something different. To students, it is just a procedure. (p. 451)

Also, Cramer, Post and Currier (1992) stated, "the cross-product algorithm is efficient, [yet] it has little meaning. In fact, it is impossible to explain why one would want to find the product of contrasting elements from two different rate pairs...The cross-product rule has no physical referent and therefore lacks meaning for students and for the rest of us as well" (p. 170).

In terms of instruction, there is consensus that the proportion equation must be introduced to students in a meaningful manner, or students must be provided with experiences to enable them to develop their own solution strategies for solving proportion equations. For example, Fisher (1988) stated, "If it is important that strategies other than the proportion formula be taught to secondary students in order that the students better understand proportion, then teachers must be made aware of the need and must accept the alternate strategies" (p. 166). Streefland (1985) stated that, "Formal procedures available as mathematical tools such as the 'rule of three' and 'cross-product' may make a provisional endpoint in this learning process but they should certainly not stand at its start" (p. 92). And, Post et al. (1988) stated, "Although it can be effectively argued that students need to automatise certain commonly used mathematical processes (Gagne, 1983) it can likewise be argued that the most efficient methods are often those that are least meaningful, and therefore should be avoided during the initial phases of instruction. Unfortunately, we sometimes confuse efficiency with meaning, and by default, even with the best intentions, we introduce a concept in the most efficient but least meaningful manner" (p. 81).

Focussing specifically on instruction, Robinson (1981) suggested that an understanding of the cross multiply procedure can be developed from a progressive approach to teaching ratio. Robinson's approach to ratio is by children constructing ratio "boxes" to correspond to the information given in a situation These ratio boxes are similar to the ratio tables advocated by Streefland (1985) and English and Halford (1995). The approach of constructing a ratio scale is used to assist children in developing understanding of the multiplicative structure inherent in the concept of ratio. For example, in the early stages of ratio instruction, according to Robinson, children would look at problems such as: *For every 2 fish John caught, Jim caught 3.* The box would be drawn as follows:

John's fish	2
Jim's fish	3

To solve the problem: *How many fish would John have if Jim caught 15?* would be solved by extending the box:

John's fish	2	2	2	2	2	=	10
Jim's fish	3	3	3	3	3	=	15

This step would then be simplified by focussing on the given information: Jim caught 15 fish, 15 3 gives 5; 5 boxes with 2 fish in each is equal to the number of fish John caught (2 x 5). Upon analysis it can be seen that from this example, in essence, the cross multiply procedure has been utilised.

John's fish	2	x
Jim's fish	3	15

Translating the above situation into a proportion equation gives the following :

2	x
3	15

Using cross multiply procedures gives

$2 \times 15 = 3 \times x$

$30 = 3x$

$10 = x$

According to Robinson (1981) this exploration guides students to generating the rule for solving proportion equations, "If you multiply the two numbers across from one another and divide by the other number, the correct answer is obtained" (p. 6).

As described in chapter 1(see section 1.1.3), the cross-multiply procedure is the Rule of Three of ancient times. Although there is a general feeling that this procedure should not be overtly taught to students (e.g., Cramer, Post & Currier, 1992; Hart, 1981; Streefland, 1985), Resnick and Omanson (1987) provided a strong argument for the study of mathematical procedures in their own right as an integral component of the study of mathematics. They stated that mathematical procedures are deeply embedded in mathematical principles, and that exploration of the principles upon which these procedures are built, can lead to deep mathematical understanding and appreciation. The fascination with the development of such procedures generates from the following questions posed by Resnick and Omanson, "What knowledge enters these constructions?" and what do "people know which permits and constrains the particular procedural variants they invented?" (p. 42). Tracing the historical development of the Rule of Three provides insight into the mathematical advancement of cultures long ago. Indeed, Swetz (1992) described the way the Rule of Three was emulated by mathematicians in the fifteenth and sixteenth century through his exploration of texts of the time, in the following:

> The 'Rule of Three', commonly known in its time as the 'Golden Rule' or the 'Merchant's Rule' was highly esteemed in the fifteenth and sixteenth century as being a powerful mathematical technique applicable to solve many problem situations. Today this rule would be recognised as a statement of simple proportion involving three quantities from which a fourth must be found. (p. 373)

Continuing his analysis of texts of this time, Swetz stated that the Rule of Three was the foremost procedure studied in schools at that time.

The simplicity and complexity of solving percent problems proportionally using the cross-multiply equation is evident. All cases of percent problems can be represented proportionally, and all proportion equations can be solved efficiently using one procedure. The implication for teachers is to consider carefully the instructional sequence for initial learners. However, when assisting learners who have experienced varying degrees of instruction in percent, the pathway is less obvious. The research study conducted by Allinger (1985) (discussed in section 2.3.2) highlights the situation of trying to maintain a balance between understanding and achievement. With the Year 10, low-achieving students in Allinger's study, the cross-multiply method for solving percent equations was presented to the students, and the students enjoyed the facility of using such a procedure to calculate solutions quickly. Such an approach to instruction for students experiencing difficulty with the study of mathematics would also find support from Noddings (1990) who stated "if it is clear that performance errors are getting in the way of concentrating on more significant problems, straightforward practice may actually facilitate genuine problem solving" (p. 15). She continued, "I'm not recommending drill and practice...rather I'm suggesting that teachers anticipate skills that students will likely need to construct important concepts and principles" (p. 15).

2.3.5 Further "helping strategies" for promoting percent knowing

Mental models

One of the recommendations suggested by Parker and Leinhardt (1995) for promoting percent knowledge is the search for a visual model of percent. As they stated, "a solid representation of percent may be one key to unlocking the door to an understanding of percent" (p. 465). The value of visual models in mathematics has been recognised by many others. For example, Post and Cramer (1989) stated, "Representations can be viewed as the facilitators which enable linkages between the real world and the mathematical world" (p. 223). Streefland (1985) stated, "The learning process should be designed with anticipating activities that account for connections with other learning sequences. To this aim schemas and visual models should be developed to support the long term learning process and within it general cognitive processes, such as abstracting, generalising and unifying" (p. 92). English and Halford (1995) stated "the essence of understanding a mathematical concept is to have a mental representation or mental model that faithfully reflects the structure of that concept" (p. 18). Hiebert and Carpenter (1992) stated, "The form of an external representation (physical materials, pictures, symbols, etc) with which a student interacts makes a difference in the way the student represents the quantity or relationship internally" (p. 66).

The most commonly used visual model for representing percent is the 10x10 grid, as it embodies the "whole divided into 100 parts" notion of percent. The use of this representation for instruction has been described previously (see section 2.1.1 and 2.2.3). The 10x10 grid has also been suggested as a means for assisting the conceptual understanding of percent calculations (e.g., Bennett & Nelson, 1994; Cooper & Irons, 1987; Weibe, 1986). According to Parker and Leinhardt (1995), one of the weaknesses of this model for such purposes is that it does not naturally lend itself to dealing with percent situations greater than 100. The difficulty is that one 10x10 grid represents one whole, which is 100%. To represent percents greater than 100 requires the use of another grid, which could be interpreted as two wholes.

The number line, or a derivative (e.g., comparison scale) is another model to assist students calculate percent problems (e.g., Dewar, 1984; Haubner, 1992; see also section 2.2.6). A number line model appears to be useful for assisting students to see the proportional nature of percent problem situations, and could also be used to assist with mental calculation and checking for reasonableness of solutions. Such a model lends itself to representation of percents greater than 100, as the number line can simply be extended past the 100% mark. Parker and Leinhardt (1995) evaluated such a model as potentially useful for representing percent as a proportional comparison, but they questioned the positioning of the number line in space and the positioning of the values on the number line. They felt that a vertical number line is at odds with typical representations of number lines, and the position of 100% at the bottom of the number line would cause conflict for students.

Problem interpretation

Research studies in percent indicated that students have difficulty in solving percent application problems, particularly percent situations involving percents greater than 100 (see section 1.2.3). For percent problem solving, Parker and Leinhardt (1995) stated that "the first stage of solving percent problems deals with reading, interpreting and defining relationships between the problem components" (p. 471). In the addition and subtraction literature, the part-part-whole notion was extensively investigated, suggesting the use of a part-whole schema to assist students identify problem types (e.g., Mahlios, 1988; Resnick, 1982; Wolters, 1983). For initial instruction, it was found that this schema assisted students to interpret the component parts of the written problem. In the percent study described by Allinger (1985) (see section 2.3.2), a similar type of analysis of the components of the problems was presented. Similarly, in the study by Mason (1975) (see section 2.3.2), the time-function procedure encouraged students to identify the events in the situation in order to interpret the problem situation. Such analysis of the component parts of percent problems appeared to assist students in word problem solving in these studies.

Holistic or sequential instruction in percent problem types?

A feature of traditional percent instruction has been the introduction of the three types of percent cases separately, with drill on each procedure followed by application in problem solving. Resnick (1992) suggested that an alternative instructional sequence to this traditional approach is where all cases of particular mathematical problems are introduced to students at once, and solving word problems occurs in the first instance, rather than after students have practised computational skills required to solve the word problems. This instructional strategy is described as presenting the whole "conceptual field" of problems to students, and the theory behind such an approach is that students can see how similar mathematical principles for solution hold across all cases. Of this approach, Resnick stated, "A program of this kind constitutes a major challenge to an idea that has been widely accepted in educational research and practice. This is the notion of learning hierarchies, specifically that it is necessary for learners to master simpler components before they try to learn complex skills" (p. 422). Of the strategy of beginning with mathematical problems presented in words (commonly referred to as story problems), Resnick stated: "It is a strategy which contrasts quite sharply with traditional methods of arithmetic drill in which practice on number proceeds independently of situations involving quantity...Rather than decontextualised drills, what is needed is extensive practice in solving well-understood quantity problems...from the teacher, or from the students" (p. 417).

2.3.6 Summary of key points

In this section, summaries of research studies exploring various instructional approaches for percent were described. Such research has not served to elucidate a single, best approach to teaching percent, but has highlighted the need to consider the student body for whom instruction is intended. In terms of helping children achieve an understanding of percent, there are two distinct bodies of learners whose needs must be considered. Students who are yet to receive formal instruction in percent need to be provided with opportunities to develop a rich concept of percent, from which intuitive, meaningful and legitimate computational procedures for percent calculation can naturally evolve. For students who have received varying degrees of instruction in percent, it appears that helping strategies for building and linking fragmented percent knowledge, and for solving percent equations meaningfully need to be provided.

The issue of developing percent knowledge through linking to the concept of proportion was seen to be an approach fraught with complexity. Proportional reasoning was seen to be embodied in the proportion equation $^a/_b={}^c/_d$, and developing students' understanding of this equation was discussed. However, the development of proportional reasoning takes a long time, which calls into question the sensibility of building percent knowledge through proportional understanding if proportional understanding and reasoning is limited. Other issues were raised in this section, pertaining to percent instruction, specifically the potential of a mental model which embodies the proportional nature of percent, strategies for interpreting percent application problems, and sequencing instruction.

2.4 Percent problem solving and percent knowledge

2.4.1 Overview

The focus of this section is percent problem solving, percent knowledge, and instruction. In section 2.4.2, a model for percent problem solving, which is presented as an efficient and effective model for guiding percent instruction for Year 8 students, is described. In section 2.4.3, guiding models for instruction to build and link mathematical knowledge are described. In section 2.4.4, a model of percent knowledge, identifying intuitive, concrete, computational and principled-conceptual percent knowledge, is proposed. A summary of the key points contained within this section is presented in section 2.4.5.

2.4.2 A proportional method for percent problem solving

Parker and Leinhardt (1995) stated that percent is fundamentally a language of proportion, and that percent instruction should focus on developing students' understanding of percent as a proportion. However, as discussed in the previous section, proportion concepts and proportional reasoning take a long time to develop, and are dependent upon the establishment of other prior knowledge (see section 2.2.3). If instruction in percent is to build from proportion knowledge, it can be argued that students will experience difficulty in linking percent to proportion if their proportion knowledge is limited, faulty or tenuous. Percent instruction, therefore, must be designed which provides students with models and strategies to experience problem solving in a successful manner. The models and strategies must encapsulate the notions of perent as a proportion, and must be accessible to students regardless of the level of their proportional knowledge.

From analysis of the literature pertaining to percent, new strategies and a model for percent problem solving is proposed. The model and strategies combine into a method to enable students to experience percent sitatuations, and is based on representing percent as a proportion. The method offers a a holistic approach to percent problem solving, which assists students interpret, analyse, represent and solve percent situations. The method draws on various suggestions across a wide range of sources, and comprises more than a procedure for percent calculations. The proportional model for percent problem solving is presented as follows.

Percent application problems are of three types, and contained within each problem are three elements: the part, the whole (total amount), and the percent. In a similar vein to a part-whole schema for interpreting addition and subtraction word problems (Mahlios, 1988; Resnick, 1982; Wolters, 1983), a part-whole-percent schema is proposed as a means of interpreting percent problems. The part-whole-percent schema may assist in the first stage of percent problem solving, which, according to Parker and Leinhardt (1995), lies in reading, interpreting and defining the relationships within percent problems. To represent percent problems, a single vertical number line is proposed. This number line is similar to the comparison scales suggested by Dewar (1984) and Haubner (1992), where an amount is compared to the percentage base of 100 simultaneously in a linear fashion on a dual-scale number line. The dual scale number line provides a clear image of the proportional relationship of percent situations (Dewar, 1984; Haubner, 1992), and can be used to model all three types of percent application problems, including increase and decrease. The number line appears to have the potential to satisfactorily represent all percent situations, however, it is acknowledged that arguments of the limitations of this model have been proposed (see Parker & Leinhardt, 1995). The representation of the percent situation on the number line enables the proportion equation to be constructed and solved using the ancient, and once highly esteemed (Swetz, 1992) Rule of Three procedure.

In this proposed method of percent problem solving, interpreting, representing and solving percent application problems occur in a series of five steps. The first step is to identify the elements given within the problem in terms of the part-whole-percent schema. The second step is to construct a dual-scale number line and label appropriately, and then to translate the information given in the problem onto the number line. The next step is to transfer the information positioned on the number line to a proportion equation. The last step is to solve the proportion equation using the Rule of Three procedure. An example of the use of the percent-schema and dual scale number line is illustrated in Figure 2.17 for a Type I problem. Type II and Type III problems are depicted in Figure 2.18.

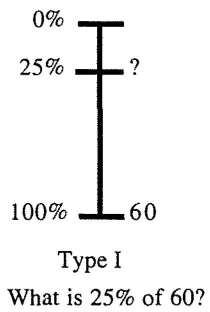


Figure 2.17. Use of the percent-schema and Rule of Three for interpreting, and representing a Type I percent problem.

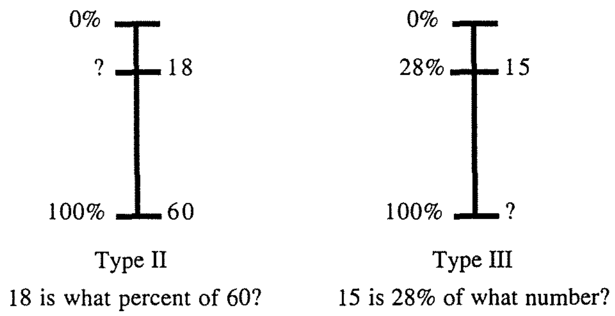


Figure 2.18. Interpreting and representing Type II and Type III percent problems.

For percent increase and decrease problems, extra analysis of information presented in the problem is required to successfully interpret and represent the problem using the number line. For example, the problem: *If I receive a 25% discount on $60, how much do I pay?*, can be interpreted two ways: subtractively or multiplicatively.

The calculations, therefore, can be carried out in either one or two steps. To solve in two steps, 25% discount is found, and this is subtracted from the original amount. To solve in one step is to interpret the 25% discount as being 75% of the original amount. Representation of the situation would occur on the number line as in Figure 2.19.

Percent increase situations can also be interpreted in two ways, in both an additive and multiplicative sense. For example, the problem: *The cost of the ticket was $50, but I had to pay an extra 15% booking fee as well, so how much did I have to pay?*, can be solved in two steps by finding 15% of the whole and adding this to the original price, or in one step by interpreting a 15% increase as 115% of the original amount. To represent this problem on the number line requires the number line to be extended beyond 100%. Representation of this problem situation is depicted in Figure 2.20.

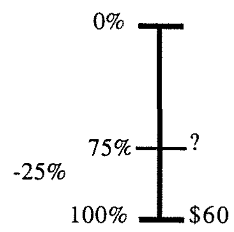


Figure 2.19. Representation of percent discount situations using the number line.

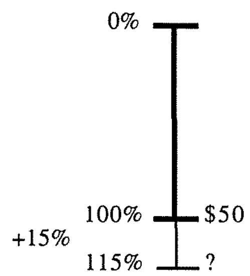


Figure 2.20. Representation of percent increase situations using the number line.

It is proposed that the benefits of this method for interpreting percent situations relate to issues of efficiency and effectiveness. The model is effective as it embodies the proportional nature of percent situations. It also provides a clear means for constructing the proportion equation which can thus be seen as an equivalent fraction. The dual scale number line also has the potential to promote estimation skills through positioning of the values on the number line using common percent benchmarks of 50% and 25%. Using the Rule of Three procedure to solve the proportion equation is an efficient method, and although it may be taught in a rote and meaningless fashion, frees up mental space required for problem solving (Sweller, 1988, 1989; Sweller & Low, 1992). The use of a single procedure for solving all types of percent problems also is efficient, and is a means for presenting the whole conceptual field (Resnick, 1992) of percent problems to students.

2.4.3 Guiding models for instruction to build and link mathematical knowledge

The proportional method for percent problem solving, proposed in section 2.4.2, focusses specifically on percent problem solving. However, percent is a multi-dimensional concept (Parker & Leinhardt, 1995) and as stated in section 1.1.3, can be a number, a comparison, a ratio, a statistic, a function. It could be argued that percent instruction which consists primarily of calculation exercises may not promote students' knowledge of percent in a global sense. There appears to be a need to define percent knowledge to inform and guide instruction.

Studies on teaching and learning multiplication (Lampert, 1986), subtraction, and fractions (Leinhardt, 1988) have shown that working definitions of mathematical knowledge, defined as comprising intuitive, concrete, computational and principled-conceptual knowledge can serve as a base for effective mathematical instruction which aims to build and link the four classes of knowledge.

To develop mathematical knowledge, Leinhardt (1988) suggested that instruction should begin with students' intuitive notions, and exploration of the mathematical topic using concrete representations and manipulations. Concrete materials are then used to represent expanded algorithms to develop computational knowledge. As Leinhardt stated, "the power of concrete representations is that they can help the students to develop an understanding of the meaning of the concepts and procedures and mathematics and they may provide a mechanism for solving new problems" (p. 121). To complete the instructional sequence, simple algorithms are practised without concrete materials, then algorithms are practised in a variety of contexts. Similarly, Lampert (1986) suggested that the goal of effective mathematical instruction is to enable children to make connections between concrete materials and principled, computational practices. The focus of instruction is on strengthening connections between intuitive, concrete, computational and principled/conceptual knowledge. Lampert also uses the four types of mathematical knowledge to guide instruction, following a model proposed by Kamii (1985) who stated that knowledge, and therefore instruction, progresses from "intuitive to concrete, concrete to computational and computational to principled/conceptual, with principled/conceptual knowledge being the ultimate goal" (Lampert, 1986, p. 313).

From the above descriptions, it could be argued that effective instruction in mathematics is instruction which builds and links intuitive, concrete, computational, and principled/conceptual knowledge. As shown by Leinhardt (1988) and Lampert (1986), planning for instruction in mathematics, therefore, can begin with applying definitions of mathematical knowledge to a piece of mathematics' curriculum to provide descriptions of intuitive, concrete, computational and principled/conceptual knowledge of that curriculum topic.

2.4.4 A model of percent knowledge

Parker and Leinhardt (1993) have called for a much broader definition of percent knowledge which is more than "conversions, computation and applications" (p. 47); a definition which embodied percent's "multiple and often embedded relational character" (p. 47) (as previously stated in section 1.1.4). The literature, however offers few definitions which encompass such a multi-dimensional definition of percent knowledge. Following suggestions that mathematical topics can be defined as comprising intuitive, concrete, computational and principled-conceptual mathematical knowledge (Lampert, 1986; Leinhardt, 1988) the following model of percent knowledge is proposed as a means for defining percent, and therefore guiding instruction. The model is based on Leinhardt's definitions of intuitive, concrete, computational, and principled-conceptual mathematical knowledge and applied to defining intuitive, concrete, computational, and principled-conceptual *percent* knowledge.

Intuitive percent knowledge

Leinhardt's definition of intuitive *mathematical* knowledge suggested that intuitive knowledge can be regarded as that knowledge usually acquired prior to formal instruction, and/or that knowledge which relates the topic to the real world situation. From this definition, intuitive percent knowledge is proposed for the model of percent knowledge to consist of: the language of percent as used in the real world; the concept of percent as it relates to 100; common percent benchmarks; and the relationship of percent equations to real world situations. Intuitive percent knowledge, therefore, is proposed to consist of the following:

1. Knowledge of the language of percent as it is used in the real world, in terms of such notions that the reduction of the whole relates to real world discount and loss situations; and the increase of the whole relates to real world profit, interest and increase situations.
2. Knowledge of the concept of percent which includes such things as the part/whole notion of percent (that a percent is part of a whole where the whole

has 100 parts), and that the sum of percent parts gives the whole (i.e., 100%), and that a percent greater than 100% is greater than the original whole.

3. Knowledge of common percent benchmarks, such as 50% is $^1/_2$, 25% is half of a half, and 10% is $^1/_{10}$.
4. Knowledge of the three types of symbolic percent statements (equations) and how they translate to real-world situations.

Concrete percent knowledge

Leinhardt (1988) stated that "*Concrete knowledge* is knowledge of the nonalgorithmic, frequently pictorial systems...that often serve as a basis for demonstration or as an explanation of an algorithm" (p. 121), and that "the power of concrete representation is that they can help students to develop an understanding of the meaning of the concepts and procedures of mathematics and they may provide a mechanism for solving new problems" (p. 121). Leinhardt added that, as concrete/pictorial representations are typically used by teachers and texts, concrete knowledge is students' understanding of those representations. This brings into the question the appropriateness of the concrete/pictorial representations selected for use in mathematics instruction. It is generally accepted that an appropriate representation for percents, decimals and fractions is the 10x10 grid (e.g., Brueckner & Grossnickle, 1953; Reys et al., 1992; Weibe, 1988), and an appropriate representation for modelling proportional situations is the number line (e.g., Dewar, 1984; Haubner, 1992; Weibe, 1988).

For this model of percent knowledge, it is proposed that concrete percent knowledge relates to: (i) understanding of the appropriateness of concrete representations of percent used during instruction, (ii) the ability to interpret pictorial representations of percent, (iii) the ability to estimate pictorial representations of percent, and (iv) the ability to represent real world percent situations pictorially.

An understanding of the appropriateness of concrete representations of percent used during instruction relates to Leinhardt's discussion of the importance of pictorial representations for concept development. Ten x 10 grids and number lines divided into 100 equal sections relate to the concept of percent meaning hundredths.

Interpreting pictorial representations is necessary concrete percent knowledge as percent instruction often relies on the use of 10x10 grids to simultaneously depict the relationship between percents, fractions and decimals. Students need to be able to interpret this interconnectedness of one pictorial model depicting three different mathematical representations of like meaning.

The ability to estimate pictorial representations of percent is important concrete percent knowledge. Exercises in shading percents of regions are for the purpose of building percent benchmarks. Percent benchmarks relate to students knowing that

50% is half of a region or set, 25% is one quarter of a region or set, and 1% is a very small amount whereas 99% is almost the whole. Students need to be able to compare and contrast a given percent with a wide range of other percents in order to understand percent (Reys et al., 1992).

The ability to represent real world percent situations pictorially relates to Leinhardt's (1988) statement as to the use of concrete representations as a "mechanism for solving new problems" (p. 121). In section 2.2, various pictorial models were described for representing and assisting problem context interpretation, solution estimation, and calculation of percent application problems. According to Leinhardt's definition of concrete mathematical knowledge, appropriate concrete models are those which serve to explain algorithms and which embody the principles of the calculation procedures. In terms of percent knowledge, the concrete model must relate to the selected algorithmic procedures chosen to solve percent problems. Thus, they are dependent upon the selected procedures. The two main models which represent all three types of percent situations are: (a) the 10x10 grid requiring part/whole fraction/percent knowledge and decimal multiplication and division calculation procedures (e.g., Bennett & Nelson, 1994) (see section 2.2.3), and (b) the dual-scaled number line/comparison scale requiring proportional reasoning and solving proportion equation procedures (e.g., Dewar, 1984; Haubner, 1992) (see section 2.2.6). Concrete percent knowledge of pictorial representations of real world percent problems thus would be dependent upon the pictorial model selected.

In this model of percent knowledge, concrete percent knowledge is proposed as consisting of the following:

1. Knowledge of concrete referent in terms of 100 parts, particularly 10x10 grids and percent number lines.
2. Ability to estimate pictorial representations of percent.
3. Ability to represent real-world percent situations pictorially.

Computational percent knowledge

Leinhardt (1988) stated that "*Computational knowledge* is the procedural knowledge of mathematics, the algorithms and procedures for operations" (p. 121). For this model, it is proposed that computational percent knowledge comprises (i) estimation and mental computation, and (ii) computational skills. For estimation and mental computation knowledge, certain percent calculations would be performed mentally through estimation and rounding. For example, to mentally find 5% of a quantity, 10% of the quantity would be found and then halved. Computational percent skills include decimal, fraction, percent conversions; changing fractions with denominators that are factors of 100 to percents; and computing the three types of percent problems, including percent increase and decrease problems. For this model of percent knowledge, computational percent knowledge is proposed as the following:

1. Ability to estimate and mentally compute percentages using common percent benchmarks.
2. Ability to convert percents to common fractions and decimal fractions, and vice-versa.
3. Ability to calculate the three types of percent application problems, including percent increase and decrease situations.

Principled/conceptual percent knowledge

Leinhardt (1988) stated that "*Principled/conceptual knowledge* is the underlying knowledge of mathematics from which the computational procedures and constraints can be deduced" (p. 121). In this model of percent knowledge, the concept of percent is the basis of a number of principles which legitimise calculation procedures. The basic concept of percent is that a percent is a part of a whole, where the whole has 100 parts; and percent is a comparison to a standard base of 100. From this concept of percent, a number of principles arise, which underlie percent calculation procedures. The principles identified have been termed: the *complement principle*; the *fraction-percent equivalence principle*; the *decimal-percent equivalence principle*; the *additive/subtractive percent increase/decrease principle*; and the *multiplicative percent increase/decrease principle*. The principles which constitute principled-conceptual percent knowledge for this model are described as follows:

1. *Complement principle.*

   A percent is part of a whole; for each percent, there is a complement percent to comprise the whole.

2. *Fraction-percent equivalence principle.*

   A fraction is also part of a whole. If the whole has 100 parts, the fraction represents a percent because of the concept of percent. If the whole is not 100, this can be attained through the principle of fraction equivalence. [The principle of fraction equivalence enables any fraction to be converted to an equivalent fraction of specified denominator or numerator through solving for the unknown. Solving for the unknown requires the mathematical principle of inverse, namely the *multiplication/division inverse principle* (that multiplication is the inverse of division and division is the inverse of multiplication). For example, to find an equivalent fraction for $^3/_7$ with a denominator of 100, the following procedure would be followed

$$x/_{100} = 3/_7$$
$$x = 3 \times 100/_7$$

It is noted that finding equivalent fractions does not solely rely on the principle of fraction equivalence. Simpler conversions can be carried out if the denominator or numerator of the original fraction is a multiple or factor of the equivalent fraction. In such cases, different mathematical principles are utilised. Thus, the principle of fraction equivalence as stated above is seen as separate from, but related to, the fraction/percent equivalence principle. The fraction equivalence principle is seen as necessary for higher order fraction, ratio, proportion, percent applications where equivalent fractions are not factors or multiples of 100.]

3. *Decimal-percent equivalence principle.*
Decimals have a base of 10. If the decimal is hundredths, the decimal is a percent because of the concept of percent.

4. *Additive/subtractive percent increase/decrease principle.*
Percent increase situations can be interpreted additively or subtractively, when only the amount of increase/decrease is the focus. For example, the term "25% increase" is seen additively as the original whole + a further 25%. A 25% decrease is seen subtractively as the original whole less 25%.

5. *Multiplicative percent increase principle.*
Percent increase/decrease situations can be interpreted multiplicatively, where the percent increase/decrease is in terms of the original amount. For example, the term "25% increase" is seen multiplicatively as 125% of the original amount; the term "25% discount" is seen multiplicatively as 75% of the original amount. The multiplicative percent decrease principle also relies on the complement principle (Principle 1).

In this model, these five principles permit the three types of percent applications: (i) finding a part or percent of a number (Type I percent application); (ii) finding a part or percent one number is of another (Type II percent application); and (iii) finding a number when a certain part or percent of that number is known (Type III percent application). Percent increase and decrease situations can be either Type I, II or III application problems, thus there are only three types of percent applications which can be solved through knowledge of the principles of percent.

For this model of percent knowledge, principled-conceptual percent knowledge is defined as consisting of the following:

1. Knowledge of the concept of percent as part of a whole, where the whole has 100 parts.
2. Knowledge of the concept of percent as a comparison to a standard base of 100.
3. Knowledge of the complement principle, namely where every percent part has a complement part to total the whole of 100%.
4. Knowledge of the fraction-percent equivalence principle, namely that every fraction is a percent if the denominator is 100.
5. Knowledge of the decimal-percent equivalence principle, namely that decimal hundredths are percents as they relate to a base of 100.
6. Knowledge of the additive/subtractive percent increase/decrease principle where percent increase/decrease situations are interpreted as an added or subtracted change to the original whole.
7. Knowledge of the multiplicative percent increase/decrease principle where percent increase/decrease situations are interpreted as a multiplicative change to the original whole.
8. Knowledge that there are only three types of percent application problems.

2.4.5 Summary of key points

In this section, a proportional method for percent problem solving was proposed as an efficient and effective means to assist interpretation, representation and solution of percent application problems. This method was seen to provide a model and strategies for percent problem solving to enable students to experience success in percent problem solving regardless of their level of proportion knowledge. The dual-scale number line within the method was proposed as reflecting the essence of percent as a proportion. Instruction in the method, therefore, is based on presenting percent as a proportion. Also in this section, a model of percent knowledge was proposed, consisting of intuitive, concrete, computational and principled-conceptual knowledge. This model was created to define percent knowledge in greater terms than conversions and calculations and thus to serve to guide instruction in percent.

2.5 Towards an integration of percent issues to inform curriculum design

In the first two sections of this chapter, teaching approaches to promote percent concept development and methods for solving percent applications problems were presented. Teaching approaches for developing the concept of percent, as described in section 2.1, included ideas for linking of percent to common and decimal fractions (e.g., Brueckner & Grossnickle, 1953; Hauck, 1954), using 10x10 grids to promote mental images of percents (e.g., Bennett & Nelson, 1994; Reys, Suydam & Lindquist, 1992), investigating the special language of percent used in society (e.g., the use of such terms as 100% attendance, 200% attendance) and building estimation skills through exploration of patterns of simple percent calculations (e.g., Cooper & Irons, 1987; Glatzer, 1984), linking percent to ratio understanding (e.g., Brown & Kinney, 1973), and studying percent expressions as statements of proportion (e.g., Schmaltz, 1977). In section 2.1.5, the potential advantages and shortcomings of the teaching approaches for percent concept development were identified, and will be reiterated here. Linking percent to common and decimal knowledge appears to be a useful starting point, assuming students have developed sufficient common and decimal fraction knowledge. However, this approach could lead to students developing a narrow view of percent if it is the only focus of teaching. The use of 10x10 grids to assist students to link common and decimal fraction knowledge to percent knowledge appears to be useful for promoting knowledge of equivalence between common fractions, decimal fractions and percents. However, the applicability of 10x10 grids can be questioned due to their seeming inability to adequately represent percents greater than 100, when the use of two grids are suggested (see Figure 2.1). Exploration of the special language of percent as used in society and the promotion of mental computation skills are useful suggestions for inclusion in a teaching program, and appear to offer students an opportunity to draw upon and clarify their intuitive notions of percent. Whilst arguments against linking percent to common and decimal fractions, and promoting the definition of percent as meaning "hundredths" are worth consideration, promoting percent knowledge through linking to ratio knowledge may be difficult. For students to develop an understanding of percent as a ratio appears to require students to have developed a solid understanding of ratio in the first instance. Similarly, to understand percent as a proportion would require students to have developed a solid understanding of proportion.

In section 2.2, various procedures, models and strategies for solving percent application problems were described, and these included the use of fraction/percent overlays, elastic strips, 10x10 grids, identification of key words, use of mnemonic strategies, and comparison scales. In section 2.2.8, potential advantages and shortcomings of each were outlined, which can be listed as follows: (a) fraction/percent overlays exemplify the part/whole notion of percent situations but can be used only for Type I and Type II percent situations; (b) Type III percent situations can be adequately modelled using an elastic strip, but the elastic strip models the proportional nature of percent situations, not the part/whole notion of the overlays; (c) 10x10 grids require extensive decimal and fraction multiplication and division procedures as well as a high level of computational estimation skill; (d) grids are time-consuming to construct and also do not appear to lend themselves to percents greater than 100; (e) mnemonics and key words appear to be very procedural methods, unrelated to percent concepts and principles, and therefore appear limited in their ability to promote students' principled-conceptual knowledge; (f) comparison scales appear to be simple to construct and powerful for representing percent situations as statements of proportion, but offer few suggestions for assisting students solve proportion equations meaningfully; and (g) procedures for calculating the proportion use relatively simple numbers, and fail to address understanding of the calculations required in proportion equations when the numbers are not multiples or factors of 100.

As seen in section 2.1 and 2.2, the percent literature offers a variety of opinions on teaching and learning percent. With such diversity of opinions and suggestions, the percent literature presents as a picture of controversy and confusion, rather than serving as a guide for the development of a comprehensive and coherent teaching program. Indeed, the studies of various teaching programs trialed in classrooms, as summarised in section 2.3.2, also indicate little consensus about percent instruction and percent teaching programs.

One particularly prominent theme in the literature is that the meaning of percent is primarily proportion (Allinger, 1985; Parker & Lienhardt, 1995; Post, Behr & Lesh, 1988). The perceived difficulty of implementing a program of percent instruction from a proportional perspective is that students typically do not have a well-developed concept of proportion at the time they meet instruction in percent (Lo & Watanabe, 1997). This issue was the point of discussion in section 2.3.

As a proportional approach to percent instruction was discussed, paradoxically as embodying the prime concept of percent, further literature was consulted in a effort to glean ideas for the development of a teaching program on percent. Pertinent points can be summarised (as in section 2.3.5) in terms of the following: (a) that mental models facilitate understanding of mathematics concepts; (b) that successful problem solving is based on the ability to analyse and interpret problems; and (c) that practise of computational skills before application to solving problems is not necessarily a useful classroom practice. In section 2.4.2, such suggestions were considered in light of developing a teaching program on percent, and can be summarised as the following: (a) that an appropriate mental model which represents percent as a proportion may assist the development of students' understanding of percent as a proportion and as well promote students' knowledge of the proportion concept itself; (b) that promotion of a part-whole-percent schema (similar to a part-part-whole schema for interpreting addition and subtraction word problems) may assist students to interpret the three elements contained in percent problems as well as enabling students to distinguish between Type I, Type II and Type III percent problems; and (c) that within the teaching program, the three types of percent problems should be presented to students simultaneously rather than treated as separate cases.

From the literature on percent, selected literature on proportion, and other literature pertaining to teaching and learning mathematics, an approach for interpreting, representing and solving percent problems, which utilised a vertical, dual-scale number line and the Rule of Three procedure for calculation of proportion equations, was presented in section 2.4.2. As stated in section 2.4.2, the method was devised in the conscious effort to provide students with strategies for percent problem solving and a model of percent as a statement of proportion which was accessible to students regardless of their level of proportional knowledge. In a further attempt to find direction for the construction of a teaching program on percent, which could be seen as more than promoting students' conversion, computation, and application skills, a model of percent knowledge was proposed in section 2.4.4. Based on Leinhardt's (1988) model of mathematics knowledge, percent knowledge was defined as consisting of intuitive, concrete, computational and principled-conceptual percent knowledge. A teaching program was developed with instruction planned to progress from intuitive to concrete, concrete to computational, computational to principled-conceptual percent knowledge, in accordance with suggestions by Lampert (1986). The proportional number line method for interpreting, representing and solving percent problems (presented in section 2.4.2) was deemed as a possible vehicle to promote students' principled-conceptual knowledge (listed in section 2.4.4) through the development of concrete and computational knowledge as students solve percent problems.

Thus faced with a percent literature which appeared to present a confusing picture of issues associated with teaching and learning percent, a teaching program was adopted that promoted principled-conceptual knowledge of percent as a proportion and provided a structure for continued knowledge growth of percent as a multi-faceted concept.

CHAPTER SUMMARY

In this chapter, issues in teaching and learning percent were presented. Summaries of instructional approaches for both developing the concept of percent in learners, and for solving percent problems, were described. Included in this chapter was a method for interpreting, representing, and solving percent problems based on a proportional teaching approach to percent, and a model of percent knowledge to promote students' percent knowledge in a broad sense. This chapter concluded with a synthesis of literature on percent in particular and selected literature on teaching and learning mathematics in general in an attempt to integrate percent issues, and to provide direction for the development of a coherent and comprehensive teaching program.

Analysis of literature on teaching and learning percent indicated the variety of strategies, methods and approaches available to teachers for developing instruction in percent, and the lack of consensus amongst authors in the field. The myriad of approaches suggests that, as students move through their years of schooling, they could be exposed to a variety of instructional approaches for percent knowledge development. Such a variety of instructional approaches could precipitate the development of confusion and misconceptions within the percent domain. Teaching programs which focus on providing assistance for students experiencing difficulty in the study of mathematics is the field of mathematics diagnosis and remediation. Diagnostic-prescriptive mathematics teaching is the focus of the next chapter.

CHAPTER 3
LITERATURE

The diagnostic-prescriptive teaching of mathematics

CHAPTER OVERVIEW

The focus of this chapter is the diagnostic-prescriptive teaching of mathematics. It is an exploration of issues pertaining to the development of learning difficulties in mathematics. In this chapter is presented a summary of research into the development of mathematical errors, misconceptions and alternative conceptions; instructional programs for overcoming such inappropriate mathematical knowledge; and factors which affect the success of such programs. Also presented in this chapter is a discussion of the role of metacognition in promoting successful mathematics learners and strategic problem solvers.

This chapter is divided into six sections. In section 3.1, diagnosis is discussed. In section 3.2, research into error patterns in mathematics is summarised. In section 3.3, remediation, or the provision of intervention programs and strategies to specifically assist students experiencing learning difficulty in the study of mathematics, is addressed. In section 3.4, specific programs of intervention are overviewed, and remediation is addressed once more and the need to redefine the term 'remediation' is discussed. In section 3.5, the Conceptual Mediation Program (CMP) (Lyndon, 1995) as an alternative framework for diagnosis and remediation is presented. In section 3.6, the Conceptual Mediation Program is discussed in terms of mathematics diagnosis and remediation, and a model of diagnostic-prescriptive mathematics teaching incorporating CMP is proposed.

3.1 Diagnosis

3.1.1 Overview

The focus of this section is on diagnosis in mathematics instruction. In section 3.1.2, the diagnostic process is described, and in section 3.1.3, the importance of determining the exact nature of the learning difficulty is discussed. This section concludes with a summary of key points presented in section 3.1.4.

3.1.2 The nature of learning problems in mathematics

It is when a student begins to exhibit symptoms of mathematical misunderstanding that those concerned with the individual's mathematical progress (e.g., parents, teachers, and the student him/herself) are alerted to the existence of a mathematics learning problem. As Kirby and Williams (1991) stated, "learning problems begin to exist only as they [the students] begin to fail" (p. 242). In the school setting, the provision of extra assistance to promote mathematical achievement is the domain of mathematical diagnosis and remediation.

Once students have been identified as experiencing difficulty with the study of mathematics, determining the nature of the difficulty is required (Ashlock, 1994; Shapiro, 1989; Underhill, Uprichard & Heddens, 1980). This is the process of diagnosis. Underhill, Uprichard and Heddens (1980) used a medical analogy to characterise the diagnostic process, where a doctor gathers as much information as possible about the nature of a patient's complaint prior to prescribing a remedy to overcome the troubling condition. In relation to mathematics, the diagnosis of a student's learning problem begins at a global level, through analysis of the student's various formal test papers; observations of the student working on mathematical problems and activities in the classroom, and in other subjects; observations of the student interacting with his/her peers; determining the home environment of the student in terms of assistance for mathematical studies and the student's preferred learning style (e.g., Fernald, 1971; Reisman, 1982; Shapiro, 1989).

Included in this information gathering process can be the administration of various psychological tests to determine whether the student has some particular physiological problem which is slowing down his/her educational progress (Fennell, 1981). If a mental process, or other disability has been eliminated as contributing to the student's difficulty in mathematics, further information must be gathered to pinpoint the specific nature of the problem. This may be accomplished through: analysis of the student's solutions to pen-and-paper test items; examination of written scripts of the student's work; interviews with, and observations of, the student as he/she works on various mathematical problems and exercises. Diagnosis, then, is characterised as a process of continual probing: providing students with a variety of mathematical tasks in relation to specific mathematical topics to determine the student's mathematical strengths and weaknesses (Ashlock, 1994; Wilson, 1976a).

A further dimension to the diagnostic process is offered by Shapiro (1989) who suggested that it should be the classroom learning environment and a teacher's teaching style which must also be analysed to build a clearer picture of the nature of a student's learning difficulty. As Shapiro (1989) stated, "Indeed, if a child fails to master an academic skill, it directly suggests potential failure in the instructional methodologies" (p. 23). Historically, diagnosing the nature of students' difficulties in any academic field was influenced by the expectation that the learning difficulty was a result of some physiological processing capacity of the learner (e.g., Kephart, 1960). Recently, diagnosis has emphasised the necessity to focus more on the students' learning environment, and the quality of the instruction the child received prior to developing the learning difficulty (MacDonald, 1972; Shapiro, 1989). As Derry (1990) suggested, "our explanations of learning difficulties are placing less emphasis on the diagnosis of structural learning processes" (p. 19). Rather than diagnosis suggesting weaknesses in the memory processing of the child for example, students' learning difficulties are more frequently diagnosed as stemming from a lack of understanding directly attributable to instruction (MacDonald, 1972; Woodward & Howard, 1994).

3.1.3 Learning difficulties and specific mathematical learning difficulties

Although the trend is away from attributing students' processing deficiencies as the cause of learning problems, the need for careful diagnosis of the nature of the student's learning difficulty has been highlighted by Kirby and Williams (1991). Kirby and Williams also acknowledged that the instructional environment may be the prime source of a learning difficulty for the majority of students in the mainstream classroom, but this may not be the case for all children. For example, quite often, students with learning problems exhibit behaviour problems in the regular classroom, such as inattentiveness and distractive behaviour. The learning environment may be the cause, but there may be another cause. According to Kirby and Williams, distractive behaviour may be a direct result of the child experiencing difficulty in a specific area, or it could be because the child's physiological make-up is causing the inattention, and hence the learning difficulty. Or, it could be that the child is suffering an emotional problem which is causing the inattentive behaviour which is causing the specific learning difficulty. Thus, Kirby and Williams have categorised learning difficulties as due to three main causes: physiological attention problems, emotional problems, and specific subject-related learning problems. They suggested that, if one of these three sources is the prime cause of the child's learning difficulty, this will cause secondary and tertiary problems. For example, a child whose learning problem is emotional in nature, will most likely exhibit attention deficit-type behaviours, which will cause a tertiary problem, in that the child will develop a specific learning problem due to lack of attention when a specific subject is being taught. Similarly, a child experiencing difficulty with learning a specific subject will develop secondary emotional problems, such as anxiety, avoidance, low self-esteem, and this may cause attention deficit behaviours.

Kirby and Williams suggested that it is learners primarily with specific learning difficulties with whom classroom teachers must deal most consistently. Classroom teachers, thus, must become acquainted with a wide range of strategies and methods for helping students with specific learning needs to achieve in the classroom.

Learners with emotional problems or physiological attention problems require more specialised assistance from others trained to deal with such problems. As Underhill et al. (1980) stated, "No matter how sophisticated the classroom teacher becomes in mathematics teaching, there still will be children with severe mathematical difficulties with which the regular classroom teacher cannot cope with in the classroom environment" (p. 67).

Kirby and Williams' (1991) analysis is useful in providing a focus for the diagnosis of students' difficulties in mathematics, and suggests that some children need other, external assistance (such as counselling, or medical assistance), but that not all overactive, inappropriate behaviour needs to be drug- or other-regulated. Much diagnosis will point at the instructional setting as a cause of learning problems, and therefore, the classroom teachers' instructional program is accountable for students' progress.

3.1.4 Summary of key points

In this section, the importance of diagnosis in determining the nature of a student's learning difficulty was discussed. It was suggested that the majority of students' learning difficulties can be traced to poor instruction, although such difficulties can result from other causes. The nature of the learning difficulty must be ascertained in order to develop appropriate intervention programs.

3.2 Error patterns in mathematics

3.2.1 Overview

The focus of this section is on error patterns and misconceptions in mathematics. In section 3.2.2, a summary of error pattern research in mathematics is presented. Errors as knowledge are discussed in section 3.2.3, and errors in terms of constructivist learning theory are described in section 3.2.4. Explanatory theories for the development of error patterns in mathematics are presented in section 3.2.5. A summary of key points in this section are listed are section 3.2.6.

3.2.2 Error pattern research

Analysis of students' errors on mathematical tasks can serve as a means for gleaning information about a student's mathematical knowledge and skill in a particular domain, and is thus a focus for diagnosis of the nature of the student's mathematical learning difficulty at a specific level. The study of error patterns in mathematics computation has revealed that, contrary to the belief that all errors are random and careless, they occur regularly and consistently (Brumfield & Moore, 1985; Cox, 1975). To determine the nature of the error, providing students with several items of similar type of computational exercises enables classification of errors as either consistent or careless. According to Cox (1975), a student demonstrating the error pattern at least three times out of five attempts indicates that the error is habitual and automatic. Technology has moved to assist the process of error pattern analysis with the advent of several computer programs designed for such a purpose (e.g., Orey & Burton, 1992; Woodward & Howard, 1994). Analysis of students' errors on pen-and-paper tests, or via computer-assisted means, can provide insight into the consistency of the error, and thus provide a focus for programs of intervention.

Error analysis can be regarded as a window through which the thinking processes of the individual in relation to construction of the error may be viewed. For example, a student may respond with a solution of $^3/_5$ to the mathematical calculation of $^1/_2+^2/_3$, and such a response may be consistent across all such fraction computational exercises. A hypothesis might be that the student is simply adding the numerators to get the digit for the numerator in the solution, and is adding the denominators to get the digit for the denominator in the solution. The creation of such a solution suggests that the student is overgeneralising whole number addition rules, and that the student has little conceptualisation of fraction addition. However, merely hypothesising the nature of the error through such means can lead to misinterpretation of a student's thinking (Orey & Burton, 1992). Probing deeper into the individual's mind to determine the nature of the error, interviewing students (Ashlock, 1994; Orey & Burton, 1992), or even having students write about their erroneous solution processes (Drake & Amspaugh, 1994) will assist the diagnostic process.

Categories of students' (and adults') patterns of mathematical error have been well documented over many mathematical domains. Ashlock (1994) provided a comprehensive historical summary of error pattern research, focussing particularly on identification of error patterns in computation. Recent studies have reported on evidence of consistency in students' (and adults) errors in mathematical skill calculation and conceptual understanding in other mathematical topics, such as Year 8 students' understanding of parallel lines (Mansfield & Happs, 1992), Year 5 students' understanding of ratio and proportion (Fong, 1995), Year 10 student's understanding of circle geometry (Borassi, 1994); high school students' skill in factoring polynomials (Rauff, 1994); secondary school teachers' concepts of group theory (Dubinsky, Dautermann, Leron, & Zazkis, 1994).

As stated in chapter 1 (see section 1.1.6), the value of error pattern research can be seen to operate on at least three levels. Primarily, the accuracy of the diagnosis will enable specific intervention strategies and activities to be developed, with a greater chance of successfully helping the student overcome the learning difficulty and progress towards mathematical achievement. At a secondary level, error pattern research has pedagogic implications. If it is known the various errors students develop in relation to particular mathematical topics, teachers can develop programs of instruction in an effort to possibly prevent the development of such errors (Maurer, 1987; Stefanich & Rokusek, 1992). The creation of appropriately rich learning environments can thus be created from an informed position with greater teacher awareness of possible student misconceptions of the teaching experience. The study of systematic errors benefits teaching in that sources can be determined and learning environments developed that inhibit errors (Behr & Harel, 1990). At yet another level, error pattern research has implications for teacher training programs. For example, Thipkong and Davis (1991) alerted educators to the influence of teacher errors and misconceptions in their teaching, and thus on student learning. In their research, they identified preservice teachers' misconceptions in interpreting and applying decimals, noting that the misconception "multiplication makes bigger, division makes smaller" was extremely prevalent. They suggested that if teachers are aware of their own errors and misconceptions in particular mathematical topics, great care will need to be taken so that such errors and misconceptions are not transferred to learners. Research on preservice and inservice teacher errors serves to thus inform mathematics teacher training programs.

3.2.3 Errors and knowledge

The study of error patterns has significantly influenced the field of diagnosis and remediation in mathematics, providing alternative perspectives on what errors indicate (e.g., Ashlock, 1994; Ashlock et al., 1983). Traditionally, students who made errors in their work were regarded as suffering from some learning disability (e.g., Kephart, 1960). From this perspective, students made errors because they lacked knowledge of a "correct" algorithm, and as such, these students needed slow and progressive re-teaching in order to repair their knowledge deficit (e.g., Valett, 1976). A deficit model of error production suggests that the student has remained completely ignorant of the correct skill performance; that nothing has been learned as a result of the original teaching effort.

Consistency in production of errors tends to negate a view that errors are indicative of a lack of knowledge. According to Ashlock (1994), the fact that errors can be systematic over certain mathematical computations indicates that they are habitual, automatic responses to specific stimuli. In contrast to random, careless errors, habitual errors are not self-detected nor self-corrected; they are conceptual and learned. The implication of errors as conceptual and learned knowledge provides an alternative perspective on what errors indicate about a student's mathematics knowledge. Errors are thus indicative of the presence rather than the absence of knowledge. The notion of mathematics learning *disabilities* suggests a difficulty in acquiring knowledge. Consistency in errors indicates that the student is, in fact, capable of learning. From this perspective, what a student has learned are merely incorrect ways of doing things. The student has somehow acquired a *learned* disability rather than a *learning* disability (Ashlock, 1994).

3.2.4 Errors and constructivism

Constructivist theories of learning state that knowledge is actively constructed by the individual. Of constructivism, Confrey (1990a) stated, "constructivism can be described as essentially a theory about the limits of human knowledge, a belief that all knowledge is necessarily a product of our own cognitive acts" (p. 67). Interpretation of mathematical errors and misconceptions as knowledge, then, is consistent with constructivist learning theory. Errors are personal constructions in the mind of the individual, and thus are meaningful, and make sense, to the individual (Confrey, 1990b; Rauff, 1994). They are an individual's interpretation of a mathematical situation at the time. Confrey (1990a) stated that the work of Piaget has served to highlight the concept of knowledge as mental constructions with Piaget's documentation of the child's view of particular mathematical situations. As she stated:

> ...a child may see a mathematical or scientific idea in quite a different way than it is viewed by an adult who is expert or experienced in working with the idea. These differences are not simply reducible to missing pieces or absent techniques or methods; children's ideas also possess a different form of argument, are built from different materials, and are based on different experiences. Their ideas can be qualitatively different, which can sometimes mean that they make sense only within the limited framework experienced by the child and can sometimes mean they are genuinely alternative. To the child, they may be wonderfully viable and pleasing. (p. 108-109)

Confrey's statements provide a clear picture of active learners operating on and interpreting mathematical and scientific situations within their own mental framework. Further, Borassi (1994) provided a comprehensive description of the growth of mathematical knowledge at a societal level to reflect mathematical knowledge growth at an individual level in terms of constructivism. Citing the philosophical contributions of explanatory theories of mathematical knowledge, including (a) Dewey, who suggested that knowledge is a process of inquiry motivated by doubt; (b) Kuhn, who described knowledge as oscillation between normal science and scientific revolution, where unacceptable results and unsolvable problems leads to new perspectives; and (c) Lakatos, who stated that mathematical knowledge occurs through a dialectic process of proofs and refutations; mathematical knowledge can thus be seen as a constructed body of knowledge, changing and evolving over time. In light of this philosophical analysis of the construction of mathematical knowledge through history, Borassi likens the growth of mathematical knowledge in students. He described a view of learning "as a generative process of meaning making that is personally constructed, informed by the context and purposes of the learning activity itself, and enhanced by social interactions" (p. 167). In terms of errors and misconceptions, Borassi's discussion shows that, at certain times in history, mathematical knowledge was erroneous, but was the socially constructed mathematical knowledge of the time. Mathematical errors and misconceptions, therefore, can be regarded as constructed knowledge. As Confrey argued, "Students are always constructing an understanding for their experiences...Students' misconceptions, alternative conceptions and prior knowledge provide evidence of this constructive activity" (p. 112). In a similar vein, Rauff (1994) described students' mathematical knowledge as constructed over time, students' mathematical knowledge is based on their beliefs, and errors and misconceptions stem from their beliefs. According to Rauff, errors are logically based from within the student's "belief-set" and are thus meaningful and logical to the owner.

3.2.5 The development of consistent errors

In relation to the development of error patterns, Resnick et al. (1989) provided a description of how students develop patterns of error in computation, which can be seen to fit within a constructivist framework. According to Resnick et al., it is through children's attempts to integrate new knowledge with established knowledge that errors often develop. In explaining this process, Resnick et al. suggested that, within the mathematics classroom, teachers provide various examples of mathematical procedures for students to learn and practise. In the classroom situation, teachers can only provide certain examples, and when students are faced with computation exercises which have not been explained by the teacher, the students must decide for themselves how to proceed. According to Resnick et al., as a result of "making these inferences and interpretations, children are very likely to make at least temporary errors. Errorful rules are a natural result of children's efforts to interpret what they are told and go beyond the cases actually presented...[therefore] errorful rules are active constructions" (p. 25); that is, they are the child's interpretation of the mathematical situation at the time.

Brown & Van Lehn's (1982) "Repair Theory" also provides an explanation of the development of patterns of error which can also be interpreted as active constructions by students. Their theory described the development of error patterns in terms of computer language. In their theory, students' errors are called "bugs", and the process through which children develop these bugs is called Repair Theory. Repair Theory explains the process of how children develop consistent patterns of error: when learners are confronted with tasks on which they are unsure of how to perform (on which they have become "stuck"), they use a simple "repair" tactic which enables them to produce a solution and become "unstuck". In this way, repairs occur as a result of learners' choosing alternative solution paths in order to produce answers.

If the repair is erroneous and left unchecked, the incorrect repair, through repetition, becomes a habit, produced in response to appropriate stimuli. The repair is now a consistent error; that is, a "buggy" solution. Some students take several alternative solution paths in response to the one stimulus, hence switching between bugs, and this is labelled "bug migration". Bug theory is an explanatory theory for the development and consistency of erroneous algorithmic procedures (buggy solutions), and the existence of several incorrect procedures for the same stimulus (bug migrations). [Bug migration does, however have implications for accurate diagnosis of consistency in errors. Some procedures yield correct solutions, thus confirming the legitimacy of the buggy procedure in the mind of the student, and hence making remediation of that error pattern all the more difficult (Ashlock, 1994)].

3.2.6 Summary of key points

In this section, a summary of error pattern research was presented. Errors were described as knowledge, and this has implications for intervention. The difficulty in intervention programs is thus in dealing with knowledge rather than assuming that errors result from a lack of knowledge. Such issues in remediation are the focus of the next section.

3.3 Remediation

3.3.1 Overview

The focus of this section is on mathematics remediation as the means for promoting successful mathematics learners. Section 3.3.2 begins with a summary of diagnostic-prescriptive models to guide the development of instructional programs to cater for students experiencing difficulties with particular mathematical topics. Included in this section is a brief review of some resources to support the diagnostic teaching of mathematics. In section 3.3.3, diagnostic-prescriptive instruction as a program of good teaching is discussed. Discussion in the following sections focuses on how the field of diagnostic-prescriptive instruction has expanded beyond programs and method for assisting students who are experiencing difficulty with particular mathematical topics to encompass models and strategies to assist students become better mathematical problem solvers, and more self-directed, self-reliant, autonomous learners. In section 3.3.4, the *Cognitive Apprenticeship Model* as a model for helping children develop their problem solving skills and strategies is described. In section 3.3.5, promoting autonomy in learning through the development of metacognitive skills is summarised. Factors influencing metacognition, as success and effort, are discussed in section 3.3.6. The issue of skill development for problem solving is discussed in section 3.3.7. A summary of key points in this section is presented in section 3.3.8.

3.3.2 Diagnostic-prescriptive teaching models

Reisman (1982) proposed a five step model for the diagnostic teaching process. After the first step of identifying the student's mathematical strengths and weaknesses, the next step is to hypothesise the nature of the difficulty and then formulate the mathematical behavioural objectives expected to be displayed by the student as a result of the intervention. The fourth step is to plan and implement the corrective remedial procedures, and the final step is the ongoing evaluation of the progress of the student as a result of the corrective instruction. The fourth and fifth steps are seen to interact and overlap.

Wilson (1976a) proposed a model for diagnosis and remediation in mathematics which simultaneously provided a guiding framework for research in this field. According to Wilson, the model operates at two levels. Primarily, the model is to be utilised to assist the planning of specific intervention strategies for students experiencing difficulty in mathematics. Guiding principles for intervention are those based on building and correcting conceptual knowledge and expanding students' awareness and repertoire of cognitive strategies. At a secondary level, the model is for research into specific remedial teaching strategies from which hypotheses can be drawn to generally inform planning of effective teaching programs and episodes. The model is based on a specific perspective of mathematics: Mathematics is a highly organised, hierarchical structure, and children experiencing difficulty with the study of mathematics lack knowledge of the structure of the topic. Interaction with children in the diagnostic/remedial situation is a means for researching activities and experiences which appear to foster children's mathematical knowledge whilst also contributing to the research of the structural organisation of mathematics itself. Wilson viewed his model as a means for unravelling the complex structure of mathematics topics to assist in the diagnostic process and hence pinpoint remediation programs. Further, Wilson believed that mathematical knowledge is the ability to apply mathematics within various situations, and to be able to "see" a topic inherent in a range of exemplars and representations. According to Wilson, this level of mathematical knowledge is the extent to which remediation should strive. Thus, in the intervention program, student's knowledge is continually probed and diagnosed over a variety of exemplars and representations. The remediation process is the continual revisiting of the topic until a student's knowledge of the topic has reached such a level of complexity. Thus the model ascribes to developing indepth mathematics concepts, principles and schemas.

Wilson described an instructional cycle as a guiding teaching model for promoting deep mathematical concept knowledge, with diagnosis being a central aspect. The model clearly described a cycle of activities for teaching: initiating, abstracting, schematising, consolidating, and transferring. Ashlock et al. (1983) incorporated this model into a description of their diagnostic-prescriptive approach to mathematics. Briefly, the model suggests that, in the initiating phase, the teacher provides activities for students to explore the new concept; in the abstracting phase, the teacher carefully structures tasks so that the key principles of the concept can be understood by students; in the schematising phase the teacher provides activities for the purpose of linking students' prior knowledge to new knowledge. Having laid the foundation for concept development through exploration and structured linking activities the teacher then provides opportunities for students to practise and consolidate their new knowledge. According to Ashlock et al., when students can "easily, habitually and accurately recall new concepts and skills" (p. 26) they are in a position to transfer their knowledge to other new situations. The transferring stage is the final phase in the model before returning to the initiating phase. Diagnosis is integral to this model, where, at every phase, diagnosis informs the teacher's decisions about further instruction. Observation, interviews and pen-and-paper tests are suggested means for gathering data on students' understanding of the topic being taught.

A further model for diagnostic-prescriptive teaching is proposed by Underhill, Uprichard and Heddens (1980) where the focus is on task analysis. Similar to the Wilson (1976a) model, this model is based on the assumption that mathematics is a hierarchically structured subject. Through task analysis, mathematical topics are broken up into component parts, thus pinpointing important steps in the learning sequence so that sequential instruction can be built. The key requirement in this model is for the teacher to become familiar with the hierarchical structure of a mathematical topic. Following this, a teacher can ascertain a student's level of knowledge in relation to that topic through interviews, where the student's knowledge is explored through use of concrete materials and representations. In this way, a teacher is gathering evidence for development of future learning experiences. The three main phases of this model are: *task analysis*, *diagnosis of student knowledge*, and *provision of learning experiences* based on both task analysis and student knowledge analysis. As Underhill et al. stated, "The diagnostic model is based upon a good teaching/learning sequence so that the outcomes lead back into the sequence in a very smooth and natural way...The diagnostic data are easily translated into needed curriculum experiences...facilitating ease of design of both tests and curriculum experiences" (p. 39-40). To support the implementation of diagnostic-prescriptive programs of intervention, various resources are available (e.g., Ashlock; 1994; Ashlock et al., 1983; Booker, Irons & Jones, 1980; Fennell, 1981; Jones & Charlton, 1992; Valett, 1978; Wilson, 1976a). Written within a framework of a diagnostic-prescriptive mathematics model for instruction, such resources offer prescriptive teaching sequences for instruction in various mathematical topics. For example, Ashlock (1994) provided a comprehensive list of activities to specifically help students understand the legitimacy of computational procedures and to overcome their own computational error patterns. The activities often involve the use of concrete materials, and the activities are designed to promote students' understanding of the place-value numeration system in which we operate. For example, the use of bundling sticks and base 10 blocks are advocated to exemplify regrouping procedures in our number system, which enable computational procedures to be performed. Another resource which focuses on computation is presented by Booker, Irons and Jones (1980). This resource describes a structured, sequential approach to teaching the four operations meaningfully, through the use of materials (particularly base 10 blocks) and appropriate language. A resource kit by Valett (1976) provides a collection of activities for developing students' understanding of number. Similarly, other authors (e.g., Ashlock et al., 1983; Booker, Briggs, Davey & Nisbet, 1992; Jones & Charlton, 1992) describe teaching approaches to various topics within the mathematics curriculum, emphasising the key concepts which need to be developed so that students will overcome their difficulties in understanding of specific mathematical topics. Such resources provide guidelines for meaningful teaching of mathematical topics for use with individual students, or with whole classes of students.

3.3.3 Diagnostic-prescriptive instruction and "good teaching"

Programs, models and strategies for assisting students who are experiencing learning difficulty in the classroom appear to share certain similarities in that they follow a spiral approach, cycling though stages of diagnosing students' prior knowledge of a topic, planning instruction based on task analysis, and monitoring students' progress to guide further instruction. Diagnosis, particularly analysis of errors and misconceptions, provides the teacher with a picture of a student's knowledge of and beliefs about a topic so that learning tasks can be planned to promote appropriate knowledge growth. The process of task analysis provides teachers with an awareness of the relative complexity of a particular mathematics topic, so that instruction can be planned to be sufficiently rich in order to minimise students' difficulties, through promoting a rich understanding of that topic. Analysis of errors and exploration of misconceptions enable a learner's mathematical knowledge and beliefs to be determined. Continual monitoring of student progress in a topic is a means of individualising instruction, and thus catering to the learning needs of all students. Thus, diagnostic teaching models appear to be based on good teaching models. Resources to support the diagnostic-prescriptive teaching of mathematics provide suggestions for building and linking students' mathematical knowledge, and the means for providing representations to give meaning to mathematical procedures.

Such resources can therefore be regarded as comprehensive summaries of the analysis of mathematical tasks (one of the elements within the diagnostic process, according to Underhill et al., 1980 - see section 3.3.2), and are thus resources for "good teaching". Teaching approaches which focus on remediation can thus be regarded as little different to strategies and methods employed in good teaching practices. If diagnostic-prescriptive approaches to teaching are simply implementing "good teaching", then it follows that if a diagnostic-prescriptive approach to teaching is taken in the first teaching effort, the need for remedial practices will be diminished. As Fischer (1989) stated, "Although diagnostic-prescriptive teaching is used to find and remedy problems that children have with the content to be learned, ideally it should be used before children have the problems. The diagnostic-prescriptive approach should have been used in the first place" (p. 7)

3.3.4 Cognitive apprenticeship

Cognitive apprenticeship models are global approaches to teaching which follow typical apprenticeship training methods (Reid & Stone, 1991; Rojewski & Schell, 1994), and are designed specifically to help students become more successful learners and problem solvers. In apprenticeship training, a trainee is apprenticed to a master craftsman, learning and acquiring skills and knowledge of the trade alongside the expert. In the cognitive apprenticeship model, the focus is not on developing expertise in execution of typically manual, trade-related skills, but on the development of high-level, mental processes and thinking skills displayed by the expert problem solver. Learners are placed in problem situations, and together with experts, solve real problems. In this collaborative problem solving enterprise, the expert models successful problem solving skills and strategies. The cognitive apprenticeship model places the teacher in a co-operative problem solving role, guiding each student's cognitive development through solving of real problems.

Rojewski and Schell (1994) proposed a four-component model of cognitive apprenticeship, of *content*, *methods*, *sequence*, and *sociology*. Each component highlights specific knowledge, skills, and teaching considerations for successful teaching. The *content component* emphasises domain knowledge, heuristics, control strategies and learning strategies: knowledge and skills required by learners. The *methods component* focuses on teaching methods, and emphasises the need for teachers to plan carefully for learning experiences, plan opportunities for guided practice, modelling, coaching, scaffolding, fading and so on. The *sequence component* is a global teaching method, focusing on sequencing of learning experiences, and such things as increasing capacity, increasing diversity and global before local skills. The *sociology component* considers the learning environment, and encompasses such principles as co-operative learning, context-based learning, establishing a community of practice, and so on. The components of this model interrelate, and the model provides a comprehensive analysis of the interaction of elements which make up a model to guide good teaching.

Rojewski and Schell's components of the cognitive apprenticeship model highlight the complexity of the task of teaching, and the many factors which must be attended to for good teaching. This model suggests that teachers must provide students with strategies and skills for successful learning, that the teaching sequence must be carefully planned and structured to nurture students' knowledge development, and that the learning environment must be conducive to interactive learning.

Reid and Stone (1991) described the value of cognitive apprenticeship models by drawing on the learning theories of Vygotsky and Piaget: from Vygotsky, the value of the social interaction in knowledge development, and from Piaget, the need for learners to internally reflect on new material presented to build new cognitive structures. Reid and Stone suggested that the power of the cognitive apprenticeship model is that it capitalises on these two elements of social interaction and development of cognitive structures as the learner interacts with the expert during problem solving, observing expert problem solving skills and behaviours. The teacher's role is to promote an interactive learning environment, drawing the students into the learning process through social interaction and knowledge sharing, and the encouragement of reflection upon action. Reid and Stone proposed that learning occurs via *prolepsis* and *reflective abstraction*. In this view, prolepsis is the bridging mechanism which enables two people to understand each other. In the teaching situation, this is through the teacher establishing a link to the students' knowledge, and propelling them forward to develop new knowledge. The construction of new knowledge is through reflective abstraction, namely, personal, internal reflection on individually held mental structures. Thus, the teacher plays a direct mediating role of building and linking student knowledge through social interaction, as Reid and Stone (1991) stated, "connecting links, links between past and current knowledge, links between participants representation of the task, links between knowledge and activity" (p. 17).

3.3.5 Metacognitive skills and knowledge and self control of learning

Characteristics of unsuccessful learners are clearly presented in the following statement by Chan (1993):

> In contrast to efficient, active, strategic, independent and self-directed learners, students with learning difficulties are found to show limited understanding of effective thinking and self-directed learning strategies, such as ways of setting a goal, making a plan, designing work tactics and routines, monitoring progress and evaluation for self-improvement. (p. 22)

By analysing what efficient learners do, it can be seen that the majority of effective learning strategies are metacognitive in nature. Metacognition, defined by Kirby and Williams (1991) is "the conscious awareness of ways of approaching tasks, of processing information, and of monitoring success" (p. 70). Students experiencing difficulty, as suggested by Chan, do not demonstrate such metacognitive strategies. Developing students' metacognitive skills appears to be a means of developing more efficient learners, and literature summaries of metacognitive training programs indicate that this is the case (Chan, 1993; Kirby & Williams, 1991).

Metacognitive training is the process of developing students' metacognitive skills (Cole & Chan, 1990). Cole and Chan suggested that explicit instruction in both cognitive and metacognitive strategies is a means of improving students' task performance. They contrast cognitive strategies as task specific strategies, applicable to only a limited number of domains, with metacognitive strategies as strategies which are highly generalisable and applicable to a wide range of situations. However, the fact that students have been provided with highly generalisable metacognitive strategies is no guarantee that they will be employed in other situations. Training in metacognitive skills must also include a focus on assisting students to see the generalisable nature of the skills they are developing; that the application of such skills is equally applicable to other learning situations (Brown & de Loache, 1983).

Brown and Palinscar (1982) described three types of training: *blind*, *informed* and *self-regulatory*. They stated that *blind training* is similar to traditional methods of instruction used in schools where children are told what to do, but not encouraged to think about why, or how the procedure relates to other situations. *Informed training* is where strategies are provided for children and reflectivity on the significance and transferability of the strategy is encouraged. In *self-regulatory training*, instruction is supplemented with training in planning, monitoring and assessing of strategies over a wide range of situations. Brown and Palinscar's discussion of training types provides useful guidelines for the depth of the training situation, and how metacognitive training must also aim to be self-regulatory training.

3.3.6 Success and effort

The development and use of cognitive and metacognitive skills is for the purpose of assisting students in successfully completing required learning tasks. The key role of success as a motivating force is well documented (e.g., Chan, 1993; 1991; Cole & Chan, 1990; Mercer & Miller, 1992). To ensure students experience success, the teacher must orchestrate situations for this to occur. Fernald (1971) stated that ensuring students experience success is a paramount requirement of remedial instruction. This process, according to Fernald, is the process of "emotional reconditioning", and ensuring the student experiences success on the very first day of remedial instruction is the means to overcome the student's negative emotional feelings towards mathematics. As suggested by Fernald, a student who has experienced repeated failure in performing mathematical tasks will only approach anything mathematical with a degree of fear and loathing. Often, negative responses to mathematical situations are in the form of avoidance behaviours, where a student shows extreme reluctance to perform any mathematical tasks, particularly those tasks with which the student has had a long history of failure. The notion of avoidance behaviour has been well-documented in the past. Over 100 years ago, James (1890) stated that avoidance behaviours are exhibited as a response to avoiding the situation: the best way to avoid failure is never to try anything new, because if there is no attempt there can be no failure, and with no failure, there can be no humiliation.

According to Cole and Chan (1990), instruction must be organised around two basic notions of "ensuring early success, and avoiding situations that lead to failure" (p. 14). Similarly, Mercer & Miller (1992) stated that, "Mathematical instruction must be designed to ensure success and promote positive attitudes" (p. 24).

To become independent, autonomous and successful learners, metacognitive awareness and control is required (Chan, 1993). In terms of metacognition for successful task performance, students can experience success by applying cognitive and metacognitive skills to purposely planned tasks. However, convincing students that such strategic skills will enhance successful performance may not be a simple task. Student's self-confidence and beliefs come into play. As Chan (1993) stated, "Students must first believe in the worth and benefits of their strategic actions and effort before they will apply those strategies and efforts in situations that require strategic learning" (p. 22).

Applying skills for successful task performance requires effort. Effort can only come from the individual, and students must become aware that they are primarily responsible for their own achievements. As Mercer and Miller (1992) suggested, students should be continually informed that what they do influences both their successes and failures, so that they "realise that their behaviour directly influences what happens to them, and consequently, that they are in control of their own learning"(p. 24). Teachers must convince students that their own efforts and persistence, together with application of learning strategies, determine successful task performance (Chan, 1993).

The focus on effort for success squarely places the onus for achievement on the individual. Learning is an active process, and as Weinstein and Mayer (1986) suggested, learning "occurs within the learner and...can be directly influenced by the learner" (p. 315). From this constructivist perspective, the learner actively influences the learning that occurs. Learning requires energy (Baird & White, 1982), and it follows that the learner is in direct control of the amount of energy to be expended.

For some learners, apparent energy expenditure may yield little result; the affective domain, and learning strategies are again called into play (as discussed above). However, as Derry (1990) stated, "advanced skill comes easily for almost no-one. Becoming an expert means very hard work" (p. 28). Becoming an expert requires practice.

3.3.7 Skill development and mathematics problem solving

The provision of opportunity to practise skills to automaticity is a necessary component of an instructional program (Derry, 1990; Mercer & Miller, 1992). However, there may be reluctance on some teachers' part to provide the opportunity for practice of mathematical skills, where the emphasis is on conceptual understanding of mathematics, rather than "drill-and-practice" of facts, skills and procedures. Traditionally, rote practice of mathematical skills was a common feature of mathematics instruction (following behaviourist learning theories, e.g., Thorndike, 1923; 1922). With calls to make mathematics more meaningful (Brownell, 1956/1987), drill-and-practice techniques became unfavourable. As Resnick and Ford (1984) stated, "computational and conceptual approaches to mathematics instruction have existed in a kind of uneasy balance as mathematicians and educators have pressed for increased conceptual understanding of mathematics" (p. 7). However, there is a need for automatic recall of mathematical skills, particularly in relation to problem solving (Anderson, 1985; Glaser & Bassock, 1989) and cognitive load (Resnick & Ford, 1984; Sweller, 1988, 1989). For example, Resnick and Ford (1984) stated that as "certain processes can be carried out automatically, without the need for direct attention, more space becomes available in working memory for processes that do require attention" (p. 31). They continue "...at least certain basic computational skills need to be developed to the point of automaticity so they can avoid competing with higher level problem solving processes for limited space in working memory" (p. 32). Similarly, Glaser and Bassock (1989) suggested that the problem solving process alternates between reliance on basic skills and higher levels of strategy and comprehension, and thus have suggested the desirability of basic skills being automatised. As they stated:

> In the development of higher levels of proficiency certain component skills need to become compiled and automatised so that conscious processing capacity can be devoted to higher levels of cognition as necessary. (p. 635)

The words of Gagne (1983) also emphasise the vital nature of computational proficiency to automaticity. As he stated, "The rules for this part of problem solving (automatic rule using to carry out mathematical operations) would be best not just learned, not just mastered, but automatised"(p. 18).

The development of skill automaticity, as Anderson suggested (1985), distinguishes novice problem solvers from experts. Familiar situations to many people, such as driving a car, learning column addition, and so on, pose real problems to people facing such situations for the first time. According to Anderson, skill learning involves three steps: (i) a cognitive stage, (ii) an associative stage, and (iii) an autonomous stage. Anderson suggested that the cognitive stage of skill learning demands the application of much cognitive effort to learn a skill or procedure. At the associative stage the procedure is practiced and more efficient means of performing the task are discovered. As a result of practice the procedure enters the third stage, the autonomous stage, and is performed with little conscious thought. Anderson stated that increased practice means a particular skill will be performed more quickly and accurately.

Anderson argued that through practice, problem situations (such as driving a car, completing column addition) become automatic processes. In describing novices and experts, Anderson suggested that expertise in a particular domain is a product of practice, memory, and the ability to recognise patterns. Similarly, Sweller and Low (1992) elaborated on how expertise in a problem situation is developed. They stated that experience in problem situations enables problem schemas to be developed. These schemas are cognitive constructs which allow "problem solvers to recognise a problem as belonging to a specific category that requires particular moves for solution" (p. 83). Hence, the better developed the cognitive construct, the more expertise shown in a problem situation. Appropriate schemas are developed through experience in the problem solving situation. The automatisation of skills is necessary to reduce cognitive load. This is the basis of cognitive load theory (Sweller, 1992; 1989; 1988). Of cognitive load theory and its implications for instruction, Sweller and Low (1992) provided the following summary:

> The theory assumes that instructional designs must be structured in a manner that focuses attentional resources on problem states and their associated moves because it is familiarity with problem states and their associated moves that provides the basis of schemas. If instructional designs require students to devote cognitive resources to other aspects of a problem or other features of instruction, a heavy extraneous cognitive load may be imposed that interferes with learning and problem solving. (p. 88)

Thus, practising certain skills to the point of automaticity appears to assist problem solving in terms of enabling more working memory space to be available to devote to higher order processes.

3.3.8 Summary of key points

In this section, mathematics remediation, or teaching approaches to assist students who are experiencing learning difficulty in mathematics, were discussed. Models for the diagnostic-prescriptive teaching of mathematics were presented, and resources for mathematics remediation were reviewed. In this section, it was argued that remedial programs and strategies are "good teaching approaches", and therefore should be a natural part of a teacher's teaching style. For developing students' problem solving skills, the cognitive apprenticeship model was presented, which was seen as a model based on the teacher taking the role of mediator in the development of students' problem solving skills and knowledge, and thus assisting students experiencing learning difficulties in mathematics. Also, in this section, instructional issues relating to metacognition, success, effort and practice were discussed in terms of promoting successful learners and mathematical problem solvers.

3.4 Specific strategies and programs of intervention

3.4.1 Overview

In this section, particular strategies and programs of intervention are described to exemplify current trends in intervention research. The programs and strategies discussed have been categorised as either general programs, based on re-teaching of the topic, or specific programs, which focus on individual students' errors/misconceptions. General programs of intervention are presented in section 3.4.2, and specific programs of intervention are presented in section 3.4.3. In section 3.4.4, the state of intervention research is summarised, and a new definition of mathematics remediation is proposed. This section concludes with a summary of key points presented in section 3.4.5.

3.4.2 Intervention programs based on re-teaching

The Strategic Mathematics Series (SMS)

Mercer and Miller (1992) described a model of mathematics instruction, designed to target the improvement of the learning environment to reduce the development of students' learning difficulties in mathematics. The Strategic Mathematics Series (SMS) follows a diagnostic model, and can be summarised as a sequence of seven steps beginning with a pretest to ascertain students' knowledge of the topic, then instruction based on building knowledge through concrete representation, to symbolic representation, to abstract application and problem solving strategies. A posttest to ascertain progress is administered, followed by practice to fluency. This program of instruction is based on principles of developing problem solving and related thinking skills (such as analysing, planning, checking problem solving attempts), strategy instruction, encouraging generalisation of skills in other domains, and attending to the affective domain by ensuring success. Thus, integral to the SMS model, is the focus on specific metacognitive skill development.

Mercer and Miller applied the SMS program to instruction for helping students with learning problems acquire, understand, and apply basic mathematics facts. Research results indicated a positive effect. The SMS is based on ten instructional components of a good teaching program, which Mercer and Miller summarised from research. The ten components of effective instruction related to the need for teachers to: (i) select appropriate mathematics content, in line with the goals of a mathematics curriculum which emphasises critical thinking, conceptual understanding and real-life problem solving; (ii) establish goals and expectancies, where the teacher and learner both are informed of the direction of the learning process, and constant monitoring and assessment is carried out; (iii) provide systematic and explicit instruction based on careful planning and lesson design; (iv) teach students to understand mathematics concepts, by using first concrete materials, and representation before proceeding to abstract symbolism; (v) monitor progress as students work on tasks, continually assessing the appropriateness of the task and making decisions as to future teaching activities; (vi) provide feedback to students so that they are constantly informed of their progress in the learning situation; (vii) teach to mastery by providing opportunities for students to practice new skills to automaticity so that such skills can be applied in problem situations; (viii) teach problem solving, including instruction on strategies, skills and metacognitive processes required for successful problem solving; (ix) teach generalisation so that skills and knowledge can be transferred across domains and situations; and (x) promote a positive attitude to mathematics, primarily by providing students with many opportunities to experience success, and having students realise that they are directly responsible for both their successes and failures in the learning situation. The ten components of effective mathematics instruction exemplify the nature of teaching as an active process. The teacher must orchestrate the learning environment so that students acquire and develop skills to succeed on academic tasks, and come to see their role as an active member of the teaching/learning process.

Conceptual Change Intervention

Connell and Peck (1993) reported on a conceptual change intervention that included a program of instruction based on creating a rich learning environment to build students' knowledge and understanding of particular mathematics topics. The four phases of this model included an unstructured, discovery phase, where students explored concepts using materials; a concrete phase, where students used concrete materials to develop meaning; an internalisations phase, where students solved problems using concrete materials; and an evaluative phase, where students' understanding of topics was probed as they worked on tasks. Connell and Peck implemented this model in the school setting over a two year period. Mathematics instruction for each topic area was guided by the model. Connell and Peck reported that this model of instruction was valuable in helping children develop rich mathematical concepts of given topics, and it was therefore a good teaching model for initial instruction. However, in implementing this model for the purpose of creating a dynamic learning environment for overcoming students' misconceptions, Connell and Peck reported that the interference of prior, inappropriate knowledge over the linkage and development of rich conceptual knowledge was powerful indeed. As they stated, "When clearly identifiable student conceptual change occurred, it had limited effect due to interference from previously acquired mental structures. Newly acquired information appeared to serve in a superordinate capacity with previously learned procedures or concepts being automatically applied" (p. 329). They also found that if concrete materials had been inappropriately used in initial teaching, this also caused interference. The importance of the initial teaching effort was evident. As a result of their study, Connell and Peck stated that the interfering nature of prior knowledge, in spite of provision of a rich, compensatory learning environment "argue strongly for extreme care in the nature of the initial mathematical experience" (p. 329).

3.4.3 Intervention programs based on errors/misconceptions

Correcting error patterns in computation

A comprehensive instructional program for overcoming students' error patterns in computation has been developed by Ashlock (1994). Ashlock has provided a resource to assist educators develop awareness and expertise in analysing students' error patterns in computation, and specific activities designed to help students overcome such errors. For example, for remedial assistance for the student whose computation (taken from Ashlock, 1994) is the following:

26	60	74	
+ 3	+ 24	+ 5	
11	84	16	(p. 133)

Ashlock suggested the provision of place value identification games, using base ten blocks or paddle pop sticks (for bundling into tens and ones) to represent each addend in the exercise, and the drawing in of place columns to display the tens and ones in each of the numbers. From such activities, it can be seen that the focus is on the particular computation in which the error pattern surfaced, and activities to build place value understanding and demonstrate the illegitimacy of the student's solution process.

A much more prescriptive approach to correction of error patterns is offered by Gable, Enright, and Hendrickson (1991). They described a three-phase model, with the first phase being the identification of the consistency of the error, and including interviewing the student. The second phase begins the intervention, and involves three stages of demonstration of the correct algorithm, selection of "error groups and appropriate corrective strategy" (p. 7), and practice of the new algorithm. The appropriate corrective strategy is through categorising the nature of the error as either conceptually-oriented or structurally-oriented. As Gable et al. stated, "Conceptually-oriented error patterns, such as regrouping errors and place value problems, require a manipulative, hands-on corrective strategy. In contrast, error patterns such as process subtraction, placement, and attention to sign can be corrected using graphically oriented strategies including the use of flowcharts or color coding to structure the work page" (p. 7). Phase two is characterised by extensive practice of the new/correct computational procedure. Phase three is the evaluative phase, and the application of the skill in the regular classroom. It has two stages, where the impact of the new skill on student performance is evaluated with the student, and the practice and maintenance of the skill is continued in the classroom context. As Gable et al. stated, this three phase model is cyclic, and can be used in the regular classroom as it integrates within a curriculum-based assessment and instruction mathematics program.

Studies of the effectiveness of programs for correcting students' error patterns in computation have reported mixed results. For example, Resnick (1982) in focusing on students' computational procedures for subtraction algorithms found that, as a result of intervention, the students in the experimental group, with intensive instruction using concrete materials and place-value games, performed only marginally better than students in the control group. Of this study, Resnick (1992) stated:

> Despite the intensive personal instruction, only half the children taught learnt the underlying semantics well enough to construct an explanation of why the algorithm worked and what the marks represented. More surprisingly, even children who did give evidence of good understanding of the semantics often reverted to their buggy calculation procedures once the instructional sessions were over. (p. 394)

Similarly, Connell and Peck (1993) (as described in section 3.4.2) found that, despite the use of concrete materials, students' prior knowledge interfered with their ability to perform computations correctly; the old, erroneous procedures continually resurfaced. Other studies specifically designed to help children overcome errors in computation have reported that students' old error patterns re-emerge despite the intensity of the remedial activities (e.g., Bourke, 1980; Resnick, 1982; Wells, 1982; Wilson, 1982). It is acknowledged, however, that particular studies have been reported, which show that the use of "good teaching" strategies will help students overcome error patterns in computation (e.g., Stefanich & Rokusek, 1992), and that student errors may naturally correct over time (Hennessy, 1993), but that commonly held errors and misconceptions by young children can resist correction and grow stronger with age (Fischbein & Schnarch, 1997).

Cognitive conflict

Cognitive conflict models of instruction are based on the premise that prior inappropriate knowledge serves as a barrier to knowledge growth and development, and that this inappropriate knowledge must be confronted (Bell, 1986-87). In such teaching situations, the environment is structured so that students' misconceptions will surface as students work on mathematical tasks deliberately developed by the teacher for that purpose. Through discussion in group situations with peers and others, students' misconceptions are brought into the open. Through discussion, the intention is that students will see the impoverishedness of their understandings, and thus conceptual change will occur.

Conflict teaching is thus based on acknowledging the power of prior learning. However, such an approach does not always result in sustained conceptual change occurring, as students' misconceptions are often in evidence after such conflict exercises (Bell, Swan, Onslow, Pratt & Purdy, 1985; Tirosh & Graeber, 1990). Even though a student can see the limitations of their own conceptualisation within a particular topic, they can develop and hold appropriate concepts without giving up their prior, inappropriate concept. The prior-held misconception continues to interfere with understanding and forward learning. This phenomenon has been described as due to knowledge compartmentalisation, where a learner holds two pieces of knowledge as separate entities in the mind, which are in conflict with each other (Posner, Strike, Hewson & Gertzog, 1982; Vinner, 1990). Posner et al. (1982) suggested that compartmentalisation is a learner's mechanism for avoiding cognitive conflict and conceptual change. This perspective suggests that human learners actively, though unconsciously, resist cognitive conflict. As Tirosh (1990) stated, "In cognitive psychology, human beings' desire to eliminate conscious inconsistencies in their thinking is regarded as a basic cognitive need" (p. 111).

Together with the fact that conflict teaching does not always lead to sustained conceptual change, is the fact that conflict teaching requires learners to openly display the extent of their inappropriate knowledge so that critical peer review and analysis can occur. This calls into question the effect of such an approach on students' self-esteem and confidence. As Tirosh (1990) cautioned: "The conflict teaching approach includes a stage in which a student realises that something in his or her way of thinking is "wrong". In certain cases, this realisation may actually be detrimental to a student's confidence or self-esteem" (p. 123).

Errors as springboards for inquiry

Borassi (1985) described how, in the field of diagnosis and remediation, errors are regarded in a negative fashion, as "signals that something has gone wrong in the learning process, and consequently remediation is needed" (p. 1). He suggested that errors should be viewed from a more positive perspective, as the means to promote students' thinking about mathematics and thus build mathematical understanding.

Motivated by the belief that errors can be used to develop students' deeper understanding of mathematics, Borassi (1994) conducted a teaching experiment, focusing primarily on using other students' errors for student inquiry. In his study, he collected students' written definitions of a circle. He then presented these definitions to other students, after asking them to write their own. The students were required to analyse each definition and compare and contrast it with their own definition, thus modifying, rejecting, arguing for, justifying, certain definitions of a circle. The teacher's role was to assist the inquiry process, prompting students to explain clearly their statements, probing their knowledge of circles, and using this to continue the growth of the definition along appropriate lines. According to Borassi, the strategy of using "errors as springboards for inquiry" appeared to, not only help students change and modify their current conceptions of the mathematical topic under study, but also engaged them in "genuine problem solving, mathematical explorations, mathematical communication, initiative and ownership in learning mathematics, constructive doubt and conflict, and the need to monitor and justify their mathematical activity, as well as more humanistic and exciting aspects of mathematics" (p. 199). Borassi also reported that the students' learning of mathematical content was increased as a result of the teaching experiment, as well as the affective domain of the students, with students feeling more positive about the study of mathematics, and their own ability to continue with the study of mathematics.

Belief-based teaching

Rauff (1994) suggested that a process of "belief-based teaching" can help students overcome inappropriate mathematical procedures, and described this in terms of students' erroneous solutions for factoring polynomials. In a process similar to Borassi (1994), Rauff suggested that, to overcome students' errors/misconceptions, beliefs about particular mathematical procedures must be determined, and the teacher's role is to assist the integration of the appropriate mathematical procedures within the student's belief set. The theoretical stance which underpins his method is that, errors and misconceptions are regarded as a "student's belief set" (p. 425), and the development of errors are logical outcomes of the belief set. According to Rauff, "The mathematics teacher who views errors in this way must discern the nature of the student's model and then attempt to modify it appropriately so that the student can work from a mathematically sound belief set"(p. 422).

In his study, Rauff (1994) reported on the relative ease with which some students modified their beliefs, and the difficulty with other students, depending on the nature of the student's current belief state. For example, one student used a particular strategy to factorise polynomials, which only worked in certain cases. The student was shown another strategy, but continued to use her own strategy first. Over time, the student came to realise that her own strategy was no longer efficient for all cases, but used it for the cases in which it yielded the correct solution. The student's initial belief set was expanded to include the new strategy, as well as her own. According to Rauff, this inclusion of the new strategy was because it "did not entail the removal of any other beliefs about factoring" (p. 424). Rauff summarised belief-based teaching thus:

> The focus of this approach into teaching and learning is student belief. An instructor using this approach to teaching factoring begins with asking the student to tell him or her what they think about factoring. The instructor then analyses their "buggy" factorisings in light of their beliefs. The students are next shown how their beliefs produce non-equivalent expressions. Finally, the students modify their beliefs appropriately. (p. 425)

Teaching by analogy

Teaching by analogy has been suggested as an approach for assisting students overcome misconceptions (Tirosh, 1990). The basis of teaching by analogy is that knowledge students have developed is linked, through analogy, to knowledge the teacher is presenting. The teaching by analogy approach can serve as a means of helping students solve analogous tasks, of helping students develop understanding through linking to analogous situations, of guiding teaching to link to analogous experiences, thus taking what is known and linking to what is new. In dealing with misconceptions, teaching by analogy involves presenting students with tasks that have been previously solved correctly, which are analogous to the tasks the student solved incorrectly. The intention is that the student will see the two tasks as analogous, and revise the approach taken for solution. Thus, students revisit a correctly performed task (the anchoring task) in order to change their approach to solving a task on which they initially experienced difficulty (the target task). This approach poses challenges for teachers, as Tirosh (1990) stated, the teacher must "find a suitable anchor task to the target task that convinces the student of the validity of the analogy" (p. 123).

Teaching for conceptual change

Castro (1998) proposed a model of teaching to promote conceptual change. The model of teaching has four phases: (i) clarification of students' ideas; (ii) activities to encourage cognitive conflict; (iii) application of new ideas to new problems and situations; and (iv) revising old ideas in light of new knowledge. The model is based on Lakatosian epistemology that new knowledge is constructed when existing knowledge is seen to be limited and limiting within a new context. That is, knowledge change occurs as a result of a "better theory". For conceptual change to occur, three conditions must be met: (i) there must be dissatisfaction with existing knowledge; (ii) the new conception must be intelligible to the knower; and (iii) the new conception must be plausible within the new context. Castro used this model of teaching for conceptual change in a teaching experiment on probability concepts with students aged 14-15 years. Results of the study showed the teaching approach positively influenced students' probability calculation and intuitive probabilistic reasoning skills, suggesting the value of this approach in mathematics classrooms.

3.4.4 Redefining mathematics remediation

As stated in section 3.2, research on error patterns suggests that errors are indicators of the presence of knowledge, not its lack. The challenge for intervention programs is often thus in confronting and dealing with knowledge. Intervention programs based on diagnostic-prescriptive teaching models which provide students with carefully planned and sequenced learning experiences for rich conceptual knowledge development should be successful in building knowledge when there is a lack. Such programs would be regarded as good teaching programs (see section 3.3.3). Remediation of systematic errors and misconceptions, where errors indicate the presence of knowledge, "good teaching" approaches incorporating the close linkage of the written representation with the concrete/pictorial representation also have been suggested (e.g., Ashlock, 1994; Booker et al., 1980; Resnick, 1982; Wilson, 1976a). Such approaches *also* appear to be based on slower, sequential and progressive reteaching and provision of rich learning experiences, and therefore appear to be no different to instruction for students who lack knowledge. In such programs, students' errors/misconceptions are acknowledged by the teacher, but are not overtly referred to in the remedial situation. Programs and methods for overcoming inappropriate knowledge have not reported clearly favourable outcomes, as prior knowledge continues to impede sustained knowledge growth (e.g. Bourke, 1980; Connell & Peck, 1993; Resnick, 1982; and also section 3.4.3).

There are two distinct aspects of remediation apparent. The first is building knowledge where there is none. The methods advocated are seen to be good teaching practices, and thus the term "remediation" appears to be seemingly inappropriate in this context; the term "reteaching" would seem to be more accurate. The second is overcoming prior inappropriate knowledge, manifest as error patterns or misconceptions, and this is where specific programs of remediation need to be applied. Such programs have reported some positive results in helping students overcome errors/misconceptions in various mathematical topics. Confronting prior knowledge appears to be the key to successful remediation. As described in section 3.4.3, there appears to be a growing emphasis on using students' errors as starting points for intervention programs.

The presence of errors and misconceptions has implications for teaching, particularly with respect to helping students achieve in academic settings. As stated by Mansfield and Happs (1992) "many students will come to the mathematics classroom with a number of misconceptions about topics to be taught. Research has clearly indicated that these misconceptions act as barriers to the acquisition of new conceptual knowledge" (p. 453). Errors and misconceptions, therefore, block students' mathematical knowledge growth, and as such, need to be overcome in order for students to continue with mathematical studies. The notion of errors and misconceptions preventing forward knowledge growth is not new. The interfering effects of prior knowledge and the need to deal with such knowledge was, almost two decades ago, encapsulated by Ausubel (1968), who stated:

> The role of preconceptions in determining the longevity and qualitative content of what is learned and remembered is crucial...[The] unlearning of preconceptions may well prove to be the most determinative factor in the acquisition and retention of ... knowledge. (p. 135)

3.4.5 Summary of key points

In this section, specific programs of intervention were described. A growing tendency in intervention programs emphasises the use of errors and misconceptions as the beginning points of remediation. In this section, it was seen that students' errors, misconceptions and alternative conceptions are extremely difficult to eradicate which underscores the paramount importance of the quality of the initial instruction. Presented in this section was a summary of current approaches, strategies and ideas for helping students overcome incorrect knowledge. The potential of the described approaches, strategies and ideas was contrasted with the inherent shortcomings of such. The varied strategies served to exemplify a changing focus from teaching which ignores students' inappropriate knowledge, to strategies which actively encourage students to explore their own inappropriate ideas and beliefs. In this section, it was argued that mathematics remediation must incorporate strategies to build knowledge, as well as strategies to overcome knowledge.

3.5 Conceptual Mediation

3.5.1 Overview

The Conceptual Mediation Program (CMP) (Lyndon, 1995), a program for overcoming students' consistent errors and inappropriate knowledge, is the focus of discussion in this section. CMP is based on a particular perspective of errors/misconceptions, and has its own theory as to why they are difficult to eliminate. The theoretical background of CMP is presented in section 3.5.2. The program offers a strategy for assisting students overcome errors/misconceptions which slow down educational progress. A description of this strategy is provided in section 3.5.3. CMP also offers a metacognitive training program to enable students to become self-empowered, autonomous learners, and this is described in section 3.5.4. A summary of key points is presented in section 3.5.5.

3.5.2 Prior knowledge and proactive inhibition

The theoretical basis of CMP states that factors which affect successful conceptual change are due to the mental phenomenon of *proactive inhibition* (PI). Such factors include lack of learning transfer from the remedial setting to the regular classroom, and recurrent appearance of error patterns and misconceptions despite intensive intervention programs; as well as the display of avoidance behaviour towards intervention programs by students. These factors, according to the conceptual framework of CMP, are a result of the activation of PI.

Proactive inhibition has been described as an information protection mechanism: "it is produced by conflicting associations that are learned prior to learning of the task to be recalled" (Underwood, 1966, p. 564). Underwood suggested that, when a person is asked to give a response to a stimulus that differs from the response the person usually gives, the brain can only do so with great difficulty. Underwood provided the following example to demonstrate proactive inhibition in practice:

> If we are told that: 2 x 2 now is 11
> 8 - 4 now is 1
> 3 + 3 now is 27
>
> we can imagine the difficulty we would have in remembering and applying this new information. Interference, indeed frustration might well occur. (p. 516)

Further, Baddeley (1990) stated that proactive interference (inhibition) occurs when "new learning is disrupted by old habits" (p. 40). Baddeley provided the following as an example of proactive inhibition: "Being taught that C means 'caldo' which means hot, but none the less 'forgetting' and turning the wrong tap would be an instance of proactive interference" (p. 40). Similarly, of proactive inhibition, Houston (1991) stated:

> Proactive inhibition is not a theory or an explanation. It is a fact, an important one. It refers to the enormous amount of forgetting that can be attributed to the interfering effects of prior learning. The more we learn, or store, the more susceptible we are to this type of interference. (p. 235)

Proactive inhibition, then, as a mechanism for protecting knowledge, is activated when new learning conflicts with prior learning. In situations where prior learning conflicts with current learning, old learning will interfere with recall of the new learning. The need for such a mechanism is apparent, as it can be seen that without such an inbuilt knowledge protection system, the human mind would be in a constant state of confusion. A person's knowledge base would be changing continually in the face of new incoming information. It can also be seen that existence of a knowledge protection system is a two-edged sword, with all prior knowledge, correct or otherwise, being protected from change. The implications of proactive inhibition for remediation in mathematics are immense. Remediation of learning difficulties in mathematics typically requires students to change their response to a particular stimulus, be it an automatic response to a number fact, a completion of an algorithmic procedure, or a conceptualisation of a mathematical topic. The teacher is providing the same stimulus, but is requiring the student to give a new response that differs to the way the student responded previously to that stimulus. In terms of proactive inhibition, the enormity of such a request is realised, and is exemplified by the examples described above. PI can also serve to explain the common characteristics of the learner in the remedial situation, encapsulated by the words of Erlwanger (1975) who questioned "why is it that remedial children often display patterns of errors, hold tenaciously to their own procedures, appear to become confused and emotionally disturbed during remedial work, and to require prolonged individual assistance and guidance?" (p. 171). In light of the discussion on PI, all characteristics described by Erlwanger can be attributed to the influence of PI.

As previously discussed, constructivist views of learning state that errors/misconceptions are knowledge (e.g., Borassi, 1994; Confrey, 1990b; Rauff, 1994). Errors/misconceptions are thus indicative of the presence, not the absence, of knowledge. In terms of diagnosis and remediation, the problem is dealing with knowledge, rather than providing learning experiences to "fill-the-gaps" or "link" knowledge as would be wont in an "absence of knowledge" perspective. Acknowledging the mechanism of PI as a part of the human mind, PI can be seen to serve as protection of errors/misconceptions from change. The role of PI is simply to prevent the cognitive conflict. As Tirosh (1990) suggested, avoidance of mental turmoil is the natural tendency of the human mind.

In view of PI as merely a knowledge protection mechanism, it can be seen that PI cannot determine appropriate knowledge from inappropriate knowledge, therefore all knowledge will be protected by PI. Psychological research studies have shown that it is initially learned knowledge which is more powerfully retained in memory over subsequent learning (e.g., Baddeley, 1990; Eysenck, 1977). The mathematics remediation literature has repeatedly stated that once acquired, students' errors, misconceptions and alternative conceptions are extremely difficult to overcome (e.g., Confrey, 1990b; Fischbein, 1987; Graeber & Baker, 1991), thus the need for carefully structured, planned and organised *initial* instruction is of paramount importance (Connell & Peck, 1993). Acknowledging the influence of PI within the remedial situation provides an explanatory theory for the persistence of errors/misconceptions.

For some students, effective reteaching programs assist in overcoming errors/misconceptions. As previously stated, this could be attributable to providing learning experiences to develop knowledge where there was a lack. In terms of knowledge and PI, it may be because of a lower level of PI. The strength of the PI mechanism is variable to the individual (Stroop, 1935). It follows then, that the lower the level of PI, the easier it will be to overcome the power of PI. Conversely a high PI level suggests the difficulty of the process of conceptual change. Studies reporting on the recurrence of errors/misconceptions despite the intensity of the remedial program can thus be reinterpreted as programs which fail to take account of the PI mechanism.

The above discussion highlights the key principles of Conceptual Mediation, which have been summarised by Lyndon (1989) into the following nine points:

1. Consistent, habitual errors indicate the presence, not the absence of learning/knowledge.
2. What the individual knows is protected from change by the proactive inhibition mechanism.
3. Proactive inhibition is an involuntary mechanism over which we have little or no control.
4. Incorrect, as well as correct, knowledge is protected since proactive inhibition cannot discriminate between what is 'right' and what is 'wrong'.
5. Proactive inhibition does not prevent learning from occurring; it merely prevents the association of conflicting ideas.
6. Considerable variation exists within the population in the level of proactive inhibition one inherits (Stroop, 1935).
7. The higher the level of proactive inhibition, the more resistant the individual will be to conventional approaches to remediation.
8. Performance also becomes cue-dependent and, in the absence of the remediator, the student reverts to the erroneous behaviour patterns.

9. In this way, transfer of learning is inhibited and errors continue to resist correction. (p. 34)

The Conceptual Mediation Program states that conventional remedial methods serve to activate the PI mechanism, causing student behaviours such as: slowness to respond; an apathetic attitude to the task; frustration and avoidance behaviour; to become evident. Avoidance behaviour can also be reinterpreted with acceptance of PI. Recurrence of systematic errors after intensive remedial instruction reinforces in learners their feelings of failure. In terms of PI, recurrence of systematic errors is the inability of intervention programs to successfully deal with PI, and a lack of knowledge on the part of the learner of the influence of PI over knowledge change. For effective remediation, CMP states that the remediator must acknowledge proactive inhibition as an inhibitor of knowledge change and growth. Remediation programs, therefore, must be structured to effectively deal with proactive inhibition.

3.5.3 "Old Way/New Way" strategy

To overcome the inhibitory influence of PI over knowledge change, a specific strategy is an integral element of CMP. The strategy is called Old Way/New Way (O/N). The essence of the O/N procedure is upon bringing the learner's "old way" to a conscious level and exchanging it for a "new way" by means of discrimination learning, followed by practice with the correct "new way". There are four steps to O/N. A simplified example of how the O/N method proceeds through the four steps is provided for the remediation of a systematic error in the subtraction algorithm. In step 1, *reactivation of the error memory*, the student is asked to complete the subtraction problem 306 - 149 in their usual way. For step 2, *labelling and offering an alternative*, the student is asked if that particular method of performing that computation can be called the "old way". When consent is given, the student is asked if a "new way" for computing 306 - 149 can be shown. Using carefully selected language, the remediator performs the algorithm the standard way. The difference between the two algorithms is then carefully pointed out. In step 3, *discrimination*, the student is asked to perform the computation the old way, then the new way, and then asked to contrast the two ways. This discrimination of the same problem (306 - 149) is repeated five times. For step 4, *generalisation*, the student is provided with six subtraction exercises and asked to complete using the new way. This sequence of four steps is called a learning trial, and takes approximately 10 minutes. According to Baxter and Lyndon (1987) the benefits of the O/N method are thus:

> O/N bypasses proactive inhibition and enables the remediator to change the child's knowledge base rapidly and permanently...The more or less instantaneous success the child experiences after one trial ensure that

avoidance learning behaviours are soon eliminated. Confidence in ability to learn is restored. (p. 8)

Analysis of the steps in O/N reveals that the student is required to repeat the old way a total of five times. Such an approach is contrary to a perception that reactivation of the error pattern will only serve to strengthen that error pattern. This perception is evident in the words of Gagne (1983) who stated, "to make students fully aware of the nature of their incorrect rules before going on to teach correct ones...seems to me...is very likely a waste of time" (p. 15). Gagne proposed that to overcome errors is to aim for "extinction" (in psychological terms) of that error, and suggested by the following comment: "The effects of incorrect rules of computation, as exhibited in faulty performance, can most readily be overcome by deliberate teaching of correct rules...This means that teachers would best ignore the incorrect performances and set about as directly as possible teaching the rules for correct ones" (p. 15). In the context of the O/N theory, the error is habitual, and more practice will not serve to make it any harder to eliminate.

The O/N procedure shares similarities with other procedures for dealing with errors/misconceptions, as described in previous sections (e.g., Borassi, 1994; Gable et al., 1991; Rauff, 1994) but its method is more prescriptive, and firmly based on psychological principles of learning. O/N can be seen to provide a vehicle for dialogic mediation, and the key element is the discrimination of differences between the student's knowledge and the mathematical knowledge presented by the teacher.

3.5.4 Conceptual Mediation and metacognition

When using O/N for the purpose of overcoming systematic computational errors, O/N can be regarded as a metacognitive strategy applied to a particular cognitive strategy; it is cognitive in that it is task specific, but simultaneously, it is metacognitive as it is a highly generalisable strategy applicable to a wide range of situations. O/N can be presented to students in all three training modes suggested by Brown and Palinscar (1982): *blind*, *informed* and *self-regulatory*. In blind training, the O/N strategy would be used with student errors, but no reason for the application of such a strategy would be given. In informed training, O/N would be used with student errors, and students would be encouraged to reflect on the potential use of the strategy in other domains with other areas of difficulty. In self-regulatory training, the theoretical basis of O/N would be shared with students to provide a clear rationale for the value of such a strategy to aid their own learning. The Conceptual Mediation Program (CMP) is a self-regulatory metacognitive training program in which O/N is presented as a key "remedial" strategy. The program takes the form of a communication with students, sharing with them ideas on what is empirically known about attention, memory, and learning, or more specifically, remembering and forgetting.

Recognition and recall memory

The first component of the metacognitive training within CMP focuses on remembering. It is generally accepted that the process of remembering is a result of two memory systems: recognition memory and recall memory. The difference between recognition memory and recall memory can be made explicit by examining the tasks performed by subjects in psychological experiments. In typical recall and recognition tasks, subjects are required to learn and remember lists of words. In recall tasks, subjects are required to remember as many studied words as possible after a given time. In recognition tasks, the word lists often comprise paired lists of words; one word invoking the memory of its pair. Subjects essentially have to be able to recognise whether certain words have been presented to them in the given context (Houston, 1991). In the metacognitive training component of CMP, recognition and recall memory are differentiated. Recognition memory is described as being externally activated and automatic. Recall memory, on the other hand, differs in that it is utilised when remembering something that is not present; it is a self-initiated event and operates at two levels. Recall memory is either automatic or effortful. Either, a memory can be retrieved automatically, or it requires a certain amount of effort for retrieval. Taking control of the remembering process is to store information in automatic recall memory where retrieval is self-initiated, rather than externally stimulated. To take control of the remembering process, practice is essential. Remembering is thus a product of practise and effort, and students need to be taught how to practise efficiently (Derry, 1990).

Natural and accelerated forgetting

The second component of the metacognitive training within CMP focuses on forgetting, of which two types are discussed with students. Forgetting can be either natural or accelerated. Natural forgetting occurs over time, as skills/knowledge learnt is not practised. As suggested by Anderson (1985), a skill that is not practised becomes a victim of the brain's process of natural forgetting. Accelerated forgetting is described to students in terms of PI as an information protection mechanism, and O/N is demonstrated as a strategy for overcoming PI and taking control of accelerated forgetting. The notion of accelerated forgetting is presented to students as a natural brain process; a process which occurs when learning a new way of doing something conflicts with an already learned procedure for doing the same thing.

In summary, the key points of the metacognitive training component within CMP, which from the basis of the communication with students, are as follows:

1. Sometimes learning seems easy and sometimes it seems hard. Learning seems hard because it is paying attention, remembering and understanding that we find hard.
2. What we pay attention to, we learn. We choose what we pay attention to. Paying attention is hard because it requires effort.
3. We have two memory systems: recognition and recall. Recognition memory just happens naturally without effort. Recall memory is naturally effortful. Recall memory can be either automatic or effortful. Recall memory only becomes automatic through practice.
4. We have two forgetting processes: natural and accelerated. We can take control of natural forgetting through use of efficient practise strategies. We can take control of accelerated forgetting by using the Old Way/New Way strategy.

Central to CMP is the notion that the brain is designed to forget. It is this key phrase: *the brain is designed to forget* which provides a rationale for the importance of teaching students how to remember. The purpose of CMP is to inform students of how their own brain works so that they can take control of their own learning. Throughout the program, the continual emphasis is on the fact that learning is a result of effort, and the effort must come from the individual. Making students aware of the process of remembering, learning and forgetting may be part way to answering Norman's (1980) statement:

> It is strange that we expect students to learn yet seldom teach them about learning. We expect students to solve problems yet seldom teach them about problem solving. And, similarly, we sometimes require students to remember a considerable body of material yet seldom teach them the art of memory. (p. 97)

3.5.5 Summary of key points

In this section, the Conceptual Mediation Program (CMP) was described. The program was presented as providing a strategy for promoting conceptual ex-change, and a metacognitive training component to empower students to take control of their own learning. The theoretical position of CMP was presented as providing a clear perspective on the longevity of errors/misconceptions and a simple approach to mathematics remediation.

3.6 CMP within the mathematics program

3.6.1 Overview

In this section, the CMP in relation to diagnosis and remediation in mathematics is discussed. In section 3.6.2, the implications of CMP for mathematics are addressed. In section 3.6.3, a model of diagnostic-prescriptive mathematics teaching incorporating CMP is presented. A summary of key points is presented in section 3.6.4.

3.6.2 Implications of CMP for mathematics intervention

The CMP challenges the notion that successful remediation in mathematics is a process of re-teaching. As the title of the program suggests, CMP is about "mediation" in the sense that mediation means "to stand between, to mediate between to reach agreement". In terms of dealing with students' error patterns and misconceptions in mathematics using CMP, the teacher is acting in a mediating capacity between the student's point of view and the mathematical point of view being presented by the teacher. The term re-mediation, in this context, refers to the process of the teacher "mediating" the two points of view a second time, or engaging in "re-mediation". This process of re-mediation can be seen to be quite different to the provision of re-teaching episodes. It is the process of acting directly upon the error/misconception to explore the differences between the appropriate knowledge and the inappropriate knowledge. Such a mediating role taken by the teacher to overcome inappropriate knowledge and to build appropriate knowledge is the teaching approach advocated in the cognitive apprenticeship model (Reid & Stone, 1991; Rojewski & Schell, 1994) (see section 3.3.4). The teacher as a mediator of knowledge has also been described by Underhill (1988) as the means to helping students overcome their inappropriate beliefs which prevent further learning. As he stated, "knowing is believing,...learning is the process of developing new beliefs and altering old beliefs...and teaching is helping others develop new beliefs and alter old beliefs" (p. 62-63). The CMP, therefore, requires that the teacher determine the beliefs of the learner in order to facilitate conceptual ex-change. Working on errors/misconceptions, the CMP is not a replacement for good teaching. Its purpose is to overcome students' errors/misconceptions which impede educational achievement. As such, it is specific in nature. Utilised in this way, CMP is a direct method for dealing with errors/misconceptions to enable forward movement of the individual in the learning process. The value of CMP is thus its direct nature for overcoming specific difficulties in mathematics. The need for such an approach is encapsulated in the words of Kirby and Williams (1992) who stated:

> We should not lose sight of the purpose of diagnosis and remediation. It is achievement which we are attempting to improve, not cognitive processes. If our goal is to improve reading skills and we succeed only in improving a child' memory skill, we have not achieved our goal. (p. 4)

3.6.3 CMP and diagnostic-prescriptive mathematics teaching

CMP, and particularly the Old Way/New Way strategy may provide the means for dealing with the protective influence of PI, but may superficially be seen as a means for replacing one mode of behaviour with another; of replacing a habit with a habit. From the example of the O/N methodology applied to systematic errors in subtraction algorithms (section 3.5.3), it would appear that O/N is useful for remediation of apparently "rote" behaviours, or automatic skills. Indeed, O/N was originally developed for use in overcoming habitual behaviours at the "rote" end of the scale, such as spelling errors, letter reversals, body mannerisms. In mathematics, O/N has shown to be a useful teaching tool in overcoming students' systematic mathematical computational subtraction errors and building subtraction knowledge in small group remedial situations (Baxter & Dole, 1990; Dole, 1993). However, O/N has also shown to be successful in changing students' alternate science conceptions in whole class situations (Rowell, Dawson & Lyndon, 1990). Thus the theoretical basis of O/N appears to relate equally well to misconceptions/alternative conceptions as it does to error patterns.

Incorporating O/N within a diagnostic-prescriptive model of mathematics teaching highlights the importance of diagnosis. Students' knowledge must be ascertained in order to make informed decisions in terms of providing further instruction to build knowledge, or to "break through" inappropriate knowledge, using O/N. In studies carried out by Dole (1993, 1995) it was found that when O/N was used to remediate systematic errors in computation, students experienced rapid success in performing procedures upon which, in the past, they could not. This success built a foundation for the development of further knowledge. Once students successfully attained correct computations via pen-and-paper methods, the task of building knowledge as to why the new computational method yielded the correct answer was open. "Good teaching" approaches for rich conceptual knowledge development suggested in the literature were implemented. The diagnostic/prescriptive model developed for remediating systematic errors in computation consisted of the following five steps:

1. Identification of systematic errors.
2. Structured interview to establish depth of principled/conceptual (Leinhardt, 1988) knowledge in relation to the mathematical content from which the error derived.
3. Implementation of O/N to remediate systematic computational errors.
4. Use of carefully chosen exemplary materials and language to link computational knowledge to concrete knowledge and thus legitimise the algorithm.

5. Utilisation of good teaching approaches for building of mathematical conceptual knowledge suggested in the literature (e.g., Ashlock, 1994; Booker et al., 1980; Booker, Briggs, Davey, & Nisbet, 1992; Jones & Charlton, 1992; Resnick, 1982; Wilson, 1976a).

A diagnostic/prescriptive model of teaching, incorporating CMP (depicted in Figure 3.1), is proposed as a dual-path model of building knowledge or changing inappropriate knowledge.

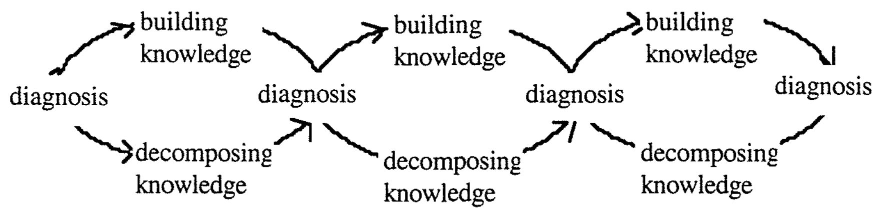


Figure 3.1. Dual-path model of diagnostic/prescriptive mathematics teaching.

The focus of the model is diagnosis. The importance of diagnosis is in the need to determine whether the learning difficulty stems from a lack of knowledge, or the presence of knowledge. If the former is the case, "good teaching" is required to build knowledge. If the latter is the case, "re-mediation" (using O/N) must take place to break through the "knowledge" barrier, followed by "good teaching" to build knowledge.

3.6.4 Summary of key points

In this section, the implications of CMP upon the notion of diagnosis and remediation in mathematics were discussed. A dual-path model for diagnostic/prescriptive mathematics teaching incorporating CMP was proposed.

3.7 Diagnostic-prescriptive models and percent instruction

The focus of this chapter was to explore issues pertaining to instruction for students experiencing difficulty with the study of academic subjects, particularly mathematics. The literature discussed was selected to present a comprehensive picture of the field of diagnosis and remediation in general, and errors and misconceptions in particular. A summary of the discussion on diagnosis and remediation presented in this chapter can be highlighted in the following points: (a) traditionally, students exhibiting errors in mathematics were deemed as having a learning disability; (b) remediation programs were typically reteaching programs consisting of representation of the mathematics content in a slower and progressive mode; and (c) models of diagnostic-prescriptive suggest that intervention is a cycle of planning, implementing, and reflecting upon teaching episodes which occurs after students' prior knowledge has been ascertained and the structure of mathematics tasks have been analysed. As well, metacognition and academic performance were discussed in this chapter, and it was suggested that students who experience difficulty with the study of academic subjects typically lack sophistication in level of metacognitive skills, and that intervention programs should aim to build students' metacognitive awareness and develop metacognitive skills.

Mathematical error patterns and misconceptions were also discussed in this chapter. A number of key points were raised, and can be summarised as the following: (a) error pattern research has suggested that errors are not an indication of students' inability to learn, but provide strong evidence of their ability *to* learn, albeit incorrect knowledge; (b) constructivist learning theory has suggested that errors and misconceptions are constructed knowledge and are part of the belief-set of the learner; (c) reteaching programs do not always lead to sustained conceptual change as errors and misconceptions resurface; and (d) current trends in intervention programs are those which consciously focus on students' errors and misconceptions rather than reteach as if the errors and/or misconceptions are not in existence.

In section 3.5 of this chapter, the Conceptual Mediation Program (CMP) was described in detail as a potential program for guiding the development of teaching programs in general as well as the tailoring of specific programs of intervention. The implications of the CMP for classroom teaching can be summarised in the following points: (a) errors and misconceptions are knowledge and knowledge is protected from change by the proactive inhibition mechanism; (b) presentation of mathematics concepts and skills in the classroom can be in conflict with students' prior knowledge and thus will be subject to accelerated forgetting; and (c) sharing with students such ideas on learning, remembering and natural and accelerated forgetting may offer students an opportunity to take greater metacognitive control of their own learning.

In terms of classroom teaching, the CMP appeared to offer some interesting possibilities. With respect to designing a teaching program on percent for Year 8 students, integrating CMP notions through metacognitive training as an integral part of the teaching program may assist students to take active control in their learning so that they may be more independent as learners, and to be metacognitive with respect to percent.

CHAPTER SUMMARY

In this chapter, issues in mathematics diagnosis and remediation were presented. The emphasis of this chapter was on instructional approaches for helping students overcome learning difficulties in academic situations. Diagnostic-prescriptive teaching models and intervention programs were described, and were presented as programs of "good teaching". Research on error patterns in mathematical computations was presented, and the concept of errors as knowledge was discussed within a constructivist framework. Intervention programs and strategies were described for overcoming systematic mathematical errors/misconceptions. The Conceptual Mediation Program was presented, in which proactive inhibition was described as the mechanism responsible for the human tendency to retain naive conceptions, alternative conceptions and error patterns in the light of rational argument. A dual-path model was proposed, based on the CMP, for the diagnostic-prescriptive teaching of mathematics.

Both this chapter, and the preceding chapter on percent teaching and learning have served to identify issues pertinent to this study. In the next chapter, the research methodology of the study is presented.

CHAPTER 4
DESIGN

CHAPTER OVERVIEW

In this chapter the design of the study is presented. The study was designed to achieve the following aims (as stated in chapter 1, section 1.3):

1. To develop a program for effectively teaching percent applications in real classrooms.
2. To draw implications and construct models for percent knowledge, percent instruction, and mathematics teaching in general.

This chapter is presented in six sections. In section 4.1, methodologies for studying the influence of instruction upon students' learning are summarised. In section 4.2, the research design and methodology for this study is presented. In sections 4.3, 4.4, 4.5 and 4.6, the subjects, data sources, procedure and analysis are described respectively.

4.1 Researching teaching and learning

4.1.1 Overview

The focus of this section is on research methodologies for studying the influence of instruction upon learning. In section 4.1.2, the teaching experiment is described. In section 4.1.3, action-research is described, and in section 4.1.4, clinical-intervention research and diagnostic-prescriptive instruction is described. Within these two sections, the nature of these research methodologies to the teaching experiment are discussed. In section 4.1.5, Hiebert and Wearne's (1991) methodology for studying learning to inform teaching is described, and in section 4.1.6, Lampert's (1992) classroom based methodology for studying teaching and learning is described. A summary of key points is presented in section 4.1.7.

4.1.2 The teaching experiment

The teaching experiment design is typically a methodology for studying teaching or, specifically, studying the influence of instruction upon students' learning (Romberg, 1992). In teaching experiments, the instructional conditions which precipitate growth and change in students' knowledge are the prime focus of the research (Uprichard & Engelhardt, 1986). According to Romberg (1992), the teaching experiment design is utilised if the study "involves gathering research about the effects of a new and different product or program" (p. 57), and is different from other qualitative approaches designed primarily to study how children think (Uprichard & Engelhardt, 1986).

Of the teaching experiment, Kantowski (1978) stated that it was designed so that researchers could study the influence of planned instruction upon children's mental processes. Thus, observing how instruction led to development of problem solving and thinking skills provided guidelines for planning of instruction to optimally influence those processes. According to Kantowski, the teaching experiment derived from research methodologies devised by Vygotsky who believed that human cognitive functioning is a result of instruction, not an inborn process. Further, that instruction takes children from "ignorance to knowledge, from one level of operation to another, from a problem to a solution" (p. 45).

The teaching experiment methodology (Kantowski, 1978; Rachlin, Matsumoto & Wada, 1987) typically requires students to be actively engaged in problem solving situations so that observations of student behaviour under such circumstances can be made. A cross-section of data is gathered from various sources including the teacher, the experimenter, an observer, and the students themselves. Selected students are often interviewed in clinical situations at various times during the teaching experiment. The students are selected on the basis of their being "strong", "average" or "weak" in the subject, according to judgment of the classroom teacher. All lessons are carefully observed and a comprehensive qualitative report of events is produced. Data from student interviews and classroom observations are combined and analysed, and conclusions as to effect of instruction on student behaviour are drawn. In light of these data, planning for future lessons is made. The lesson planning is contingent, but general course outline and content to be covered is determined in advance.

In a teaching experiment, the researcher assumes an executive role, one where, as Hunting (1980) stated, the researcher is "a witness and interpreter of events" (p. 17), one who will "control the sequence of events and remain conscious at an 'executive' level" (p. 17). Rachlin et al. (1987) further expanded upon the teaching experiment and delineated roles for each stage of the teaching experiment design. According to Rachlin et al., the teaching experiment involves a research team, comprising a *researcher* to provide a theoretical framework for the study; a *curriculum developer* to translate the framework into course materials; a *teacher* to implement the curriculum and adapt instruction to suit the program; an *evaluator* to provide feedback on the implemented program, and a *disseminator* to prepare others to use the new instructional program.

4.1.3 Action research

Action research (Kemmis, 1982; Kemmis & McTaggart, 1990) also shares similar goals to the teaching experiment. Kemmis and McTaggart (1990) described action research as "trying out ideas in practice as a means of improvement and as a means of increasing knowledge about the curriculum, teaching and learning" (p. 5) which is a way of "working which links theory and practice into one whole: ideas-into-action" (p. 5). Kemmis and McTaggart described the methodology of action research as a spiral of self-reflection, "a spiral of cycles of planning, acting (implementing plans), observing (systematically), reflecting...and then replanning, further implementation, observing and reflecting" (p. 22). Thus, action research is classroom based, cyclical, and on-going. Its similarity to the teaching experiment is that it is a process of reflection upon practice. Action research provides a structure for guiding reflection upon practice.

4.1.4 Clinical intervention research

Clinical intervention research (Wilson, 1976b) and diagnostic-prescriptive teaching (Uprichard & Engelhardt, 1986) offer a methodology for researching the influence of instruction upon students' learning. As described in section 3.3.2, clinical intervention research (Wilson, 1976a) is a methodology for research where instruction is specifically designed to help individual children overcome learning difficulties in mathematics. It can be seen to parallel the teaching experiment design in that treatment evolves in a spiral specific to the needs of the individual at the time, and the methodology involves extensive systematic descriptions of treatments and the recording of apparent effects. Individual cases are taken as a whole to draw generalisations from the patterns emerging so that they may form testable hypotheses for future research. Clinical intervention research and diagnostic-prescriptive instruction differ from the teaching experiment in that instruction is presented to individual cases, rather than whole class groups. However, there appears to be a trend away from using intact classes for teaching experiments, to presenting instruction to small, select groups of students (Thompson, 1994; Uprichard & Engelhardt, 1986). With small groups of students, a focused analysis of the influence of instruction can be undertaken, which, is the context for diagnostic and prescriptive mathematics (Uprichard & Engelhardt, 1986).

4.1.5 A methodology for studying learning to inform teaching

Hiebert and Wearne (1991) (see also Wearne & Hiebert, 1988) described a specific methodology for studying learning to inform teaching. This methodology can be summarised into the following four steps:

1. selecting the content domain and defining it clearly;

2. identifying the cognitive processes that are critical for successful performance in the domain;
3. designing instruments to promote the acquisition and use of the key processes; and
4. examining the relationship between instruction and cognitive change and assessing the extent of cognitive change.

The focus of the methodology, according to Hiebert and Wearne, is the identification of the key processes required for successful operation within the mathematical domain; of designing instruction which directly facilitates the development and use of such key processes. The study of instruction upon the development of domain specific key processes may serve to identify and clarify key processes, thus the methodology may develop in a spiral, cyclical approach of "identifying processes and designing instruction to support their acquisition" (p. 165). The cyclical nature of the methodology simultaneously enables the influence of instruction upon the development of key processes to be determined, together with identification of further key processes, or identification of other key processes as the students are engaged in the learning process. Models can be built of the cognitive processes required to operate in that domain as well as models of teaching that facilitate the acquisition of such processes. To determine the extent of change, the methodology utilises two measures, direct and transfer measures. Direct measures assess how well students use the key processes on instructed tasks; transfer measures assess the degree of transfer of the key processes to other tasks not presented to students in the original instruction.

4.1.6 Classroom based research

One of the criticisms aimed at educational research is the limited degree to which research results filter to the real world of the classroom, or the degree to which results can be implemented directly to the classroom (Lerman, 1990). Hiebert and Wearne (1991) contended that their methodology is a means for placing research in the school situation to directly inform teachers. They advocate the use of whole class groups, as "full classroom settings afford greater ecological validity" (p. 163). A similar methodological approach was used by Lampert (1992), where instruction is presented to the entire class, and curriculum and instruction is developed simultaneously through studying the thinking processes of children as they interact within a mathematical domain. Lampert argued that the classroom situation, with the teacher as researcher, provides a totally different learning context to teaching in a clinical setting. As she stated, "it seems problematic to define understanding based on research done outside of the classroom and then to assess whether classroom instruction is successful in producing that kind of understanding in the social setting of school activities" (p. 246). One of the problems of authentic classroom research is in determining students' thinking within a mathematical domain, and to assess the level of change as a result of instruction. To overcome this methodological problem, Lampert advocated the collection of data from many sources, but also confining the gathering of data to methods usually employed in the classroom situation. As she stated, children behave differently in problem solving situations conducted in clinical situations, and also during test situations. Therefore, data collection should be unobtrusive and "natural".

4.1.7 Summary of key points

In this section, the teaching experiment design was described. The teaching experiment was presented as a general term encompassing classroom-based research into the influence of instruction upon learning. In this section, action-research, clinical intervention research and diagnostic-prescriptive teaching were described as research methodologies which share similar features to the teaching experiment. Other research methodologies were also described which were classroom based with a focus on instruction and learning.

4.2 Research design and methodology

4.2.1 Overview

In this section, the design and methodology of the study are described. In section 4.2.2, the "hybrid" (Schulman, 1990) nature of the design and methodology is presented, in section 4.2.3, the sequence of, and activities within the teaching experiments are described, and in section 4.2.4, data gathering techniques to achieve the outcomes of the study are listed. A summary of key points is presented in section 4.2.5.

4.2.2 A "hybrid" design

The design of the study primarily followed that of the teaching experiment (Hunting, 1980; Kantowski, 1978), but methodological procedures were adopted from other qualitative designs for studying teaching. The design was thus a "hybrid" (Schulman, 1990). According to Schulman (1990), mixes of various research designs are "exciting new developments in the study of teaching" (p. 3).

For the study, a series of teaching experiments was conducted to develop an efficient and effective instructional sequence and to explore students' understanding of percent. The "research team" (Rachlin et al., 1987) for the teaching experiment consisted of one person assuming the roles of *researcher*, *curriculum developer*, *teacher*, *evaluator*, and *disseminator*. The teaching experiments were performed on intact classes to afford greater ecological validity (Hiebert & Wearne, 1991). An authentic research context was created with instruction being presented to students during their normal school routine in a manner similar to that advocated by Lampert (1992). The teaching program was presented to the students by the researcher who assumed the role of the classroom teacher. The researcher had expertise in the instructional sequence to be trialled. The classroom teacher acted as observer to the lessons, and therefore assumed a minimalist role of "evaluator" (Rachlin et al., 1987) in the research team. Teaching episodes were planned and modified in light of analysis of students' written work and actions during instruction. The methodology thus incorporated a diagnostic-prescriptive approach to instruction (Ashlock, Johnson, Wilson & Jones, 1980). The action-research self-reflection spiral (Kemmis & McTaggart, 1990) provided a framework for analysis of each teaching episode. The teaching episodes were developed through identification of key processes for successful operation within the domain of percent, with instruction planned for promotion of these key processes in students (as per Hiebert & Wearne, 1991). The teaching episodes were written as systematic descriptions of treatments with the recording of apparent effects in accordance with clinical intervention research (Wilson, 1976b). Simultaneously, the appropriateness of the key processes, and the instructional sequence to optimise acquisition of the key processes, were studied following the methodology for studying learning to inform teaching (Hiebert & Wearne, 1991).

4.2.3 Sequence of, and activities within, the teaching experiments

Four teaching experiments were conducted. The first teaching experiment was conducted in a school selected for its typicality; later experiments were conducted in a more controlled environment to enable a greater focus on instruction and cognitive change rather than behavioural change (see section 4.3). Each teaching experiment consisted of a series of teaching episodes which were modified during each of the experiments in response to the students' reactions. The initial teaching episodes were developed from the literature (see section 2.4) and were modified across the experiments. The focus throughout the experiments was on the relation between instruction and cognitive change (as per Hiebert & Wearne, 1991). Changes in teaching were implemented to maximise learning. The experiments took place in actual classrooms following the schools' existing timetables. Each teaching experiment was, by necessity, compressed into a short time line. Major reflection upon each teaching experiment therefore tended to occur between experiments when time was available to adequately analyse data collected. Major modifications of teaching episodes occurred between experiments, as depicted in Figure 4.1. The experiments can be seen to flow together as a series of hermeneutic cycles (Guba & Lincoln, 1989), where results of each previous experiment informed the planning of the following experiment.

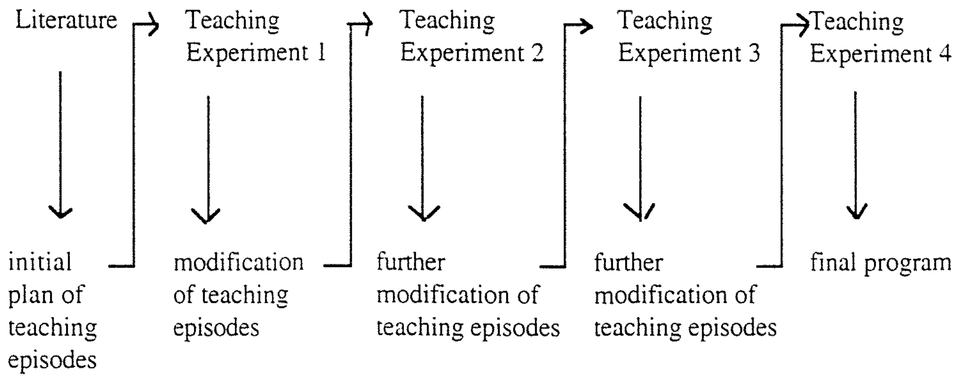


Figure 4.1. Development of teaching experiments.

Within the teaching experiments, the teaching episodes were planned prior to implementation, but modified upon reflection of student reaction, observer feedback, and analysis of field notes upon implementation. Planning for future instruction was thus contingent upon preceding episodes. A diagnostic-prescriptive approach to instruction also guided the planning of the teaching episodes. Following the dual-path model of diagnostic-prescriptive mathematics teaching presented in section 3.6.3, students' knowledge of the topic was ascertained prior to instruction via a pen-and-paper test. Results of this test influenced development of particular teaching episodes. Each teaching episode required careful planning, implementation, and monitoring so that subsequent episodes could be refined. The CMP (Lyndon, 1995) was also included as a teaching episode within each teaching experiment, and planning for CMP occurred as for all other episodes. The teaching episodes evolved in a spiral-manner through cycles of planning, acting, observing and reflecting (Kemmis & Mctaggart, 1990) upon individual teaching episodes. The development of the teaching episodes is depicted in Figure 4.2.

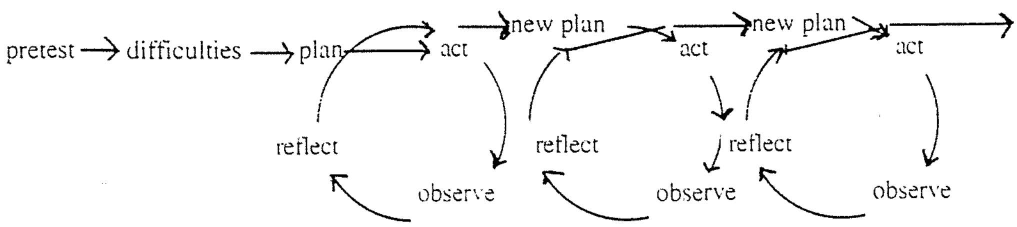


Figure 4.2. Development of teaching episodes

4.2.4 Data collection and outcomes

The study was designed to produce two outcomes. First, the experiments were sufficiently related so that instructional actions that produced learning could be cumulated, resulting in the documentation of an effective instructional program for teaching percent, and a description of incorporation of CMP into the mathematics classroom. Second, the experiments were sufficiently open so that data gathered provided insight into students' percent knowledge structures and percent problem solving, as well as a measure of the effectiveness of the CMP technique and the information on conditions under which CMP can be effectively introduced and implemented.

To achieve the outcomes of the study, and to overcome the methodological problem of validity of results when conducting classroom-based research (Lampert, 1992), data were collected from a variety of sources. The lesson plans were written, and evaluations made at the conclusion of each lesson. Comprehensive field notes were written upon completion of each teaching episode. The classroom teacher's observations were recorded at the same time. Ad hoc interviews were conducted with the students and the classroom teacher throughout each experiment, and included in the report. Students' work-samples were collected, and students were asked to write about various lessons in a reflective journal. Students were pre- and posttested, and results of the pretest were used for diagnostic purposes within the design of various lessons. Results also served to assess change in performance within the mathematical domain. The pre- and posttests were developed for direct and transfer measures (Hiebert & Wearne, 1988). A delayed posttest was administered to two of the four student groups approximately 8 weeks after instruction.

4.2.5 Summary of key points

In this section, a description of the design was presented. The design, broadly categorised as a series of teaching experiments, can be seen to incorporate elements of other designs for studying the influence of instruction upon learning. The overriding theme of the design is authenticity with the study carried out in the complicated context of the classroom. The desire for authenticity was seen to constrain the design in one sense, but also to give power to transferability of results.

4.3 Subjects

Four groups of students participated in the study. Each group of students was an intact class of approximately 30 Year 8 students (approximate age of 12-13 years). The students in all groups were not ability streamed, therefore a range of ability levels were represented in each group. Groups 1 and 2 students attended a co-educational suburban state secondary school; Groups 3 and 4 students attended an inner-city private girls' school. Upon implementation of the teaching program with Groups 1 and 2, it was found that instruction was hindered due to environmental factors associated with whole class instruction in naturalistic settings with an unfamiliar teacher. Groups 1 and 2 students reacted to an unfamiliar teacher, exhibiting behaviours which made it difficult to fully implement the teaching program as planned. The school environment of Groups 3 and 4 assisted the implementation of instruction by an unfamiliar teacher, and ensured that the planned teaching program was implemented in its entirety. Implementation of instruction for Groups 3 and 4 students required less emphasis on management than Groups 1 and 2, and enabled greater field data to be gathered.

4.4 Data sources

4.4.1 Overview

Several data sources were utilised in the study. These included a pen-and-paper test, field notes, ad hoc interviews, student diaries, student artefacts and worksamples. A description of the pen-and-paper Percent Knowledge Test used for pre- and posttesting purposes, and the delayed post-test, is presented in section 4.4.2. Researcher-generated data in the form of field notes and a self-reflection journal are described in section 4.4.3. Subject-generated data in the form of student journals, work samples and artefacts are described in section 4.4.4.

4.4.2 Pen and paper test

Pre- and posttests

The pen and paper Percent Knowledge Test was inspired by Leinhardt's (1988) descriptions of intuitive, concrete, computational, and principled-conceptual mathematical knowledge, and developed to relate to percent knowledge (see section 2.4.4 for a description of proposed intuitive, concrete, computational, and principled-conceptual percent knowledge). Using Leinhardt's theoretical basis of mathematics knowledge, test items were constructed in light of percent knowledge categories proposed in section 2.4.4. The validity of the test was maintained through checking links between percent knowledge categories and the definitions of mathematics knowledge proposed by Leinhardt. For this study, the test results were to serve as indicative of knowledge growth after instruction, and also for diagnostic purposes. In this study, the test results were not used for quantitative analysis of an inferential nature, so reliability measures were not undertaken. The test was constructed in three sections; with Section I focusing on percent concepts and principles; Section II focusing on percent conversions and benchmarks; and Section III focussing on percent problem solving.

Section I contained items relating to intuitive and principled/conceptual percent knowledge. Intuitive knowledge of percent was interpreted as the use of percent in the real world, relating to such concepts of bank interest, profit, loss, discount; as well as the use of common benchmarking percentages of 50% is a half, 25% is a quarter, and 10% is one-tenth. Principled/conceptual knowledge of percent was interpreted as the mathematical structures related to the domain of percent, such as: (i) the relationship between decimal and common fractions and percent; (ii) that 100% is the whole, and that a number of percentage parts can total the whole; (iii) interpretation of the additive and multiplicative language of percent increase and decrease situations; and (iv) construction of real-world examples of the three types of percent problems. Section I of the test consisted of 21 multiple-choice items and 3 written response items.

Section II contained items relating to computational percent knowledge, specifically: equivalence of percent, common and decimal fractions; and mental computation of amounts using percent benchmarks. For Section III, the test items related to concrete and computational percent knowledge, specifically: computation of percent exercises and application problems for the three types of percent problems, and the use of appropriate diagrams for representing percent problems. Section II and III of the test required students to calculate solutions to the questions, and record all working on the test paper. A taxonomy of items within the Percent Knowledge Test is presented in Appendix A.

Delayed posttest

A shortened and modified Percent Knowledge test consisted of 10 items. The first two items were multiple choice questions relating to interpreting the additive and multiplicative language of percent increase/decrease problems. These two items related to intuitive, principled/conceptual percent knowledge. The other eight items required calculation of percent exercises and problems. These items related to concrete/computational percent knowledge. A taxonomy of test items within the delayed posttest is presented in Appendix B.

4.4.3 Researcher Data

Researcher developed field notes

A daily diary of events for each lesson was compiled. During each teaching session, brief notes were jotted by the researcher (identified from now on as the teacher-researcher) if the students were actively engaged with a task and did not require assistance. Jottings were transcribed in detail at the earliest convenience after each teaching session, and thus a snap-shot of events in each lesson was constructed.

Ad hoc interviews

At the conclusion of every teaching session, the researcher discussed the lesson with the classroom teacher. Through this informal interview situation, the researcher gathered data on the lesson from the observer's perspective. Informal interviews were also conducted by the teacher-researcher on an ad hoc basis with the students themselves at various times, including during the lessons as the teacher-researcher moved about the room checking students' progress, and during out-of-school times such as at recess or before or after class.

Self-reflection journal

The daily diary entries were incorporated into a "self-reflection" journal using the three of the four key elements in action research (Kemmis & McTaggart, 1990) of *plan, action,* and *reflection.* Under the subheading *plan,* the planned action for the daily teaching session was detailed. The subheading *Action* provided a description of the extent to which the session followed the pre-planned sequence. Details of notable features of the session were listed, particularly students' reactions to the implemented action. Descriptions were (i) anecdotal, where accounts of individual's reactions were detailed, and included the context and events preceding and following particular incidents; and (ii) of field note type, which included subjective impressions and interpretations (Kemmis & McTaggart, 1990). Under the subheading of *Reflection,* the teacher-researcher reflected upon the session, using subjective feelings noted at the conclusion of each teaching session and data from informal interviews with the classroom teacher and from ad hoc interviews with the students. Reflections included suggestions for subsequent lessons.

4.4.4 Subject-generated data

Subjects' personal log books

Ellerton (1989) suggested that mathematical understanding is enhanced by encouraging children to reflect on daily mathematical lessons through writing. To this end, Ellerton suggested that children write entries into log books for such purposes. Ellerton's suggestions were incorporated within this study, primarily as a data gathering technique, but also for the purpose of increasing student's self-reflective skills. Wardrop (1993) described a means for developing children's writing, and therefore reflective skills in mathematics, by providing focusing starting sentences which children must use to begin their writing. Wardrop's suggestions were adopted for this study, and consequently, all subjects in the study were given a note book which served as their mathematics journal. At various times during the instructional period, the subjects were expected to write an entry into their journal given specific guidelines. These specific guidelines are presented in Figure 4.3.

Mathematics Journal

1. Write your thoughts about today's maths lesson.
2. Put the date at the beginning of each journal entry.
3. Each journal entry must be at least $^1/_2$ a page in length.
4. Each journal entry must have at least 2 sentences beginning with the following sentence starters:
   One thing I learned in maths today was ...
   I was pleased that I ...
   I found out that ... means...
   I think I am getting better at ...
   I'm still confused about...
5. Rule off after every entry.
6. You may include diagrams or illustrations or decorate your journal in any (acceptable) way you wish.

Figure 4.3. Instructions to students for completing mathematics reflection journals.

Student artefacts and worksamples

Student artefacts and worksamples included any pieces of work students completed during the instructional unit, such as their notebooks, homework sheets, class worksheets, written reports.

4.5 Procedure

For each group, the teaching experiment was conducted in four steps, as follows:

Step 1 - Pretest

Step 2 - Metacognitive training

Step 3 - Instruction

Step 4 - Posttest

For Groups 3 and 4, a fifth step was included, which was the administration of the delayed posttest. Between each teaching experiment, planning for subsequent teaching experiments occurred.

Step 1 - Pretest

Prior to commencement of teaching, the Percent Knowledge Test was administered to students during timetabled mathematics classes. The test required two separate mathematical lessons for completion. Students were permitted to use calculators for the concrete/computational items of the test if they so desired. The test was administered by the students' usual classroom teacher. The tests were analysed for diagnostic purposes. Specific items with which the majority of students experienced difficulty were noted to thus guided the development of instruction.

Step 2 - Metacognitive training

The metacognitive training component of the instructional sequence was adapted from the Conceptual Mediation Program (CMP -see section 3.5 for a full description of this program). Two lessons were planned. The first session was planned to explore issues of attention, learning and remembering, (and to introduce a strategy called Look-Say-Cover-Write-Check (LSCWCh) which is a suggested means for taking control of remembering in the CMP). The second session was planned to explore natural and accelerated forgetting, (and to introduce the Old Way/New Way (O/N) strategy, an integral component of the CMP, for overcoming accelerated forgetting). The metacognitive training episode was planned to make the following points:

1. The nature of the brain
   The brain is designed to forget.

2. Learning and attention
   Learning is easy, remembering and paying attention is hard.
   We can choose what we pay attention to.
   What we pay attention to, we learn.
   Paying attention is hard because it requires effort.

3. Recognition and recall memory
   Our brain has two quite different ways of remembering, called recognition memory and recall memory. Knowing how these are different will help us take control of remembering.
   Recognition memory just happens naturally without effort.
   We can't improve our recognition memory by practice because it's already automatic.
   Recall memory is naturally effortful.
   Recall memory only becomes automatic through practice.
   We can avoid problems of effortful recall by using a good recall strategy.
   "Look, Say, Cover, Write, Check" (LSCWCh) is a good recall strategy.

4. Natural forgetting

Natural forgetting is normal; it just happens.

We can take control of natural forgetting by using a good recall strategy and practice.

When we have forgotten something, we go through a process of relearning.

Relearning is generally fast once practice recommences.

5. Accelerated forgetting

Accelerated forgetting is natural and only occurs when someone has their own way of doing something before they try to learn a new or better way.

Accelerated forgetting is quite rapid.

We can take control of accelerated forgetting by using the Old Way/New Way (O/N) strategy.

Within the metacognitive training program, two strategies are presented. The LSCWCh strategy is a sequence of actions for committing something to memory. If used, for example, to learn the spelling of a new word, the first action is to LOOK at the word, noting the interesting features; then SAY the word; COVER the word; WRITE the word from memory, and then CHECK that the written word matches the targeted word. As a strategy within the CMP, LSCWCh is presented as a suggested means to take control of remembering, with emphasis on COVERING and WRITING the "word" in order to activate recall memory. In CMP, LSCWCh is described as a generic strategy, not just for learning to spell new words. The O/N strategy is presented in the metacognitive training episode as a strategy for taking control of forgetting. This strategy is described in detail in section 3.5.3.

One other main activity is incorporated within the metacognitive training program, and this is the Colour Card activity. The purpose of this activity is to demonstrate the existence of the proactive inhibition (PI) mechanism. For this activity, a set of "Colour Cards" are distributed. The colour cards are a set of three cards on which a series of words printed in various coloured inks are displayed, to cause maximum degrees of proactive interference (e.g., the word "green" printed in blue ink). In this activity, the sets of words are read, and reading time recorded. Results of this activity are displayed as a focus for discussion on the PI mechanism and accelerated forgetting.

Step 3 - Instruction

Instruction was designed around a number of teaching episodes. The teaching episodes were planned in consideration of two factors: (i) the need to balance presentation of key percent knowledge as suggested in the literature together with equipping students to succeed in solving percent problems in school situations; and (ii) the time allocated by the school for teaching the topic of percent. Development of curriculum is guided by the mathematics syllabus, and, as stated in section 1.2.2, specific learning experiences for Year 8 in the topic of percent relate to analysing the relationship between percentage and decimal and common fractions, estimating and calculating to find a percentage of a number or quantity (Type I percent problems), estimating and calculating to increase and decrease by a given percentage, and solving and creating problems involving practical application of percent, including discount and simple interest. As also stated in section 1.2.2, the syllabus states that Type II and III percent problems be introduced to students in a staggered manner, in Years 8, 9 and 10 respectively. As suggested in section 2.5, instruction in percent would focus around analysing and interpreting percent problems through development of a part-whole-percent schema, representation of percent problems on a vertical, dual-scale percent/quantity number line, and the Rule of Three procedure for proportion problems (see section 2.4.2 for a detailed description of this method). Instruction was designed to follow suggestions for the development of mathematics knowledge (Kamii, 1995; Lampert, 1986; Leinhardt, 1988) that instruction should proceed from intuitive to concrete, concrete to computational, computational to principled-conceptual. For the first teaching experiment, instruction was planned into seven teaching episodes, including metacognitive training. The seven teaching episodes are outlined as follows:

1. Metacognitive training
   (See description of this episode previously presented in step 2)

2. Concept of percent
   Percent use in the real world
   The multifaceted nature of percent

3. Fraction equivalence and the Rule of Three
   Practice Rule of Three procedure

4. Interpreting percent problems
   Interpreting percent problems in terms of part-whole-percent (percent schema)
   Creating percent problems to match the three types of percent equations

5. Solving percent problems
   Representing elements of percent problems on vertical number line
   Constructing proportion equation from representation
   Solving percent word problems using Rule of Three

6. The language of percent increase and decrease
Interpreting percent increase and decrease problems in terms of additive and multiplicative language

7. Percent, fraction, decimal equivalence
Analysing the relationship between percent, common and decimal fractions
Practice in conversions between percents, fractions and decimals

Percent instruction per se, was planned to commence in Teaching Episode 2, with the concept of percent being the focus of the episode. This episode was planned to draw out students' intuitive percent knowledge. In subsequent episodes it was planned that students would represent percent situations on a vertical number line and develop skill in solving proportion equations using the Rule of Three procedure. The dual-scale, vertical number line would be presented as a mental model of percent as a proportion on which percent situations could be visually represented. These episodes were to promote students' concrete and computational knowledge of percent as a proportion. Development of such concrete and computational knowledge associated with percent was to lay the foundation for the development of principled-conceptual percent knowledge, as listed in section 2.4.4.

Step 4 - Posttest and delayed posttest

At the conclusion of the instructional sequence, the posttests were administered by the Researcher acting as classroom teacher. For Groups 3 and 4, the delayed posttests were administered approximately 8 weeks after instruction.

Planning of subsequent teaching experiments

Upon completion of the teaching experiment with Group 1, the instructional sequence was modified in light of reflection of implementation. The instructional sequence developed for Group 2 was thus contingent upon instruction implemented with Group 1. The instructional sequence for Group 3 was modified in a similar fashion. Analysis of the third teaching experiment indicated positive results, thus the fourth teaching experiment was conducted with little modification to the instructional sequence used with Group 3. Teaching Experiment 4, therefore, was simply a further trial of the instructional sequence under conditions similar to the experiment conducted with Group 3.

4.6 Analysis

4.6.1 Overview

Data analysis occurred at two levels: (i) within and between episodes in each teaching experiment, and (ii) across the four teaching experiments. These two levels of analysis are described in sections 4.6.2 and 4.6.3 respectively.

4.6.2 Within and between episodes in each teaching experiment

For each episode in the teaching sequence, the teacher-researcher's field notes and the classroom-teacher's observations were combined with information on student performance to produce a description of lesson actions and student responses. Episode descriptions were presented in terms of the four steps of the action research cycle (Kemmis & McTaggart, 1990): plan, action, observation and reflection. Reflection upon each episode informed the plan of the following teaching episode. Teaching episodes were evaluated by reflecting on events within the lesson. Each episode was rated as successful or unsuccessful through analysis of students' responses and behaviour, and through discussion of the episode with the classroom teacher. Episodes were rated as unsuccessful through consideration of many factors, such as: the lesson was not implemented as planned; anticipated task completion by students did not occur; student behaviour indicated disinterest, lack of attention and/or motivation; students did not appear to grasp the concepts and ideas being presented; reflection upon implementation of the lesson, through discussion with the classroom teacher, suggested the episode was not successful. Evaluation of episodes primarily was on the basis of the "feel" through consideration of many factors (as per design-experiments of Hawkins & Collins, 1992).

For each teaching experiment, pre- and post-tests were scored and the results of these were combined with teaching episode information and the students' reflective journal writing to produce a description of the total teaching experiment which documented relationships between teaching action and student response across the experiment. This was evaluated in terms of the changes in test results (overall effectiveness) and the relative effectiveness of the episodes. This was particularly achieved in two ways. First, for percent knowledge, changes in test performance for different topics were related to impressions of the effectiveness of the teaching episodes for those topics. As a result of this, episodes which appeared successful in terms of their evaluation and related test results were retained, episodes for which results were mixed were modified, and episodes which were unsuccessful in terms of evaluation and test results were reconstructed. Second, for CMP, later episodes were observed for evidence of the CMP actions taught in early episodes. As a result, the CMP teaching actions which appeared successful in terms of their evaluation and the later use of the their teaching were retained, actions for which results were mixed were modified, and actions which were unsuccessful in evaluation and later use were reconstructed.

At the conclusion of each teaching experiment, summative evaluation occurred at two levels: (i) reflection upon implementation of teaching episodes, and (ii) reflection upon pre- and posttests and instruction. The focus was on modifying the experiments so that a larger number of episodes was deemed effective and test score results showed greater improvements. Thus, as shown in Figure 4.4, the analytic process became a series of hermeneutic cycles (Guba & Lincoln, 1989) through which instruction was refined (Hawkins & Collins, 1992). As well, episodes that retained effectiveness across experiments became the mainstay of the final teaching plan.

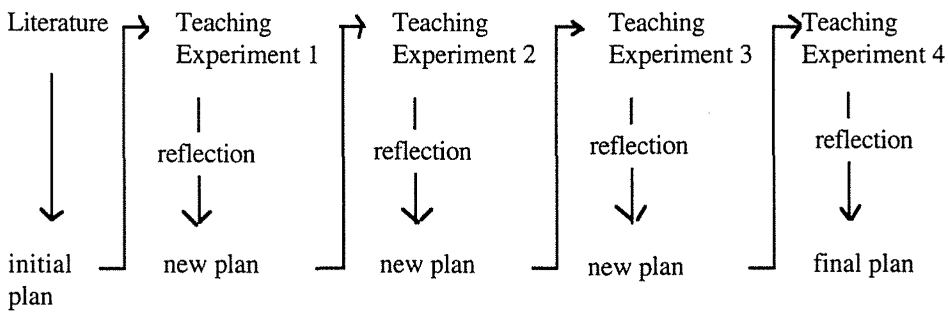


Figure 4.4. Planning of teaching experiments following a hermeneutic cycle.

4.6.3 Across the four teaching experiments

Upon completion of the four teaching experiments, results were analysed collectively. Key processes in teaching and learning percent and the CMP had been developed from the literature (see sections 2.4.2 and 3.6.3). For each of these key processes, data from relevant episodes, test results and reflective journals were combined within each teaching experiment and compared across experiments (taking account of teaching differences). The focus here was on constructing categories for student responses with respect to percent knowledge, percent problem solving and use of CMP techniques, and in developing explanations for these responses in relation to the teaching program.

CHAPTER SUMMARY

In this chapter, the design of the study was described. The focus of the study was to research a teaching program for influencing Year 8 students' percent knowledge and problem solving skills, as well as metacognition, in authentic classroom situations. The design of the study followed the teaching experiment, but as presented in this chapter, was seen to be a "hybrid" design. The subjects, data sources, procedure and analysis were described. The results of the study are presented in the next chapter.

CHAPTER 5
RESULTS

CHAPTER OVERVIEW

In this chapter the results of the study are presented. There are five sections in this chapter. The four teaching experiments within the study, are described individually in sections 5.1 to 5.4. In section 5.5, results across the four teaching experiments are discussed. The description of each teaching experiment includes pre- and posttest results, an overview of the planned teaching sequence, and a detailed description of the implementation of each teaching episode within the planned sequence. The contingent nature of the study is shown as the report on each teaching experiment concludes with a statement of reflection, which can be seen to provide direction for the following teaching experiment.

The report contained in this chapter is a narrative. The primary description of events is from teacher-researcher field notes, shaped by taking account of student reactions and responses, and classroom-teacher observations. Initially in the report, there is minimal reference to student diary entries and classroom teacher observations. A widening field of data sources is utilised as the narrative progresses with student diary entries and classroom-teacher observations increasingly being incorporated within the report, to provide depth to descriptions presented. The narrative shows the evolutionary nature of the four studies, initially with teacher-researcher self-reflection being used to mould and shape subsequent teaching episodes, to gradual utilisation of student and observer feedback to confirm teacher-researcher observations and reflections.

5.1 Teaching Experiment 1

5.1.1 Overview of report on Teaching Experiment 1

The results of Teaching Experiment 1 are reported in this section. The pre- and posttest results are presented in section 5.1.2. An overview of the planned teaching sequence, comprising seven teaching episodes, is presented in section 5.1.3. In sections 5.1.4 to 5.1.10 inclusive, implementation of each of the seven teaching episodes is described. Each episode is presented as a new section. Each episode description is presented under the subheadings *Plan*, *Action*, *Observation* and *Reflection*. A reflection upon the implementation of the unit of work is presented in section 5.1.11. A reflection upon pre- and posttest results and instruction is presented in section 5.1.12. This chapter section concludes with a statement on directions for Teaching Experiment 2 in section 5.1.13.

5.1.2 Pre- and posttest results

As stated in section 4.4.2, the pre- and posttests were parallel forms, containing items relating to intuitive, concrete, computational and principled-conceptual knowledge (Leinhardt, 1988) of percent. The test comprised three sections, with Section I focussing on intuitive and principled-conceptual percent knowledge, Section II focussing on percent, common and decimal fraction equivalence notions and percent benchmarks, and Section III relating to percent calculations and problem solving. Group 1 scores on each section of the pre- and posttests are presented in table 5.1.

Table 5.1

Pre- and Posttest Means (%) for Group 1 Students on the Percent Knowledge Test in Total and in Each Test Section

	Components of the Percent Knowledge Test			
Test	Total	Section I	Section II	Section III
Pretest	43%	60%	60%	10%
Posttest	52%	69%	54%	33%

From Table 5.1, it can be seen that Group 1 test scores showed a slight positive change with a pretest score of 43% and a posttest score of 52%. Within each section of the test, there is a slight positive change for Section I (intuitive, principled/conceptual knowledge), a slight negative change for Section II (conversions and benchmarks) %, and a positive change for Section III (percent calculations and problem solving). From Table 5.1, it can be seen that Group 1 students' intuitive and principled-conceptual percent knowledge, and proficiency in percent conversions and benchmarking, prior to instruction, was much greater than their percent calculation and problem solving skills. It also appears that, as a result of instruction, Group 1 students' intuitive and principled/conceptual percent knowledge (Section I), and computational knowledge (conversions and benchmarks - Section II) did not change markedly as a result of instruction. Computational and concrete knowledge (percent calculations and problem solving - Section III) did change as a result of instruction, but did not increase to a satisfactory level for the whole group. In terms of the total test performance, there is a limited positive increase overall. Graphical representation of the pre- and posttest scores are presented in Figure 5.1, highlighting the marked change in pre- and posttest scores for Section II Part II of the test, and minimal change in other sections, and in the test overall.

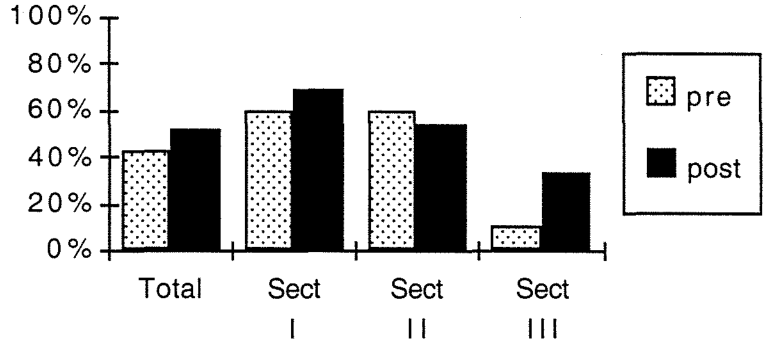


Figure 5.1. Graphical representation of Group 1 students' pre- and posttest means (%) in total, and in each test section.

Each section of the test was also scored for diagnostic-prescriptive purposes, so that particular areas of weakness could be identified and thus planning for instruction informed. Within the three parts of the test, the number of incorrect student responses for each item was tallied. Test items showing a high level of incorrect scores were identified from the Taxonomy of Percent Knowledge Test items (see Appendix A). Representation of the number of incorrect responses to each item on Section I (intuitive, principled-conceptual knowledge), Section II (conversions and benchmarks) and Section III (percent calculations and problem solving) of the Percent Knowledge Test are presented in Figures 5.2, 5.3 and 5.4 respectively.

In Figure 5.2, it can be seen that, prior to instruction, the majority of students in Group 1 experienced difficulty with intuitive, principled-conceptual percent knowledge items 1d, 6c, 7a-c and 8a-c. From the taxonomy (see Appendix A) these items relate to understanding the real world percent transaction of bank interest, the additive and multiplicative language of percent increase and decrease, and the relation of real-world percent situations to percent equations. Specifically, item 1d relates to the interest charged on a credit card where the purchase cost via a credit card is greater than the initial purchase price. Item 6c relates to the additive language of percent increase situations where a 150% increase is the original amount plus 150%. Items 7a-c relate to the multiplicative language of percent increase and decrease situations, where a) a 25% discount is the same as 75% of the original; b) a 25% increase is the same as 125% of the original; and c) a 125% increase is the same as 225% of the original. Item 8a-c relates to posing of real-world percent problems for: a) Type I; b) Type II; and c) Type III percent situations.

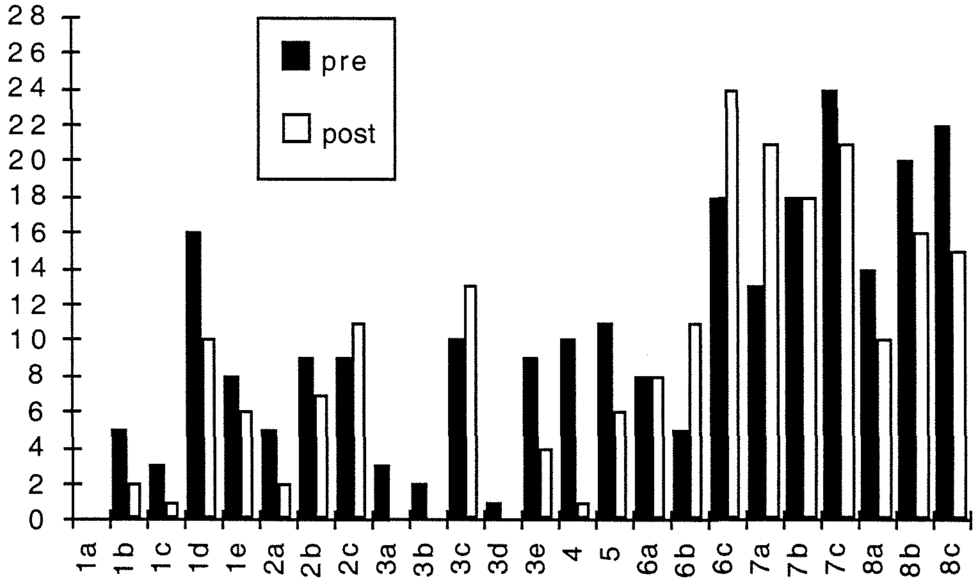


Figure 5.2. Group 1 pretest (n=25) and posttest (n=27) *incorrect* scores on Section I (intuitive, principled/conceptual percent knowledge) of the Percent Knowledge Test. Graph indicates number of students *incorrectly* responding to particular items.

Comparing pre- and posttest errors on Section I of the test, it can be seen that, after instruction, Group 1 students continued to experience difficulty with the same types of items as in the pretest. That is, many students still experienced difficulty with items relating to bank interest and loans, the multiplicative and additive language of percent increase, and the ability to pose real world percent problems from percent equations (items 1d, 6c, 7a-c, and 8a-c), although students' ability to pose real world percent situations showed improvement (item 8a-c). They also appeared to have difficulty with the percent benchmark item of 10% as one in ten (item 2c), and the concept of percent as a part/whole relationship (item 3c).

In Figure 5.3, showing incorrect responses on Section II (conversions and benchmarks) of the Percent Knowledge Test, it can be seen that, on the pretest, the majority of students incorrectly responded to items 1 and 4, which relate to percent-to-fraction conversions and fraction-to-percent conversions. Percent benchmark items (items 5a-j) were generally answered correctly. On the posttest, there appears to be a reversal of scores, with students incorrectly responding to items involving percent benchmarking (items 5a-j) and correctly converting percents to fractions (item 1). Students' performance on conversions of fractions to percents (item 4) remained unchanged.

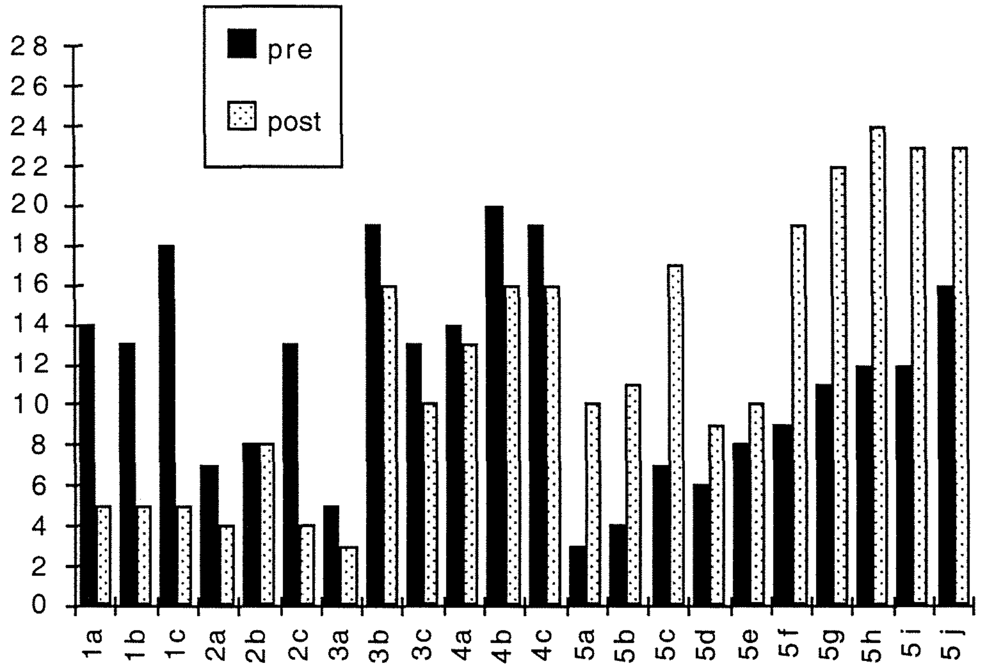


Figure 5.3. Group 1 pretest (n=28) and posttest (n=27) *incorrect* scores on Section II (conversions and benchmarks) of the Percent Knowledge Test. Graph indicates number of students *incorrectly* responding to particular items.

In Figure 5.4, it can be seen that, on the pretest, all students experienced difficulty with percent calculations and solving percent problems (Section III). Results indicate that students experienced less difficulty with item 1a and 2a, which relate to calculation of Type I percent equations, and solving Type I percent word problems respectively. Figure 5.4 also indicates that no students drew diagrams to assist percent problem solving (item 3), and most students experienced difficulty writing percent problem solutions in words (item 4). Compared to pretest scores, posttest scores indicate that less students experienced difficulty in solving percent word problems (item 2a-c) after instruction, and slightly more than half of the students were using diagrams to assist percent problem solving (item 3). Slightly more students were also successfully tackling multi-step percent word problems (item 5a-d).

Analysis of students' calculation procedures for items on Section III of the pre- and posttest indicate change in methods used by students. In the pretest, students who successfully completed Type I problems, utilised the decimal multiplication procedure (e.g., 24% of 15 = 0.24 x 15), or keyed the sequence into the calculator (e.g., 15 x 25 %), using the calculator "percent" button. Other students simply wrote the correct solution, therefore giving no indication of their solution strategy. Of the students who got Type I problems wrong, most students made no attempt at solution, or provided a single answer with no working out. One student used an incorrect subtraction strategy (e.g. 15% of 24 = 15 - 24), which indicates little understanding of percent calculation procedures.

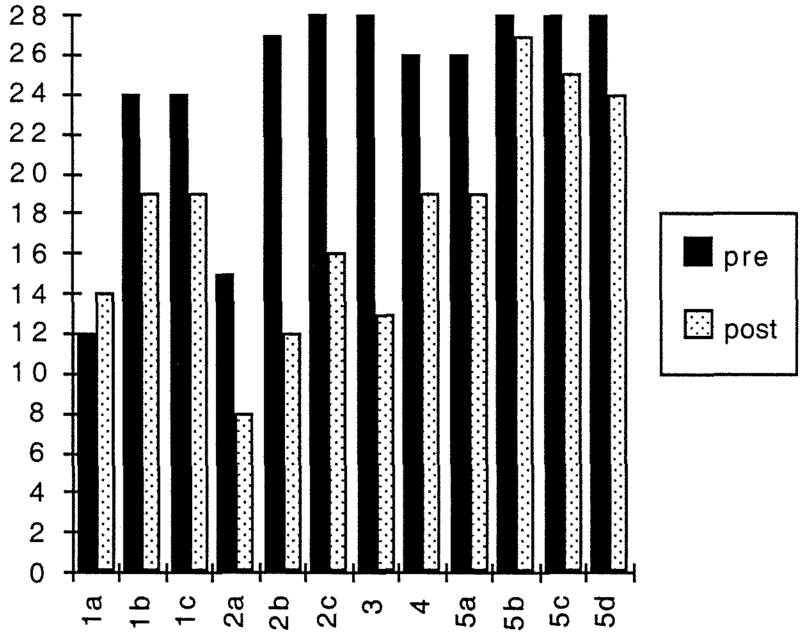


Figure 5.4. Group 1 pretest (n=28) and posttest (n=28) *incorrect* scores on Section III (percent calculations and problem solving) of the Percent Knowledge Test. Graph indicates number of students *incorrectly* responding to particular items.

For Type II problems, of the students who successfully answered such items, the division strategy was used (e.g., express 3/7 as a percentage = 3 ÷ 7 = 0.4285 = 42.85%). For students who scored Type II problems incorrectly, there was no solution attempt made, or an answer only was given showing no working. For Type III problems, the four students who correctly answered these items gave a numerical answer and no working. Of the students who did not score on these items, the majority of students made no attempt at a solution. Two students used multiplication (e.g., 72 is 8% of what number = 0.08 x 72), and one student continued to use the subtraction procedure as for Type I problems (72 is 8% of what number = 72 - 8 = 66%). Students' solution procedures for percent calculations therefore, showed utilisation of various strategies, but the majority of Group 1 students made no attempt to answer these items.

On the posttest, utilisation of the proportional number line method, presented in the teaching experiment, was evident for many students. Of the students who correctly responded to such items on the posttest, the strategy used was the number line, or the proportion equation without the number line. Some students provided answers only which were correct. One student continued to use the decimal multiplication procedure. Incorrect responses showed use of the proportional number line method with placement of numbers in incorrect positions, or incorrect placement of numbers into a proportion equation. One student used decimal multiplication for all three types of percent problems, and thus successfully solved Type I problems only. Other incorrect responses included no solution attempt by some students. Comparing pre- and posttest results in Figure 5.4, it can be seen that, on the pretest, more students experienced difficulty with Type II and III percent calculations (items 1b and 1c) than with Type I percent calculations (item 1a). Solving percent word problems (item 2a-c) was more successfully attempted by students after instruction, through use of the proportional number line method.

5.1.3 Planned teaching episodes

For Group 1, seven teaching episodes were planned, to fit within an allocated time frame of two weeks (10 class periods each of 40 minutes duration). The planned teaching episodes, described in section 4.5, are presented in Table 5.2. The table provides the number and title of each teaching episode, together with the number of actual lessons taken for implementation, and the chapter section in which discussion of each episode appears.

Table 5.2

Overview of Teaching Episodes for Group 1

Episode	Topic	Lessons	Location
1	Metacognitive training	2	5.1.4
2	Concept of percent	1	5.1.5
3	Fraction equivalence and the Rule of Three	1	5.1.6
4	Interpreting percent problems	1	5.1.7
5	Solving percent problems	2	5.1.8
6	The language of percent increase and decrease	3	5.1.9
7	Percent, common and decimal fraction equivalence	2	5.1.10

A total of 16 class periods were spent with Group 1. Ten lessons were directly on the topic of percent, with a further 2 lessons used for metacognitive training. Pre- and posttesting occupied 2 days prior to and immediately following implementation of the unit of work.

5.1.4 Episode 1: Metacognitive training

Plan

For the metacognitive training, 2 lessons were planned. As described in section 4.5, the first lesson was to explore issues of attention, learning and remembering. The Look-Say-Cover-Write-Check (LSCWCh) strategy was to be presented in this lesson as a strategy for taking control of remembering. The second lesson was to focus on natural and accelerated forgetting. It was planned that the Colour Card activity would be implemented to demonstrate the PI mechanism and the O/N strategy described as a means of taking control of accelerated forgetting. The main points of the episode, being: the nature of the brain; learning and attention; recognition and recall memory; natural and accelerated forgetting; would be printed on coloured cards, and displayed on the board as they were addressed (see section 4.5 for list of key points). The students would be directed to take notes on the key points as they were presented.

Action

This episode spanned 2 lessons. In the first lesson, *the nature of the brain* (that the brain is designed to forget), and *learning and attention,* were discussed. The students took notes on the key points in their journals as directed by the teacher-researcher. Key points on recall and recognition memory were described. The recall strategy LSCWCh was not presented to the students. Midway through the lesson, a student arrived late to class, and told other students the reason for his lateness. At the end of the lesson, the students were instructed to write an entry in their diary.

The second lesson was effectively 30 minutes long as the students arrived late from their previous class. Natural and accelerated forgetting was described to the students, and the Colour Card activity presented. The students did not finish the Colour Card activity. The notions of proactive inhibition (PI) and accelerated forgetting were described, with the teacher-researcher drawing upon students' experience with the Colour Card activity. The Old Way/New Way (O/N) strategy was not presented to the students.

Observation

In the first lesson, subdued laughter came from the student body as they were told that: *The brain is designed to forget.* It appeared that the students were interested in this notion, and some students whispered such things as: "Now I know why I don't remember to do my homework". When students were asked to think of instances of their brain forgetting, the majority of students volunteered to share their experiences. During the discussion on *attention* and *control of attention* the students appeared attentive and interested.

With the late arrival of the student during this lesson, control of the lesson appeared to be momentarily lost, with audible questions directed to the student as to the reason for his lateness. As the lesson progressed, students appeared to be less attentive; there were increased whisperings between students, several students were observed gazing out the window, and audible comments of "What's' she going on about?"; "What's all this for?"; "I thought this was a maths lesson!" were noted. The students appeared to be clearly confused about the purpose of this lesson. Student diary entries support this observation, with students writing such things as:

> *One thing I learnt in maths was that the brain is designed to forget. I don't know why we did this.*
>
> *I thought this was mathematics, not English.*
>
> *I'm still confused about all this brain stuff we did today.*
>
> *One thing I learnt in maths was that we had a new teacher.*
>
> *I was pleased I learnt about remembering. I am still confused about why we didn't do maths today.*

The classroom teacher (acting as observer of these lessons), stated that he thought the lesson progressed well, and that the students appeared attentive and well-behaved. He noted that he thought the students became concerned when they were told that they were responsible for their own actions. As he commented: "They [the students] didn't like being told that it was their responsibility to pay attention."

During the second lesson, the students appeared to enjoy the Colour Card activity, as they were cooperative and animated. All students used this activity in a competitive fashion, trying to better their friend's score. However, attention appeared to be difficult to maintain as the teacher-researcher tried to describe the nature of PI and taking control of accelerated forgetting. Students' mumblings indicated that only a certain amount of the students were paying attention, and audible comments of "this is not maths", were heard. The fact that this was not a mathematics lesson during mathematics time appeared to confuse the students. The classroom-teacher stated that he thought the students had understood very little of the colour-card activity, although he stated that they appeared to enjoy it.

Reflection

The whole planned metacognitive program was not presented to the students in the two lessons. The students' unsettled behaviour appeared to make it difficult to present the lesson in a coherent manner. The students were not presented with the strategies for recall memory and accelerated forgetting due to the fact that it was perceived that the students would not understand the purpose of such strategies. Student data, classroom-teacher comments, together with observations of student behaviour during the lesson, confirmed the teacher-researcher's perceptions that the episode required further planning. It was perceived that the rationale for the lesson should have been clearly presented to the students, and that more student activity should have occurred. The students appeared to have difficulty determining the difference between recognition and recall memory, thus it was perceived that an activity to demonstrate the difference between these two memories needed to be designed, so that the need to use a good memory strategy for storing information into recall memory would be more apparent to the students.

The implementation of this teaching episode was unsuccessful. Several factors appeared to influence this outcome, including: lack of sustained student attention; poor planning and sequencing of instruction; non-establishment of clear goals and expectancies; poor alignment of the episode to the context of the class; and non-achievement of assisting students to see the generalisable nature of the skills presented.

The purpose of this teaching episode was to teach students about learning, and to teach students the "art of memory" (Norman, 1980). It was to provide explicit instruction in metacognitive strategies for improved task performance (Cole & Chan, 1990) via self-regulatory metacognitive training (Brown & Palinscar, 1982). In this episode, students' lack of attention made it difficult for teaching to occur as planned. Students' lack of attention provides supporting evidence to key points of the CMP, namely, "What we pay attention to, we learn"; "We choose what we pay attention to"; and "Paying attention is hard because it requires effort" (Lyndon, 1995). In terms of effort, applying effort to learn comes from, and is controlled by, the learner (Weinstein & Mayer, 1986). In this episode, the students were choosing not to pay attention, and thus it appeared that minimal learning occurred. And, although developing metacognitive skills is considered the means through which independent, autonomous learners develop (Chan, 1993), convincing students that strategic skills will enhance task performance is not a simple task. According to Brown and de Loache (1983), metacognitive training must enable students to apply metacognitive skills to other learning situations. In this episode, this was not achieved, as the metacognitive strategies planned were not presented to the students, which may have further contributed to students' confusion as to the purpose of the episode, and thus led to their stating that "this is not maths". No opportunity was provided for students to see how descriptions about learning, memory and forgetting related to mathematics.

Students' lack of attention during this episode cannot totally be held responsible for the failure of this episode. The design of the lesson also was a contributing factor. This episode appeared inadequately planned as it was unable to capture and maintain students' attention. Careful planning of a teaching sequence is an element of successful teaching (Rojewski & Schell, 1994). Integral to this is the importance of informing learners of their role in the learning process through establishment of the goals of the teaching (Mercer & Miller, 1992), as well as considering the sociology of the classroom (Rojewski & Schell, 1994). Although the metacognitive training episode was carefully planned and sequenced, it was apparent upon implementation that modification and further planning was necessary. Rather than attempt to replan and implement a modified version of this episode with Group 1, it was decided to implement the next planned episode to maintain some continuity of instruction.

5.1.5 Episode 2: Concept of percent

Plan

As an introductory episode for eliciting students' notions of percent, the episode was planned as follows:

1. Brainstorm percent words/notions/uses in the real world. Students copy into books.
2. Students form groups to search through newspapers, cutting out examples of real-world percent use.
3. Students to create a poster of newspaper clippings, writing a descriptive meaning of the use of percent indicated on each example.
4. Each group to present poster to rest of class, describing examples of percent uses found.

Action

The episode proceeded as planned. Not all groups completed their posters in the time allowed. All posters were presented.

Observation

The brainstorming session about percent resulted in suggestions such as: "of 100; discount; profit; out of 100, test scores; sport; shops". These suggestions were written on the board. The students were informed of the task for the lesson, and the cardboard, scissors, glue and newspapers were distributed. The students formed groups and began the newspaper search. The lesson progressed noisily as students moved around the room, organising themselves into groups or talking to other students in the classroom. Throughout the lesson, the teacher-researcher walked amongst the groups, checking that the task was being completed. As the episode progressed, all students appeared involved in the activity. However (similar to the view of the classroom teacher) it was perceived that this lesson did not actually help students build on their knowledge of percent usage in the real world. Throughout the lesson, the teacher-researcher reminded individual groups that the focus of the task was to find a variety of examples of percent usage in the real world, not many of the same. For example, one group of students cut out five advertisements showing discounted prices of goods. The teacher-researcher assisted these students by searching the newspaper for percent used in a different way, such as survey data presented in a report.

When the student groups presented their poster to the class, they simply stated what the advertisements were showing. For example, one group pointed to a cutting on their poster and stated, "Here is an ad for bank interest." Examples of percent used in statistical form, for example, to describe research data reported in the paper, were not located by the students (except with teacher-researcher assistance). There appeared to be reluctance on the students' part to scan various articles to identify such percent usage. Through informal interviews with the students, there appeared to be general consensus that the students enjoyed the "cutting up" activity, and that the activity was something they had not done for a long time during mathematics. The classroom-teacher stated that he thought the students had enjoyed the relaxed nature of the activity, however, he also stated that stimulation of student knowledge of percent was not great during this activity.

Reflection

As an introductory activity, this lesson was perceived as a satisfactory beginning to a unit, where the topic was introduced in a relatively "free" setting. However, the lesson was perceived as relatively nondescript, as it neither appeared to promote development of the concept of percent, nor exemplified to students the multifaceted use of percent in our society. The primary concept of percent is its relationship to a base of 100, and instruction must promote this notion (Cooper & Irons, 1987; Reys et al., 1992). However, as stated by Parker and Leinhardt (1993, p. 47), knowing percent "means to understand its multiple and often embedded meanings and its relational character." The purpose of this episode was to promote students' awareness of the multiple forms of percent use in the real world. From students' responses during the brainstorming session, it was apparent that students were familiar with the basic notion of percent and its relationship to 100. From students' actions during the episode, it was apparent that students were not aware of the multifaceted nature of percent used in the real world. The episode did not appear strongly to promote awareness, although discussion between teacher and students may have assisted this awareness. In terms of lesson design, the episode was rated as requiring more structure. It was perceived that the rationale for the episode needed to be communicated to the students, as well as activities which directed students to the focus of the lesson. For students to see the multifaceted nature of percent using media required careful selection of examples to be given to students for analysis. This episode was primarily an introductory lesson to provide an opportunity for the teacher-researcher to listen as students engaged in the activity, determining their knowledge of percent use in the real world. The next episode was the beginning point for presentation of the proportional number line method for percent problem solving, thus implementation began in the next lesson.

5.1.6 Episode 3: Fraction equivalence and the Rule of Three

Plan

This lesson was planned to introduce students to the Rule of Three as a method for solving proportion equations where proportional equations are equivalent fractions. The plan for this lesson was as follows:

1. On board, write: $\frac{3}{4} = \frac{?}{20}$

   Ask a student to volunteer the solution, and to explain his/her solution procedure. Other methods used by students to be discussed (if any).
2. Demonstrate solution using Rule of Three procedure.
3. Ask students to check that 20x3÷4 equals 15 using calculator.
4. Present students with another pair of equivalent fractions, with the unknown in a different place. Ask students to use calculator to check answers.
5. Provide a third example with the unknown in a different place to the first two examples.
6. Provide 10 practice examples. Correct in class, repeating procedure.
7. Provide students with 30 practice examples. Teacher-researcher to individually correct students' work whilst moving around the room, monitoring progress.

Action

The lesson proceeded as planned. After the Rule of Three had been related to equivalent fractions, sets of practice exercises were written on the board. All planned blackboard exercises were completed. At the end of the lesson, the students were provided with further practice exercises for homework.

Observation

The discussion on the equivalent fraction procedure was completed rapidly, with all students indicating that to find $^3/_4$ as equivalent to $^{15}/_{20}$, they multiplied by $^5/_5$. The students were attentive as the Rule of Three procedure was demonstrated. As each set of practice exercises was written on the board, the teacher-researcher individually marked students' answers as they worked. The students appeared to enjoy the rapid feedback on their progress, as there were continual requests for further marking of work completed. For the duration of the lesson, the classroom noise was minimal. The students appeared to work consistently for the whole lesson. The solid nature of the students' progress during that lesson, perceived by the teacher-researcher, was confirmed by the classroom-teacher. He remarked that he felt that the students had worked extremely hard and were on task for the majority of the lesson. As students exited the room at the end of the lesson, several students approached the teacher-researcher and showed the amount of work they had completed in that lesson, and commented: "Look at how many I got right!"

Reflection

Implementation of this episode was perceived as successful. One of the features of this episode was the fact that it contributed to students' feelings of success. Designing activities to ensure successful performance is a key element of a teaching program (Cole & Chan, 1990; Mercer & Miller, 1992) and a strong motivating force (Cole & Chan, 1990). The episode also considered the sociology of the classroom (Rojewski & Schell, 1994) to provide a mathematics lesson during mathematics time. The design of the episode also enabled the teacher to provide immediate feedback to students as they completed the exercises, as well as providing students with the opportunity to develop a skill to automaticity. These are two components of effective instruction (Mercer & Miller, 1992). In this episode, the students practised the Rule of Three procedure; the provision of practice was to enable the procedure to reach automaticity to assist in later problem solving (Anderson, 1985; Glaser & Bassock, 1989; Resnick & Ford, 1982; Sweller, 1989). The procedure was presented to the students in a manner which is not a recommended practice in mathematics instruction (see, for example, Cramer, Post & Currier, 1992; Post, Behr & Lesh, 1988; Streefland, 1985). However, the students in this group did not question the purpose of learning the Rule of Three, or why it "works".

At this point in the teaching program, several pathways were apparent to provide students with meaning to the Rule of Three procedure. The procedure could have been explored from a historical perspective, searching out the embedded mathematical principles from which it is derived (Resnick & Omanson, 1987). The Rule of Three could also have been developed through linking to ratio and proportion (Hart, 1981; Robinson, 1981). In this sequence of instruction, the Rule of Three was presented as a "means to an end"; a procedure to aid in solving problems. Thus, like Allinger (1985), the Rule of Three was presented, and practised, to enable students to achieve future success in percent problem solving.

5.1.7 Episode 4: Interpreting percent problems

Plan

The focus of this episode was on introducing the dual-scale number line for solving percent equations. The plan was to introduce the three types of percent problems, demonstrate the number line for representing all types of percent problems, and solve percent problems utilising the Rule of Three. A set of three worksheets was prepared (see Appendix C), each containing 5 examples of percent word problems and 5 examples of percent equations of either Type I, II, or III. The planned sequence of instruction was to begin with the Type I worksheet, printed on an overhead transparency (OHT), demonstrating to the students the steps for analysing, representing and solving the three types of percent problems. The lesson sequence was planned as follows:

1. Identify the elements of the word problem, as either *part*, *whole* or %.
2. Draw the number line vertically and label with 100% on the bottom left.
3. Position the elements of the problem onto the number line representation.
4. Transfer the numbers on the number line to a proportion equation.
5. Use the Rule of Three to solve the equation.

It was planned that the teacher would work through an example on the board. The students would copy the number line, proportion equation, and solution into their books. After working through a Type I percent word problem, the matching Type I percent equation (of the form: $\Delta\%$ of $\Delta=\Delta$) would be shown. The students would be directed to write a story problem to match the equation, using the previous word problem as a guiding model. A second Type I problem would then be presented, with students copying the solution procedure demonstrated by the teacher. The students would then be directed to solve the problem. The students would then be handed the first of the 3 worksheets, and directed to complete the examples. After the majority of students had completed the worksheet of Type I problems, the same procedure would be adopted for the Type II and Type III problems printed on the other two worksheets.

Action

The lesson progressed as planned, but not all planned activities were implemented. Type I percent problems and exercises were presented. Only one Type II percent problem was discussed. At the end of the lesson, the students were given a worksheet containing Type II problems to be completed for homework.

Observation

At the beginning of the lesson, a Type I percent problem was presented, and several students called out "I know how to do that - we did it last year." Individual students presented their solution procedure to the class. Allowing individual students to describe their methods for solving Type I problems reduced instructional time for the lesson, and appeared to make many students restless and inattentive, as they chatted to friends during other students' explanations. Attention was then refocussed on the percent situation presented, and the five steps outlined in the plan (above) were followed. A Type I percent equation was then shown to the students, and students were asked to write a story problem to match. A second Type I percent problem was presented on the OHT, and interpreting, representing and solving the problem was demonstrated. The students copied the procedure in their books. A further three percent problems were presented, and students were directed to interpret, represent and solve as shown. Faster finishing students were directed to write real-world story problems for the four percent exercises presented on a worksheet. When the majority of the class had completed the initial task, a Type II problem was displayed and the procedure for analysing, representing and solving the problem was repeated.

During the lesson, the teacher continually reminded students to pay attention; to watch the blackboard; to listen to the explanation. The students appeared to find it tedious to identify and record: part = __, whole = __, % = __, and then to transfer this information to the number line. Observations of students' workbooks showed that some students made errors in setting up the number line, and in representing the numbers on the number line; some students were ignoring the number line representation and completing the problem their own way. One student asked: "Why are we doing this? I already know how to do this." Writing of story problems for percent equations was not completed by many students. The students appeared unable to do this task, although some students did create appropriate story problems, which were written on the board for the rest of the class to see and copy. Students' reaction to this task was interpreted as indicating the difficult nature of the task, and one which required further time for development. This task appeared to interrupt the focus of the lesson. As the focus of the lesson was on solving percent word problems, writing of word problems was abandoned midway through the lesson.

Reflection

This lesson was perceived as unsuccessful. Instruction on solving percent problems, and creating percent problems within one lesson simultaneously was a difficult task. This was a case of instruction overload due to insufficient planning and lesson design (Mercer & Miller, 1992). The number line procedure required many steps, to which the students did not pay close attention, thus learning did not occur (Lyndon, 1995). As a result, it was anticipated that many errors would begin to emerge, and once established, errors are difficult to eradicate (Confrey, 1990a). The structure of the lesson also appeared to trigger students' lack of attention, in that many students stated they knew how to do the percent problems presented. Interference of prior knowledge influenced instruction (Lyndon, 1995). Prior knowledge was not taken into account in this lesson.

Upon reflection of this lesson, a further episode was planned which focused primarily on interpreting, representing and solving percent word problems. For this episode, a worksheet was created which contained all three types of percent problems in story situations, where (i) the words *part, whole* and *percent* were provided requiring students only to write the values for each; (ii) the number line was provided, requiring students only to transfer the numbers from the problem onto the diagram; (iii) space to write the proportion equation in order to solve the problem was provided, and (iv) the "shell" of a percent equation in the form of: p% of p = p was provided for students to write the elements of the problem in a traditional percent equation form. The worksheet developed is in Appendix D. To consolidate the number line approach for interpreting, representing and solving percent problems, a second worksheet was developed, less structured in design than the first worksheet. It contained percent problems with space for students to show their working (this worksheet is located in Appendix E). This worksheet was designed so that students would not be constrained by the structure of the worksheet if they possessed their own efficient methods for solving percent equations.

5.1.8 Episode 5: Solving percent problems

Plan

For this episode, the two worksheets, described in section 5.1.7 (Appendix D and E), were planned to be implemented over 2 lessons. The first worksheet was to be given to the students during the first lesson in this episode. The first three problems would be worked together as a whole class. The students would then have the rest of the lesson to work through the other problems; any unfinished problems were to be finished at home. The second worksheet was planned to be handed out to students during the second lesson in this episode.

Action

The episode was implemented as planned. At the end of the first lesson, the students were instructed to complete the first worksheet for homework. At the end of the second lesson, no student finished the 15 problems on the worksheet in the lesson period. The students were instructed to complete a further 5 problems for homework.

Observation

During the two lessons, the class appeared to remain on-task, and the noise level was low. The students worked on the problems at their own pace while the teacher-researcher moved around the classroom, monitoring students' solution strategies, and correcting individual student's responses. The first worksheet appeared to help students organise the steps required to solve the problems. Analysis of students' work samples revealed that the majority of students followed the steps appropriately, however there was evidence of error patterns being developed. For example, one student's worksheet showed an inconsistent placement of the part, whole, and% elements on the number line; another student positioned the numbers on the incorrect sides of the number line. During the second lesson, as students worked independently, the teacher-researcher was able to provide individual attention to students experiencing difficulty. Through discussion with the two students making errors in their work, it appeared that these students had little concept of the whole being 100%, and that a percent is a part of the whole. With extra assistance provided, the students identified the elements of the problems and completed several problems correctly by following the steps.

Productivity during the second lesson was perceived as not as great as for the first lesson. The rate of progress appeared to be increased as the teacher-researcher stood behind students, watching them as they worked. Analysis of students' responses as they worked on the second worksheet revealed that many students did not draw the diagram; they simply constructed the proportion equation from analysis of the problem. Upon questioning, the students stated that they knew where to place the numbers without drawing the diagram. General consensus was that the number line took too long to construct. Two students were identified as constructing the proportion equation incorrectly. Both students could quickly identify the elements of the problem (in terms of *part*, *whole*, *percent*), and with guidance, managed to translate this information to the number line. The students stated that they did not want to use the number line because their friend (sitting beside them) did not use a number line. Through discussion, the students eventually agreed that the number line helped them map the problem, and that the number line was useful.

Reflection

The first worksheet appeared to greatly assist students in internalising the steps for solving the percent problems using the proportional number line method. The second worksheet appeared to be useful for consolidation purposes, as many students dispensed with the number line and progressed straight to construction of the proportion equation, although errors were becoming apparent. Some students had not practiced the procedure sufficiently, and the skill was still in the cognitive or associative stage (Anderson, 1985), meaning that the skill would be effortful to recall later. The students' tendency to not engage in sufficient practice appears to support Lyndon's (1995) notion that remembering and paying attention is hard. Becoming an expert and attaining an advanced level of skill requires hard work (Derry, 1990). The students in this group obviously were choosing not to afford themselves the opportunity to become experts. Students' rate of progress through the second worksheet was much slower than for the first worksheet, and three hypotheses are tendered: (i) the students were becoming bored with more of the same type of exercise, (ii) the students felt that they knew how to "do" percent problems, and more practice was of little benefit, or (iii) the first worksheet, which minimises the amount of writing required, encouraged students to complete more exercises. The last two hypotheses find support through analysis of students' responses in the lessons.

The structure of this episode followed a diagnostic-prescriptive teaching approach to instruction (Ashlock et al., 1983) with instruction cycling through activities of initiating, abstracting, schematising, consolidating and transferring. In this episode, the structure of percent word problems was described, and activities provided for students to become familiar with this. Representation of percent word problems was provided and the procedure for solving percent word problems was linked to the Rule of Three activity. With the assistance of the worksheet, consolidation and practice was provided. In implementing the episode, the teacher-researcher had time to analyse students' work, and to identify those students requiring individual assistance. The second lesson enabled the teacher-researcher to provide individual assistance, therefore, diagnosis was an integral component of this episode. At the end of the second lesson, particularly with students showing reluctance to complete the worksheet, the teacher-researcher diagnosed a need to plan further learning experiences to stimulate the majority of the class and thus allow for transfer of new knowledge as well as provide extra assistance to individuals. A conscious decision was made against this, however, due to the need to advance through the unit of work, and to present the whole class with the next episode.

5.1.9 Episode 6: Language of percent increase and decrease

Plan

The focus of this episode was on developing the concept of percent increase and decrease, and of building students' awareness and understanding of the language associated with percent increase and decrease. The episode was planned to take two lessons. The first lesson was preplanned; the planning of the second lesson was contingent upon implementation of the first lesson.

The first lesson was developed from analysis of the pretest results. Analysis of the pretest (see Figure 5.1.2 item 7c) indicated that many students in this class selected the incorrect response to the following multiple choice item:

> At 6am, there were 100 people lined up to buy State of Origin rugby tickets.
> At 9:00 am, the crowd had increased 400%. The crowd size now is

a) 3 times the original size (i.e., 300)

b) 4 times the original size (i.e., 400)

c) 5 times the original size (i.e., 500)

d) none of the above

The most popular incorrect answer was b). The plan for this lesson was to re-present this problem to students on a piece of paper handed to them at the beginning of the lesson, and to ask them to select the correct response. On the paper, they would also be asked to provide a reason for their selection. Anticipating that students would continue to select b), the lesson was planned to help students develop a concept of percent increase greater than 100%. The procedure for the lesson was as follows:

1. Brainstorm words associated with increase and decrease, e.g.

INCREASE	DECREASE
profit	loss
raise	discount
more	less

Students to write these lists into their books

2. Using props, including 1kg bags of sugar and a small blanket, construct a "baby" by wrapping the 1kg bag of sugar in the blanket, thus producing a "baby" with a mass of 1kg. On the board, display a card to represent the mass of the baby:

1kg baby

Tell students the story of the baby growing, with the baby's mass increasing 50%. Add a 500g packet of sugar to the "baby". Students to describe the mass of the baby in terms of percent. On the board, present the following diagram:

|-----100%---||-50%--| increase

| 1kg baby | | 500 g |

Description language to include:

The baby's mass increased 50%

The baby's mass is 150% of its original size.

Direct students to copy these two sentences from the board.

3. Example 2: A dog weighs 10kg. After 12 months, its weight increased 90%.

|-----100%---||----90% --| increase

| DOG |

Discuss the two ways to write about this situation; e.g., (i) the dog's weight is 90% more; the dog's weight increased 90%; the dog's weight showed a 90% improvement; and (ii) the dog's mass is now 190% of its original mass. Stress the difference between these two statements: one being the amount of change in growth, the other being total growth in relation to original size.

4. Example 3: *A baby weighs 1kg. After 6 months, its weight increased 100%.* Use the baby prop; show the baby as having a mass of 1kg. Increase the baby's mass 100%, with the "baby" being wrapped in a blanket and another 1kg bag of sugar added. (The purpose of the demonstration is to assist students visualise that a 100% increase is an increase of the same amount as the original.) On the board, present the following diagram:

   |-----100%---||-----100%---|

1kg baby	1kg

   Students to describe the mass of the baby, writing two different sentences to tell the same story (as in example 2).

5. Example 4: Last year's crowd was 2500. This year there was a 160% increase.

   |-----100%---||----100%----||---60%--| increase

   Direct students to write two sentences describing this situation (as in example 2).

6. Set homework: *A tree measured 50cm when planted. Within 3 months, its height increased 200%.* Direct students to describe the change in two different sentences as practiced during the lesson.

Action

The multiple choice activity was completed. The first three planned demonstrations of percent increase situations were presented.

Observation

The students were handed the multiple choice task as they entered the room. The teacher-researcher moved around the room scanning students' responses. Initial scanning indicated that the majority of students, as predicted, incorrectly selected b). Students' written responses for why they selected b) included:

> *The crowd was originally 100, that is 100%. If its 400%, it would be because the original size was 100=100%.*
>
> *I don't know.*
>
> *I guessed.*

Only one student provided a response which indicated some understanding of the additive and multiplicative nature of percent increase situations. Her response is presented below:

> *Because 100 people = 100%, 400 people = 400%, 400 + 100 = 500.*

The multiple-choice item appeared to give a focus to the lesson, and students quickly settled in their seats after entering the room. As a result of this task, the teacher was provided with an indication of the number of students who had misinterpreted this percent increase situation. During the brainstorming session the students copied the list of increase and decrease words as they were suggested. The students were cooperative for this activity. The "baby" example was presented to the students, and the students copied the two different sentences from the board. For the second example shown, the students were asked to write the situation in two sentences in a similar fashion to the first example. The students appeared to have difficulty in constructing their own sentences of the situation, and many students sought clarification of this task from the teacher, or began talking to their peers. The teacher asked various students to read their responses, and then they were displayed on the board. The third example was presented, and time allowed for students to construct two sentences of the situation. Asking for answers to be volunteered, the teacher then wrote the sentences on the board. Although the students were attentive during the description of each percent increase situation, only a limited number of students could construct their own statements. Time was spent re-explaining the task, and writing suggestions on the board for students to copy. The number of examples shown was less than planned for this lesson.

Reflection

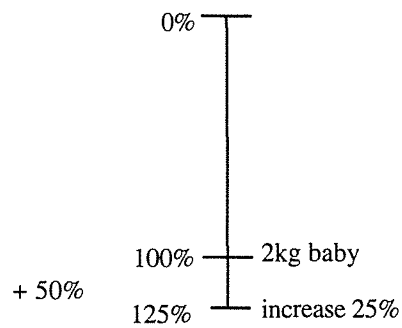
At the end of the lesson, it was perceived that many students had not been able to make the connection to 100% increase as 200% of the original amount. From students' relative difficulty in constructing their own percent increase situations, it was perceived that the lesson had not been successful in building students' awareness of percent increase. The classroom-teacher agreed, stating that he thought many students did not appear to be actively engaged in thinking of the different percent increase language, and were waiting to copy from the board. He also stated that percent increase situations were notoriously difficult for students to interpret, and that he found it a difficult topic to teach every year. Dealing with percents greater than 100 is a topic in which students experience great difficulty (Parker & Leinhardt, 1995). As shown in chapter 2, there is a paucity of literature on effective approaches to developing students' concept of percent increase, contrasting the complicated myriad of approaches for solving percent problems. Of this episode, the classroom teacher stated that he found the props an interesting way of presenting the material, and one which managed to gain students' attention. The modelling of percent increase situations using cards of various sizes may or may not have assisted students in linking percent increase situations to a comparison of the whole and the new amount. Directly representing percent increase situations on the number line, which was now familiar to the students, may have been more appropriate. A visual mathematical model should enable linkages of mathematics to the real world (Post & Cramer, 1989).

The model also must reflect the structure of the mathematical concept (English & Halford, 1995), and promote cognitive processes such as abstracting, generalising and unifying (Streefland, 1985). The number line model was proposed as a model which reflects the proportional nature of percent situations (Dewar, 1984; Haubner, 1992). For consistency, the number line representation, extended beyond 100%, may have assisted students to abstract and generalise about percent increase situations in relation to the model. A second lesson was planned for this episode, in which further percent increase situations, as well as percent decrease situations, would be modelled with props, including groups of jellybeans. Each percent increase situation would be represented on the extended number line, with a similar representation in reverse for percent decrease situations.

Re-Planning of percent increase lesson

The lesson sequence was planned as follows:

1. Example 1: *A 2kg baby increased its mass by 25%.* Diagram generated to model this situation:



Students to copy diagram and write two sentences:

25% increase/growth/improvement

IS THE SAME AS

125% of the original

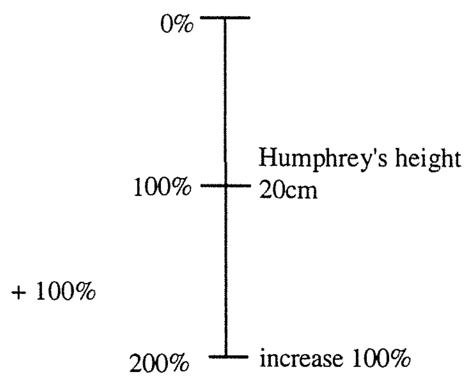
2. Example 2: Humphrey Bear (a small toy bear) is 20cm tall. He grew, and his height increased 100%.

Students to copy diagram and write two sentences:

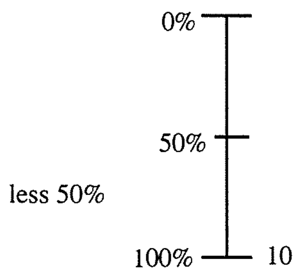
100% increase

IS THE SAME AS

200% of the original



3. Example 3: *A group of 4 jellybeans was increased 100%.* Place 4 jellybeans on the overhead projector (OHP). Students to draw diagram and write two sentences.
4. Present other examples using jellybeans:
   Example 4: *A group of 3 jellybeans was increased 200%.* Students to draw diagram and write sentences.
   Example 5: *A group of 3 jellybeans was increased 300%.* Students to draw diagram and write sentences.
   For each of the examples, it was planned that the teacher-researcher would provide the example, then move around the room observing students' responses and check responses for accuracy. It was planned that the jellybeans would be given away to various students.
5. Present example of percent decrease.
   Example 6: A group of 10 jellybeans was reduced by 50%. Draw diagram on board:



Stress positioning of the "less 50%" on the number line and explain how this provides two ways of interpreting the decrease situation. Write on board:

50% discount IS THE SAME as 50% of the original.

Action

The lesson was implemented as planned, but the percent decrease situation was not presented.

Observation

During the lesson, the students were cooperative and followed instructions. When the number line representation was drawn on the board, all students quickly copied the number line correctly. When the jellybeans were used, the students appeared eager to describe and present their solutions/solution procedures in class, and all other students were attentive as their peers spoke in class. The jellybeans appeared to be a strong motivating force behind students' willingness to participate, and this was confirmed by the classroom teacher. Two sentences were generated for each percent increase example:

a 100% increase

IS THE SAME AS 200% of the original

When directed to construct the sentences for each example presented, many students were given continual encouragement to complete the sentences in their books. There appeared to be reluctance by some students to continually write these sentences. As the lesson proceeded, it was apparent that the majority of students could construct the number line appropriately, but were experiencing difficulty in the descriptive sentences without assistance.

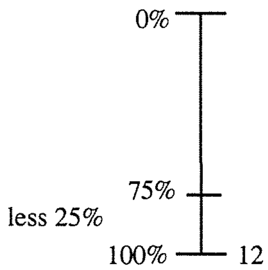
Reflection

The lesson was perceived as successful in promoting students' awareness of percent increase language. In terms of lesson design, Rojewski and Schell (1994) suggested that effective instruction is where provision is made for guided practice, modelling, coaching, scaffolding and fading. The design of this episode on percent increase was perceived as well-planned in this respect, but possible information overload contributed to the students' hesitancy to construct percent increase sentences, and resulted in minimal "fading" of teacher assistance. The extended number line model was readily adopted by the students thus suggesting that this representation was more appropriate than using cards (as in the first lesson). As two lessons had now been spent on percent increase, the next lesson was planned to explore the language of percent decrease, and to link this to percent increase via the number line representation. Practising using the language of percent increase and decrease was also necessary, and this was also planned for the next lesson.

Plan for percent decrease lesson

The plan for this lesson was to build from the previous lesson on percent increase, using the number line to represent percent decrease situations, and to introduce both the multiplicative and subtractive nature in which percent decrease situations can be interpreted. An activity for the purpose of enabling students to discuss the language of percent increase and decrease situations was developed for this lesson. The plan for this lesson was as follows:

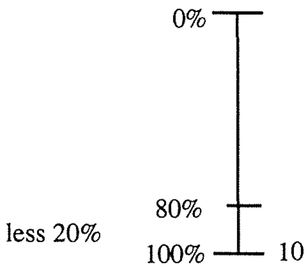
1. Place 12 jellybeans on the OHP. State: " Of the 12 jellybeans, 25% were eaten. How many left?" Construct number line:



Stress subtractive nature of the 25% decrease. Direct students to describe the situation in a single sentence.

25% discount IS THE SAME AS 75% of the original.

2. Place 10 jellybeans on OHT. State: "Of the 10 jellybeans, 20% were eaten. How many left?" Construct number line. Stress the subtractive nature of the situation.



Direct students to write a single sentence of the situation:

20% reduction IS THE SAME AS 80% of the original.

3. Distribute worksheet. (The worksheet, located in Appendix F, contains various percent increase/decrease situations, together with a number of statements of interpretation). To complete the worksheet, students select correct interpretations of the various percent increase and decrease situations presented. Students to work in groups.

Action

The lesson proceeded as planned, with all percent discount examples presented. The worksheet was handed to students after they formed groups. The lesson ended before all students had sufficient time to complete the worksheet. The students were instructed to complete the sheet at home and submit next lesson.

Observation

The students were attentive during presentation of the jellybean examples, with all students wishing to volunteer the 2 different interpretations for percent decrease situations. The jellybeans appeared to be a tremendous incentive to the students; order and control was maintained throughout presentation of the examples of jellybean decrease. The students completed the number line representation and sentences as directed, but some students were seen to be copying from others around them. The students readily formed groups for the worksheet activity, and appeared to enjoy the opportunity to work in groups of their own choosing for the multiple choice sheet. The students were told that the group with the highest number of correctly identified increase and decrease statements would win a prize. This information appeared to motivate students to begin the worksheet activity. As the students worked through the worksheet, the teacher-researcher visited each group, monitoring progress. Students were observed identifying one correct statement, and then moving on to the next example. Midway through the lesson, the teacher informed the students of the number of correct responses for each increase and decrease situation (this information was subsequently added to later printings of the worksheet). Moving from group to group, the teacher continually informed the students that there was more than one correct statement for each example. Students then checked their work. Analysis of the worksheets revealed that many students were correctly interpreting the percent situations. Some students, however, were still not identifying both the additive and multiplicative sentences.

Reflection

In terms of attention, cooperation and enjoyment, this lesson was perceived as successful. The students were attentive during instruction, and thus favourable conditions for learning (Lyndon, 1995) were achieved. The activity enabled students to work in groups, with the opportunity for interaction and knowledge sharing, which is an element of a cognitive apprenticeship model of instruction (Reid & Stone, 1991). The worksheet appeared to motivate students to work in groups, and to cause them to discuss the sentences, and thus become familiar with the language of percent increase and decrease. The structure of the worksheet, however, did appear to influence performance as the majority of students identified one correct interpretation of the percent increase and decrease situations, and then moved on to the next example. This may have been due to students' past experiences with multiple choice items, where the selection of only one response is required. This could be explained in terms of proactive inhibition (Lyndon, 1995) interfering with performance. In light of this observation, the worksheet was amended to include the number of correct sentences contained to signal to students that more than one correct response was possible.

To consolidate the building of students' knowledge of percent increase and decrease situations, one of the components of effective instruction is the provision of opportunity for students to apply their new skills and strategies in a variety of situations (Mercer & Miller, 1992). Providing this opportunity, however, would require further class time, which was not available. To adhere to the time frame of the work program, the next planned episode was on fraction, decimal and percent conversions.

5.1.10 Episode 7: Percent, common and decimal fraction equivalence

Plan

To develop and consolidate the link between percent, common and decimal fraction equivalence, and to follow the theme of the number line representation, one lesson was planned in which students would explore the equivalent forms of fraction, decimal and percent numbers, and then practice their skills in fraction, decimal and percent conversions. In this episode, the students would construct a number line by gluing the five, A5 size sheets of paper together. When assembled, the length of paper would show a number line with 100 calibrations, designed to be read vertically. On the number line, percent, fraction and decimal equivalent forms would be listed (as in Figure 5.5).

The purpose of the number line was to assist students in developing their understanding of equivalence and skill in carrying out percent, fraction and decimal conversions, and in developing mental computation of percent amounts. The lesson was planned in the following steps:

1. Students to construct number line.
2. Students to mark in fraction hundredths (and also in simplest form) along the number line up to $^{20}/_{100}$. Students to mark in the corresponding decimal fraction representation, then the percent representation:
3. Students to mark in commonly used fractions: $^{50}/_{100}$ ($^{1}/_{2}$), $^{25}/_{100}$ ($^{1}/_{4}$), $^{75}/_{100}$ ($^{3}/_{4}$), $^{100}/_{100}$ (1); and every tenth fraction: $^{10}/_{100}$, $^{20}/_{100}$, $^{30}/_{100}$, $^{40}/_{100}$, $^{60}/_{100}$, $^{70}/_{100}$, $^{80}/_{100}$, $^{90}/_{100}$.
4. Students provided with instructions on how to use the number line to determine percent, common and decimal fraction equivalence.
5. Students to use their constructed number line to complete the prepared worksheet for practicing percent, common and decimal fraction conversions (worksheet located in Appendix G).

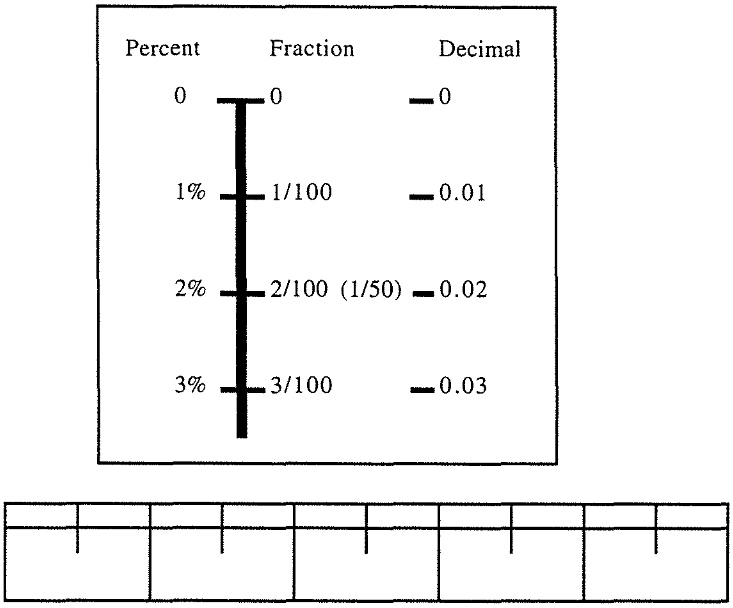


Figure 5.5. Number line constructed from 5, A5 size pieces of paper. (Inset: calibrations on the number line).

Action

The students were shown how to construct the number line. Instructions for assembling and calibrating the number line were provided. Instructions on how to use the number line for percent, common and decimal fraction conversions were not provided, and the worksheet was not handed out.

Observation

The students appeared to listen to the instructions for constructing the number line, but once students were allowed to begin, the room became noisy and chaotic. To construct the number line, students had to join the five sections of paper with sticky-tape (glue, although available, was not effective). The room was extremely noisy as students moved around the room, chasing sticky-tape, and calling out for the return of the sticky-tape. Many students appeared to experience difficulty lining-up the A5 sheets of paper. As students began recording the numbers on their number lines, errors surfaced. The first major error was in placing the number $^{1}/_{100}$ on the beginning point of the number line, in the position for "0". The second common error was students' writing of decimal fractions. When writing the decimal fractions, many students wrote: 0.08, 0.09, 0.010, 0.011. As the teacher-researcher provided assistance to individual students who displayed errors in decimal place value, many other students continually interrupted the teacher, asking for the instructions to be repeated.

Reflection

This lesson was perceived as unsuccessful, as the number line construction appeared to take up too much class time, which decreased the amount of time students could spend developing their conversion and mental computation skills. However, for diagnostic purposes, the lesson was perceived as useful, with students requiring further assistance in decimal place value being identified. As a result of students' task performance, teachers can identify areas of weakness and plan instruction accordingly (Reisman, 1982; Ashlock et al., 1983). In light of students' performance in this lesson, it was apparent that a second lesson was required in order for students to practise decimal, fraction and percent conversions using the constructed number line, if necessary. In light of diagnosis of students' decimal knowledge, a second lesson would also enable the teacher to work individually with students experiencing difficulty.

Plan for second lesson in this episode

The plan for the second lesson in this episode was as follows:

1. Provide instruction on how to use the number line.
2. Distribute practice worksheet for building students' skills in decimal, fraction and percent conversions, and direct students to work at their own pace on the exercises presented.

Action

The students' usual classroom teacher implemented the planned lesson.

Observation

The classroom teacher stated that he was quite happy with the structure of the lesson, and that the number line appeared to help some students who were unsure of percent, common and decimal fraction conversions. Analysis of students' worksheets revealed that no student completed all worksheet questions, but many students showed evidence of working solidly in class. Errors were apparent, particularly writing decimal tenths as percentages (for example, many students wrote 0.8=8%).

Reflection

This episode was perceived as unsuccessful as too much time was spent on constructing the number line, and therefore too little time was available to assist students consolidate understanding of equivalence and conversions. Students' performance on constructing the number line indicated that this activity was unsuccessful in promoting students' understanding of percent as a number which could be located on a number line with an equivalent decimal and fraction form.

Students' errors on the worksheet also confirmed that the equivalence notions were not being consolidated. Other student responses on the worksheet indication that, for approximately 50% of the class, skill in percent conversions and benchmarking was satisfactory. For students who did not have such sufficient skill, the number line appeared to have little influence upon development of equivalence knowledge.

This particular episode, in terms of the whole unit, was perceived as inconsistent. Decimal, fraction and percent conversion are primarily taught as a foundation for percent calculations using a decimal fraction approach (e.g., Breuckner & Grossnickle, 1953; Cooper & Irons, 1987; Hauck, 1954; Reys et al., 1992). In this unit of work, the proportional nature of percent situations was the focus, thus skill in conversion was a non-vital element. Although knowing percent means to know its multidimensional nature, including understanding that percent is a number (Parker & Leinhardt, 1995) and that developing mental computation skill in percent comes from percent benchmarks in relation to fractions and decimals (Glatzer, 1984; Reys et al., 1992) this episode was perceived as unsuccessful in promoting such knowledge, and the timeframe for implementation was too tight to enable this episode to be given appropriate attention. As fractions, decimal and percent conversion are however, an integral part of the school work program for percent (see section 1.2.2), this episode was planned for authenticity purposes (to cover objectives of the syllabus), and therefore more careful planning may render the episode more successful.

5.1.11 Reflection upon implementation of teaching episodes

Upon reflection of implementation of this unit of work with the Group 1 students, it was perceived the unit was not well implemented, and was only minimally effective in promoting students' understanding and access of percent knowledge in problem solving. The exploratory nature of the episodes within the unit of work is highlighted by the modifications which occurred during implementation. Many of the episodes within the unit were perceived as unsuccessful in achieving their planned purpose. In Table 5.3, a summary of the success rating of each episode is presented.

Table 5.3

Success Rating of Teaching Episodes for Group 1

Episode	Topic	Success rating*
1	Metacognitive training	U
2	Concept of percent	U
3	Fraction equivalence and the Rule of Three	S
4	Interpreting percent problems	U-S
5	Solving percent problems	U-S

Episode	Topic	Success rating*
6	The language of percent increase and decrease	U-S
7	Percent, common and decimal fraction equivalence	U

* S = successful
U = unsuccessful

From Table 5.3, the metacognitive training episode was rated as unsuccessful, as well as the concept of percent, and the percent, decimal and fraction equivalence episodes. The episodes of interpreting percent problems, solving percent problems and the language of percent increase and decrease were rated as midway between unsuccessful and successful. Episode 3, on fraction equivalence and the Rule of Three, was the only episode rated as successful.

As discussed in the reflection section of 5.1.4, the metacognitive training episode was rated as unsuccessful. Metacognitive training assists students to become more efficient learners (Chan, 1993; Kirby & Williams, 1991), as successful learners are those who display self-directed learning strategies, such as goal setting, worktactics, self-monitoring and self-evaluation (Chan, 1993). The metacognitive training program in this unit of work was to inform students of their role in the learning process; to inform students that their actions directly influence their successes and failures (Mercer & Miller, 1992); that successful task performance is a product of students' own efforts and persistence (Chan, 1993); that learning occurs within the learner and can be directly influenced by the learner (Weinstein & Mayer, 1986); and that learning requires energy (Baird & White, 1982) and is hard work (Derry, 1990). Such key notions are encapsulated in key points of the CMP, which state that learning occurs as a result of paying attention; and that paying attention is hard because it requires effort, but that the individual is in control of the amount of attention he/she chooses to pay (Lyndon, 1995). Upon implementation of the metacognitive training program, the majority of students were choosing not to pay attention to the teacher-researcher, and this was compounded by several disruptions to normal routine which occurred. The episode was difficult to implement primarily due to the difficulty in maintaining students' attention. Although students' lack of attention contributed to poor implementation, the design of the lesson was also at fault, in that students' attention was not maintained. As stated in section 3.3.4, Rojewski and Schell (1994) described four elements which must be considered for successful teaching, being content, method, sequence and sociology. In this episode, the social context of the classroom was not taken into consideration sufficiently, as students questioned the nature of the metacognitive training program during mathematics lessons. Although the strategies and ideas within the CMP are generic across all learning situations, convincing students of this was not a simple task. This finding is supported by Chan (1993) who stated that, students must be convinced that application of strategic actions which require effort on the individual's part will yield strong positive results, before students will demonstrate such strategic behaviour. Implementation of this metacognitive training program with Group 1 required further planning in order to "sell" the product to the students. Modification of this episode was required.

The second episode in this unit on the concept of percent was also rated as unsuccessful. Its purpose, as stated in the reflection upon this episode in section 5.1.5, was to promote students' understanding of percent usage in the real world. In this episode, students worked in groups. Allowing for group work in learning situations provides a means through which students can discuss ideas together and build their own knowledge, and which enables the teacher to mediate in the process of knowledge growth (Reid & Stone, 1991; Rojewski & Schell, 1994). In this episode, the activity did not appear to engage students in thinking deeply about the use of percent in the real world, except when the teacher-researcher interacted with individual groups. Further planning of this episode was required to steer students directly into finding real examples of the multi-faceted nature of percent used in society.

The seventh episode, on decimal, fraction and percent equivalence, was rated as unsuccessful. The construction of the number line was time-consuming. Instruction for this needed to be more sequentially presented. More explicit instruction was also required to develop students' mental computational skills.

As described in section 2.4.2, this unit of work was primarily based around presenting to students the proportional number line method for interpreting, representing and solving percent problems. Teaching of this method was planned in four episodes, which occurred in episodes 3-6 in this unit. Episode 3, on fraction equivalence and the Rule of Three was the first episode directly related to presentation of the method. Implementation of this episode was rated as successful. As stated in the reflection upon this episode in section 5.1.6, the design of this lesson appeared to contribute to successful implementation, particularly in building students' feelings of success. One concern of this episode was the fact that it appeared to be a rote teaching approach to skill development. However, the overarching goal was to provide students with a skill to enable them to be successful percent problem solvers. Providing students with skill can be the basis for construction of important principles and concepts (Noddings, 1990). To present students with investigations into the meaning of the Rule of Three may have actually caused students to lose sight of the topic. In the introduction to this episode, the Rule of Three was presented as a means for generating equivalent fractions. Although a superficial link, an attempt was made to relate this to prior experiences of the students.

Episodes 4 and 5, which linked to the skill of the Rule of Three, were rated as moderately successful. Proactive interference was encountered by presenting students with a Type 1 problem initially. Preparation of the structured worksheet met with successful task performance and increased productivity from students. However, students did not allow themselves time to practice the procedure for representing and solving percent problems to automaticity. Students' resistance to practice suggested that they were not prepared to take responsibility for their own learning; a component of metacognitive knowledge. Time limitations prevented further practice and consolidation exercises being implemented.

Episode 6 on the language of percent increase and decrease underwent change and modification upon implementation. Modelling of percent increase situations maintained students' interest, but use of jellybeans maintained attention. Provision was made for students to share their understanding of the language of percent increase and decrease situations through the group activity, but time limitations prevented presentation of further consolidation and application activities.

Reflecting upon episodes within the teaching experiment, five main factors which influenced implementation can be identified. They are: teacher factors; student metacognitive factors; time factors; school context factors; and curriculum factors. Teacher factors included poorly sequenced instruction, lack of provision for practise of new skills, lack of activities to stimulate and maintain student interest and attention, insufficient acknowledgment of prior knowledge in planning instruction, poor lesson design. Student factors were students' non-application of effort to maintain attention, lack of worktactics, goal direct-behaviour and self-directed learning strategies, non-practise of skills to mastery, and resistance to learning about metacognitive skills. Time factors included lack of time to completely implement lessons/episodes, lack of sufficient time to re-implement lessons/episodes upon reflection, limited amount of time allocated to this topic in the school program. School context factors included disruption to episode implementation due to late arrival of students to class, change of room situations, and student reactions to a new teacher. Curriculum factors were the need to include a teaching episode on decimal, fraction and percent conversions as a requirement of the Year 8 syllabus. These 5 factors did not work in isolation, but were seen to impinge upon each other. For example, lack of provision of opportunity for students to practise new skills (teacher factor) was due to lack of time allocated to the topic (time factor) and students' reluctance to apply effort to practise skills to mastery (student metacognitive factor) as well as reacting to disruptions to the school situation (school factor). Also, more careful planning and lesson design (teacher factor) may have led to greater student attention and application of effort (student metacognitive factor) which would ensure sufficient time (time factor) for students to consolidate new knowledge.

5.1.12 Reflection upon pre- and posttests and instruction

Reflecting upon pre- and posttest scores and implementation of this unit, parallels in results emerge. Implementation of this unit of work overall, was rated as unsuccessful. Pre- and posttest scores overall, indicate little change in student performance, with a prestest score of 43% and a posttest score of 52%. Minimal change in test scores would be anticipated from an unsuccessful rating of the teaching sequence upon implementation. Relating test scores on the sections of the Percent Knowledge Test to teaching episodes, it can be seen that promoting students' intuitive and principled-conceptual percent knowledge (Section I of the Percent Knowledge Test) was the focus of the entire teaching sequence, but particularly episodes 2 and 6; promoting students' conversions and benchmarking skills (Section II of the Percent Knowledge Test) was the focus of episode 7; and promoting students' percent calculations and problem solving skills (Section III of the Percent Knowledge Test) was the focus of episodes 3, 4 and 5. Students' performance on Section I of the posttest improved from the pretest (60% - 69%), but mainly in students' ability to pose real world percent problems from equations. Episode 2, rated as unsuccessful, did not appear to influence students' intuitive and principled-conceptual percent knowledge. Episode 6, which specifically focused on the language of percent increase, was rated as moderately successful, and also appeared to have had little influence on students' intuitive and principled-conceptual knowledge of the additive and multiplicative language of percent increase. Students' posttest performance on Section II of the test was a negative change (60% pretest - 54% posttest). Episode 7 was rated as unsuccessful, and thus instruction appears to have had a negative influence on students' performance on this section of the test. Students' posttest performance on Section III was a positive change (10% to 33%), and episodes 3, 4 and 5 were rated as moderately successful. Instruction appears to have influenced students' ability to perform percent calculations and solve percent problems, but instruction appears to have been beneficial for only approximately one-third of the class.

5.1.13 Direction for Teaching Experiment 2

Reflections upon implementation of the teaching episodes, suggest that the unit of work could potentially be effective, but modifications to particular lessons would be required. Apparent changes within various episodes were identified as follows: the metacognitive training episode required contextualisation and/or inclusion of activities to sustain interest and attention; the concept of percent episode required focus on the multidimensional nature of percent in the real world; the episode on the number line model for interpreting, representing and solving percent problems and equations, although modified during implementation, required further, sequential planning; the episode on percent, fraction and decimal equivalence required specific instruction to be planned to build knowledge and understanding of percent as a number. These considerations were taken into account for the episodes implemented with Group 2.

Elaborating on replanning the episodes on interpreting and solving percent problems, these episodes present to students the proportional number line method for solving percent problems and exercises. This method was devised prior to instruction as a key process for operating within the domain of percent. Following the research methodology for studying learning to inform teaching (Hiebert & Wearne, 1991), in this first teaching experiment instruction was devised to promote acquisition and use of the key process in percent problem situations. Analysis of students' methods for solving percent problems (as described in section 5.1.2) indicates that students abandoned their own methods, and adopted the number line model presented during the teaching sequences. Instruction, therefore, appears to have directly influenced the acquisition of the method as a key process, but only for approximately one-third of the class. Particular modification of episodes relating to the proportional number line method as a key process were considered for Teaching Experiment 2.

5.2 Teaching Experiment 2

5.2.1 Overview of report on Teaching Experiment 2

The results for Teaching Experiment 2 are reported in this section. The pre- and posttest results are presented in section 5.2.2. An overview of the planned teaching sequence, comprising six teaching episodes is presented in section 5.2.3. In sections 5.2.4 to 5.2.9, implementation of each of the six teaching episodes is described, under the headings of *plan*, *action*, *observation* and *reflection*. In Section 5.2.10, a reflection upon implementation of this unit of work is presented, followed by a reflection upon pre- and posttest results and instruction in section 5.2.11. This section concludes with a statement on direction for Teaching Experiment 3 in section 5.2.12.

5.2.2 Pre- and posttest results

Group 2 scores in total and for each section on the pre- and posttests are presented in Table 5.4. From Table 5.4, it can be seen that the Group 2 posttest score overall increased from 37% to 66%. Within each section of the test, there is a slight positive change for Section I (intuitive, principled-conceptual knowledge), a stronger positive change for Section II (conversions and benchmarks) and Section III (percent calculations and problem solving).

From Table 5.4, it can be seen that Group 2 students' intuitive and principled-conceptual percent knowledge, and proficiency in percent conversions and benchmarking, prior to instruction, was much greater than their percent calculation and problem solving skills. These results are similar to those of Group 1 (see Table 5.1 in section 5.1.2). Positive change was most dramatic on Section III of the test on percent calculations and percent problem solving. Graphical representation of the pre- and posttest scores are presented in Figure 5.6, highlighting the change in pre- and posttest scores in total and in all sections.

Table 5.4

Pre- and Posttest Means (%) for Group 2 Students on the Percent Knowledge Test in Total and in Each Test Section

	Components of the Percent Knowledge Test			
Test	Total	Section I	Section II	Section III
Pretest	37%	64%	41%	7%
Posttest	66%	79%	70%	48%

As for Group 1, Group 2 students' test results were analysed for diagnostic purposes. Pre- and posttests were scored to identify specific items on which students responded incorrectly. Within the three parts of the Percent Knowledge Test, the number of incorrect student responses for each item was tallied. Representation of the number of incorrect responses to each item on Section I (intuitive, principled-conceptual percent knowledge), Section II (conversions and benchmarks), and Section III (percent calculations and problem solving) of the Percent Knowledge Test are presented in Figures 5.7, 5.8 and 5.9 respectively.

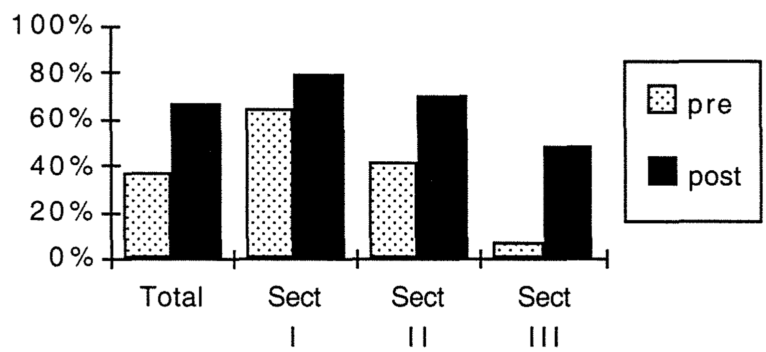


Figure 5.6. Graphical representation of Group 2 students' pre- and posttest means (%) in total, and in each test section.

In Figure 5.7, it can be seen that, similar to Group 1, Group 2 students experienced most difficulty with items 6c, 7a-c, and 8a-c on Section I of the pretest, and also item 3e, which relate to the interpretation of the multiplicative and additive language of percent increase, and posing of real world percent problems from percent equations, and the concept of percent increase as more than 100%. Posttest scores indicate a positive change in performance after instruction, but also indicate that students continued to experience difficulty in interpreting the multiplicative and additive language of percent increase situations (item 6c, 7a-c), and in posing real world percent problems from percent equations (item 8a-c).

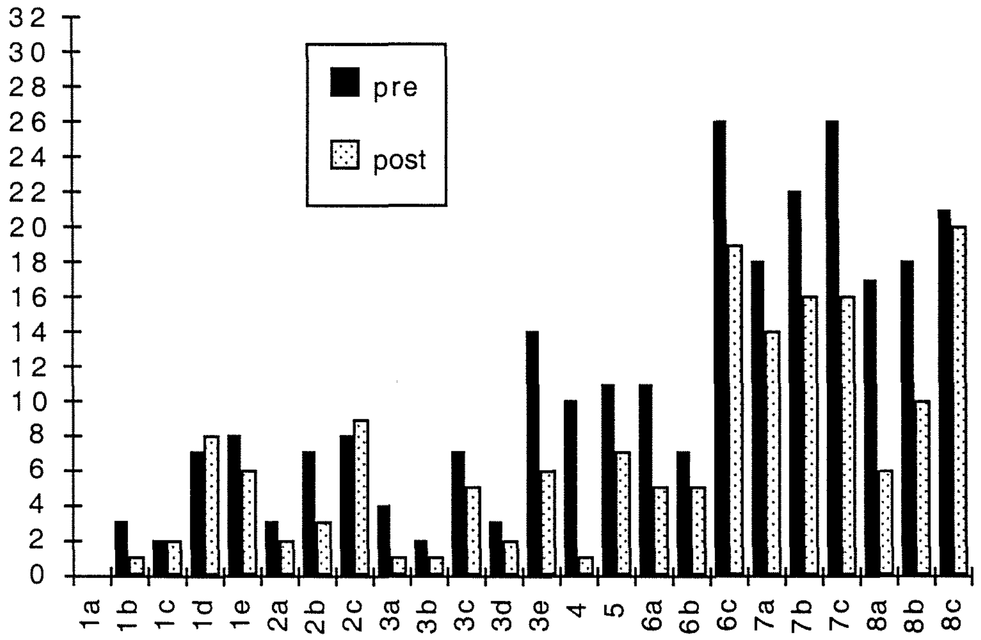


Figure 5.7. Group 2 students pretest (n=31) and posttest (n=29) *incorrect* scores on Section I (intuitive, principled/conceptual percent knowledge) of the Percent Knowledge Test). Graph indicates number of students *incorrectly* responding to particular items.

In Figure 5.8, it can be seen that, on Section II of the pretest, many students experienced difficulty in fraction, percent and decimal conversions (items 1a-c, 2a-c, 3a-c, 4a-c), and in using percent benchmarks, particularly benchmarks of 15% (as 10% + 5%), 30% (as 3 x 10%), 60% (as 6 x 10%) and $33^1/_3$ (as $^1/_3$) (items 5g-j). Compared to pretest scores, posttest scores indicate an improved student performance on conversions and use of benchmarks, most notably on percent-to-fraction conversions (item 1a-c), and percent to decimal conversions (item 2a-c). On the posttest, more students also were using percent benchmarks successfully (item 5a-j).

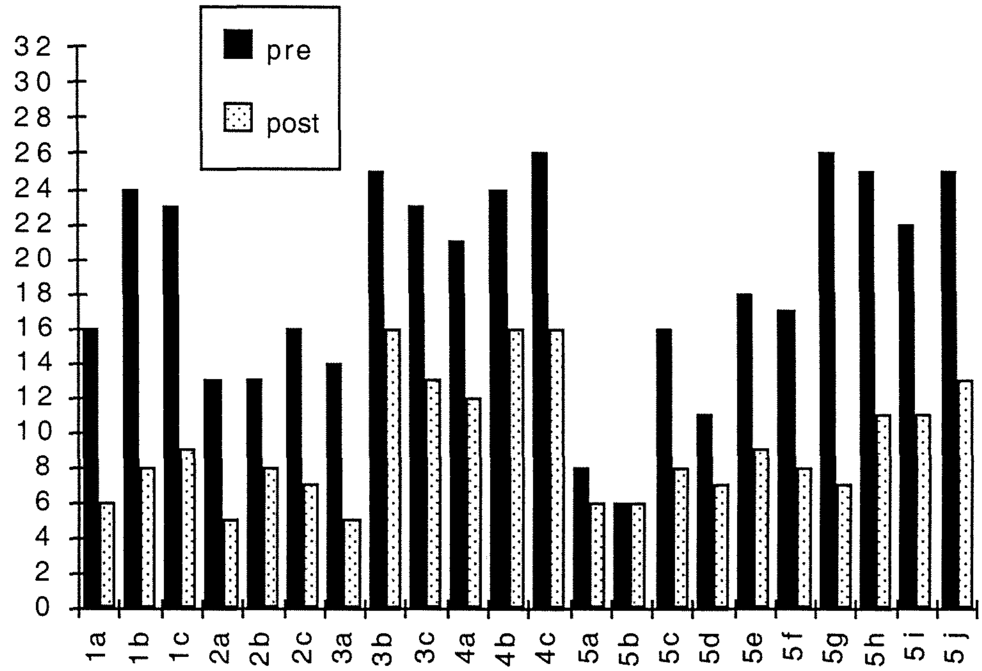


Figure 5.8. Group 2 students pretest (n=32) and posttest (n=31) incorrect scores on Section II (conversions and benchmarks) of the Percent Knowledge Test. Graph indicates number of students *incorrectly* responding to particular items.

Figure 5.9 indicates that Section III of the pretest (percent calculations and problem solving) was poorly attempted by students. From Figure 5.9, it can be seen that prior to instruction, some students successfully performed calculations of percent equations, but only those of Type I (item 1a). Type II and Type III percent equations (item 1b and c), and all three types of percent problems (item 2a-c) were not successfully completed by students. In Figure 5.9, it can also be seen that no student used diagrams to assist percent problem solving (item 3), and no student could successfully express percent problem solutions in words (item 4). Posttest results indicate greater student facility in solving percent problems and performing percent equations after instruction. On the posttest, more than half the students successfully performed percent calculations of all three types (item 1a-c), and also successfully solved percent problems (item 2a-c). Approximately one-third of the students were using diagrams to solve percent problems (item 3), and could express solutions to percent problems in words (item 4). Facility in solving multistep word problems showed minimal improvement (item 5a-d).

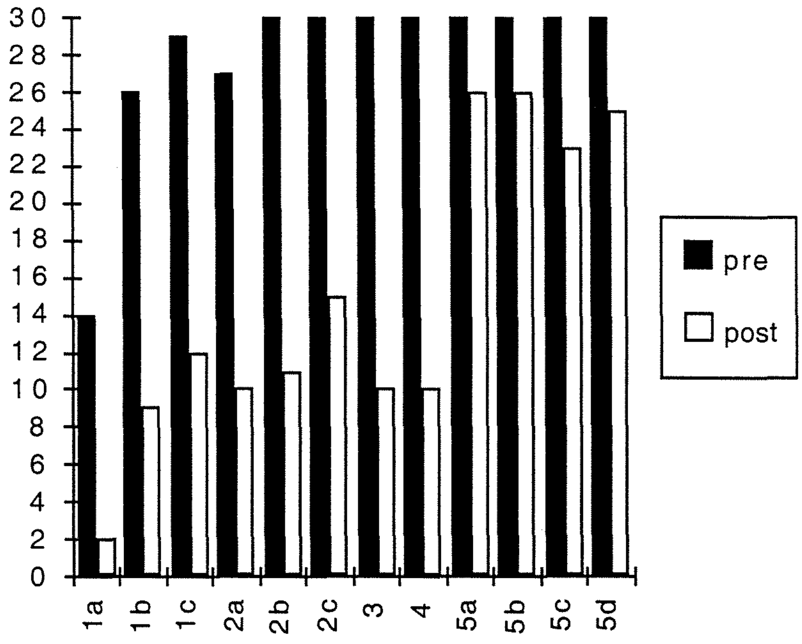


Figure 5.9. Group 2 students pretest (n=30) and posttest (n=30) *incorrect* scores on Section III (percent calculations and problem solving) of the Percent Knowledge Test. Graph indicates number of students *incorrectly* responding to particular items.

Analysis of students' solution procedures for solving percent equations and problems indicates change in strategies utilised by students after instruction. As with Group 1, on the pretest Group 2 students employed a variety of procedures for solving percent equations and problems. For Type I problems, the most common correctly used strategy was decimal multiplication, followed by the strategy of using the percent key on the calculator. For Type II problems, successful students used long division equations. For Type III problems, correct responses were single answers only, therefore solution strategy could not be categorised. The majority of students who were unsuccessful on these items made no attempt at a solution, or presented a single answer providing no evidence of their solution strategy.

On the posttest, similar results were gained as for Group 1. However, more Group 2 students than Group 1 students, utilised the proportional number line method. Of students who successfully completed percent calculations and problem solving items, the number line strategy was used, or the proportion equation without the number line. Some students wrote answers only. Of the students who incorrectly responded to these items, errors stemmed from incorrect placement of numbers on the number line, or incorrect placement of numbers on the proportion equation. One student utilised an incorrect sequence when solving the proportion equation. Other students made no attempt at solution.

5.2.3 Planned teaching episodes

Six teaching episodes were planned for Group 2. Two of the episodes used with Group 1 were collapsed into one episode. The planned teaching episodes for Group 2 are presented in Table 5.5. The table provides the number and title of each teaching episode, together with the number of actual lessons taken for implementation, and the chapter section in which discussion of each episode appears.

Table 5.5

Plan of Teaching Episodes for Group 2

Episode	Topic	Lessons	Location
1	Metacognitive training	2	5.2.4
2	Concept of percent	1	5.2.5
3	Fraction equivalence and the Rule of Three	1	5.2.6
4	Interpreting and solving percent problems	3	5.2.7
5	Percent, common and decimal fraction equivalence	2	5.2.8
6	The language of percent increase and decrease	2	5.2.9

As can be seen from Table 5.5, the order of presentation of the episodes changed from that with Group 1. The episode on percent, common and decimal fraction equivalence (episode 5) was presented after the episode on interpreting and solving percent problems (episode 4). The episode on the language of percent increase and decrease was moved to last place (episode 6). The reason for this was to create a "break" in the teaching sequence, where students' attention was turned to a rather dissimilar topic in percent after working with the proportional number line method for interpreting and solving percent word problems. It was felt that, with Group 1, the episode on interpreting and solving percent word problems had demanded a lot of students' attention, and that revisiting the topic of percent, decimal and common fraction equivalence (in which students would have had prior experience) would be less cognitively demanding than the topic of percent increase.

A total of 15 class periods were spent with Group 2. Nine lessons were directly on the topic of percent, with a further 2 lessons used for metacognitive training. Pre- and posttesting occupied 2 days prior to, and immediately following, implementation of the unit of work.

5.2.4 Episode 1: Metacognitive training

Plan

In line with reflection of this episode with Group 1 (see section 5.1.4), this episode was replanned. The intention of the episode was to make the same key points, but to supplement with activities that would reduce the use of teacher monologue, and engage the students' attention. As with Group 1, this episode was planned to occupy 2 lessons. Attention and remembering would be the focus of lesson 1, and forgetting the focus of lesson 2. Lesson 1 was modified with the inclusion of activities for use when discussing the two types of memory. Lesson 2 was not modified as this lesson was not fully trialled as planned with Group 1. A description of the activities planned to assist students to understand the difference between recall memory and recognition memory, and thus justify the need for a good recall memory strategy to promote remembering is presented below.

1. Direct the students to write the answers to these questions in their books:

    Spell: cat

    Spell: stop

    What is: 2x2, 2x3, 4x4

    What did you have for breakfast?

    What did your teacher wear yesterday?

    Discuss that these are instances of recall memory at work.

2. Introduce the terms automatic and effortful recall memory.

    Provide students with a variety of stimulus sounds/pictures:

    a) music from Sesame Street

    b) music to the ABC News report

    c) picture of our Prime Minister

    d) picture of the McDonald's arches

    Ask students if they recognised any of these. Explain that these are instances of recognition memory at work.

3. Discuss the difference between recognition and recall memory:

    (a) Provide students with a list of 10 familiar mathematics words. Give them one minute to study the words on the list, and then have them write as many from memory as possible.

    (b) Provide students with a different list of 10 familiar mathematics words and direct students to memorise them. Then show students cards with words on them, one at a time, and ask them to determine which of the words on the cards were on the original list memorised for this activity.

    Explain to the students that the first activity relied on recall memory, and the second activity relied on recognition memory.

4. Discuss the difference between effortful and automatic recall memory: refer to the spelling of the word *cat* and to the activity of listing what the teacher wore yesterday; ask students to determine which was easier to recall; discuss the desirability of having things stored in automatic recall memory; discuss strategies for committing information to recall memory volunteered by the students.
5. Demonstrate the Look Say Cover Write Check (LSCWCh)strategy.

Action

The first lesson in this episode proceeded as planned. The LSCWCh strategy was presented using the word Pythagoras. The importance of the steps in this strategy was stressed, especially the need to write from memory, rather than "copying" the information 5 times. At the end of the lesson, the students were asked to write a reflection of this lesson in their journal.

For the second lesson in this episode, a Mathematics Competition was held at the school on this day, which entailed a room change. The colour card activity was completed, but the discussion on proactive inhibition and accelerated forgetting was not completed as planned. The Old Way/New Way strategy was not demonstrated to the students.

Observation

In the first lesson in the episode, the students appeared interested, and participated willingly in the activities. The classroom teacher also commented that the students appeared cooperative and attentive throughout the lesson. The recognition memory activity using "sounds" appeared to generate interest, as several students asked for more "sounds" to be played. Upon presentation of the LSCWCh strategy, one student stated that she was familiar with the strategy, and volunteered to demonstrate it to the rest of the class. All students practised the strategy. It was observed that many students were taking care to check each letter, and to cover the word completely, forcing memory to be activated. Some students questioned quietly why they were doing "spelling" in mathematics. Of the lesson, the classroom teacher expressed doubt about the value of the lesson, in that she expected few students to be able to connect this "spelling" lesson to helping the learning of mathematics. This perception was confirmed by students' diary entries. Analysis of students' diary entries revealed that many students questioned the purpose of a "spelling" lesson in mathematics time.

During the second lesson, several students were absent as they were involved in the Mathematics competition. The switched classroom for this lesson was a Science laboratory. The noise of the stools on the floor appeared to make it difficult for the teacher-researcher's voice to carry. The discussion on natural forgetting was interrupted several times due to the late arrival of students. The colour card activity was completed, and the students were attentive as the data was recorded on the board. As the teacher-researcher began to explain the results of the colour card activities, several students enquired about the purpose of the lesson. After the lesson, students asked the teacher when instruction in percent would begin.

Reflection

The first lesson in this episode was perceived as successful. The inclusion of the activities for recognition and recall memory appeared to contribute to sustaining student interest. Students' careful use of the LSCWCh strategy indicated the purpose of this strategy (for controlling memory) was well understood by students. Lesson design thus appeared to assist students in maintaining attention. The second lesson was not perceived as successful. The change of classroom clearly influenced the degree to which the planned instruction was implemented. However, even though the students appeared to enjoy the Colour Card activity, their questioning about when they were going to "do some maths" suggested that they were not paying attention to the discussion on PI and controlling of accelerated forgetting. The interruptions to this lesson prevented the key issues being presented to the students, and the natural flow of the lesson was lost. Although implementation of this episode appeared much smoother than with Group 1, the same issues presented themselves, in terms of context of the lesson, maintaining students' attention, and in "selling" the notion that metacognitive strategies assist in successful task performance. Once again, impressing upon students how strategic behaviour as a result of effort from the individual influences learning, was not an easy task.

5.2.5 Episode 2: Concept of percent

Plan

As this episode, used with Group 1, was perceived as relatively unsuccessful in building students' concept of percent used in the real world, a new lesson on developing the concept of percent was planned for Group 2. The whole class brainstorming session on percent usage in the real world, as used with Group 1 (see section 5.1.5) would be included, but the students would be given a box of candy-coated chocolate drops (Smarties) and asked to investigate their box of Smarties. The purpose of this introductory lesson was for the teacher-researcher to assess the extent to which students spontaneously used percent notions. The lesson sequence was planned as follows:

1. Brainstorm percent words/notions/uses in the real world; students copy into

their books.

2. Students form groups, with each group of students provided with a box of Smarties.
3. Students given sheets of poster paper, felt pens, and a worksheet containing the instructions of: "Describe the contents of your box of Smarties." (On the worksheet, the words *fraction, decimal, percent, ratio, proportion, graph,* were listed as stimulus ideas - see Appendix H).
4. Students present their completed posters to the class.

Action

The lesson began with brainstorming percent notions which students copied into their books as they were written on the board. The instruction sheet was distributed to students, and the task explained. The lesson finished before students could present their posters to the class. The students' work was collected.

Observation

From the brainstorming session, the following student suggestions were listed on the board: "shopping, sales, % discount, bank interest rates, out of 100". The use of the Smarties appeared to cause excitement amongst the students, with the teacher-researcher being asked many questions about when the Smarties could be eaten. The students were quiet and attentive as instructions for the activity were given. When the students were instructed to organise themselves into groups, some groups took longer to form than others. The majority of students began the task by recording the number of Smarties of each colour contained in the box; some students counted the contents of the box to determine the number of Smarties each would get to eat. Most student groups recorded the number of Smarties of each colour as a fraction of the total number, and then began constructing a graph of this information. One student's work revealed descriptions of the various Smarties colours as ratios, and this student had been observed describing her ratio calculations to her partner.

Reflection

The Smarties assisted classroom management, as all students were actively engaged in the task. However, this lesson was perceived as unsuccessful in assisting the teacher-researcher to gain insight into the students' understanding of percent, or in helping students build their own concepts of percent. Only the student who described the Smarties as ratios provided the teacher with insight into her ratio knowledge. The brainstorming session did suggest that the students related percent to "one hundred", and that discount was a common use of percent known by the students. The worksheet was perceived as too open for the purposes of this lesson, however, students' responses in this lesson indicated that they were more comfortable working in fractional amounts rather than percentage amounts. The classroom teacher stated that she felt the students had not been challenged during this lesson due to the lack of time, which, she stated, was a major difficulty with group work. In a similar vein to reflections upon this episode with Group 1, the lesson was unproductive in developing students' understanding of the multifaceted nature of percent. The students were unchallenged, but had enjoyed operating in group situations. Students' relative inexperience with group activities was reflected in the amount of time students took to form groups. In light of the students' limited productivity in the lesson, it was decided to begin the next teaching episode rather than replan this lesson using valuable class time.

5.2.6 Episode 3: Fraction equivalence and the Rule of Three

Plan

With Group 1, this episode was rated as successful, thus with Group 2, the plan for the episode was modified only slightly. For this episode, students' understanding of the diagrammatic representation of fraction equivalence was to be explored. In this episode, the students would be instructed to draw a rectangle and represent one-third on it. The students would then be directed to use their diagram to show that $^1/_3$ is equivalent to $^2/_6$. From this, students' procedures for generating equivalent fractions would be discussed, and the Rule of Three presented as an alternative method. The remainder of the lesson would proceed as with Group 1, where students practised exercises using the Rule of Three to generate equivalent fractions.

Action

The episode was implemented as planned. Students were provided with 10 equivalent fraction exercises to complete for homework.

Observation

At the beginning of the lesson, the students were asked to draw a diagram to represent $^1/_3$. Students' drawings revealed some errors, particularly in relation to drawing "equal parts". The teacher drew an appropriate diagram, discussing equal parts and the nature of equivalence. The students copied the diagram into their books. Upon presentation of the equivalent fraction example in symbolic form, many students volunteered to explain their solution, thus indicating their familiarity with the mathematical procedure for determining equivalent fractions. The practice examples for finding equivalent fractions using the Rule of Three appeared to be well-received by the students as they worked solidly on completing all exercises in a quiet manner.

The classroom teacher commented that she thought that the students had been very productive during that lesson as she felt that all students had worked solidly on the exercises.

Reflection

In this episode, the introductory activity of drawing a diagram to represent the equivalent nature of $^1/_3$ and $^2/_6$ did not appear to be useful, and may have served to complicate the main focus of the lesson. It is hypothesised that some students may not have had much experience representing fractions diagrammatically. In the lesson, insufficient time could be devoted to exploring students' knowledge of diagrams for representing equivalent fractions. Students appeared to be much more comfortable interpreting equivalent fractions at the symbolic level. This mode of presentation, of concrete representation prior to abstract symbolism is a key component of effective teaching and lesson design (Mercer & Miller, 1994). In light of students' responses in this lesson, familiarity with working at the symbolic level appeared more beneficial and less confusing to the students than use of the diagrammatic representation. It appears that, the "concrete before abstract" teaching approach may not always be the best way when students have already been exposed to the concept. Connell and Peck (1993) also discussed the difficulty of using concrete representations to link previously learnt symbolic representations. At the symbolic operation level, this lesson was perceived as successful in providing students the opportunity to experience success, and in practicing a skill to be used in percent problem solving, as with Group 1.

5.2.7 Episode 4: Interpreting and solving percent problems

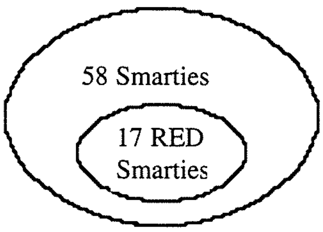
Plan

With Group 1, two separate episodes were planned for (i) interpreting percent problems and (ii) solving percent problems. In light of reflections upon these two episodes, the plan for Group 2 was to restructure episodes 4 and 5 used with Group 1 into one episode on interpreting and solving percent application problems. Three major modifications were made to this episode. First, the notions of part, whole and percent as elements within percent situations would be related to the students' experience with the Smarties activity of Episode 2 (see section 5.2.5). Second, only Type II problems would be shown initially to the students on an overhead transparency (OHT) for identification of part, whole or percent elements. This deliberate modification was an effort to discourage students volunteering to demonstrate their calculation procedures for the familiar Type I problems. Third, writing real situations to match percent exercises in the standard symbolic form of: $\Delta\%$ of $\Delta = \Delta$ would be omitted, in an effort to not distract students' attention from the method presented.

This episode was planned to span 2 lessons, with instruction on interpreting, representing and solving percent word problems presented in lesson 1, and practice of this method in lesson 2. The plan for this episode was as follows:

1. Introduce the terms *part, whole, percent* as elements of percent application problems, referring to the contents of the box of Smarties used in a previous lesson (see below for specific activities).
2. Present Type II problem situations on an OHT.
3. Help students identify elements in the given percent problem.
4. Continue with other examples of Type I and III problems.
5. Introduce the vertical percent number line.
6. Demonstrate the procedure for transferring information given in the problem onto the number line.
7. Demonstrate construction of the proportion equation. Link to equivalent fraction solution procedure of the Rule of Three.
8. Provide students with the structured practice worksheet (Appendix D) used with Group 1.
9. Provide students with the consolidation worksheet (Appendix E) used with Group 1.

For the first part of the episode (point number (1) above), it was planned students would be told that, in a box of 58 Smarties, 17 were red. On the board, this information would be displayed as shown:



The students would be asked to identify which was the *whole* and which was the *part* in this situation. Three separate cards with the words *whole*, *part*, *percent* would be displayed, and students would be asked to volunteer to place the correct card on the Smarties information displayed on the board. A second example would then be presented: *There were 73 Smarties in a box, and 14 were blue.* Students would be asked to identify the *part*, *whole* and *percent* elements of this situation. It was expected that students would state that there was no percent given in these two situations. The students would then be asked to create a percent problem, asking the rest of the class to determine the percent in this situation.

Action

The first seven steps of the planned episode were implemented during the first lesson. The terms part, whole and percent were introduced in relation to the contents of the Smarties box. Percent word problems were presented on an OHT. The students were directed to write in their books: part:___, whole:____, %:____, and to then record the three elements of each problem presented, placing a question mark next to the unknown element in the problem. The vertical number line procedure for representing and solving percent problems was demonstrated. The lesson ended before the worksheet could be handed out. In the second lesson, the technique for interpreting, representing and solving percent word problems was revised. The worksheet was handed to students, and students were instructed to work at their own pace on the sheet. The second worksheet was available for faster workers.

Observation

As the Smartie demonstration was presented, several students questioned the need for the percent term. No student volunteered to describe the Smartie part/whole situation into a problem requiring calculation of percent. Students' behaviour indicated that this segment of the lesson was causing inattention. The students were presented with a Type II word problem on an OHT, describing how percent word problems could be identified in terms of part, whole and percent elements. Unlike students' reactions to this part of the lesson with Group 1, no student stated that they knew how to solve the Type II problem presented. The teacher-researcher walked around the room, observing students as they identified the part, whole, percent, elements of each problem presented, noting that all students completed the task as required.

The students completed the number line representation quickly. When students were asked to draw the number line in their books, a quick tour of the room by the teacher-researcher indicated that all students could place 0%, 100% and 50% at appropriate places on their diagram. No student questioned the vertical position of the number line, or the positioning of the 0% point on the number line. The students watched quietly as the procedure for transferring the information onto the number line was demonstrated. As the proportion equation was set up, the link between this equation and the fraction equivalence equations practised in Episode 3, was drawn. The students were asked to describe how they would solve the proportion equation. Many students volunteered to speak. Once the solution process was demonstrated to the class, the teacher-researcher stressed again how this procedure was identical to the one practised in the previous lesson. When the proportion equation was taken from the number line, there was audible positive recognition of the link. One student muttered: "Oh, now I get it.", and other students collectively said "Aaahh".

At the commencement of second lesson in this episode, the students were not lined up outside the classroom, and were scattered along the verandah. An "incident" involving two members of the class had occurred during the lunch-break prior to this lesson, and the class took approximately 8 minutes to move into the classroom and settle down at their desks. The lesson started with the teacher-researcher demonstrating the solution procedure for percent application problems, but one of the students involved in the incident continually recounted the events of the incident to students around him. The teacher-researcher handed out the worksheets, and asked the students to work through the examples given. Many students required individual assistance in completing the exercises on the sheet, as they could not recall the sequence of steps demonstrated on the board. The student involved in the incident remained off-task all lesson, as did the students around him. As the teacher-researcher moved around the room observing students working, it was apparent that many students were confused about the steps involved. The teacher asked the students to hand in their worksheets at the end of the lesson. Analysis of students' responses indicated that errors were occurring, particularly in relation to positioning numbers on the number line, and also in the cross-multiplication technique. The identification of *part*, *whole*, *percent* elements was generally correctly performed. All work planned for this episode was not fully implemented.

Reflection

The interpretation of percent problems in terms of the three elements was perceived as a successful part of this episode, but the use of the Smarties demonstration was not. In an effort to link to prior experiences (i.e., with the Smarties) it was found that students did not readily see the link to the fractional notation of the contents of a box of Smarties to percentage parts and wholes. Although students had prior experience with fraction, decimal and percent conversions, as indicated by their satisfactory performance on the pretest (Section II) it appeared that they did not readily relate fraction-to-percent conversions to real collections. Therefore, this introductory section of the episode was not a relevant link to the students' percent knowledge. The fact that all students could readily identify a percent problem in terms of part, whole, percent suggested that this was a relatively simple task. The use of the Smartie situation did not appear to have added any meaning to this part of the episode. The link between the proportion equation and the fraction equivalence method appeared to help students see the purpose of the previous lesson on finding equivalent fractions using the Rule of Three technique. The positive murmurings when the proportion equation was solved using the Rule of Three were interpreted as students seeing a link between the two procedures.

The structure of the first lesson in this episode appeared to assist the successful nature of the lesson. Apart from the unsuccessful attempt to link fractions to percent via the box of Smarties, the lesson instructions were sequential, gently guiding students to solving percent problems. After identifying the three elements in percent problems, the students drew a horizontal number line and marked in 0%, 50% and 100%. The teacher-researcher could thus quickly check students' understanding of this representation for percent. The rotation of the number line to a vertical position appeared to enable students to see how this representation linked to the horizontal format, and provided the structure for percent problem interpretation. The fact that the vertical number line positioned thus yielded a proportion equation which matched the positioning of numbers in equivalent fraction form was perceived as linking percent equations to proportion equations and thus to equivalent fractions. According to Parker and Leinhardt (1995), the positioning of the number line in this way may cause confusion as the scales showed greater amounts (100%) at the bottom of the scale, rather than at the top, as is usual in graphical representations. With students in this group, no such reaction occurred, and it actually linked to their new knowledge of the Rule of Three, as indicated by their responses. Also interesting with this group was the fact that presenting a Type II problem did not meet with resistance and lack of attention as was the case when Group 1 students were presented with a Type I problem. This "repatterning" approach to teaching, advocated by MacDonald (1972) appears to be a useful strategy to overcome the interfering influence of prior knowledge.

In the second lesson, the lunchtime "incident" clearly affected lesson implementation. It was felt that re-presentation of the explanation for solving percent application problems was inadequate and caused students to begin to make errors. Analysis of students' worksheets indicated that two distinct categories of learners emerged: - those who could complete the percent application problems correctly, and those who could not. It was apparent that grouping students for instruction was necessary. To enable all students to practise solving percent problems, some students required further instruction in the number line procedure. A further lesson was planned in this purpose.

Replanning for third lesson in this episode

For the third lesson in this episode, it was planned to group the students on the basis of correct performance of percent application problems. Students successfully using the method would work individually, completing the problems on the worksheet, and then continue on with the further practice worksheet (Appendix E). Students experiencing difficulty, would receive demonstrations of the steps prior to working independently on their worksheets. At the beginning of the lesson, it was planned that the students would be presented with a brief historical timeline of percent. The purpose was to show the students the ancient nature of the Rule of Three via the historical timeline (this timeline is in Appendix I).

Action

This lesson began with students instructed to sit in specified areas as they entered the room. The teacher-researcher returned students' worksheets to them, and the format of the day's lesson was described. The percent timeline was presented on an OHT, the Rule of Three descriptions were read to the students. The lesson continued as planned.

Observation

The students quietly moved to their allocated groups. The students listened quietly and appeared interested in the historical note of the Rule of Three. (The classroom-teacher commented during the lesson that she had not realised that percent was "such an ancient notion". She thought the Rule of Three was a useful procedure which she had not previously applied for solving percent problems.) With the class grouped in two, the teacher-researcher spent time with the students experiencing difficulty whilst the rest of the class worked individually on the worksheet. All students worked relatively quietly although some chatting amongst students occurred. From observations of students' work, some students were experiencing difficulty in interpreting the elements of Type III problems. Only 5 of the 30 students completed all 15 problems on the worksheets during the lesson. When working on the second worksheet, one student did not draw the diagram, or show evidence of working out. However, all solutions were correct. This student stated that he did most of the working in his head. The teacher-researcher asked the student what method he used, and the student replied that it was the proportion equation for Type II and III problems, but the percent button and multiplication for Type I problems. Another student did not draw a number line, but wrote the problem information as a proportion equation. This student's work showed evidence of misinterpretation of Type III percent problems. The teacher-researcher asked the student why he did not draw a number line, and he replied "But if I do, I'll never finish. The number line takes too long." The teacher-researcher stated that it was better to complete only two problems correctly than to finish all 8 and get most of them wrong. The teacher-researcher suggested that the number line would be useful to check the position of the numbers in the proportion equation. The student then proceeded to construct the number line.

Reflection

Student responses to practising solving percent problems were similar to those observed with Group 1 students. Once students had completed all problems on the structured worksheet, they found ways to cut corners, and thus errors developed.

The students were not allowing themselves time to practise strategic procedures for successful performance. The students had been presented with a cognitive strategy (Cole & Chan, 1990) to ensure successful percent problem solving, but their metacognitive knowledge had not been sufficiently developed to enable them to understand the benefit of practising the strategy to expert stage. The interplay of cognitive and metacognitive knowledge, and the difficulty of convincing students that they are in control of the learning situation were highlighted once again.

In this episode, students' productivity in terms of the number of problems completed was not great. It was perceived that all students would benefit from another class period of further practice examples, and the students who were developing expertise in this method, could be presented with more open investigations. Due to the timetable restrictions a further lesson could not be afforded at this point.

5.2.8 Episode 5: Percent, common and decimal fraction equivalence

Plan

The plan for this episode was the same as for Episode 7 used with Group 1 (see 5.1.10), with some modifications. In view of the difficulty some Group 1 students experienced in constructing the number line, it was planned that instruction for assembling the number line would be more directed and sequential. Two lessons were planned for this episode, with the first lesson taken up in construction of the number line, and the second lesson for practise of percent, decimal and fraction conversions. The planned sequence of instruction in this episode was as follows:

1. Hand out a pre-printed A5 sheet of paper. Instruct students to begin counting in common fraction hundredths, marking the position of each hundredth on the number line, to $^{19}/_{100}$. Simplest form fractions to be written in brackets.
2. Instruct students to write decimal fraction symbols for each corresponding common fraction symbol on the number line, to 19 hundredths (0.19).
3. Ask students to identify $^{1}/_{100}$ as a percent, and write the corresponding percent values on the appropriate place on the number line, to 19%.
4. Instruct students on how to join 4 more A5 sheets of paper to complete the number line to the number 1 (100%, $^{100}/_{100}$, 1.00).
5. Direct students to mark commonly used decimal and common fractions, and percents - $^{25}/_{100}$, $^{50}/_{100}$, $^{75}/_{100}$, $^{100}/_{100}$, and every tenth common fraction $^{10}/_{100}$, $^{20}/_{100}$, $^{90}/_{100}$.
6. Students to use the constructed number line to assist in completion of the worksheet (see Appendix G). For example, if the item on the worksheet asked students to convert $^{4}/_{5}$ to a percent, from the number line, locate this simplest form fraction as $^{80}/_{100}$. The fraction is equivalent in value to 80%.

Action

The first lesson proceeded as planned, with all students successfully constructing their number lines. At the beginning of the second lesson, the teacher demonstrated how to use the number line to assist in determining decimal and common fraction and percent equivalences. The worksheet was handed to the students, and the activities explained. The students were instructed to work through the activities at their own pace.

Observation

The step-by-step instructions for constructing the number line appeared to be a means for maintaining calm within the class, as all students completed the activity in a quiet manner. When directing students to fill out the decimal counting numbers, the teacher-researcher wrote the sequence of numbers 0.07, 0.08, 0.09, ____, on the board and asked the question: "What number comes next? Is it 0.10 or 0.010?" A student stated that "0.10" would come next, as "0.09 is nine-hundredths, and if we wrote 0.010 we have ten thousandths, not ten hundredths". By the end of the lesson, all students had successfully constructed the number line, to show equivalent forms of decimal, fractions and percent numbers.

The worksheet was presented in the second lesson. As students worked through the worksheet (Appendix G), some students required individual assistance in writing decimals as equivalent percents. For example, to assist the conversion of 0.29 as a percent, students were instructed to read the decimal as "twenty-nine hundredths" so they could "hear" the decimal as "hundredths" and thus as a percent. Most students experienced little difficulty in changing percentages (e.g., 52%) to common fraction form as they simply wrote the percent as a fraction of 100 (i.e., $^{52}/_{100}$). However, only a few students reduced such fractions to the simplest form. Despite the fact that the students' number line were marked with common fractions expressed in simplest terms, many students did not use the number lines as a reference to help them change, for example, $^{4}/_{5}$ to a percentage. Not all students had completed all activities on the worksheet by the end of the lesson.

Reflection

Instruction in this episode appeared to overcome students' counting errors in decimal, fraction and percent, and also ensured construction of the oversized number line was completed efficiently. As with Group 1, students with limited knowledge and skills of percent, common and decimal fraction conversions did not appear to develop such new skills (that is, in developing skills in reducing fractions to simplest form; in finding fraction equivalence to denominators of 100; and in converting decimal tenths to percent). These students did not refer directly to their constructed number line to assist them in completing the worksheet exercises. For the students who already possessed satisfactory percent conversion and benchmarking skills, this episode appeared to provide an opportunity for practise of those skills. Construction of the number line for such students may have been a superfluous exercise.

The purpose of having students construct a number line was to show that any percent value can be located on a number line; its equivalent common and decimal fraction form can also be located at the same point, thus reflecting the equivalent nature of the three number forms. Students' understanding of equivalence is generally weak (Vance, 1992), and this activity was designed to strengthen such understanding. In a similar vein to the reflection of this episode with Group 1 (section 5.1.10) this episode was perceived as requiring more direct instruction to draw students' attention to this fact. The episode therefore is rated as between successful and unsuccessful, as it did not overtly appear to build new understanding of equivalence, but did provide for practise of conversion skills.

5.2.9 Episode 6: Language of percent increase and decrease

Plan

Given the "success" of using jellybeans to demonstrate percent increase situations (as used with Group 1), it was decided to use only jellybeans for demonstration of percent increase and decrease situations in this episode. The plan was to use jellybeans to demonstrate the effect on the whole when the number of jellybeans is increased by a certain percent. As in Episode 6 with Group 1 (see section 5.1.9), each percent increase/decrease situation would be represented diagrammatically on the board, using the number line representation. Percent decrease situations would be modelled using jellybeans as with Group 1. The jellybean "decrease" situation would be represented diagrammatically using the number line, and additive and multiplicative language used in two different sentences. The multiple choice worksheet (Appendix F) would be used to enable students to practise their understanding of the language of percent increase and decrease situations. Thus, the plan for this episode was as follows:

1. Present students with the multiple-choice item from the pretest on a small slip of paper. Direct students to select the appropriate response and provide a reason why they selected that response.
2. Scan students' responses to ascertain the approximate number of students who selected the appropriate response.
3. Demonstrate percent increase situations using jellybeans in the following sequence:
    (a) 6 jellybeans, increased 50%
    (b) 8 jellybeans, increased 25%

(c) 10 jellybeans, increased 20%

(d) 3 jellybeans, increased 100%

(e) 3 jellybeans, increased 200%.

For each situation, represent on a number line, and write sentences to describe the situation using (i) additive language, and (ii) multiplicative language.

4. Demonstrate percent decrease situations using jellybeans in the following sequence:

(a) 10 jellybeans, 50% eaten

(b) 12 jellybeans, 25% eaten

(c) 10 jellybeans, 20% eaten

For each situation, represent on a number line, and write sentences to describe the situation using (i) subtractive language and (ii) multiplicative language.

5. Present students with multiple-choice worksheet (Appendix F). Instruct students to work in groups to identify correct interpretations of given increase and decrease situations.

It was planned that percent increase situations would be the focus of the first lesson in this episode, and percent decrease situations and the worksheet would be the focus of the second lesson.

Action

In the first lesson, the students completed the multiple choice item. The students were asked to leave the multiple choice item on their desks until the end of the lesson. The percent increase situations were presented to the students as planned. All percent increase demonstrations were completed during this lesson.

The second lesson in this episode proceeded as planned. The delivery of the decrease examples was completed in less time than with Group 1, and thus the students had more class time to work on the worksheet. Many students completed the worksheet, which the teacher scanned, and provided students with immediate feedback if they had not identified all correct percent increase and decrease statements. For the students who did not complete the worksheet, the opportunity to complete the sheet in their own time was offered.

Observation

At the beginning of the first lesson, a quick scan of students' responses to the multiple choice item indicated that the majority of students had misinterpreted the question. As the lesson progressed, several students changed their initial response as various percent increase situations were demonstrated. New meaning of percent increase situations, as a result of the demonstrations, appeared to be developing for these students.

The students remained quiet and were cooperative during the whole lesson. The organisation of the lesson (demonstration, representation, writing additive and multiplicative sentences) enabled the teacher-researcher to scan students' work in between demonstrations. The students waited quietly as the teacher-researcher checked all students' written interpretations of each situation presented.

The second lesson in the episode also appeared to proceed smoothly, and students remained cooperative and attentive. As students worked in groups on the worksheet, the teacher-researcher moved amongst the groups, listening to students' discussion of the percent increase and decrease sentences. Students who were still experiencing difficulty with such interpretations were identified, and extra assistance was provided by the teacher. From analysis of students' responses to the worksheet, it appeared that many groups of students correctly identified the majority of percent increase and decrease sentences.

Reflection

This episode was perceived as successful. This episode was implemented in a smooth manner as a result of reflection upon this episode with Group 1. Lesson design appeared to maintain students' attention, and thus students were assisting themselves to learn. The two-way nature of the learning process is highlighted here. The student is in control of the learning situation in that the amount of attention he/she pays influences what they learn (Lyndon, 1995). However, the teacher must plan carefully to assist students to maintain attention (Mercer & Miller, 1994) as maintaining attention is effortful and hard work (Lyndon, 1995).

5.2.10 Reflection upon implementation of teaching episodes

Reflection upon implementation of this unit of work was that it was more successful than with Group 1, but only moderately. Some episodes, as a result of reflection upon implementation with Group 1, were more successful than with Group 1, but not sufficiently so in terms of an efficient and effective unit of work. In Table 5.6, the success rating for each episode is presented.

From Table 5.6, it can be seen that the metacognitive training episode was rated as between successful and unsuccessful, which is a change from its totally unsuccessful rating for Group 1. The other episode to be rated similarly was the percent, common and decimal fraction equivalence episode (episode 5), which also had been rated as unsuccessful for Group 1. The concept of percent episode (episode 2) was rated as unsuccessful, which is the same rating as for Group 1. The episodes which focused specifically on the proportional number line method for solving percent equations (episodes 3, 4 and 6) were all rated as successful, which is a change in a positive direction compared with these episodes for Group 1.

Table 5.6

Success Rating of Teaching Episodes for Group 2

Episode	Topic	Success rating*
1	Metacognitive training	U-S
2	Concept of percent	U
3	Fraction equivalence and the Rule of Three	S
4	Interpreting and solving percent problems	S
5	Percent, common and decimal fraction equivalence	U-S
6	The language of percent increase and decrease	S

* S = successful

U = unsuccessful

The metacognitive training episode was modified from that used with Group 1, and this appeared to contribute to a more successful implementation with Group 2. However, the entire metacognitive training episode was not implemented entirely as planned with Group 2, and thus it failed to strongly inform the students of their role in the learning process, and the effort required on their part. The episode also did not provide students with strategies for successful task performance, or with sufficient explanation of how they could be applied for successful performance in mathematics. Thus, at times when the O/N strategy could have been applied within the unit of work, it was not, due to lack of this shared knowledge and understanding between the students and the teacher.

The last episode in this unit, on fraction, decimal and percent equivalence, was also rated as between successful and unsuccessful. It did not appear to build students' knowledge of equivalence, as students did not refer to the constructed number line when completing conversion activities. However, the episode did provide the opportunity for those students with sufficient skills in percent conversions and benchmarking to practice those skills. The episode therefore was of benefit to some students but not to others.

The second episode in this unit, on the concept of percent, was rated as unsuccessful. In a similar manner to this corresponding episode with Group 1, this episode was perceived as unsuccessful in promoting students' understanding of the multifaceted nature of percent, and its uses in describing a group of objects. Although the students had appeared to enjoy the activity, no challenging notions were presented to the students.

The episodes which specifically focused on the proportional number line method for percent problem solving (Episodes 3, 4 and 6) were rated as successful, due to modifications as a result of reflection upon these episodes with Group 1.

The major difficulty encountered once again, was the lack of time available in the real school situation to enable students to practise and consolidate skills and develop rich concepts. This unit of work was implemented with Group 2 in 11 lessons which was a reduction of one lesson compared to Group 1, and many new concepts and skills were covered. In terms of efficiency, this unit was more efficient of teacher time, due to careful planning and structuring of each episode in light of reflection of implementation with Group 1. Teacher factors associated with lesson design and sequencing were thus more controlled in Teaching Experiment 2.

5.2.11 Reflection upon pre- and posttests and instruction

Reflecting upon pre- and posttest scores and implementation of this unit, some interesting comparisons can be seen. The implementation of the unit of work was rated as moderately successful, and overall pre- and posttest scores for Group 2 showed moderate positive change from 37% to 66% respectively. Within the unit, the episodes on interpreting and solving percent problems were rated as successful, and Group 2 test scores on that section of the posttest are 48% compared to 7% pretest score. The number line model was the predominant strategy used by students on this section of the posttest. Thus, instruction appears to have contributed to more students' successful performance in interpreting and solving percent problems, therefore supporting the teacher-researcher's perceptions of the successful nature of these episodes in the unit, and the value of the proportional number line method as a key process. On Section II of the test (conversions and benchmarks), a positive change was recorded between pre- and posttests with scores of 41% and 70% respectively. Implementation of the episode on conversions and benchmarking was rated as between unsuccessful and successful, and students' test scores can be interpreted as reflective of this rating. As stated in the reflection upon implementation of this episode (see section 5.2.10), students who had sufficient skill in percent conversions and benchmarking completed class activities in a satisfactory manner during the episode; the constructed number line, however appeared to be of minimal assistance for other students in building their knowledge and skill in such conversions. The episode, therefore appears to have positively assisted students to practise conversion skills, but not for helping other students develop such skills. In terms of intuitive and principled-conceptual knowledge, the students in Group 2 performed at a satisfactory level on the pretest, thus indicating that their percent knowledge in this category was satisfactory. The positive change on performance on this section of the posttest indicates that the unit of work may have raised students' awareness of percent notions, with pre- and posttest scores of 64% and 79% respectively. As with Group 1, Group 2 students experienced difficulty with particular items on the pretest relating to the multiplicative and additive language of percent increase (see Figure 5.7, items 6a-c and 7a-c in section 5.2.2). As also noted with Group 1, Group 2 students' posttest scores indicate that students continued to experience difficulty in interpreting such items after instruction. Intense instruction on percent increase therefore appears to have been minimally beneficial for students in this group, although this episode in the teaching sequence was rated as successful.

5.2.12 Directions for Teaching Experiment 3

Reflections upon implementation of the teaching sequence with Group 2 indicated that teacher factors were more controlled in Teaching Experiment 2, but that other factors, such as student metacognitive factors, time factors, school context factors and curriculum factors were difficult to control. In Teaching Experiment 2, often the reasons for unsuccessful episode implementation was due to these external factors. In an attempt to overcome such factors, Teaching Experiment 3 was trialled in a different school to Group 1 and 2. It was anticipated that student factors and school context factors may be more controllable to enable the planned teaching sequence to be fully implemented, and thus the time factor would be less interfering. Teaching Experiment 3 was thus conducted to enable evaluation of the teaching sequence to be undertaken.

5.3 Teaching Experiment 3

5.3.1 Overview of report on Teaching Experiment 3

The results of Teaching Experiment 3 are reported in this section. The pre-, post- and delayed posttest results are presented in section 5.3.2. An overview of the teaching sequence comprising six teaching episodes is presented in section 5.3.3. In sections 5.3.4 to 5.3.9, implementation of the six teaching episodes is described under the subheadings of *plan*, *action*, *observation* and *reflection*. In section 5.3.10, a reflection upon implementation of this unit of work is presented, followed by a reflection upon pre- and posttest results and instruction in section 5.3.11. This section concludes with a statement on direction for Study 4 in section 5.3.12.

5.3.2 Pre-, post- and delayed posttest results

Group 3 scores on each section of the pre- and posttests are presented in Table 5.7 and show that Group 3 test scores overall increased from a pretest score of 44% to a posttest score of 76%. Within each section of the test, there was a slight positive change on Section I (intuitive, principled-conceptual knowledge); a stronger positive change on Section II (conversions and benchmarks), as well as a similarly strong positive change on Section III (percent calculations and percent problem solving). From Table 5.7, it can be seen that, like Group 1 and 2 students, Group 3 students' intuitive and principled/conceptual percent knowledge, and proficiency in percent conversions and benchmarking, prior to instruction, was much greater than their percent calculation and problem solving skills. Graphical representation of the pre- and posttest scores are presented in Figure 5.10 highlighting the change in pre- and posttest scores in total, and on all test sections.

Table 5.7

Pre- and Posttest Means (%) for Group 3 Students on the Percent Knowledge Test in Total and for Each Section

	Components of Percent Knowledge Test			
Test	Total	Section I	Section II	Section III
Pretest	44%	72%	42%	19%
Posttest	76%	86%	75%	67%

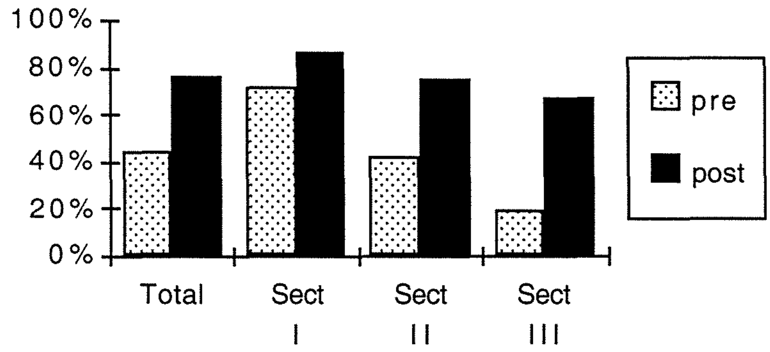


Figure 5.10. Graphical representation of Group 3 students' pre- and posttest means (%) in total, and in each test section.

As for Groups 1 and 2, Group 3 students' test results were analysed for diagnostic purposes. Pre- and posttests were scored to identify specific items on which students responded incorrectly. Within the three parts of the Percent Knowledge Test, the number of incorrect student responses for each item was tallied. Representation of the number of incorrect responses to each item on Section I (intuitive, principled-conceptual percent knowledge), Section II (conversions and benchmarks) and Section III (percent calculations and problem solving) are presented in Figures 5.11, 5.12. and 5.13 respectively.

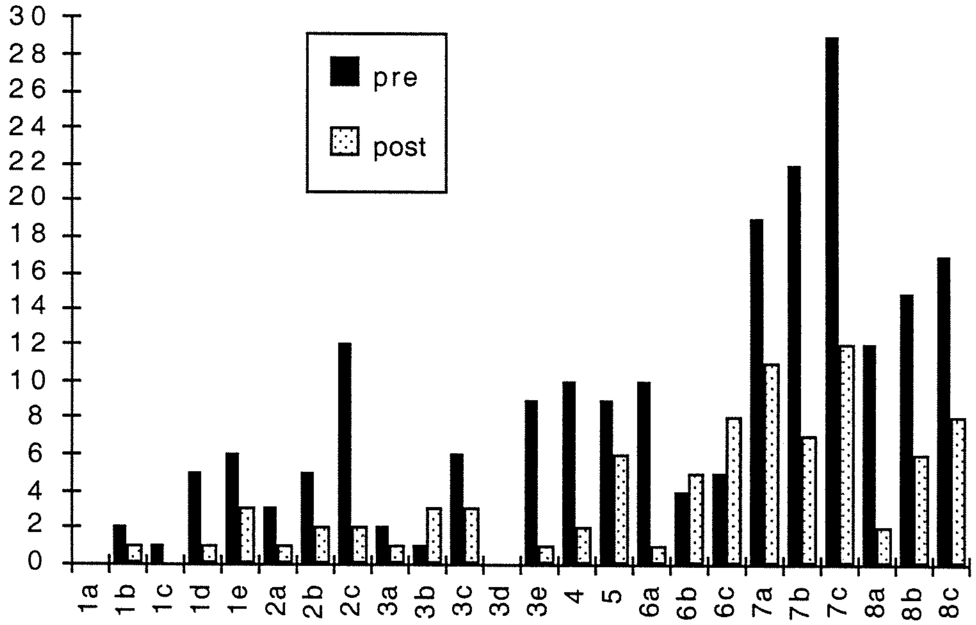


Figure 5.11. Group 2 students' pretest (n=30) and posttest (n=26) *incorrect* scores on Section I (intuitive, principled-conceptual percent knowledge) of the Percent Knowledge Test. Graph indicates number of students *incorrectly* responding to particular items.

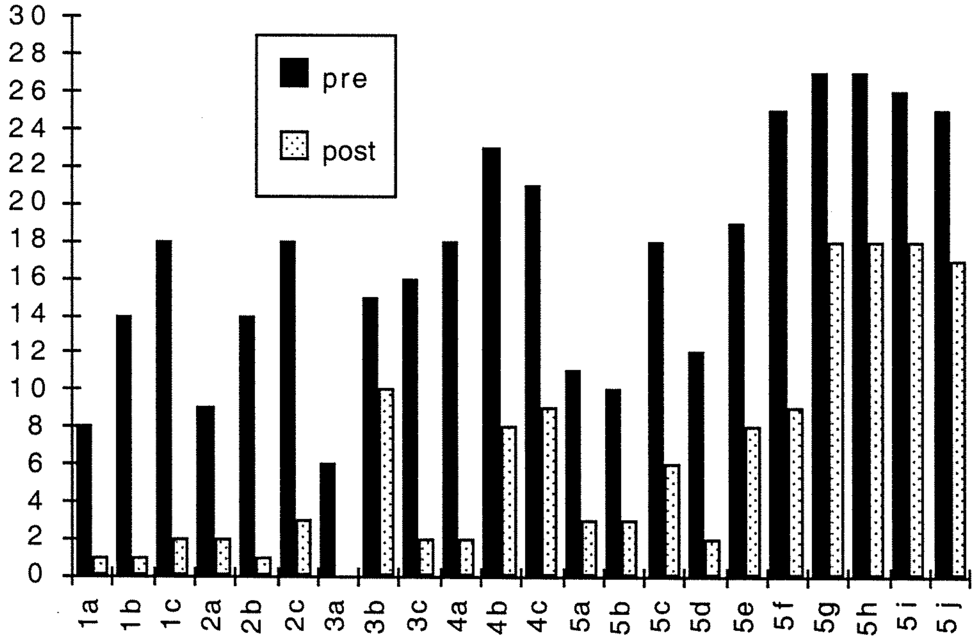


Figure 5.12. Group 3 students pretest (n=30) and posttest (n=25) *incorrect* scores on Section II (conversions and benchmarks) of the Percent Knowledge Test. Graph indicates number of students *incorrectly* responding to particular items.

In Figure 5.11, it can be seen that, on the pretest, Group 3 students experienced most difficulty with items 7a-c, and 8a-c, which relate to interpretation of the multiplicative and additive language of percent increase, and the posing of real world percent problems from percent equations and also item 2c, which relates to the percent benchmark that 10% is one tenth. Posttest scores indicate positive change in performance after instruction, but also indicate that students continued to experience difficulty in interpreting the multiplicative and additive language of percent increase situations (item 7a-c). Students' ability to pose real world percent problems from percent equations improved on the posttest (item 8a-c).

Figure 5.12 indicates that, prior to instruction, many students experienced difficulty in fraction, percent and decimal conversions (items 1-4), and in using percent benchmarks (item 5), particularly benchmarks of 15% (as 10% + 5%), 30% (as 3 x 10%), 60% (as 6 x 10%) and $33^1/_3$ (as $^1/_3$) (item 5f-j). On the posttest, improved student performance can be seen, most notably on percent-to-fraction conversions (item 1a-c), and percent-to-decimal conversions (item 2a-c). After instruction, more students also were using percent benchmarks successfully, however percent benchmarks of 15%, 30%, 60% and $33^1/_3$ were still causing difficulty for some students (items 5g-j).

Figure 5.13 indicates that, Section III of the pretest (percent calculations and problem solving) was not well-attempted by the students.

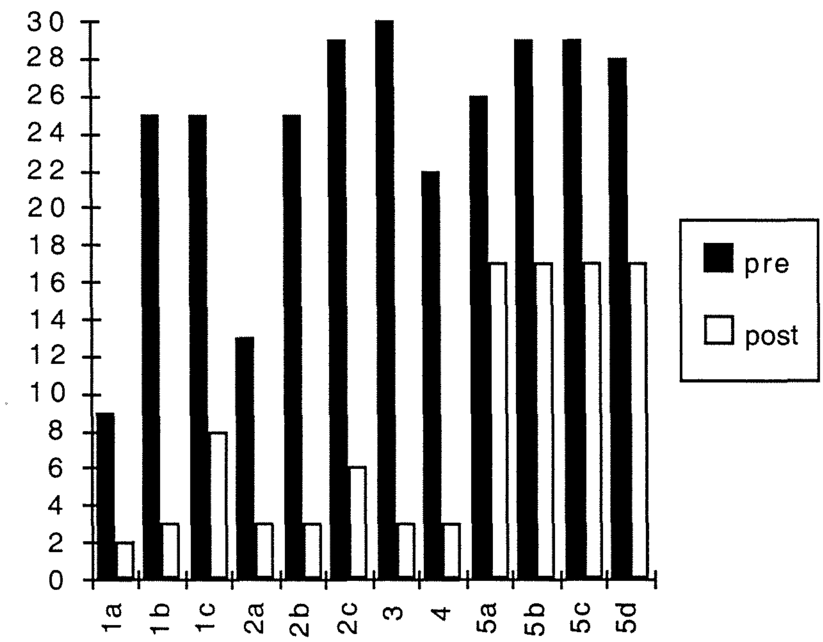


Figure 5.13. Group 3 students pretest (n=30) Posttest (n=25) *incorrect* scores on Section III (percent calculations and problem solving) of the Percent Knowledge Test. Graph indicates number of students *incorrectly* responding to particular items.

From Figure 5.13, it can be seen that, on the pretest, some students could successfully perform calculations of percent equations and solve percent problems, but mainly those of Type I (items 1a and 2a). Type II and Type III percent equations (items 1b and 1c), and percent problems (item 2b and 2c) were not successfully completed by students. In Figure 5.13, it can also be seen that no student used diagrams to assist percent problem solving (items 3), and only a minority of students could successfully express percent problem solutions in words (item 4). Posttest results indicate greater facility in solving percent problems and performing percent equations after instruction, with over two-thirds of the students successfully performing percent calculations of all three types (item 1a-c), and also successfully solving percent problems (items 2a-c) on the posttest. The majority of students used diagrams to solve percent problems (item 3), and could express solutions to percent problems in words (item 4). Facility in solving multistep word problems also showed some improvement (items 5a-d).

Analysis of students' solutions to percent calculations and problem solving items on the pre- and posttests indicate change in strategy, similar to results for Groups 1 and 2. On the pretest, Group 3 students used decimal multiplication or the calculator sequence for Type I problems, and the long division procedure for Type II problems. Of the students who successfully performed Type III problems, answers only were given, thus categorisation of solution strategy was unavailable. Some students also gave answers only for each of the three types of percent problems; some solutions were correct and some solutions were incorrect. On the posttest, all students who revealed their solution procedure used the proportional number line method. Of the students who utilised this method, but gave an incorrect response, errors could be seen to emerge from incorrect placement of numbers on the number line. Of the students who successfully attempted all three types of percent calculations and word problems on the pretest, posttest results showed use of the proportion equation as their solution strategy. Thus, as with Groups 1 and 2, there was a greater use of the proportional number line method for percent calculation and problem solving by all students as a result of instruction, and utilisation of this strategy led to successful task performance.

As stated in section 4.5, a delayed posttest was administered to students approximately 8 weeks after instruction. In Table 5.8, results for this test are displayed, by individual item. From Table 5.8, it can be seen that students were still a showing high level of mastery in percent calculations (items 3a-c) and in solving percent problems (items 4a-c). Procedures used were predominantly the proportional number line method, or simply the proportion equation. Performance on all items relating to additive and multiplicative language of percent increase (item 2, 5a and 5b) was minimal, with scores of 29%, 10% and 14% for items 2, 5a and 5b respectively.

Interpreting the subtractive language of percent discount (item 1) was generally well handled by students (82%).

Table 5.8

Delayed Posttest Means (%) on Individual Test Items for Group 3 Students

Item	1	2	3a	3b	3c	4a	4b	4c	5a	5b
Score	82%	29%	86%	71%	71%	93%	86%	71%	10%	14%

Students' scores on individual items of the delayed posttest are graphically displayed in Figure 5.14, and serve to highlight the particular items with which students experienced difficulty.

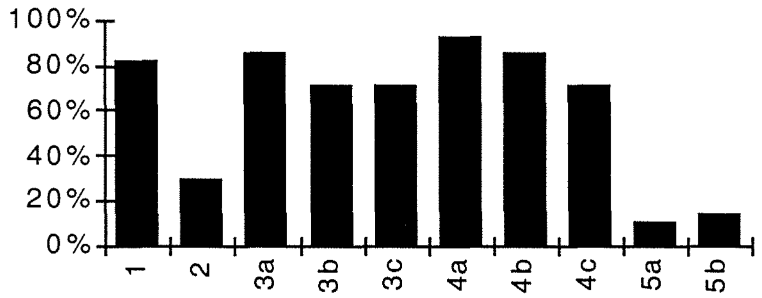


Figure 5.14. Graphical representation of Group 3 students' delayed posttest means (%) by item.

5.3.3 Planned teaching episodes

The sequence of six teaching episodes implemented with Group 3 was the same as implemented with Group 2. In Table 5.9 each episode, and the number of lessons required for implementation with Group 3 is presented. The table provides the number and title of each teaching episode, together with the number of actual lessons taken for implementation, and the chapter section in which discussion of each episode appears.

As can be seen from Table 5.9, the order of presentation of the teaching episodes is the same as with Group 2. Upon reflection of implementation with Group 2, the sequence of episodes was rated as satisfactory and thus change of order was perceived as unnecessary. A total of 15 class periods was spent with Group 3. Nine lessons were directly on the topic of percent, with a further 2 lessons used for metacognitive training. Pre- and posttesting occupied 2 days prior to and immediately following implementation of the unit of work.

Table 5.9

Plan of Teaching Episodes for Group 3

Episode	Topic	Lessons	Location
1	Metacognitive training	2	5.3.4
2	Concept of percent	1	5.3.5
3	Fraction equivalence and the Rule of Three	1	5.3.6
4	Interpreting and solving percent problems	3	5.3.7
5	Percent, common and decimal fraction equivalence	2	5.3.8
6	The language of percent increase and decrease	2	5.3.9

5.3.4 Episode 1: Metacognitive training

Plan

For this Group, the metacognitive training episode was planned to be implemented as described with Group 1 (see section 5.1.4) with the inclusion of the additional activities for recognition and recall memory as used with Group 2 (see section 5.2.4). It was planned that attention, memory and recall memory strategies would be presented in the first lesson, and natural and accelerated forgetting, proactive inhibition (PI), and the O/N strategy presented in the second lesson.

Action

The metacognitive training episode was implemented as planned. Attention and memory were discussed in the first lesson and the LSCWCh strategy demonstrated. Forgetting and PI were discussed in the second lesson and the O/N strategy demonstrated. At the end of both lessons in this episode, the students were instructed to write a diary entry in relation to this lesson for homework.

Observation

The students remained attentive and appeared interested in the material presented in the episode. The students remained attentive during presentation of the LSCWCh strategy, with several students stating that they knew of the strategy. One student stated that she had used the strategy before, although not in the totally prescriptive manner described, but that she now understood the need for all the steps to control memory. The Colour Card activity was readily completed by the students, with several students commenting that the activity had been "fun", but really "annoying" due to the interference felt. The O/N strategy was demonstrated with a student volunteer and her misspelling of the word "seperate" (separate). After the lesson, several students commented as they left the room, that the lesson had been "fun".

The classroom-teacher commented that she had enjoyed both lessons in the episode, and that she had learnt a lot about her brain as a result. She commented that the students appeared very interested in the lessons. She also commented that, when the students were informed that they were in control of their attention, she noted that there appeared to be a visible, startled, reaction by the students.

The students' diary entries indicated that, of the first lesson, the key idea for the lesson, that *the brain is designed to forget,* was well noted by the students. Students' diary entries also confirmed the classroom-teacher's and teacher-researcher's perception that the students found the metacognitive training episode extremely interesting and useful. The following comments, taken from students' diaries, are categorised as key points from the episode, or general statements about the episode. The following comments are student's interpretations of the key points of the metacognitive training episode:

> *"One thing I learned in maths today was that our brains are made to forget. Another thing that I learned was that if you go over something 5 times that thing will go into auto recall."*
>
> *"I learnt that paying attention requires effort."*
>
> *"There are many types of remembering, but you have to practise things 5 times before they get into the part of our brain where we want them to get into which is called the automatic recall memory. We want them there because then we can just bring it up when ever we want, and won't have to think too much."*
>
> *"I found out that recognition memory happens naturally without effort.*
>
> *"I found out that learning is easy but paying attention is hard."*
>
> *"This lesson has actually taught me more about the brain and how it works. It has now made me realise how my attention gets distracted very easy. I was pleased that I found out about how my attention flies away."*
>
> *"One thing I learnt today in maths was about accelerated forgetting is natural and I learnt there is a way to learn something again which I had forgot before."*
>
> *"Today I learnt how to relearn something if I have learnt it incorrectly."*
>
> *"One sort of forgetting is Accelerated Forgetting. Accelerated forgetting is natural and it shows the creativity in our minds because say if we forget how to spell a word but then we come up with our own and no-one corrects well we keep on spelling it that way."*

"One thing I learned in maths today was how to reprogram an old way of doing something (e.g. an incorrect way of spelling something) to a correct way of doing something. The process is called "old way, new way". I think I will use it a lot. It will be handy."

"One thing I learned in maths today was about the two types of forgetting. I found out that there are two types of forgetting, which I didn't know before and how to take over accelerated forgetting and improve on it."

The following comments are students' statements of the episode in general:

"In maths today we learnt that our brain is designed to forget. That is good, - that means my brain is working! I was paying attention because the lesson was fun and interesting."

"I learned that you could train your brain to learn things. I thought you either were smart or not smart. I think I am getting better teaching my brain to take things in better."

"I was pleased that it was such a different and exciting lesson, and although it doesn't seem like I was learning, I was actually learning and having fun at the same time."

"Today's maths lesson was more interesting than most because we did something that I like."

"It was really interesting learning about the memory."

"I felt I have learnt something during our maths lesson yesterday. For I learnt to listen and concentrate to other people."

"I enjoyed this lesson and I understood it fully."

Reflection

This episode, implemented as planned, was perceived as successful. The lesson plan appeared to enable the key points of the episode to be presented in a flowing manner, with activities and discussion providing a balance between "listening" and "doing" for the students. Delivery of this episode did not meet with the resistance observed with Groups 1 and 2. Although the metacognitive training did not provide a mathematical context for the memory strategies, the students paid attention during the episode. Convincing students of the need for strategic behaviour in the learning situation was a simpler task for this group than with Groups 1 and 2. Students' diary entries indicated that the key points of CMP were well-noted. It appears that, for successful presentation of the metacognitive training a relatively controlled environment is required, particularly with uninterrupted amounts of time for successful implementation.

5.3.5 Episode 2: Concept of percent

Plan

Upon reflection of this episode with Group 1 and 2 students, this episode was planned to stimulate students' knowledge of percent in the real world in a similar fashion to the manner with Groups 1 and 2, but in a more structured way. A large worksheet/poster was printed, containing examples of percent taken from a newspaper. To begin the lesson, it was planned that brainstorming percent notions would be done with the whole class. In groups of four, the students would then be directed to discuss the uses of percent presented on the worksheet, and write an explanation underneath. The lesson was for the purpose of providing students with examples of percent used to describe discounts, profits, data, interest rates.

Action

The lesson began with brainstorming of percent notions which students copied into their books as they were written on the board. The worksheet was distributed to students, and the task explained. The students finished the task before the end of the lesson. The students' work was collected. The teacher conducted a discussion on percent language used in the real world using a pre-prepared OHT (see Appendix J) on which notions, such as 110% effort, 75% full and 25% empty, were displayed. The students were asked to use the O/N strategy at home on a spelling word for homework.

Observation

The brainstorming session was completed quickly with the teacher-researcher writing many percent notions on the board similar to those presented by Group 1 and 2 students. The students were informed of the activity and moved into groups quickly and quietly. The students discussed each example as directed, and the teacher-researcher visited each group of students, listening to students' discussions, and encouraging students to give better explanations. For example, one advertisement for home units stated that 66% of units had been sold. The teacher asked students what percent were unsold. After group discussion, the students wrote an explanation which described the complement notion of percent on the sheet. One group of students asked the teacher-researcher to explain bankcard interest, indicating that this "real world" application of percent was not "real" to the world of all Year 8 students.

The class discussion on percent language used in the real world occupied approximately ten minutes. The students were attentive during the discussion, and many students volunteered other examples of similar percent expressions for each expression presented. This activity, although impromptu, appeared to provide a good stimulus for generating students' thinking on the language of percent used in our world.

Reflection

The lesson was rated as useful in providing the teacher-researcher opportunity to talk to students, which enabled the teacher-researcher to build students' understanding of the multifaceted nature of percent in various situations. The group activity enabled students to talk about percent notions, and the fact that all groups remained on task suggested that students need to be familiar with group activities in order for this instructional strategy to promote learning. The Group 3 students were more familiar with group work that students in Groups 1 and 2.

how percent is used in our society (e.g., use of the expression 110% effort). Because of students' interest and contributions in this discussion, it appeared that structuring this into an introductory lesson on the concept of percent would be useful.

5.3.6 Episode 3: Fraction equivalence and the Rule of Three

Plan

This lesson on fraction equivalence and the Rule of Three was planned to follow the format used for Group 2 in Episode 3 (see 5.2.6), but without reference to the diagrammatic representation of equivalent fractions.

Action

This lesson was implemented as planned.

Observation

All students remained on task throughout the lesson. The lesson appeared to progress at a steady pace. All students completed the exercises, and appeared to enjoy the immediate feedback and success in completing such exercises. The students' diary entries also confirmed the teacher-researcher's perception of the productive nature of the lesson. The following comments on this lesson were taken from students' diaries:

> *"One thing I learnt was how to do equivalent fractions an easier way."*
>
> *"One thing I learnt in maths today was how to do equivalent fractions on the calculator. It was very helpful."*
>
> *"One thing I learned in maths today was how to use the calculator to find equivalent fractions. I was pleased that I could do that exercise."*
>
> *"I was pleased that I learnt how to work out the answer of missing fractions because it is a quick and easy way. We did a couple of these fractions so I now know how to do them properly all the time."*
>
> *"Today in maths we learnt more about equivalent fractions and a quick way to work them out. I was pleased that I already knew how to do this well."*

Reflection

As with Groups 1 and 2, this lesson was perceived as successful. The lesson appeared less complicated without reference to diagrammatic representation of equivalent fractions; the students appeared familiar and comfortable with the symbolic procedure. The "skill-drill" nature of the episode appeared to be enjoyed by the students, and one which appeared to contribute to the rapid acquisition, and correct use, of this skill. Lesson design, and planning for success, appeared to ensure this was a successful lesson.

5.3.7 Episode 4: Interpreting and solving percent application problems

Plan

As with this episode for Groups 1 and 2, the plan for this episode was to orientate students to the three elements of percent word problems, and to provide skill consolidation and practice in solving percent word problems. Taking account of the reflection upon this episode with Groups 1 and 2, that students required further time for consolidation of the technique, this episode was planned to span 3 lessons. For the first lesson, the plan for interpreting and solving percent problems was to follow that used with Group 2 in Episode 4 (see section 5.2.7), except that Smarties would not be used to represent part, whole, percent, elements of the problems. It was planned that students themselves would be used to demonstrate the elements of percent problems. For example, 8 students would be asked to stand at the front of the room and the teacher-researcher would give such directions as:

"50% of the group move towards the door, how many is that?"

"4 of the group move, what percent is that?"

"100% of the group move, how many is that?"

The purpose of this activity was to familiarise students with the terms *part, whole, percent,* reinforcing that the whole amount is 100%, and a part of the whole amount is a percent of the whole. To assist students develop skill in identifying the elements contained in percent problems, a series of percent problems was prepared on an overhead transparency (OHT), accompanied by a student worksheet in which each question was numbered, and the terms: part___, whole___, percent___, listed. As each percent problem was displayed on the OHT, students would fill in their worksheet, stating the part, whole and percent values given in each problem. (The worksheet of percent problems and accompanying student sheet is located are located in appendix K.) For representing and solving percent problems on the number line, the lesson was planned to follow steps 5 to 9 in the sequence described in Episode 4 with Group 2 (see section 5.2.7).

The second lesson was planned to provide students with the opportunity to practise interpreting and solving percent word problems by completing the structured worksheet (Appendix D) distributed in lesson 1, and beginning the second worksheet (Appendix E). Students would be provided with instruction on how to write percent word problems as percent equations in the form: p% of p = p at the beginning of lesson 2 so that they could formally complete that task on the first worksheet. Students would be given instruction on how to write story problems for percent equations at the beginning of the lesson 3.

Action

All lessons proceeded as planned. For the second lesson, the students' usual mathematics classroom was being used by another class for examinations, necessitating a room change for these students, and 7 minutes less for the lesson. Instruction on how to write percent word problems as percent equations in the form: Δ% of Δ = Δ was presented. The second worksheet was available for students when they completed the first worksheet. In the third lesson, students continued to work on the problems presented on the worksheets. Four students completed all worksheets, and were provided with activities involving percent calculations on more investigative problems.

Observation

At the beginning of the first lesson, the majority of seated students raised their hands to answer such questions as: "Is 50% the part or the whole?" All students appeared to be readily able to interpret percent situations in terms of the elements: part, whole, percent. All students correctly identified the elements of the percent problems as they were presented on the OHT. After students were provided with a demonstration of the number-line procedure for solving percent problems, the worksheet was distributed, and the students began working quietly. The teacher-researcher checked all students' work during class, offering individual help to students still experiencing difficulty. The students appeared to readily adopt the proportional number line method for solving percent problems. Students' diary entries tended to confirm the teacher-researcher's positive perceptions of the lesson. The diary entries also provided insight into students' reactions to the proportional number line method. The following comments were taken from students' diary entries, and provide evidence that the students found the proportional number line method simple and easy to use; that they found the worksheets useful; and that they enjoyed the speed with which they could solve percent problems using the number line method:

> *"One thing I learnt in maths today was how to work out percent problems. I found out that part means %, well they mean the same thing. The new way we learnt to do percent problems is a lot easier than the way we learnt last*

year, so now I can do % problems a lot easier and quicker."

"I think I'm getting better at percentages. I understand them better and I'm getting quicker at them. Using the new way is getting easier. The classes are interesting as I've haven't learnt % before."

"Today I learnt how to do even more % percents. I'm really happy I'm getting much better at percent, and I feel much more confident."

"I am pleased that I have learnt more about percentages. I understand them more now and can work them out faster. After practising how to work percentages out so much the way is stuck in my head and I am sure I will use the ways."

"One thing I learned today was an easier way of finding the percentage of something. I have used that way instead of using my own way because it is so easy."

"One thing I enjoyed in maths today was doing percent sheets. I think I am getting better at percent because it is easier now."

"Percents are getting easier, and I can understand better now. That worksheet really helped."

"Percentage is so easy. I really enjoy it. I am pleased we are doing percentage because before I didn't have a clue what I was talking about but now it has become a lot clearer."

"I feel that I have learnt heaps more from when I first began percentage. I feel 100% more confident in my maths."

"One thing I learned in maths today was how to work out percentage fractions by using a new way. I think I am getting better at percent and working out percentage problems."

"I was pleased that I was doing fractions and percentages today because I am not very good with percentage. I was taught percentages last year in grade 7 just a little but I have forgotten. I have a little idea on how to do them. As you can probably see I'm not very good in maths."

One student's diary entry indicates that this student found this lesson confusing. The following extract presents the student's response, and the teacher-researcher's written reply, together with the student's response after the following lesson:

"In maths today we have begun learning about percentages. I find it pretty hard at the moment because we haven't been retaught how to do percentage yet, and we are just getting sums to do. We were given a sheet each and it had many problems to solve on it which most of I didn't understand."

"Dear M

Please let me know if it is still too hard."

"No, I'm not still having trouble with percent. The new way you showed us works really well for me - I think I'll remember it really well it has helped me a lot. I'm now quite sure of percent."

For the second lesson, the students worked steadily at their own pace. This lesson was perceived as useful to the students for consolidation of the technique. The students' responses in their diaries appeared to confirm that many students were gaining confidence in solving percent problems. The following comments were taken from students' diaries:

"I think I am getting better at percent because I doing it easier and find it easy."

"Today we did more practise on our new way of doing percentages. We also learnt how to do percent equations. Percent equations is just using part, whole, %."

"Today was a very good revision lesson for those percent questions. Going over and over different types of questions like that has made me more confident in doing those types of problems. I think I am getting better at percent questions and I am pleased that I have learnt this quick and efficient way. Thank you very much."

"I'm really confident with percent now so this has helped."

"I think I am getting better at percentage now because I find it easier to work them out with the number line."

In third lesson, the students continued to work steadily at their own pace, as with lesson 1 and 2. Students' diary entries indicated that this lesson was useful to the students. The following comments were taken from students' diaries:

"One thing I learned was how to write stories to go with equations. I was pleased that I could finish both sheets and that I knew what I was doing."

"Today in maths we learnt how to write percentage problem sentences."

Reflection

This episode, presented as a sequence of three lessons, was perceived as successful, providing students with sufficient time to develop and consolidate skills in interpreting and solving percent problems and in developing confidence. The successful nature of this episode can be seen as due to effective lesson design, together with students taking control of their learning by paying attention during instruction, and in practising problem solving skills to mastery.

5.3.8 Episode 5: Percent, common and decimal fraction equivalence

Plan

This teaching episode on percent, common and decimal fraction equivalence was planned to follow the format used in the teacher-directed manner with Group 2 in Episode 5 (see section 5.2.8), with the construction of the number line taking place in the first lesson, and the worksheet for practising percent, decimal and fraction conversions being implemented in the second lesson.

Action

The episode proceeded as planned.

Observation

The students constructed the number line in an orderly fashion. Some students began working on the exercise sheet during the first lesson as they had completed construction of their number line. During the second lesson, with all students working on the worksheet, the teacher-researcher provided individual assistance to students as required. Many students completed the worksheet; some students experienced difficulty with some of the exercises, especially in applying percent to fraction conversions for benchmarking and mental computation.

Reflection

As with Group 2, this episode appeared to be a useful practice session for students who already were proficient in percent, common and decimal fraction conversions, but unsuccessful for promoting conversions and benchmarking skills in students with limited knowledge of such. The lesson therefore was rated once again as between successful and unsuccessful. The episode was also perceived as "out of place" in the teaching sequence, and this perception was confirmed by the classroom teacher. The classroom teacher's observations of the episode were that the students had not achieved as much during this episode in comparison to the previous episode. She also stated that she thought the students did not enjoy this episode as much as they had when working on percent problems in the previous episode.

5.3.9 Episode 6: Language of percent increase and decrease

Plan

This teaching episode on the language of percent increase and decrease, was planned to take 2 lessons, following the format used with Group 2 in Episode 6 (see section 5.2.9), with the language of percent increase the focus of the first lesson, and the language of percent decrease and the worksheet the focus of the second lesson.

Action

The episode was implemented as planned.

Observation

On the multiple choice item presented to students at the beginning of the lesson, the majority of students selected the incorrect response. During the course of the first lesson, many students altered their response to this item. The jellybeans appeared to be well-received, and a strong motivational force. The students worked cooperatively in the group activity, with the activity generating a lot of discussion on percent increase and decrease situations. The students' diary responses indicated that the pacing of the episode was suitable, and that the jellybeans were well-enjoyed. The following comments were taken from students' diaries:

> *"One thing I learned in maths today was how to do increase percent problems. I won 6 jellybeans because I answered a question right. We also wrote down some words that mean to increase and decrease so we can use them in the sentences we have to write. We never learnt how to do increase/decrease % problems last year so this is something new to me, but it is not that hard so that is good."*
>
> *"I learnt a lot in maths class today. I now know how to write 3 plain and simple sentences which explain about percent. I even got some jelly beans because I knew the answer to one of the questions and I finished first. I really enjoyed class today."*
>
> *"Today I learnt quite a lot about increase and decrease. I was given an excellent understanding of how to draw a diagram and write sentences to show an increase or decrease in something from the original amount."*
>
> *"I was having a bit of trouble with increase and decrease but I now understand them a lot better using the new way."*
>
> *"Today we learnt a new sort of percentage problem. And how to put it out. We also learnt some more words for increase and decrease. These problems are written in two ways. The first way is to say how much it went up by and the second is to look at it from the whole."*
>
> *"I think I am getting better at doing increase problems and doing diagrams. One thing I learned in maths today were other words that you could use instead of increase and decrease I really like doing the problems and they are fun."*

Reflection

This episode was evaluated as successful in familiarising students' with the varying language used in percent increase and decrease situations. The successful nature of the episode appears to be attributable to both careful lesson design and sustaining student interest and attention. Students' diary entries support the successful rating of this episode.

5.3.10 Reflection upon implementation of teaching episodes

Reflection upon implementation of this unit of work was perceived as highly successful, with only one episode in the sequence rated as less than successful. In Table 5.10, the success rating of each episode is presented.

Presentation of the metacognitive training was perceived as successful, and appeared to provide a common language which the teacher could use to focus students' attention upon the tasks at hand. The key points noted in student diaries upon completion of the metacognitive training episode indicated that the purposes of the episode were achieved. The students were taught the "art of memory" (Norman, 1980); they were provided with explicit instruction in metacognitive strategies for improved task performance (Cole & Chan, 1990); they were aware that they were in control of the amount of effort they expended in learning (Weinstein & Mayer, 1986). Students' diary entries also showed evidence of the use of metacognitive related language as a result of this section of the teaching sequence with students describing things as "old ways", "own ways" and "new ways". The students' own spontaneous differentiation between their old ways and new ways indicated that they were thinking about their own learning in a language shared with the teacher-researcher. The mediating role of the teacher in linking students' prior knowledge and new knowledge through language is a key factor in cognitive apprenticeship teaching models (Reid & Stone, 1991).

Table 5.10
Success rating of teaching episodes for Group 3

Episode	Topic	Success rating*
1	Metacognitive training	S
2	Concept of percent	S
3	Fraction equivalence and the Rule of Three	S
4	Interpreting and solving percent problems	S
5	Percent, common and decimal fraction equivalence	U-S
6	The language of percent increase and decrease	S

* S = successful
U = unsuccessful

The concept of percent episode was rated as successful in that it enabled the teacher to focus directly on various concepts of percent used in the real world, and to interact with students in small group situations. The inclusion of the discussion of the language of percent used in the real world appeared valuable with many students contributing their ideas. This additional activity within the lesson supports the suggestion that instruction which capitalises on students' intuitive notions of percent promotes understanding of percent (Glatzer, 1984).

Episodes 3, 4 and 6 which directly related to proportional number line method, were rated as successful. The lessons were structured in accordance with successful teaching principles, in that: the lessons were designed to enable students to experience success whilst avoiding situations that lead to failure (Cole & Chan, 1990); opportunity was provided to enable successful task performance so that positive attitudes could be promoted (Mercer & Miller, 1992); opportunity to practice skills to automaticity was provided (Derry, 1990); automaticity of skill led to successful problem solving performance through limiting cognitive load (Resnick & Ford, 1984; Sweller, 1988, 1989, 1992). The lessons in these episodes also can be seen to align the ten components of effective instruction (Mercer & Miller, 1992 - see section 3.4.2), particularly in relation to monitoring students' progress on tasks, providing immediate feedback and provision of systematic and explicit instruction based on careful planning and lesson design. The episodes were also rated as successful due to the students' own decisions to apply effort, and to enable themselves sufficient opportunity to practice skills to automatic level (as per Anderson, 1985). The students' diary entries show their understanding of the need to practise new skills to take control of forgetting. The metacognitive training appears to have had a positive influence on students' effort applied during these episodes.

The episode on conversions between percent, fractions and decimals (episode 5) was rated as midway between satisfactory and unsatisfactory. Similar to reflections upon this episode with Group 1 and 2 students, this episode appeared out-of-context, and did not appear useful in building students' percent knowledge, other than to jog memories of conversion learnt in prior mathematics lessons.

In this teaching experiment, the entire teaching sequence was implemented as planned. Teacher factors and student metacognitive factors appeared to harmonise with Group 3. Teacher factors which influenced successful implementation were that lesson design and sequence was appropriate, variety in instruction assisted students to maintain attention, and opportunities were given to enable students to experience success. Student metacognitive factors which influenced successful lesson implementation included students' application of effort to maintain attention, utilisation of goal-directed behaviour, practise of skill to mastery. Teacher and student factors in combination appeared to minimise the influence of other factors of time, school environment and curriculum issues. Due to applied student effort and streamlined instruction, sufficient time was available to implement all teaching episodes and to provide students with more opportunity to apply and consolidate skills. Student metacognitive factors also controlled school context factors, in that disruptions to normal school routine (for example, a room change) did not disrupt implementation of instruction.

Of implementation of the teaching sequence, the classroom-teacher expressed interest in the proportional number line method trialled, and stated that the students had learnt a lot in a short time. She stated that the metacognitive program was an extremely valuable experience for the students, and noted that many students adopted the language of the program (such as old way, paying attention, forgetful brain, self-control) throughout the unit. Of the percent unit, she felt that many students had achieved a good understanding of percent, and were much more confident in their approach to mathematics. She expressed her interest in the Rule of Three method as a means for solving all percent calculations. She acknowledged that there is very little time available to develop the concept of percent as proportion, as well as provide students with opportunities to practise solving percent problems, and to develop an understanding of percent increase. Of the percent increase procedure, the teacher stated that, if there had been opportunity for the students to practise solving percent increase problems, the notion of percent increase may have been consolidated. She stated, however, that she felt the diagrammatic representation of percent increase situations was a useful visual image.

5.3.11 Reflection upon pre-, post- and delayed posttests and instruction

Implementation of this unit with Group 3 was rated as successful. Pre and posttest scores also indicate an overall successful performance on the posttest (76%) compared to the pretest (44%). The successful nature of implementation of the unit of work perceived by the teacher-researcher, and the classroom teacher, appears to be reflected in the posttest scores. Similar to test scores for Group 2, Group 3 test scores on each section of the test show improved performance after instruction. Students' proficiency in conversions and benchmarking (Section II) increased from 42% on the pretest to 75% on the posttest. The teaching episode directly related to this topic was rated as between unsuccessful and successful, and, similarly to Group 2, test scores can be interpreted as reflective of this rating. Implementation of the episode was seen as unsuccessful for helping students develop understanding of equivalence and therefore skill in percent conversions and benchmarking, but successful in providing students who already possess such skills the opportunity to practise those skills. Students' test scores on intuitive and principled-conceptual items of the test (Section I) improved as a result of instruction, with pretest scores of 72% and posttest scores of 86%. Students' performance on items relating to the use of percent increase language changed in a positive direction on the posttest, with more students successfully interpreting the language of percent increase (see Figure 5.11 in section 5.3.2). The episode on the language of percent increase was rated as successful, and test scores appear to align with this perception. The most dramatic increase in performance on the posttest for Group 3 was in Section III (percent calculations and problem solving) with a posttest score of 67% compared to a pretest score of 19%. The proportional number line method was the predominant strategy used by students on the posttest. The episodes directly relating to developing this method as a key process were rated as successful, and test results appear to support the perceived successful nature of these episodes. Results on this section of the test are similar to those of Group 2 students.

Reflection upon delayed posttest scores and instruction indicate that the successful rating of implementation of the teaching sequence with Group 3 students is reflected in the delayed posttest results. The primary focus of instruction was to assist students become proficient in percent calculations and percent problem solving. Delayed posttest results indicate that students have retained this knowledge and continued to use the proportional number line method as a key process. The delayed posttest scores do indicate decay of knowledge relating to percent increase. Even though instruction on percent increase within this unit was rated as successful, and posttest scores (Section I - intuitive, principled-conceptual percent knowledge) show greater positive performance on items relating to percent increase, delayed posttest results indicate that the influence of instruction upon retention of this knowledge was not permanent.

5.3.12 Directions for Teaching Experiment 4

In Teaching Experiment 3, the use of O/N in the classroom upon a specific mathematical difficulty did not occur. In Teaching Experiments 1 and 2, full implementation of the metacognitive training episode did not occur, and thus O/N was not discussed with the students. In Teaching Experiment 3, the O/N strategy was presented to the students, but the tightness of the timeframe did not allow an O/N lesson to be trialled. For Group 4 students, percent, decimal and fraction conversions had already been covered in class, thus this episode within the unit of work was not necessary. It was decided to implement the unit of work with Group 4 students to compare results with Group 3 students, and to trial O/N in a whole class situation.

5.4 Teaching Experiment 4

5.4.1 Overview of report on Teaching Experiment 4

The results of Teaching Experiment 4 are reported in five sections. The pre-, post- and delayed posttest results are presented in section 5.4.2. In section 5.4.3, an overview of the teaching sequence comprising six teaching episodes is presented, with implementation of the teaching episodes described in section 5.4.4. Implementation of O/N is described in section 5.4.5. Reflection upon implementation of the teaching episodes is presented in section 5.4.6, with reflection upon pre-, post and delayed posttests and instruction presented in section 5.4.7.

5.4.2 Pre-, post-, and delayed posttest results

For Group 4 students, only two parts of the Percent Knowledge Test were utilised. Section II of the Percent Knowledge Test was deleted, due to the deletion of the episode on decimal, fraction and percent conversions. Tests scores in total and on Section I (intuitive, principled-conceptual knowledge) and Section III (percent calculations and problem solving) for Group 4 are presented in Table 5.11.

Table 5.11

Pre- and Posttest Means (%) for Group 4 Students on the Percent Knowledge Test in Total and for Section I and Section III

	Components of the Percent Knowledge Test			
	Total	*Section I*	*Section II*	*Section III*
Pretest	*44%*	*75%*	*N/A*	*12%*
Posttest	*84%*	*90%*	*N/A*	*77%*

From Table 5.11, it can be seen that Group 4 test scores overall and within each section increased, with a dramatic positive change in posttest scores for Section III (percent calculations and problem solving). From Table 5.11, it can be seen that, like Groups 1, 2 and 3 students, Group 4 students' intuitive and principled-conceptual percent knowledge, prior to instruction, was much greater than their percent calculation and problem solving skills. Graphical representation of the pre- and posttest scores are presented in Figure 5.15 highlighting the change in pre- and posttest scores in total and in the two test sections.

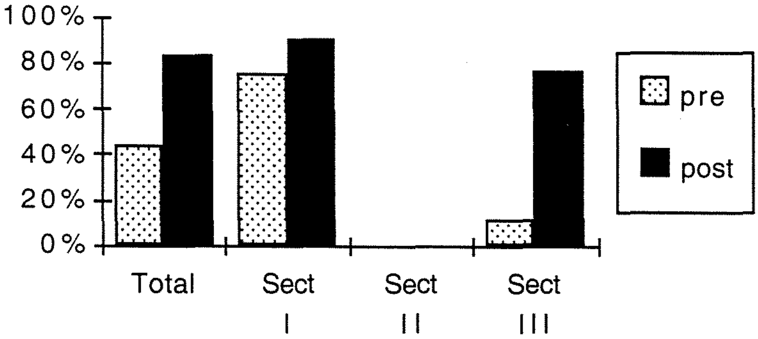


Figure 5.15 Graphical representation of Group 4 students' pre- and posttest means (%) in total, and in each test section.

As for Groups 1, 2, and 3, Group 4 students' test results were analysed for diagnostic purposes. Pre- and posttests were scored to identify specific items to which students responded incorrectly. Thus, for each section of the pre- and posttest, the number of students incorrectly responding to individual items, was tallied. Representation of the number of incorrect responses to each item on Section I (intuitive, principled-conceptual percent knowledge) and Section III (percent calculations and problem solving) are presented in Figures 5.16 and 5.17 respectively.

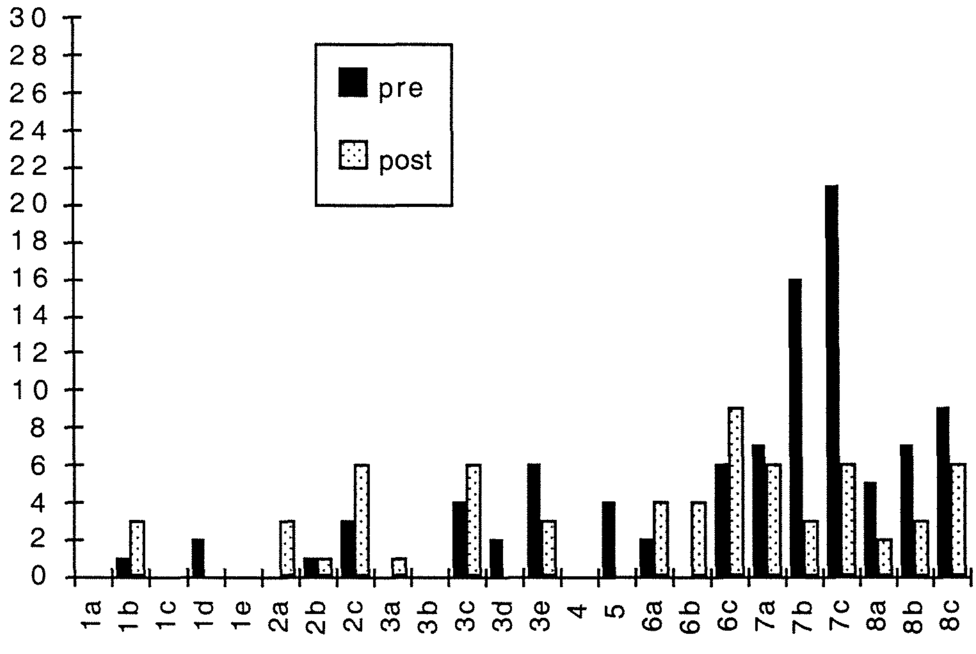


Figure 5.16. Group 4 students pretest (n=23) and posttest (n=29) *incorrect* scores on Section I (intuitive, principled-conceptual percent knowledge) of the Percent Knowledge Test. Graph indicates number of students *incorrectly* responding to particular items.

In Figure 5.16, it can be seen that, on the Pretest, Group 4 students experienced most difficulty with items 7a-c, which relate to interpretation of the multiplicative and additive language of percent increase. Posttest scores indicate positive change in performance after instruction, but also indicates that students continued to experience difficulty in interpreting the additive and multiplicative language of percent increase situations (items 6c and 7c).

Figure 5.17 indicates that, Section III of the pretest (percent calculations and problem solving) was not well-attempted by the students.

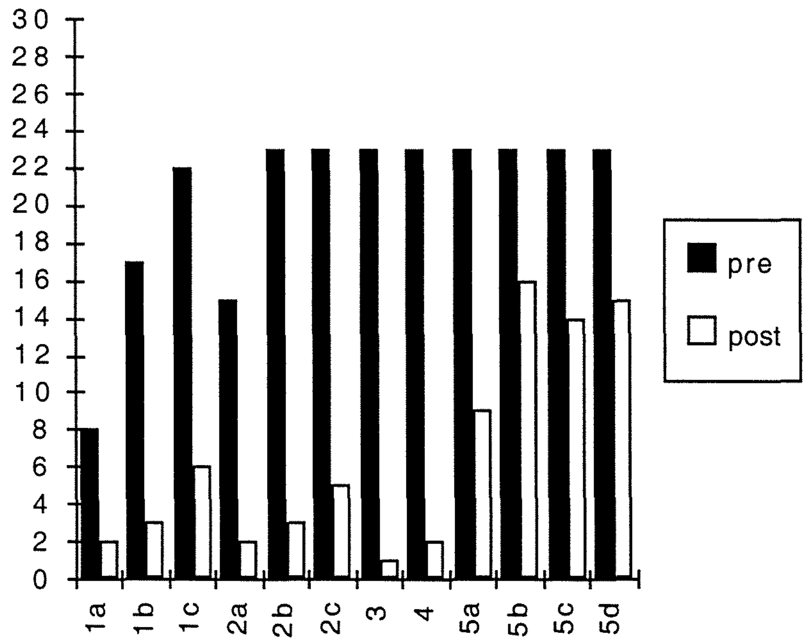


Figure 5.17. Group 4 students' pretest (n=23) and posttest (n=29) *incorrect* scores on Section III (percent calculations and problem solving) of the Percent Knowledge Test. Graph indicates number of students *incorrectly* responding to particular items.

From Figure 5.17, it can be seen that, on the pretest, some students could successfully perform calculations of percent equations and solve percent problems, but mainly those of Type I (items 1a and 2a). Type II and Type III percent equations (items 1b and 1c), and percent problems (items 2b-c) were not successfully completed by students. In Figure 5.17, it can also be seen that no student used diagrams to assist percent problem solving (item 3), and only a minority of students could successfully express percent problem solutions in words (item 4). Posttest results indicate greater facility in solving percent problems and performing percent equations after instruction. The majority of Group 4 students successfully performed percent calculations of all three types (items 1a-c), and also successfully solved percent problems (items 2a-c) on the posttest. All Group 4 students, except 1, could use diagrams to solve percent problems (item 3), and all students except 2 could express soltuions to percent problems in words (item 4). Facility in solving multistep word problems showed improvement (item 5a-d).

As with Groups 1, 2 and 3, there was a change of strategy employed by Group 4 students in solving percent calculations and problems. On the pretest, students utilised a variety of procedures for successfully solving Type I problems, including decimal multiplication, use of the calculator percent button, and fraction multiplication (e.g., 15% of 24 is? = $^{15}/_{100}$ x $^{21}/_{1}$ = 3.6). For Type II problems, one student wrote the solution in approximate terms (e.g., 27 is what percent of 350 = approximately 7.7); other answers showed no working and thus solution strategy could not be determined. For the majority of students, no attempt was made on any of the items, and thus performance was rated as unsuccessful. Other unsuccessful responses showed an incorrect fraction procedure for a Type II equation (27 is what percent of 350 = $^{27\%}/_{100}$ x $^{350}/_{1}$); an incorrect combination of procedures for a Type III problem (53% of 65 = 3% of 65 = 1.95; (65 x 2) + 1.95 = 131.95). On the posttest, successful performance on all three types of problems came from students using the proportional number line method or from directly using a proportion equation. Unsuccessful performance came from incorrect placement of numbers on the number line, or incorrect placement of numbers directly in the proportion equation. Of the two students who successfully completed all three types of percent calculations and problems on the pretest, they used a proportion equation on the posttest. Thus, their posttest results showed adoption of the number line method over their own other methods (which were difficult to determine as they responded to these items on the pretest with an answer only, and no working).

Delayed posttest scores for Group 4 are presented in Table 5.12 where student performance on every item of the test is displayed.

Table 5.12

Delayed Posttest Means (%) on Individual Test Items for Group 4 Students

Item	1	2	3a	3b	3c	4a	4b	4c	5a	5b
Score	100 %	96 %	100 %	92 %	92 %	100 %	96 %	92 %	46 %	29 %

From Table 5.12, it can be seen that test performance was high for all items, except items 5a and 5b. Students' proficiency in percent calculations (items 3a-c) and interpreting and solving percent problems (items 4a-c) remained high after a delay of 8 weeks after instruction. Solution strategy was the proportional number line method or proportion equation without the number line representation. Group 4 students performed well on items 1 and 2, which related to interpreting the language of percent increase and decrease (100% and 96% proficiency respectively), but not so well on applying that knowledge to problem situations (item 5a and 5b). Compared to Group 3 students, Group 4 performed better on the delayed posttest indicating greater retention of knowledge.

Students' scores on individual items of the delayed posttest are graphically displayed in Figure 5.18, and serve to highlight the particular items with which students experienced difficulty.

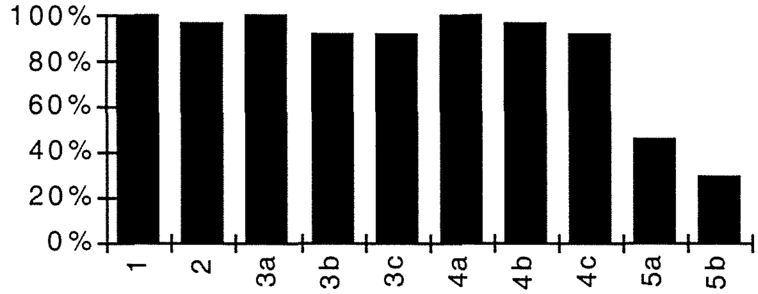


Figure 5.18. Graphical representation of Group 4 students' delayed posttest means (%) by item.

5.4.3 Planned teaching episodes

The planned teaching episodes for Group 4 are presented in Table 5.13. The table provides the number and title of each teaching episode, together with the number of actual lessons taken for implementation. The sequence of episodes for Group 4 was the same as for Groups 2 and 3, however, the percent, common and decimal fraction equivalence episode was deleted, moving the episode on percent increase and decrease to episode 5. A new episode on O/N was created, and implemented as the final episode in this unit.

A total of 12 class periods were spent with Group 1. Seven lessons were directly on the topic of percent, with a further 2 lessons used for metacognitive training and 1 lesson being used for an Old Way/New Way trial. Pre- and posttesting occupied 1 day prior to and immediately following implementation of the unit of work. The deletion of the episode on percent, common and decimal fraction equivalence resulted in deletion of Section II of the Percent Knowledge test (conversions and benchmarks), thus reducing the amount of time required for pre- and posttesting.

Table 5.13

Plan of Teaching Episodes for Group 4

Episode	Topic	Lessons
1	Metacognitive training	2
2	Concept of percent	1
3	Fraction equivalence and the Rule of Three	1
4	Interpreting and solving percent problems	3
5	The language of percent increase and decrease	2
6	Old Way/New Way trial on the multiplicative language of percent increase	1

5.4.4 Implementation of teaching episodes

The episodes implemented with Group 4 followed the same plan as corresponding episodes implemented with Group 3 students. In all episodes, student and classroom-teacher data supported the teacher-researcher's perceptions that all episodes were implemented successfully. Students' diary entries on various lessons within the unit reflected similar sentiments to those written by Group 3 students (see *Observation* sections 5.3.4 through to 5.3.9). As such, a report on implementation of this unit of work with Group 4 students will not be presented here. In episode 6, O/N was trialled as a whole group exercise. The focus of the O/N trial was upon the concept of percent increase. This episode is the only episode not trialled with any other groups. A detailed description of implementation of this episode is presented in the next section under the four subheadings of *plan*, *action*, *observation* and *reflection*.

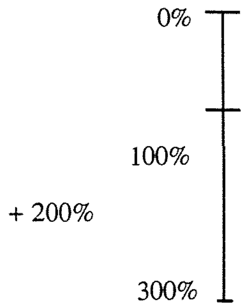
5.4.5 Episode 6: Multiplicative language of percent increase using Old Way/New Way

Plan

This lesson was planned to provide students with an opportunity to revise the topic of percent and also to trial a whole class "Old Way/New Way" on interpretation of percent increase problems. Anticipating that many students could still be confused in interpreting percent increase problems, this lesson was planned to begin with the multiple choice item being presented to students on an OHT. The students would be asked to select the appropriate response, writing either a, b, c, or d in their notebooks. The students would then be grouped according to those who selected the correct response and those who did not. Those who selected the appropriate response would be provided with a revision sheet of percent problems; the other students would engage in an O/N trial, of the following steps:

1. Hand out a worksheet upon which *Old Way* and *New Way* columns are written. Simultaneously, present this worksheet as an OHT.

2. Direct students to complete steps in O/N trial:
OLD WAY - *200% increase is the same as 200% of the original*
NEW WAY - draw diagram,



and write: *200% increase is the same as 300% of the original.*

3. Direct individual students to differentiate between the Old Way and the New Way a total of 5 times.
4. Provide students with 5 practice examples.
5. Students then continue on the extension/revision activities as the other students.

Action

The lesson proceeded as planned. The students quickly moved to their two groups so that the teacher could address the "O/N" students together as one group as the other students worked independently. During the O/N procedure, all students were attentive and followed the directions as given by the teacher-researcher, completing the appropriate section of the worksheet as directed. For the difference between the old way and the new way, various students were asked to describe the difference in their own words. The differences stated were that: in the new way, a diagram is first drawn to represent the situation; that the number line is extended past 100% for increase; the increase was added on to the diagram; the diagram can be read in relation to the amount of increase or in relation to the original; the old way links 300% increase to 3 times increase; the old way does not consider the original whole. At the generalisation phase, the students worked individually, and the teacher provided immediate feedback on individual student's progress. All students completed the generalisation problems appropriately, although in some instances, the teacher-researcher had to ask various students: "Is that your old way or your new way?" The students then would quickly correct themselves. The O/N students then worked on the worksheets being completed by the other students in the class. At the end of the lesson, all students handed in the sheets they had been working on.

Upon analysis, one student's worksheet showed unexpected evidence of proportional reasoning. This particular student had been working on representing data on a pie chart. The student had calculated data as percentages, and then calculated the segment of the circle needed to represent each percentage. On the worksheet, the student had written: $^{65}/_{100} = {^x}/_{360}$, thus indicating a proportion equation for converting percent to degrees in a circle.

Reflection

Prior to implementation of this episode, the teacher-researcher felt that presenting the O/N sequence would be "strange" to the students, and that there was a need to "sell" the procedure to students to encourage them to participate and pay attention. However, during this episode, all students remained attentive and cooperative. The worksheet was designed to minimise the amount of writing required of the students, and this appeared to help students to "see" the sequence of steps to follow in the O/N trial, and its structure appeared to assist them to remain on task. Having various students describe the difference between the *old way* and the *new way* appeared to be useful to the whole class as it provided the opportunity for all students, who shared a common misconception to hear how other people "mis"-interpreted percent increase situations; that is, to share their personal "old ways". Discriminating differences a total of 5 times was perceived by the teacher-researcher as rather tedious, however all students remained cooperative during this phase. During the generalisation stage, the students appeared to be challenged to pay attention to the differences between the old way and the new way, as they needed to interpret the problems on their own. The O/N trial at a whole class level, was perceived by the teacher-researcher as well-received by the students, although the teacher-researcher did feel slightly uncomfortable "telling" the students the new way 5 times. The whole class O/N trial appeared to be at odds with the teacher-researcher's usual teaching style, and hence the feelings of uncomfortableness. This reaction can be interpreted within the CMP framework, which states that old knowledge interferes with new knowledge. In this instance, old teaching styles were interfering with trialling of a new style. The teacher-researcher clearly was engaging in a personal O/N trial, to continue with the episode, consciously differentiating between an old style of teaching and a new style required when implementing an O/N lesson. The teacher-researcher could feel the tendency to lapse into a re-teaching mode; to present to students the activities to represent percent increase, rather than advancing through the differentiation of the old way and new way 5 times. The teacher-researcher was experiencing proactive inhibition (Lyndon, 1989), protecting a usual style of teaching from change. The students, however, participated well in this activity, and negative feelings towards the O/N trial were not exhibited by the students.

The O/N activity was trialled in a relatively experimental and tentative fashion.

Guidelines for such trials are virtually non-existent in mathematics, with prescriptive descriptions for conducting O/N trials available, thus far, only for spelling errors (Lyndon, 1989) and erroneous subtraction algorithms (Dole, 1993); which may both be regarded as focusing on remediation of procedural skills rather than conceptual knowledge. There is a strong call in the literature for teaching approaches that overcome students' inappropriate knowledge at a conceptual level, encapsulated in the words of Mansfield and Happs (1992, p. 453) who stated that, "Many students will come to the mathematics classroom with a number of misconceptions about topics to be taught. Research has clearly indicated that these misconceptions act as barriers to the acquisition of new conceptual knowledge." The interfering effects of prior knowledge were evident with the Group 4 students in this study. The students had been presented with instruction on the language of percent increase and decrease 2 days prior to presentation of the O/N activity. At the beginning of that particular percent increase episode, the majority of students failed to correctly interpret the language of percent increase situations, however, at the end of the two lessons, students were demonstrating greater facility with such language. The O/N episode took place 2 days after that episode. In 2 days, 12 students failed to correctly interpret percent increase situations, and thus participated in the O/N trial. The preconception or misconception of percent increase language was the focus for unlearning and hence the O/N trial. Unlearning of prior knowledge may be the key to laying a pathway for acquisition and retention of knowledge (Ausubel, 1968; 1985). The fact that students still had difficulty interpreting the language of percent increase so soon after instruction indicates the longevity of prior knowledge and the power of proactive inhibition, as emphasised by Lyndon (1989; 1995) and serves to underscore the words of Ausubel relating to unlearning. After conventional teaching, retention of knowledge on the language of percent increase for many Group 4 students was decaying. Experimenting with O/N for changing conceptual knowledge appears to be a worthwhile pursuit.

At the beginning of the O/N trial, students' errors in interpreting percent increase situations were identified rapidly. The students were informed that an O/N would be trialled and students participated appropriately, with no indication of negative feelings towards exposing errors in front of their peers. Due to the metacognitive training received by the students, the purpose of the O/N trial could be linked directly to the function of memory and the controlling of forgetting. The trial thus proceeded with the teacher-researcher mediating between the students' own ways and the new way, using appropriate language; an approach which links to the cognitive apprenticeship model (Reid & Stone, 1991). The utilisation of errors in the O/N strategy as the beginning point to affect knowledge change is similar to procedures suggested by others, such as Borassi (1994) and Rauff (1994). Presenting students with metacognitive training prior to use of O/N, appeared to enable the O/N trial to proceed in a positive and non-threatening way, as suggested by students' cooperative behaviour throughout this trial. Trialling of O/N therefore has presented some interesting points for discussion.

Also within this lesson, unexpected evidence of the development of proportional reasoning for one student was obtained. This student's response supports the value of the Rule of Three procedure being used to promote proportional reasoning. The spontaneous link made by this student suggests that instruction which presents a procedure for problem solving in a relatively "rote" fashion may not be perceived as a meaningless procedure for all students. Such a finding conflicts with the strong words of Cramer et al. (1992), who stated that "the cross product algorithm is efficient, [yet] it has little meaning. In fact it is impossible to explain why one would want to find the product of contrasting elements from two different rate pairs...The cross-product rule has no physical referent and therefore lacks meaning for students and for the rest of us as well" (p. 170). In light of the student's response in this study, it appears that the cross-product rule can be meaningful to students, and thus should possibly be a consideration for promoting meaning for all students in future proportion instruction.

5.4.6 Reflection upon implementation of teaching episodes

Implementation of this unit of work was perceived as proceeding in a smooth manner, with all episodes rated as successful. The unit of work, as a whole, appeared sequential, and the deletion of the percent, decimal and common fraction episode appeared to assist in contributing to the sequential flow of the unit. The metacognitive training was implemented effortlessly, and was perceived as enjoyable for the students, the classroom-teacher and the teacher-researcher. The O/N trial was readily engaged in by the students. O/N may have proved difficult to trial if the students had not been involved in the metacognitive training prior to this trial. The justification for the trial had been provided through these introductory sessions. The O/N trial itself was perceived by the teacher-researcher to be "uncomfortable" to implement, as asking students to do "old ways", then "new ways" in a sequence 5 times, was an unusual request. The students, however, did not seem to react in the same way. The O/N trial was deemed as successful as all students were readily differentiating between "old ways" and "new ways" during the O/N generalisation phase.

Throughout implementation of the unit of work, the students remained on-task and attentive. Their diary entries revealed a level of confidence in their ability to perform percent calculations. Also, informal interviews with students suggested that they found the method for solving percent equations easy and useful. At various times during the teaching episodes, several students showed the teacher-researcher their own methods for solving percent equations, but all stated that they wanted to use the method shown in class, because, as one student said, "I don't have to try to guess whether to multiply or divide, I just know where the numbers go."

The classroom teacher stated that she felt the whole unit, including the metacognitive training, had been extremely successful, and that she would like to continue experimenting with the procedure with further classes. Of the whole class O/N trial, the teacher said she would be interested in the effect such a procedure had on the students' performance of such problems, but was not confident to trial the method herself. The teacher felt that the students' percent problem solving performance, as well as their understanding of percent, had greatly improved since implementation of the unit.

5.4.7 Reflection upon pre-, post- and delayed posttests and instruction

Implementation of this unit of work was rated as successful, and comparison of pre- and posttest results indicates positive improved performance as a result of instruction. The overall prestest score for Group 4 was 44%, and the posttest score was 84%, which reflects the teacher-researcher's perception of the successful nature of this unit of work. Students' pretest score for Section I (intuitive, principled-conceptual knowledge) was 75%, indicating a high level of real-world percent knowledge prior to instruction. After instruction, the posttest score for this section of the test was 90%, which reflects the successful ratings of all episodes in this teaching sequence, particularly the episodes on the language of percent increase. Students' pretest score for Section III (percent calculations and problem solving) was 12%, indicating minimal proficiency prior to instruction. After instruction, the posttest score on this section of the test was 77%. Such a large positive change indicates the successful nature of the unit of work in assisting students to build computational knowledge of percent. The posttest results aligns with the successful rating of the episodes on percent calculations and interpreting percent problem situations.

Reflection on delayed posttest scores and instruction indicate that, similar to Group 3 students, the successful rating of implementation of the teaching sequence is reflected in the delayed posttest results. After a delayed period, students' proficiency with percent calculations and solving percent problems was high, and development of this knowledge was the primary focus of the teaching episodes. Group 4 students' proficiency with interpreting and solving problems relating to percent increase situations was also relatively high on the delayed posttest. When this result is compared to Group 3 students, difference in instruction between the 2 groups is the inclusion of the O/N episode on the language of percent increase with Group 4 students. Delayed posttest results appear to indicate that the O/N episode influenced Group 4 students' greater positive performance with percent increase language, and limiting the decay of this knowledge. The inclusion of the O/N episode appears to have lasting effects on conceptual development of percent increase, and therefore contributed to the successful rating of implementation of the unit, and students' positive performance on the posttest and delayed posttest.

5.5 Results across Teaching Experiments 1, 2, 3 and 4

5.5.1 Overview

Results of the four teaching experiments collectively, are discussed in this section. In section 5.5.2, the success rating of particular teaching episodes is overviewed, and highlights the value of reflection upon implementation to create a successful teaching program. A summary of the structure of the teaching program is presented in section 5.5.3. In section 5.5.4, pre- and posttest results across the four teaching experiments are compared, and discussed in terms of instruction. This section highlights the influence of instruction upon knowledge change and growth, and successful task performance. In section 5.5.5, implementation of instruction in real classrooms is addressed. A summary of key points from this section is presented in section 5.5.6.

5.5.2 Implementation of the teaching episodes across the four groups

A summary of the success rating upon implementation of the teaching episodes across the four groups is presented in Table 5.14.

Table 5.14

Summary of Success Rating of Each Teaching Episode for Each of the Four Groups

	Groups			
	Group 1	Group 2	Group 3	Group 4
Metacognitive training	U	U-S	S	S
Concept of percent	U	U	S	S
Fraction equivalence and the Rule of Three	S	S	S	S
Interpreting and solving percent problems	U-S	S	S	S
Percent, common and decimal fraction equivalence	U	U-S	U-S	N/A
The language of percent increase and decrease	U-S	S	S	S

S = Successful
U = Unsuccessful

Tracking each teaching episode across the four groups shows that episodes which were rated as unsuccessful upon first implementation with Group 1, were finally rated as successful with Group 4, or earlier with Groups 2 and 3. Throughout the study, it was seen that several teaching episodes underwent modification as a result of reflection upon implementation. In the discussion to follow, modifications to each teaching episode are described across the four teaching experiments, revealing the influence of planning and reflection upon successful implementation.

The first episode within each teaching program was the metacognitive training episode. For the metacognitive training component of the teaching program, the CMP as suggested by Lyndon (1995) was adapted into a teaching episode comprising of two lessons. The two lesson sequence enabled the two main topics within the CMP (remembering and forgetting) to be addressed separately. Presenting these two topics to the students was to exemplify the need for two categories of learning strategies: strategies to assist remembering, and strategies to overcome accelerated forgetting. The two strategies incorporated into the CMP for these purposes were the Look-Say-Cover-Write-Check (LSCWCh) strategy and the Old Way/New Way (O/N) strategy, for remembering and accelerated forgetting respectively. The metacognitive training episode in this study was thus designed around providing students with a rationale for the need for such particular learning strategies, through discussion of issues relating to human remembering and forgetting.

In this study, the metacognitive training episode was rated as unsuccessful for Group 1, unsuccessful-successful for Group 2, and successful for Groups 3 and 4. With Group 1, the entire metacognitive training program was not fully implemented as planned. As described in section 5.1.4, contributing factors included disruptions to instruction, students' lack of attention, and students' resistance to a non-mathematical content lesson during timetabled mathematics classes. Students' lack of attention was also attributed in part to poor lesson design and lack of activities to assist students understand the psychological concepts associated with human memory and forgetting. In the episode, students experienced difficulty differentiating between recall and recognition memory, and also the function of proactive inhibition (PI) in protecting prior knowledge. For Group 2 students, instruction was modified to exemplify recognition and recall memory. The LSCWCh strategy was well-received by students after these activities. The second lesson in the episode, dealing with forgetting, was not fully implemented with Group 2. The contributing factor appeared primarily to be the necessary room change as required for another school activity, and also shortened lesson time as a result. For Groups 3 and 4, the episode was fully, and successfully, implemented following the original plan for Group 2.

As a result of trialing CMP in this study, it was seen that CMP incorporated into a metacognitive training episode was most successfully implemented when instruction was carefully planned and sequenced to incorporate exemplary activities. Although the CMP states that learning is dependent upon paying attention, and that the individual is in control of the amount of attention he/she chooses to pay, careful lesson design and planning of instruction by the teacher assisted students to maintain attention. The metacognitive training episode in this study followed the planned sequence described in Chapter 4 section 4.5 (metacognitive training), with additional activities for promoting understanding of the concepts presented.

The second episode in the teaching program related to the concept of percent. The concept of percent episode, presented to Group 1 students, commenced with students being asked to explore percent uses presented in newspapers. Upon reflection, this episode did not appear to promote students' understanding of percent in the real world, as the students did not analyse the ways percent was presented in the media. For Group 2 students, an entirely different episode was presented where students were asked to explore a real-world collection in a variety of mathematical ways (i.e., as fractions, decimals, percents, graphically, ratio, and so on). This episode also did not appear to promote students' real-world percent knowledge, although students actively engaged in the activity, as Group 1 students did in their corresponding activity. For Groups 3 and 4 students, the episode used with Group 1 students was replanned. Various uses of percent in newspapers were presented to students, and in groups, they were required to discuss questions relating to real-world transactions, such as bank interest, profit, loss, discount and statistical uses of percent. The replanned episode with these groups appeared to be useful in promoting discussion of percent amongst students, and in providing the teacher-researcher opportunity to individually ask students questions relating to percent.

The third episode in the teaching program was on fraction equivalence and the Rule of Three. The aim of the episode was to introduce students to a procedure for finding equivalent fractions. The episode was implemented without any linkage made to the topic of percent. The episode began with demonstrations of the traditional method for changing a fraction into an equivalent form. The students' attention was then drawn to the placement of the numbers in the two equivalent fractions, where the initial fraction is given and the required denominator of the equivalent fraction form. The Rule of Three procedure was demonstrated, and stated in the words of Robinson (1981), "Multiply the two numbers across from one another and divide by the other number" (p. 6). Practice of the Rule of Three was then provided, with students given sets of 10 exercises with the unknown in various positions within the equivalent fraction pairs.

In the study, this episode was rated as successful upon initial implementation with Group 1 students, and the structure of the episode remained unchanged in all four teaching experiments. Of all episodes in the study, this particular episode was the only one which was rated as successful from first implementation. Reflections upon implementation of this episode with all groups suggested that the successful nature of the lesson was due to several factors, including that the episode enabled students to experience success and thus they felt confident about their mathematical ability; the lesson was structured to provide students with immediate feedback on performance and thus provided students the opportunity to develop a skill to automaticity, thus reducing cognitive load; the episode contributed to students' sense of normality and expectations in that they "did maths" during a mathematics lesson. Instruction in the episode could be described as clear, clean, and direct. The lesson was not cluttered with superfluous description and was without mathematical analysis of the procedure. As a means to an end, this episode served its purpose in providing students with a procedure to solve proportion equations.

Interpreting percent problems was the focus of the fourth episode in the teaching program. Instruction in this episode was modified after implementation with Group 1. Initial instruction with Group 1 began with students being presented with a Type I percent application problem in which the elements: part, whole, percent, were identified. The vertical number line was then drawn, and positioning of the elements on the number line demonstrated. A proportion equation as an equivalent fraction was constructed corresponding to placement of elements on the number line representation. The Rule of Three was suggested to complete the calculation and thus solve the problem. The students were then provided with a symbolic Type I percent problem (e.g., 16% of 58 = Δ) and were asked to construct a real world problem to match the given equation.

Upon implementation of this episode, it was apparent that the sequence of instruction required modification. Contributing factors were that students actively resisted paying attention to a new procedure for solving familiar Type I percent problems, and that too much information and too little time for consolidation of steps involved in the new procedure, was given. During the episode, modifications were made, where only unfamiliar Type II problems were exhibited to draw students' attention away from the familiar Type I problems. The steps in the proportion number line method were demonstrated, but students were not required to write real-world situations to match the percent exercises. Upon reflection, this episode was recreated for Group 1 into an episode on solving percent problems. To assist students internalise the sequence of steps in the proportional number line method, a worksheet (Appendix D) was created to serve as a scaffold. The worksheet appeared to serve this purpose well for Group 1 students, who solidly completed the problems presented. A second worksheet (Appendix E) was designed to enable students to practise their skill in the proportional number line method without providing the scaffolding assistance.

The two episodes on interpreting percent problems and solving percent problems implemented with Group 1, were condensed into one episode on interpreting and solving percent problems for Groups 2, 3 and 4. To assist students identify elements of percent situations as comprising a part, a whole and a percent, a real world situation was presented. A series of Type II and III problem situations was shown to students, and students were provided with practice in identifying the three elements given in each situation. Type I problems were also shown. In the next phase of the episode, a Type II problem was presented, and the students identified the two given elements and the unknown element. For introduction to the number line representation, students were asked to draw a number line indicating 0% and 100% positions. Students were then asked to rotate their number line from its traditional horizontal position to a vertical position where 100% was at the bottom of the number line. Rotation of the number line from a horizontal to a vertical position is depicted in Figure 5.19.

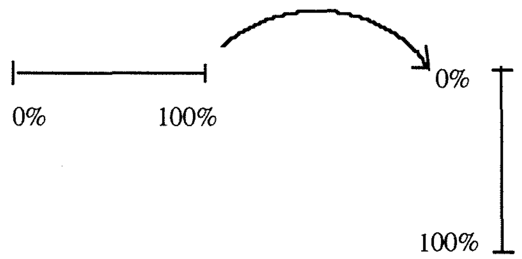


Figure 5.19. Horizontal to vertical rotation of the 0-100% number line.

From Figure 5.19, it can be seen that through rotation of the number line, the 0-100% scale falls naturally on the left side of the vertical number line. Students' attention was then drawn to the dual-scale nature of the number line with the percent scale falling on the left-hand side, and the quantity on the right-hand side. Instruction then focussed on placing the information given in the problem onto the number line, constructing the proportion equation and solving the problem using the Rule of Three. To assist students consolidate the steps in the proportion number line method, the worksheet (Appendix D) was presented, followed by the further practice but less structured second worksheet (Appendix E). For Group 2, 3 and 4 students, this episode was rated as successful.

The fifth episode in the teaching program, on percent, common and decimal fraction equivalence was modified slightly across the three groups. but mainly in instruction given in the first part of the episode on construction of the oversized number line. The episode was perceived as useful in providing students with a visual model of fraction, percent and decimal equivalent forms, and in providing students with an opportunity to practise their percent conversions and benchmarking skills.

The episode was perceived as not so useful for assisting students, with limited knowledge of, and skill in, equivalence and conversions, to develop such knowledge and skill. The episode was perceived as "out-of-place" in the context of the teaching program. It could be argued that conversions link more closely to the notion of percent as it relates to fractions and decimals, and traditionally, instruction in such was for the purpose of assisting students perform percent calculations (Parker & Leinhardt, 1995). As percent calculations were taught via the proportional number line method, percent to decimal and fraction conversions were unnecessary in this teaching program. As a peripheral component of the teaching program, instruction was not modified significantly to focus on promoting notions of equivalence. The episode was included for completeness, and to fulfil requirements of the syllabus. In light of reflection upon implementation, it may have been better to omit this episode, and slot it into another teaching program for developing equivalence concepts.

The last episode in the teaching program, on the language of percent increase and decrease, was created as an extension to the fourth episode on interpreting and solving percent problems. Upon implementation with Group 1, this episode was modified for Groups 2, 3 and 4. With Group 1, the first phase of this episode began with diagnosis of students' misinterpretation of percent increase situations. This appeared to be a useful introduction to the lesson, as it immediately enabled the number of students who misinterpreted the percent increase situation to be identified and also served to draw students' attention to the focus of the lesson. To help students visualise a percent increase as a change to the original whole, a number of situations were presented, relating to real-world increase situations of weight and height increase. For Group 1 students, such situations were demonstrated using various props, and representation of each situation via cardboard sections (see section 5.1.9). As the episode progressed, such situations were then modelled using a collection of objects (jellybeans) with the situation being represented on the proportional number line, extended beyond 100%. Percent discount situations were modelled the same way. To provide students with practice in identifying the multiplicative and additive/subtractive nature of percent increase and decrease situations, a worksheet was created (Appendix F) to encourage students to discuss the language of percent increase and decrease. For Group 1, this episode was rated as unsuccessful-successful. After implementation of the episode with Group 1, the episode was replanned in a more streamlined manner for Groups 2, 3 and 4, and a successful rating of this episode was achieved with these groups.

In this study, the teaching program was implemented across the four groups following the action-research spiral (Kemmis & McTaggart, 1990). Through the actions of planning, acting, observing, reflecting and replanning, the episodes within the teaching program were modified and finally implemented in a totally successful manner. A successful program of instruction was thus created.

5.5.3 Summary of the teaching program

Overall, the teaching program can be seen to consist of both metacognitive training, and instruction on the mathematical topic of percent. The primary focus of the instruction on percent was on developing students' percent problem solving skills through application of the proportional number line method. Secondary to this was instruction for developing percent conversions and benchmarking skills, and the concept of percent as it relates to the real world. Thus, the teaching program consisted of three instructional components: (i) metacognitive training; (ii) essential percent instruction for percent problem solving; and (iii) ancillary instruction for percent conceptual and associated skills development. Essential percent instruction for percent problem solving was regarded as instruction pertaining to the proportional number line method and included the teaching episodes of: (i) fraction equivalence and the Rule of Three; (ii) interpreting and solving percent problems; and (iii) the language of percent increase and decrease. Two other episodes specific to the topic of percent which were seen as ancillary to the main instruction in the proportional number line method were: (i) the concept of percent; and (ii) percent, common and decimal fraction equivalence. These episodes were included in the teaching program for the purposes of building students' real-world knowledge of percent use in our culture, and in promoting understanding of percent in a mathematical sense in relation to its equivalence to common and decimal fractions. The metacognitive training component was seen as a blanket surrounding the program of percent, but also separate to the teaching program, with elements of the CMP feeding into and supporting the teaching sequence at various points. The three components and associated teaching episodes of the teaching program used in this study are represented in Figure 5.20.

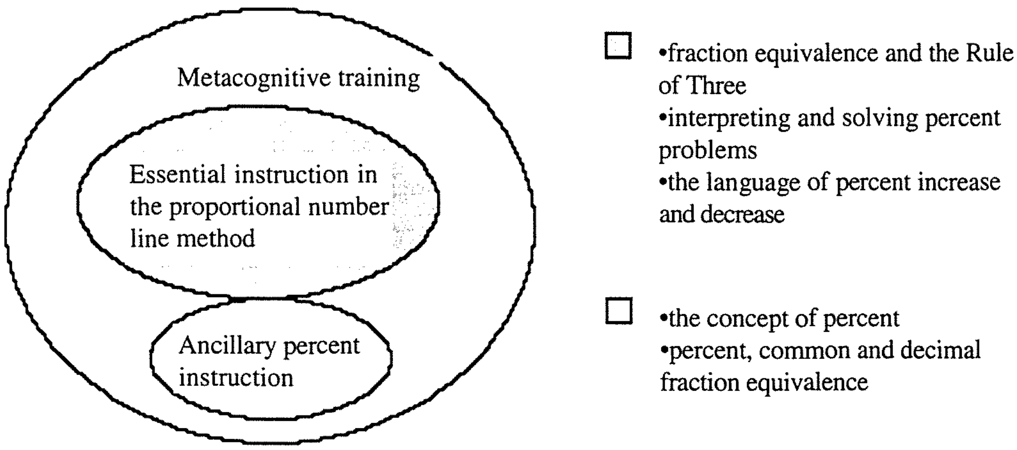


Figure 5.20. Diagrammatic representation of the components of the teaching program for the study.

5.5.4 Pre- and posttest results and instruction across the four groups

Pre- and posttest scores on the Percent Knowledge Test for the four groups, were combined to compare test performance for each group prior to instruction, and to analyse performance after instruction. These combined results are presented in Table 5.15. These results are graphically presented in Figure 5.21 to highlight change across each group.

Table 5.15

Pre- and Posttest Results by Group Across the Four Teaching Experiments

		Components of the Percent Knowledge Test			
Group	Test	Section I	Section II	Section III	Total
Group 1	pretest	60%	60%	10%	43%
	posttest	69%	54%	33%	52%
Group 2	pretest	64%	41%	7%	37%
	posttest	79%	70%	48%	66%
Group 3	pretest	72%	42%	19%	44%
	posttest	86%	75%	67%	76%
Group 4	pretest	75%	N/A	12%	44%
	posttest	90%	N/A	77%	84%

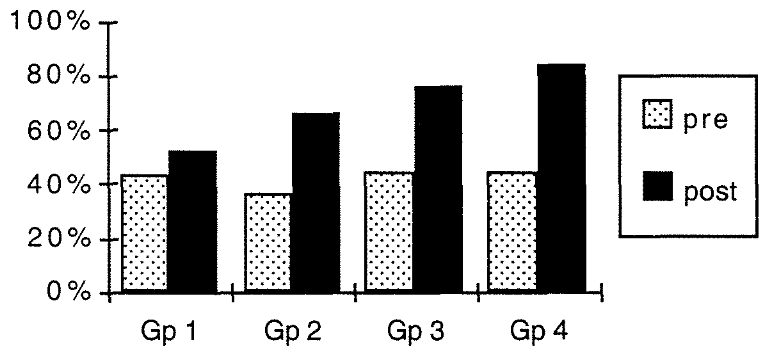


Figure 5.21. Pre- and posttest means (%) in total for Groups 1 - 4 students on the Percent Knowledge Test.

From Figure 5.21, it can be seen that the students within each of the four groups in the study, performed similarly on the Percent Knowledge pretest. It appears, therefore, that prior to instruction, all four groups of students began from a similar knowledge base. Looking at the posttest scores overall, in Figure 5.21, the change in posttest results between the four groups shows a positive trend in scores, with Group 4 scoring a higher posttest score overall to the other three groups.

Pre- and posttest scores for each of the three sections (intuitive, principled-conceptual percent knowledge; conversions and benchmarks; and percent calculations and problem solving) of the Percent Knowledge Test for each group are presented in Figures 5.22, 5.23 and 5.24 respectively.

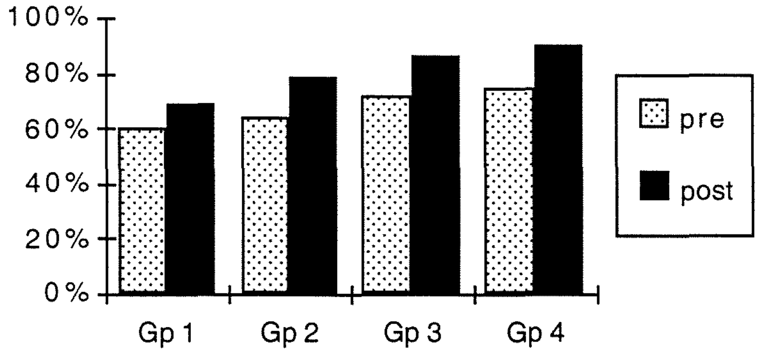


Figure 5.22. Pre- and posttest means (%) on Section I (intuitive, principled-conceptual percent knowledge) of the Percent Knowledge Test for each of the 4 groups.

From Figure 5.22, it can be seen that, for each of the four groups, pretest performance was similar and also quite satisfactory (60% or better for each group). Thus, all four groups' intuitive and principled-conceptual percent knowledge was similar prior to instruction. After instruction, posttest results show an increased test performance for all groups, with greatest increased performances from Group 1 (60% - 69%) and Group 4 (75% - 90%). Diagnostic analysis of specific items on this section of the test for each group indicated that students in each of the groups experienced difficulty on similar test items (compare Figures 5.2, 5.7, 5.11 and 5.16). These items related to understanding of bank interest, the additive and multiplicative language of percent increase and decrease, and posing of real world percent problems. Posttest results indicate that all four groups still experienced difficulty with these intuitive, principled-conceptual knowledge items, but these items were handled considerably better by Group 4 students than all other students (see Figure 5.16). One episode in the teaching sequence specifically focussed on developing students' understanding of the multiplicative and additive/subtractive language of percent increase and decrease, and implementation of this episode was rated as moderately successful with Group 1, and successful with Groups 2, 3 and 4. Groups 1, 2 and 3 did not perform significantly better on such items in this section of the posttest; Group 4 students, however, did. The difference in instruction for Group 4 between Groups 1, 2 and 3 was that an Old way/New Way trial was implemented with Group 4 on the language of percent increase and decrease. Group 4 student' better performance on these items within the intuitive, principled-conceptual section of the posttest appear to indicate that O/N contributed to greater understanding of percent increase language.

Figure 5. 23 shows pre- and posttest scores for each Group on Section II (conversions and benchmarks).

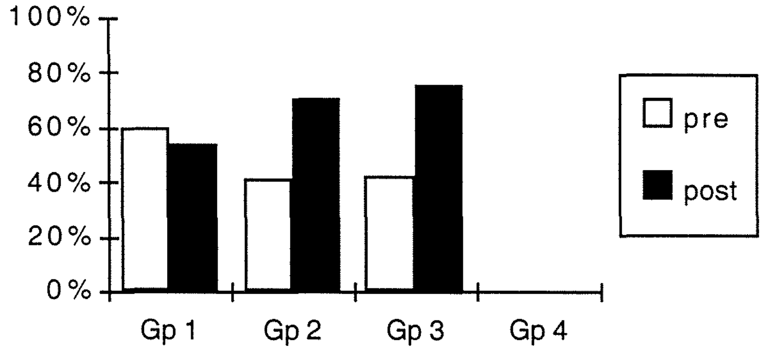


Figure 5.23 Pre- and posttest means (%) on Section II (conversions and benchmarks) of the Percent Knowledge Test for each of the 4 groups.

From Figure 5.23 it can be seen that pretest scores shows similar performance by Groups 2 and 3 (41% and 42% respectively) prior to instruction compared with Group 1 (60%). After instruction, posttest scores for Group 2 and 3 show considerable and similar positive change (70% and 75% respectively) and a minimal but negative change for Group 1 (54%). Looking at success ratings of the episode focussing on this section of the test, there is an unsuccessful rating for implementation with Group 1, and a middling rating for implementation with Groups 2 and 3. Success ratings of this episode appear to reflect test scores. The episode was rated as of little value to Group 1 students, and posttest results show little change. For Groups 2 and 3, the episode was rated as useful for students who already possessed skills and knowledge of fraction equivalence, but not for students who did not possess such skill and knowledge. Test results show positive change but for only approximately two-thirds of the class. The teacher-researcher's reflections on implementation of this particular episode with each group indicate dissatisfaction with the episode. The teacher-researcher continually stated that the episode interrupted the flow of the teaching sequence, and appeared to be at odds with the proportional nature of percent being presented in all other episodes. Such reflections suggest that this episode may need contextualising to cater to the needs of all learners, possibly as an episode in a separate teaching program.

Figure 5.24 shows pre- and posttest results for each group on Section III (percent calculations and problem solving) of the Percent Knowledge Test.

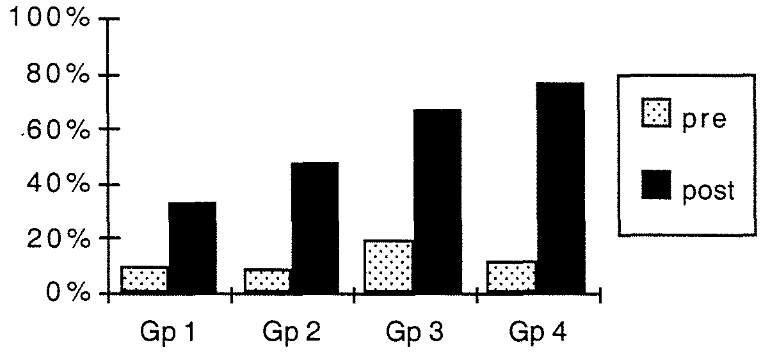


Figure 5.24. Pre- and posttest means (%) on Section III (percent calculations and problem solving) of the Percent Knowledge Test for each of the 4 groups.

From Figure 5.24, it can be seen that, prior to instruction, all four groups performed similarly, and unsatisfactorily, on this section of the test (all scores less than 20%). It appears that, prior to instruction students in all four groups had little success in performing percent calculations and solving percent problems. After instruction, students' performance across all four groups showed great improvement. The greatest improvement was for Group 4, with a pretest score of 12% and a posttest score of 77%. Posttest scores for Groups 1 and 2 did not reach 50% (33% and 48% respectively) but the score for Group 3 did (67%). Success ratings of implementation of teaching episodes relating to this section of the test compare with test scores. With Group 1, the episode on interpreting and solving percent problems was rated as only moderately successful. Group 1 results show moderate improvement as compared to pretest scores (10% to 33%). For Groups 2, 3 and 4, episodes relating to interpreting and solving percent problems were rated as successfully implemented; posttest results show greater performance with such items on the posttest for each of these groups. Successful implementation of the episode, therefore, appears to have contributed to greater test performance.

5.5.5 Real-world school context

Implementation of the teaching episodes within authentic classrooms, factors beyond control of the teacher-researcher were seen to impinge upon the success rating of particular episodes. Three main factors were (i) school timetable and time constraints; (ii) distractions outside the classroom; and (iii) the curriculum.

Throughout the teaching experiment, implementation of the teaching episodes was constrained by the school timetable and lesson time allocation. The teaching program was designed to fit within allocated time for teaching percent in the school year. For all groups in the teaching experiment, this was two weeks, or a total of ten class periods. For Group 1, instruction directly on percent spanned the allocated ten lessons; for Groups 2 and 3, instruction spanned nine lessons, and for Group 4, instruction spanned seven lessons (the episode on percent, common and decimal fraction equivalence was not included with Group 4 - see section 5.3.12). The teaching program thus conformed to timetable requirements. However, the timetable constraints did impinge upon delivery of the program in that little time could be afforded to enable students to fully consolidate their new knowledge and skills, and for the teacher-researcher to carry out informal discussions with students through group problem solving activities. The need to advance through the program was a frequent reason for moving on to the next teaching episode without proper implementation of the prior episode.

Related to the timetable constraints was allocated class lesson time. For each group within the study, the allotted lesson time was 40 minutes. Given that students took approximately 5 minutes to settle into the mathematics classroom, instructional time was thus effectively 35 minutes. Lesson planning therefore had to be clear and precise to make valuable use of every minute in the lesson period. Through reflection upon events in every teaching episode, instruction was seen to be modified throughout the study to ensure that instruction was sequential, clear and precise. As the study progressed, better lesson planning contributed to successful delivery of instruction within the allotted lesson time.

As the study was conducted in authentic classrooms, typical disruptions occurred which influenced implementation of the teaching program. Such disruptions included room changes, students' lateness to class, interactions amongst students prior to mathematics classes. Such disruptions were more frequent and more distracting with Groups 1 and 2 students, than with Groups 3 and 4. On several occasions with Group 1 and 2, lessons failed due to such disruptions as a student's late arrival to class (Episode 1, section 5.1.4), switching of room from a mathematics classroom to a science lab (Episode 1, section 5.2.4) a lunchtime "incident" (Episode 4, section 5.2.7).

Another factor constraining the development of the teaching program was the constraint of the curriculum. For the mathematical topic of percent, the school mathematics syllabus states that developing students' skill in percent, common and decimal fraction conversions is required in the teaching program. For authenticity, such an episode was included in the teaching program for this study, but was seen to "interfere" to an extent, with the sequential flow of the program. Inclusion of such an episode was regarded as inhibiting engagement of students in more real-world percent problem solving investigations where their new knowledge of percent applications could be utilised.

5.5.6 Summary of key points

Results across the four teaching experiments were discussed in this section. The teaching program as it was moulded and refined through implementation across the four groups was summarised. Pre- and posttest results across the four groups were compared and results were analysed in terms of the success ratings of episodes within the program. Through modification of the teaching program across the four teaching experiments, a successful program was developed which appeared to influence the positive change in students' percent knowledge and problem solving performance. Factors which impinged upon implementation of the teaching episodes within the study were also outlined thus highlighting the factors which influence classroom instruction and research.

CHAPTER SUMMARY

In this chapter, results of the study were presented. Results of each teaching experiment, comprising pre- and posttest scores, and a detailed description of implementation of teaching episodes, were given. This chapter concluded with comparison of results across the four teaching experiments. In this chapter, results highlighted the influence of instruction upon students' percent knowledge change and growth, and the real classroom factors which influence implementation of instruction. Discussion of issues emerging from the study is the focus of the next chapter.

CHAPTER 6
DISCUSSION

CHAPTER OVERVIEW

From the results of the study, a multitude of issues pertaining to teaching and learning percent in particular, and teaching and learning mathematics in general, have arisen. In this chapter, four main issues have been selected for discussion: percent problem solving; percent knowledge and instruction; metacognition; and diagnostic-prescriptive instruction. These issues are discussed in sections 6.1, 6.2, 6.3, and 6.4 respectively.

6.1 Percent problem solving

6.1.1 Overview

The focus of this section is on issues associated with the proportional number line method in relation to the development of percent problem solving proficiency. In section 6.1.2, prior research in the field of percent instruction and its relation to the advent of the proportional number line method is discussed. The effectiveness and efficiency of the proportional number line method for percent problem solving is described in sections 6.1.3 and 6.1.4 respectively. A model of percent problem solving incorporating the proportional number line method is presented in section 6.1.5. A summary of key points is presented in section 6.1.6.

6.1.2 The proportional number line method and prior research

The proportional number line method as a procedure for percent problem solving consists of four steps:

1. Interpretation of percent problems and exercises as comprising three elements: *part*, *whole*, *percent*.
2. Placement of part, whole, percent amounts on the dual-scale number line, stressing the relationship of the *whole* to 100%, and *part* to *percent*.
3. Construction of the proportion equation.
4. Solving of the proportion equation using the Rule of Three.

Using this method, successful completion of percent problems relies on application of the Rule of Three procedure. Instruction in the Rule of Three must precede instruction in the proportion number line method. Results of the study have shown that the proportional number line method enabled students to operate competently and confidently on percent problems, and to successfully carry out percent calculations. This study has indicated that the proportional number line method is a valuable method for percent problem solving.

Previous research into instruction in percent has provided no such methods. Of studies into the teaching of procedures for percent problem solving, Parker and Leinhardt (1995) concluded that, to date, no single best method for instruction was apparent. However, previous research has provided suggestions for important elements of effective percent instructional programs. Three particular studies (Mason, 1975; Maxim, 1982; Tredway & Hollister, 1963), reviewed by Parker and Leinhardt, provided useful insights into percent instruction, and directions which research into this field should take. Tredway and Hollister's (1963) study showed how holistic instruction of the three types of percent problems and the use of visual representations were a positive inclusion for instruction over drill and practice methods. Mason's (1975) study suggested how the multiplicative and additive relationships in percent situations could be emphasised, and proposed a method where conversions between percent and decimals could be avoided. Maxim's (1982) study proposed the key areas of difficulty for students in solving percent problems, as poor fraction knowledge, difficulty of the percent symbol, and interpreting word problems, and suggested that these need to be taken into consideration when planning percent instruction.

In this study, findings and suggestions from the above studies were taken on-board. This study has provided a clear sequence for instruction in percent which can be seen to be a study culminating from prior research. Like the Tredway and Hollister (1963) study, this study utilised a powerful visual representation, but it was not the 10x10 grid adopted by those researchers. Also, like the Tredway and Hollister study, this study suggests the value of presenting all three types of percent problems to students intermingled, rather than in a lock-step fashion. Like the Mason (1975) study, this study has shown how the additive and multiplicative relationships of percent increase situations can be described, but in a much clearer and simpler fashion than the method shown by Mason. This study has also shown how to overcome students' poor fraction knowledge, percent problem interpretation, and difficulty with the percent symbol, which hindered students' performance in the study by Maxim (1985).

Results of this study can be seen to link and build on the results of the study conducted by Allinger (1985) (see section 2.3.2). Allinger's instructional sequence for percent problems included construction of proportion equations from percent situations, emphasis of the notion that percent means hundredths and that the whole relates to hundredths, and the use of the cross-multiply technique via a calculator to solve proportion equations. Allinger's instructional sequence, however, only related to Type I and Type II percent situations. Thus, the instructional sequence presented in this study goes beyond Allinger's study in that all three types of percent problems were dealt with, as well as exposing students to the additive and multiplicative nature of percent situations, which Allinger did not.

The relatively recent study by Lembke and Reys (1994) on percent instruction, does not relate to this study. Their findings suggested that intuitive percent notions should be fostered to enable students to spontaneously develop their own computational procedures for percent situations. Year 5 students in the Lembke and Reys study did this, however, Year 7 students did not. The Lembke and Reys study offered important suggestions for instruction prior to students being presented with formal procedures, but did not assist students who have already received such instruction, like the students in this study.

The sequence of instruction developed in this study can be seen to be an all-encompassing sequence, drawn and built on research from previous studies to clarify the components of effective percent instruction.

6.1.3 The effectiveness of the proportional number line method for percent problem solving

As presented in Chapter 5 (particularly section 5.5.4), all students in the study displayed poor percent problem solving and calculation skills prior to instruction (all groups scored less than 20% for this section of the Percent Knowledge Test). On the posttest, all students in all groups displayed greatly improved performance on such percent problem solving and calculation activities. The predominant method utilised by students for percent problem solving was the proportional number line method. Results of the study, therefore, indicate that instruction in the proportional number line method led to greatly enhanced student problem solving skill. This study has shown that, the proportional number line method proposed in Chapter 2 (section 2.4.2), is an effective method for percent problem solving for Year 8 students.

The effectiveness of the proportional number line method can be attributed to its valuable features which respond to stated requirements for percent instruction: (i) that instruction must be developed which assists students to read, interpret and define relationships between percent problem components; (ii) a solid representation of percent is required; and (iii) instruction in percent must aim to build students' understanding of percent as a proportion (Parker & Leinhardt, 1995).

One of the difficulties students experience in percent problem solving is in problem interpretation and understanding of the relationship between the component parts of the problem (Parker & Leinhardt, 1995). To help students read, interpret, and define relationships between percent components, in this study students were instructed to interpret the elements of percent situations in terms of *part*, *whole* and *percent*. This step in the proportional number line method was adapted from research on the use of a part-part-whole schema to assist students in identifying addition and subtraction problems (Mahlios, 1988; Resnick, 1982; Wolters, 1983). Students were presented with a part-whole-percent schema for interpreting percent problems, and for identifying the relationships between the elements within problems. In this study, students readily developed proficiency in identifying the part, whole and percent components of percent situations (see sections 5.1.7, 5.2.7 and 5.3.7) and in analysing the relationships between elements of the percent situations, encapsulated by the following diary entry from a Group 3 student: *"One thing I learnt in maths today was how to work out percent problems. I found out that part means percent, well they mean the same thing."* For this student, the part-whole-percent schema assisted her to see the relationship between the quantity whole in problem situations with 100%, and the quantity part as a percent. The part-whole-percent schema, therefore appears to provide a clear and powerful linkage to the components of percent situations and interpretation of those components as a comparison to a standard base of 100.

Within the proportional number line method, the dual-scale number line was proposed as a solid representation of percent. The number line model was selected as a visual model from suggestions that visual models are a vital link in developing students' problem solving skills (e.g., Post & Cramer, 1989; Streefland, 1985). Leinhardt (1988) stated that appropriate mathematical models are those which serve to explain algorithms; which embody the principles of the calculation procedures; and which may provide students with a mechanism for solving new problems. As used in the study, the dual-scale number line was seen to fulfil all such requirements. It very simply, but very powerfully served to represent all three types of percent situations as statements of proportion; it enabled percent situations to be interpreted as part/whole situations comparing to an equivalent fraction with a denominator of 100; it assisted students perform the algorithm in its ability to organise the elements within the percent situation into a statement of proportion and therefore a proportion equation. Most importantly, the dual-scale number line simply lent itself to dealing with percentages greater than 100%. By extending the number line beyond the 100% mark, the change to the original whole could be represented and easily visualised on the extended number line. The dual-scale number line used in this study provided a solid representation of percent as a proportion, and easily lent itself to representing percent increase and decrease situations. Not only that, it was readily adopted by students in the study, and therefore was seen to be an accessible and viable model for percent problem solving with Year 8 students.

The dual-scale number line is a relatively unexplored representation for percent situations, with the 10x10 grid more commonly used for developing percent conceptual understanding (Cooper & Irons, 1987; Reys et al., 1992), and for representing percent situations for solution (Bennett & Nelson, 1994; Cooper & Irons, 1987; Weibe, 1986). The number line model used in this study appears to be superior as a representation for percent problems, and for visualising solution procedures as the number line caters to all three types of percent problems. The 10x10 grid model suggested by Weibe (1986) caters only to Type I and II problems, and that proposed by Cooper and Irons (1987) caters only to the simpler Type I problems. Bennett and Nelson (1994) described how the 10x10 grid could represent all three types of percent situations, and assist solution of such problems, but their approach is extremely complicated, relying predominantly on students' mental computational and conversion skills for successful use. One of the criticisms of the use of the 10x10 grid for solving percent problems is that it does not readily represent percents greater than 100% (Parker & Leinhardt, 1995). To show, for example, 120% using a 10x10 grid, two grids are required, which would thus show 2 wholes rather than a change to the original whole. In this study, it was seen that the dual-scale number line readily catered to percents greater than 100%. Other representations for percent suggested in the literature, include the triangular representation (Boling, 1985; Teahan, 1979) and the "is-of" percent mnemonic (McGivney & Nitschke, 1988 - see section 2.2.5). Both these representations require students to "fit" particular values within the percent situation into the representation. These representations can be seen to be somewhat limited when compared to the number line or even the 10x10 grid, as they do not accurately represent the "hundredthsness" of percent. They also do not appear to have the potential to develop students' principled-conceptual knowledge as a result of their use.

As a representation to assist percent problem solving, the value of the dual-scale number line can also be seen in its power to embody the concept of percent as a proportion. According to English and Halford (1995), the essence of understanding a mathematical concept is to have a "mental representation or model that faithfully reflects the structure of that concept" (p. 18). The dual-scale number line provides a clear picture of percent as a statement of proportion (Dewar, 1984), and it was through experience with such a number line that, at least one student, developed understanding of percent as a proportion (see section 5.4.5). As described in section 5.4.5, one student utilised the proportion equation to solve a problem relating to degrees in a circle to construct a pie graph. The student's written response indicated that the student saw a relationship between 65% as $^{65}/_{100}$ and $^{x}/_{360}$ to convert 65% to number of degrees to shade on the pie graph. As a result of this study, it was seen that the dual-scale number line as a representation for percent situations can lead to development of proportional understanding. In this study, it was also seen that student utilisation of the dual-scale number line was high after instruction, with students abandoning their own previous methods for percent problem solving and calculation.

Posttest results also showed that many students began to shorten the number line model to straight construction of the proportion equation. It is hypothesised that this short version reflects the growth of proportional knowledge in students, in that they were seeing a percentage situation as equivalent to a part/whole fraction situation. Although students' understanding of the proportional nature of the dual-scale number line could not be gleaned from pen and paper test results, the number line model appeared to be internalised by students in that many students dispensed with drawing the number line and simply constructed the proportion equation. The students positioned the numbers in the proportion equation as they would be positioned on the number line (although it is acknowledged that a small percentage of students did place numbers incorrectly in the proportion equation). Hiebert and Carpenter (1992) suggested that the physical representation of a mathematical notion experienced by students influences the way that notion is internalised.

In this study, the number line model was internalised as a proportion equation for the majority of students, with the percentage part/whole being equivalent to the quantity part/whole. The number line therefore provided a representational link between the elements within the percent situation and the proportion equation, and as the number line appeared to faithfully reflect the structure of percent as a proportion, it promoted students' understanding of percent as a proportion that could be linked at a later time to students' prior knowledge of the proportional representation.

As the students in this study internalised the number line as a model for percent as a proportion, it can be argued that the number line model contributed to their conceptual understanding of percent as a proportion, although this was not directly determined. The number line did, however, lead to the growth of proportional understanding for the student mentioned above who exhibited transfer of knowledge when she spontaneously constructed a proportion equation in a non-percent but proportional situation.

This evidence of knowledge transfer as a result of internalisation of a mental model strongly suggests the appropriateness of this model for representing percent as a proportion, and may answer the call by Parker and Leinhardt (1995) when they stated a "solid representation of percent may be one key to unlocking the door to an understanding of percent" (p. 465).

6.1.4 The efficiency of the proportional number line method for percent problem solving

The efficiency of the proportional number line method can be seen in its ready adoption by the students in this study, and its utilisation of the Rule of Three procedure for percent calculations.

Analysis of students' responses on pre- and posttests on Section III show that, after instruction, there was a change in students' procedures for performing percent calculations and solving percent problems, with the majority of students in all groups using the proportional number line method. Across all four groups, there was a greater utilisation of the proportional number line method, or simply the proportion equation for solving the three types of percent problems. This strategy is different to the various strategies employed by students prior to instruction, which included decimal multiplication, use of calculator percent button, fraction multiplication, and trial and error. In this study, it was seen that students appeared to abandon their own strategies for solving the three types of percent problems and to adopt the proportional number line method. Group 3 students' diary responses indicate that the number line method was a preferred method for percent equations and problem solving, as students found it quick and easy. The following responses provide this evidence:

> *"The new way we learnt to do percent problems is a lot easier than the way we learnt last year, so now I can do percent problems a lot easier and quicker."*
>
> *"Using the new way is getting easier."*
>
> *"I am pleased that I have learnt more about percentages. I understand them more now and can work them out faster."*
>
> *"I think I am getting better at percentage now because I find it easier to work them out with the number line."*
>
> *"I use that way (number line method) instead of using my own way because it is so easy."*

Clearly, the number line strategy was readily adopted by students, particularly Group 3 and 4 students, and was seen by students as a simple procedure for successful calculation of percent equations and problems.

For solving proportion equations, and therefore percent problems, the students in this study were taught the Rule of Three procedure. Students developed facility with this procedure in a short period of time, and enjoyed the speed with which it enabled them to complete calculation of percent equations. These reactions are similar to those of students in the research study conducted by Allinger (1985) who provided Year 10, low achieving students with the procedure for the same reason.

The Rule of Three taught to students simply as a procedure has been questioned due to its seeming meaninglessness (Cramer et al., 1992). Others have suggested that it should not be directly taught, but should be borne from experiences for conceptual development of ratio and proportion (Hart, 1981; Streefland, 1985; Robinson, 1981). However, building students' proportional knowledge and reasoning skills is a slow and gradual task as such knowledge and skill is dependent upon various rational number concepts, such as, equivalence, division, ratio (Post et al., 1988). Development of proportion knowledge, therefore, is not a simple and straight forward task, which could be "added on" prior to instruction in percent. The dilemma here is that, in this study the Rule of Three enabled successful operation in the domain of percent, but meaningful application would depend upon building conceptual knowledge of proportion. Thus, like the study by Allinger (1985), the decision was made to teach the Rule of Three for successful percent problem solving. In this study, it was seen that application of the Rule of Three contributed to successful performance. The provision of procedures for successful problem solving performance is supported by Noddings (1990) who stated that concentration on procedures may actually interfere with students' construction of important concepts and principles. In this study, the procedure was seen as peripheral to representation of percent as a proportion.

Inclusion of practice of the Rule of Three in the instructional sequence may appear to work against Parker and Leinhardt's (1995) calls for percent instruction to be away from practise of mechanical procedures. However, in this study, the Rule of Three was a minor, but essential component of the instructional sequence to enable completeness in percent problem solving. Other components of the proportional number line method provided students with a wider understanding of percent as a proportion; thus the Rule of Three procedure was not the sole component of the instructional sequence. The proportional number line method added a dimension to the instructional sequence in that students were provided with a very old and once esteemed mathematical procedure (Swetz, 1992), and investigation into the origins of the procedure potentially would provide for an interesting investigation (Resnick & Omanson, 1987). Also, inclusion of the Rule of Three enabled successful problem computation, and thus students were immersed in solving a variety of percent problems using a proportional method. Immersion in problem solving tasks can actually lead to the development of underlying key concepts (Lo & Watanabe, 1997).

6.1.5 A model for percent problem solving incorporating the proportional number line method

The proportional number line method can be seen to incorporate four elements: a percent schema for interpreting percent situations in terms of their component parts; a representation which embodies percent as a proportion; a structure for symbolisation of the percent situation as a proportion equation and hence an equivalent fraction; and a procedure for generating the equivalent fraction. These four elements are incorporated into a model for percent problem solving as follows:

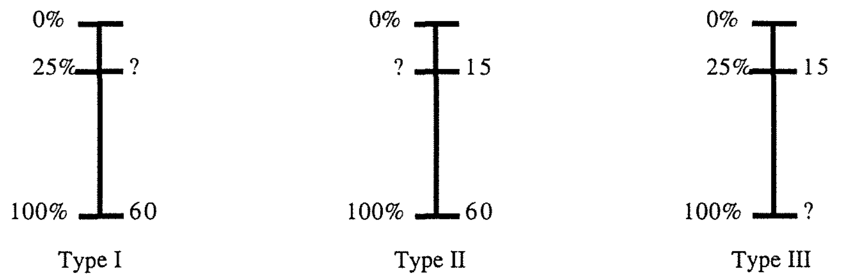
1. Percent situations contain three elements, which can be identified as part, whole, percent. In any percent problem, two elements are given, and solution requires finding the third. The three types of percent problems give the three different combinations of the three elements:

   Type I e.g. 25% of 60 = Δ — part =Δ, whole=60, percent = 25%

   Type II e.g. Δ% of 60 = 15 — part = 15, whole=60, percent = Δ%

Type III e.g. 25% of Δ= 15 part = 15 , whole=Δ, percent = 25%

2. A dual-scale number line can represent the elements of the percent problem. For each of the three different types of percent problems, the elements are located in different places:



3. The percent situation can be written as a proportion equation taken directly from the number line:

Type I	Type II	Type III
$\frac{25}{100} = \frac{?}{60}$	$\frac{?}{100} = \frac{15}{60}$	$\frac{25}{100} = \frac{15}{?}$

4. The Rule of Three procedure enables solution of the proportion equation, and therefore the percent problem:

Type I	Type II	Type III
$\frac{25}{100} = \frac{?}{60}$	$\frac{?}{100} = \frac{15}{60}$	$\frac{25}{100} = \frac{15}{?}$
$25 \times 60 \div 100 = 15$	$15 \times 100 \div 60 = 25\%$	$15 \times 100 \div 25 = 60$

In the above description, the three different types of percent problems (Type I, II and III) are given to show the different placement of the elements on the number line according to the problem types. To use the proportional number line method, students do not need to identify whether the percent problem is a Type I, II, or III problem; they merely need to identify elements of the problem in terms of part, whole or percent. For placement of elements on the number line, students need to recognise that the "whole" corresponds to 100% and thus these two amounts are on the same level either side of the number line. The percentage and part are on the same level, with the percentage located on the scale between 0% and 100% and the corresponding part at the same level on the other side of the number line. The proportional number line method, therefore is not a traditional/case procedure (see section 1.1.5) for percent where solution depends on application of one of three rules for solving each Type or Case.

The dual-scale number line of the proportional number line method lends itself to representation of percent increase problems. When used in this way, it clearly exemplifies the additive and multiplicative nature of percent increase situations. Figure 6.1 shows representation of the percent increase situation of: *A baby's mass increased 25% in 3 months from its birthweight of 3156g.*

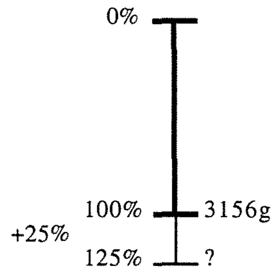


Figure 6.1. The dual-scale number line representing the additive and multiplicative nature of percent increase situations.

From Figure 6.1, it can be seen that this percent increase situation can be interpreted two ways. The baby's new mass can be seen as its original mass plus 25% more. It's new mass can also be seen as 125% of its original mass. Therefore, two calculation procedures are possible. Either, by finding 25% of the original mass and adding this to the original mass to determine the new mass. Or, by calculating 125% of the original mass (using the Rule of Three: ${}^{100}/_{125} = {}^{3156}/_{?}$) to determine the new mass.

The proportional number line method can also be used for percent decrease situations. In Figure 6.2, the number line depicts the problem: *A shirt costing $75 was reduced 35% in a sale. How much would the shirt cost now?*

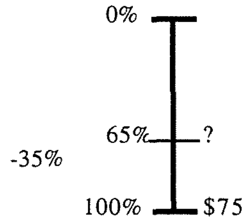


Figure 6.2. The dual-scale number line representing the subtractive and multiplicative nature of percent decrease situations.

To place 65% on the number line, the problem needs to be clearly interpreted using complementary percent principle knowledge: a 35% discount is the same as 65% of the original amount. A subtraction takes place, and thus the number line model shows the subtraction. The situation can thus be interpreted in two ways: the shirt is discounted 35%; or, the shirt now costs 65% of the original amount. Similarly, calculation can also proceed in two ways. Either, 35% of the original cost can be found, and subtracted from that cost, or calculate 65% of the original cost using the Rule of Three ($^{65}/_{100} = {^?}/_{75}$). Using a subtractive application takes percent discount problems from a two-step procedure to a one-step procedure. The number line thus organises elements within percent problems, particularly increase and decrease problems, so that interpretation can occur.

The proportional number line method appears to be a simple, yet powerful representation for percent situations as proportions, and more importantly, for developing understanding of the additive (and subtractive) and multiplicative nature of percent increase (and decrease).

6.1.6 Summary of key points

In this section, the proportional number line method and percent problem solving were discussed. The power of the proportional number line method was seen in its effectiveness to assist students interpret percent problems, to provide a solid representation of percent, and a means to develop students' understanding of percent as a proportion. The proportional number line was also seen as an efficient means for percent problem solving as students in this study readily adopted the method, and the inclusion of the Rule of Three procedure enabled efficient solution of percent calculations. Also in this section, results of this study were compared to other studies into percent instruction, and it was seen that the model of percent problem solving proposed, builds and extends results of previous studies.

6.2 Percent knowledge

6.2.1 Overview

In this section, percent knowledge is discussed. In section 6.2.2, the growth of students' percent knowledge through utilisation of the proportional number line method for percent problem solving is discussed. In section 6.2.3, change in students' principled-conceptual percent increase and decrease knowledge is discussed in relation to instruction. In section 6.2.4, permanence of students' percent knowledge after instruction is discussed through presentation of results of delayed posttest scores. A model of percent instruction and a model of percent knowledge is presented in section 6.2.5 and 6.2.6 respectively, generated through analysis of results of this study. A summary of key points is presented in section 6.2.7.

6.2.2 The proportional number line method within the model of percent knowledge

To use the proportional number line method to assist percent problem solving requires both concrete and computational percent knowledge. Concrete percent knowledge associated with the proportional number line method is the ability to construct the dual-scale number line to represent real-world percent situations, and to use this scale to estimate percentages. Computational percent knowledge associated with the proportional number line method is the ability to solve the percent problem (proportion equation) using the Rule of Three procedure.

In the teaching program, with its focus on developing students' problem solving skills, students' concrete and computational percent knowledge associated with the proportional number line method were actively and overtly promoted through instruction. Via the Percent Knowledge Test, it was determined that instruction contributed to the development of such concrete and computational percent knowledge as students adopted the proportional number line method for percent problem solving.

Principled-conceptual percent knowledge associated with the proportional number line method was not overtly promoted during the teaching program, and was not directly determined via the Percent Knowledge Test. Principled-conceptual percent knowledge associated with the proportional number line method is (i) that percent is a comparison to a standard base of 100; (ii) that every fraction is a percent if the denominator is 100; and (iii) that any fraction can be converted to an equivalent fraction of specified denominator or numerator through solving for the unknown (or using the Rule of Three).

Descriptions of concrete, computational, and principled-conceptual percent knowledge associated with the proportional number line method suggest the embedded and specific nature of principled-conceptual percent knowledge in relation to the concrete and computational knowledge of particular percent procedures. In accordance with Leinhardt's (1988) definitions of concrete, computational and principled-conceptual mathematical knowledge, defining principled-conceptual percent knowledge is dependent upon the concrete representation and the computational procedures selected for percent calculations and problem solving. As Leinhardt stated, concrete knowledge is "the frequently pictorial systems...that often serve as a basis for demonstration or explanation of an algorithm" (p. 121), computational knowledge is the "procedural knowledge of mathematics, the algorithms and procedures for operations" (p. 121), and principled-conceptual knowledge is the "underlying knowledge of mathematics from which the computational procedures and constraints can be deduced" (p.121).

By definition, students' concrete and computational knowledge would be relatively simple to determine, as it would reflect in their ability to complete particular mathematical calculations and procedures. Students' principled-conceptual knowledge, however, would be more difficult to determine because of its covert nature. The underlying principles and concepts associated with the proportional number line method were not overtly the focus of instruction in the teaching program, however, it was seen that, for one student at least, the underlying principles of the proportional number line method were gleaned as a result of developing concrete and computational percent knowledge associated with the proportional number line method. The student referred to here spontaneously transferred knowledge of the proportional number line method to a non-percent, but proportional, situation. This student's response indicates that, as a result of being presented with a model for representing percent situations as proportions, the student developed the principled-conceptual percent knowledge described previously. The proportional number line method thus directly influenced this student's construction of knowledge about proportion which transferred to other situations. Instruction in the proportional number line method, unexpectedly enabled a transfer measure (Hiebert & Wearne, 1991) of the effect of this key process upon cognitive activity, with this student transferring the key process spontaneously to another task not presented in the original instruction. Such a spontaneous transfer indicates the power of the proportional number line method to promote principled-conceptual percent knowledge.

6.2.3 Principled-conceptual percent knowledge of percent increase

In this study, students received direct instruction to promote their principled-conceptual percent knowledge of percent increase and decrease situations. As stated in the model of percent knowledge proposed in Chapter 2 (section 2.4.4), principled-conceptual knowledge of percent increase and decrease language consists of: the additive/subtractive percent increase principle where percent increase/decrease situations are interpreted as an added or subtracted change to the original whole; and the multiplicative percent increase/decrease principle where percent increase/decrease situations are interpreted as a multiplicative change to the original whole. Students' concrete percent knowledge of the dual-scale number line was built upon to promote students' understanding of the additive/subtractive and multiplicative knowledge of percent increase/decrease situations. Within the teaching program, one entire episode was devoted to the language of percent increase and decrease. In the Percent Knowledge Test, particular items were constructed specifically to provide a snap-shot of students' principled-conceptual knowledge of the additive/subtractive and multiplicative language of percent increase/decrease, before instruction, and directly after instruction. On the Percent Knowledge Test, these items were items 6a-c and items 7a-c of Section I. Students' scores on the additive/subtractive items (items 6a, 6b and 6c) were combined to give a singe score for students' additive/subtractive percent knowledge and similarly with the multiplicative items (items 7a, 7b and 7c). Scores for these items across the four groups are presented in Table 6.1.

Table 6.1

Pre- and Posttest Means (%) for Additive and Multiplicative Language of Percent Items on the Percent Knowledge Test Across the Four Groups

	Group 1		Group 2		Group 3		Group 4	
	pre-	post-	pre-	post-	pre-	post-	pre-	post-
add.	59%	47%	53%	67%	79%	82%	88%	80%
mult.	27%	26%	28%	48%	23%	62%	36%	83%

Looking across the table at pretest scores for each group it can be seen that students' interpretation of the additive language of percent increase situations is generally satisfactory (greater than 50%), with Group 3 and 4 students displaying greater interpretation of such items. Posttest scores for all groups are generally similar to pretest scores, indicating little change in interpretation of additive percent situations after instruction. Looking across the four groups at items relating to multiplicative understanding of percent increase, it can be seen that students in all four groups experienced difficulty with such language, with pretest scores around 30%. Posttest scores, however, show change in interpretation of multiplicative percent items for groups 2, 3, and 4, but not for Group 1. Group 1 posttest scores reflect little change compared to pretest scores. Comparing posttest scores between Groups 2, 3 and 4, there appears to be a positive trend in scores with Group 2 showing a 20% increase, Group 3 showing a 39% increase, and Group 4 showing a 47% increase. This trend is similar to the trend in test scores for all groups overall (refer Chapter 5, section 5.5.4), and as discussed in Chapter 5, appears to reflect success ratings of instruction. The teaching episode specifically focussing on the language of percent increase and decrease was rated as unsuccessful-successful for Group 1, and successful for Groups 2, 3 and 4.

In terms of knowledge change and growth with respect to the additive/subtractive and multiplicative language of percent, the results of this study appear to suggest that students' knowledge of additive percent language was relatively unchanged as a result of instruction, although students' knowledge of this language was generally well-developed prior to instruction. Results do appear to suggest that instruction was effective for promoting students' understanding of the multiplicative language of percent increase, with greatest knowledge increase occurring with Group 4. Group 4 students not only were provided with instruction directly related to the language of percent increase and decrease as were Groups 2 and 3, but also engaged in an O/N trial (to be discussed in section 6.4). Instruction received by Group 4 students appears to have promoted students' knowledge of the multiplicative language of percent increase to a much greater extent than instruction received by Groups 2 and 3, although instruction for Groups 2 and 3 was effective in promoting students' multiplicative understanding to a satisfactory level.

As described in section 5.1.9, the teaching episode for the language of percent increase and decrease implemented with Group 1, began with physical representation of percent increase situations (using 1 kg bags of sugar to simulate a baby's growth, and so on), and cardboard strips to represent the original plus the percent increase. This method of representing percent increase situations was replaced with the number line (familiar to the students for percent problem solving) extended beyond 100%. This change in representation and focus of instruction appeared to assist students to interpret percent increase situations, as the extended number line was the only representation for percent increase situations used with Groups 2, 3 and 4, who were the students who exhibited greater facility in interpreting the language of percent increase and decrease. The results reported here appear to further support the dual-scale number line as a powerful representation for percent situations.

The focus of this episode within the teaching program reflects the attempt to develop students' multiplicative understanding of percent situations as proportions. As discussed in Chapter 2 (section 2.3.3), development of multiplicative structure is a necessary element of proportional reasoning (Behr et al., 1992). In Chapter 2, Behr, Harel, Post and Lesh's (1992) suggested activities for the development of multiplicative structures were summarised, describing activities to enable students to explore "change". The example given was that students should explore change to numbers in both additive and multiplicative ways, where, for example an additive and multiplicative change to 4 can result in 8 (add 4 or multiply by 2), but that such a multiplicative change will not hold for 13 changing into 17, but an additive change will. Behr et al. stated that representation of change (or difference) in both additive and multiplicative terms will promote proportional reasoning. The dual-scale number line can be seen to serve this function, through presentation of the proportional number line method as described in section 6.1.5. It can be argued that the dual-scale number line to represent percent increase situations, used in this study, successfully and simultaneously modelled both additive and multiplicative change in percent increase and decrease situations. Changes to the whole which occur in percent increase and decrease situations can be readily represented on the dual-scale number line, with the number line extended beyond the 100% mark for increase situations, and travelling backwards from 100% on the number line for percent decrease situations. Visually, the additive or subtractive change and the multiplicative change in percent increase and decrease situations is thus facilitated by the number line representation. The dual-scale number line therefore, is a representation which can be seen to faithfully reflect the structure of the concept; a vital aspect to evaluating the appropriateness of a representation (English & Halford, 1995).

In this study, students' ability to interpret the multiplicative nature of percent increase and decrease situations was greatly enhanced through instruction, but students' understanding of the additive/subtractive nature of percent increase and decrease situations remained relatively unchanged. This suggests that further explicit instruction is required to develop and link such knowledge to promote further understanding of percent as a statement of proportion, and therefore develop students' proportional reasoning.

6.2.4 Permanence of percent knowledge

To provide an indication of the permanence of percent knowledge change after instruction, a delayed posttest was administered to Groups 3 and 4 approximately 8 weeks after instruction. Results of these delayed posttests for Group 3 and 4, presented previously in Chapter 5 (Figure 5.14 in section 5.3.2 for Group 3, and Figure 5.18 in section 5.4.2 for Group 4) have been combined in Table 6.2. As listed in the Taxonomy of delayed posttest items (see Appendix B), the test contained items relating to 5 areas of percent knowledge: the additive language of percent increase; the multiplicative language of percent increase; percent calculations; percent problem solving; and two-step increase problems. Students' scores for items in these 5 categories correspond to the 5 scores for each group given. It is also seen from Table 6.2, that there is one score for item 3 and item 4, whereas in the figures presented in Chapter 5, three separate scores for each item (3a, 3b and 3c and 4a, 4b and 4c) were presented. Items 3 and 4 correspond to percent calculations and percent problem solving of the three types of percent situations, and as students' scores on these items were similar for the three types of percent situations, scores were combined to indicate performance in general on these two items. Combination of scores also occurred for item 5 to give one score for two-step percent problem solving.

Table 6.2
Delayed Posttest Means (%) for Group 3 and 4

Item	1	2	3	4	5
Group 3	82%	29%	76%	83%	12%
Group 4	100%	96%	95%	96%	38%

From Table 6.2 it can be seen that, for percent calculations and problem solving of the three types of percent situations (items 3 and 4), student performance was high. Group 4 students' performance was much greater than Group 3 with scores of 95% and 96% respectively for items 3 and 4. Students' methods for percent calculations and problem solving was predominantly the proportional number line method. Group 4 students' two-step problem solving was generally better than group

3 (item 5), although neither group scored 50%. For additive and multiplicative language of percent increase items (items 1 and 2), Group 4 scored higher than Group 3 on both items. Group 3 students performed better on the additive language item (item 1) than the multiplicative item (item 2).

Analysis of the delayed posttest results suggests that the proportional number line method was well retained by students several weeks after instruction, and students continued to successfully perform percent calculations and solve percent problems using the method. Students' principled-conceptual knowledge of the additive and multiplicative percent increase principles was greater overall for Group 4 than Group 3, particularly the principle of multiplicative percent increase. As previously stated (section 6.2.3, section 5.5.4), the difference in instruction between Group 3 and 4 was that Group 4 underwent an O/N trial on the multiplicative language of percent increase, whereas Group 3 students did not. The O/N trial, therefore, appears to have contributed to students' permanence of this knowledge over time.

6.2.5 A model of percent instruction

As stated in section 5.5.2, the teaching program consisted of metacognitive training and percent instruction. Percent instruction comprised essential percent instruction for percent problem solving, and ancillary instruction for percent concept and associated skill development. Essential percent instruction for percent problem solving related to the proportional number line method. The embodiment of the proportional number line method is the dual-scale percent number line. As discussed in section 6.1.3, the power of the dual-scale number line lies in the fact that it provides a solid representation of percent; it faithfully reflects the nature of the concept of percent as a proportion; it is simple to construct and to extend as necessary; its dual-scale organises the elements of percent situations; it represents all percent situations within the conceptual field; and it simultaneously can represent the additive and multiplicative change inherent in percent increase and decrease situations. In this study, the dual-scale number line was introduced to students as a means to represent percent situations after analysis of a number of percent problems of the three types. The other steps in the proportional number line method were also introduced to students during this time. As a result of this sequence of instruction, students developed proficiency in solving the three types of percent problems, and thus developed concrete and computational percent knowledge. It was through the provision of a concrete model and a prescriptive method for solving percent problems that students developed such knowledge. This sequence of instruction also appeared to lay the foundation for the development of principled-conceptual percent knowledge as well as proportional reasoning skills.

This approach to instruction is at odds with instructional approaches which advocate providing students with experiences to enable them to develop their own percent problem solving skills, computational procedures, and principled-conceptual knowledge (Behr et al., 1992; Hiebert & Carpenter, 1992; Streefland, 1985). Also this approach of introducing students to percent problem solving of the three types of percent problems before practising computational skills via engagement in percent exercises, is at odds with the type of instruction which, as suggested by Resnick (1992), appears to be based on learning hierarchies and the notion that simpler skills must be learned before students can apply these skills to problem situations. Of these two approaches to instruction, it can be seen the first "discovery" approach would be a valuable approach for initial instruction in percent, and the second "skills before application" approach would possibly stifle students' ability to develop principled-conceptual links amongst procedures presented (as suggested by Resnick, 1992). For the Year 8 students in this study, who had received prior instruction in percent and engaged in activities to develop their conceptual understanding of percent, the approach to instruction presented appeared to be rich in its power to promote students' percent problem solving skills and broaden their knowledge of percent.

A model of percent instruction is proposed, depicted in Figure 6.3. The teaching program in this study commenced with instruction in the proportional number line method; this led to the development of concrete and computational percent knowledge for percent problem solving, and appeared to lead to the development of proficiency in percent problem solving. Through percent problem solving experience, it is suggested that principled-conceptual percent knowledge was developed, or at least students were exposed to the following percent principles (as outlined in the model of percent knowledge proposed in section 2.4.4): (i) the concept of percent as part of a whole where the whole has 100 parts; (ii) the concept of percent as a comparison to a standard base of 100; (iii) the complement principle where every percent part has a complement part to total the whole of 100%; (iv) the fraction-equivalence principle that every fraction is a percent if the denominator is 100; (v) the additive/subtractive percent increase/decrease principle where percent increase/decrease situations are interpreted as an added or subtracted change to the original whole; (vi) the multiplicative percent increase/decrease principle where percent increase/decrease situations are interpreted as a multiplicative change to the original whole; and (vii) there are only three types of percent problem situations. The instructional sequence, depicted in Figure 6.3 was seen to flow and draw from students' prior knowledge of related mathematical topics of decimal-fractions, ratio, proportion. For this model of percent instruction, it is proposed that through instruction in the proportional number line method, concrete and computational percent knowledge will develop, students' experience in percent problem solving will increase, which will lead to the development of principled-conceptual percent knowledge. In this model of percent instruction, it is proposed that through instruction on the proportion number line method, knowledge of associated mathematical topics of fractions, decimals, ratio, proportion is enhanced, and knowledge of the Rule of Three is developed. Future instruction can then build on this knowledge.

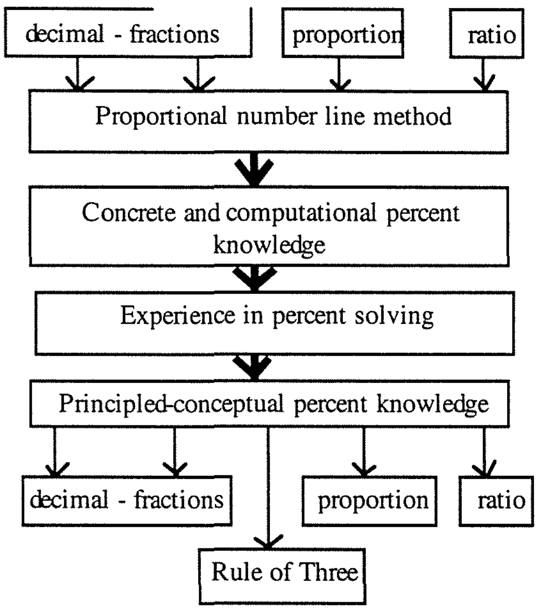


Figure 6.3. A model of percent instruction as the foundation to promoting percent knowledge and associated mathematical knowledge.

This model is based on the belief of teacher efficacy. Smith (1996) stated that teacher efficacy is the degree to which teachers believe they directly influence their pupils' achievement. High teacher efficacy, according to Smith, results in higher student outcome; the creation of learning environments by the teacher which are responsive to students; teachers who spend more time with students experiencing learning difficulty; classrooms which make productive use of group work; and teachers who are willing to try new ideas and innovations in the classroom. The down side of high teacher efficacy, according to Smith, is that such teachers regard themselves as the holders of subject knowledge, and that they see their role as one of transmission of knowledge to students. According to Smith, teachers thus tend to use a "telling" model of teaching, where the students' role is to listen, watch and practise, and the teacher's efficacy is measured through student achievement on computational tasks. Smith suggests that an "action model" of teaching is required to align curriculum reform, where students not only learn computational skills, but also mathematical reasoning and number sense. Action models are those which engage students in problem situations, and the teacher is a collaborator, facilitating students' construction of knowledge. However, the route to developing action models of teaching for teachers with a high teacher belief efficacy, according to Smith, is not clear.

The model of percent instruction presented in Figure 6.3 is an intermediate view to the models provided by Smith, and it is proposed that it does provide direction for teachers with high teacher efficacy beliefs. The model in Figure 6.3 shows how direct teaching of a method for problem solving enables students to construct and enhance their own knowledge of percent and related mathematical topics through engagement in percent problem solving. The proportional number line method, created in response to the need for instruction to develop students' understanding of percent, can be seen to act as a bridge between teaching for computation and problem solving proficiency, and teaching to assist principled-conceptual percent knowledge development. Smiths' *telling* teaching model versus the *active* teaching model can be regarded as two extremes of a continuum. A *telling model* for percent would be seen to promote students' percent computational proficiency. An *action model* would take more time than a school curriculum could afford. The model for percent instruction presented in Figure 6.3 is a clean, direct model midway between two extremes.

The model of instruction in Figure 6.3 can also be seen as a type of pedagogical mediation model as advocated by Vygotsky (Schmittau, 1993a, 1993b). The essence of pedagogical mediation, according to Schmittau (1993a) is that instruction focuses on exposing the structures of mathematical topics so that students can see interrelationships amongst concepts and thus engage in problem tasks which require higher order thinking. Instruction is focussed, presenting students with carefully sequenced learning tasks and activities to enable them to extract the mathematical structure of the topic. This approach is different to a "discovery learning" approach of a constructivist nature, and is based on the belief that students do not need to "discover" or rediscover mathematical concepts which already exist and have been historically and culturally constructed. The pedagogical mediation philosophy states that scientific concepts (as distinct from everyday concepts; hence mathematical concepts are regarded as scientific concepts) "objectively exist and need only to be individually appropriated" (Schmittau, 1993b, p. 16). The essence of pedagogical mediation is summarised by Davydov (cited by Schmittau, 1993b) in the following manner:

> Formal education must find a mode of presentation within which, through adequately constructed activity, the child can appropriate the objective essences of scientific concepts in their essential systematic interrelatedness. In the absence of such pedagogical mediation scientific concepts can be expected to be appropriated only with difficulty or not at all. (p. 16)

Pedagogical mediation, therefore, provides pathways for all students to enable them to extract the essence of mathematical concepts. Pedagogical mediation links to the notion of teacher efficacy and belief in teacher influence upon learning. The proposed model of percent instruction can be seen to be a model of pedagogic mediation in the topic of percent as it enabled students to become proficient percent problem solvers, it assisted students to make connections between the three types of percent problems and to link percent to other mathematical topics. The model of percent instruction gives rise to a model of percent knowledge, as presented in the next section.

6.2.6 A model of percent knowledge

led to the development of students' percent knowledge. It was seen that the proportional number line method, and particularly the dual-scale number line led to the development of percent problem solving proficiency and conceptual percent knowledge. From results of this study, and from percent literature, a model of percent knowledge is suggested.

Percent knowledge consists of two main elements, or nodes, these being: (i) percent concepts and (ii) percent applications. In the literature, suggestions are provided for (i) promoting the concept of percent in learners (as presented in Chapter 2, section 2.1), and (ii) assisting students to solve percent application problems (as presented in Chapter 2, section 2.2). Models and strategies for solving percent applications frequently are proposed to promote students' principled-conceptual percent knowledge so that percent problems can be solved meaningfully (e.g., Bennett & Nelson, 1994; Cooper & Irons, 1987; Dewar, 1984; Haubner, 1992; Weibe, 1986). However, some models and strategies appear to exist in isolation and are seen to link not to principled-conceptual knowledge of percent [see, for example, models and strategies suggested by Boling (1985); Teahan (1979); McGivney & Nitschke (1988); presented in Chapter 2, section 2.2.5). It is suggested therefore that particular models and strategies, firmly embedded within principled-conceptual percent knowledge, may serve to promote knowledge of percent applications and percent concepts, and thus serve as a bridge between these two percent knowledge nodes. It is also suggested that solving of percent problems increases experience with real-world applications of percent, and this may serve to then draw principled-conceptual percent knowledge to a higher level of understanding. Therefore, a model of percent knowledge is proposed, consisting of the following four elements:

1. Two knowledge nodes (percent concepts and percent applications);
2. Connections between the two nodes;
3. Models and strategies;
4. Real-world experience and principled-conceptual knowledge.

Pictorially, the components of the proposed model of percent knowledge are presented in Figure 6.4. The model of percent knowledge is presented in further detail in section 6.3.4 to incorporate metacognition.

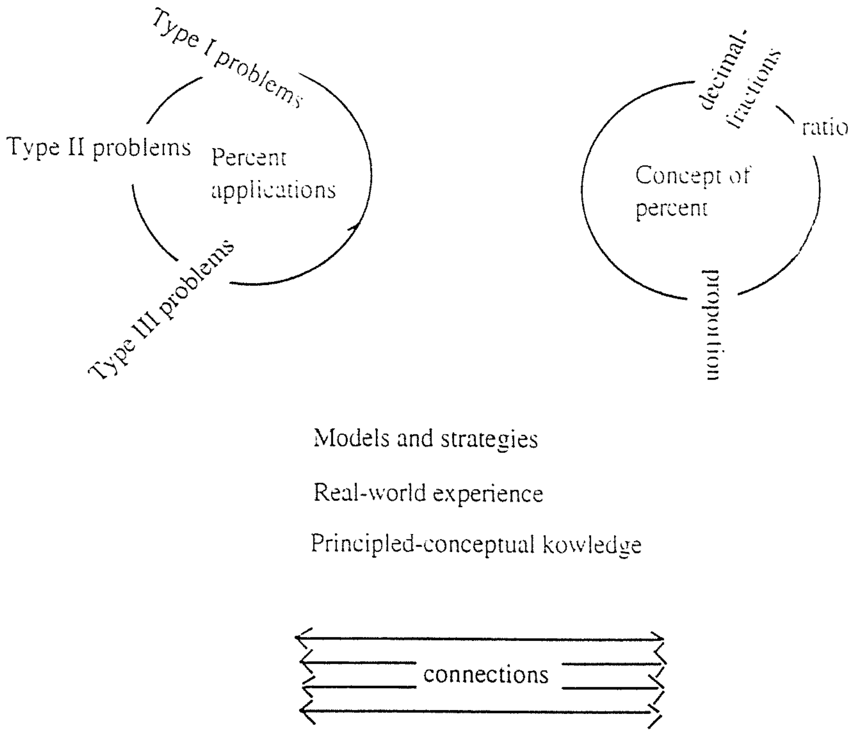


Figure 6.4. Components of a model of percent knowledge.

6.2.7 Summary of key points

In this section, students' development of percent knowledge was discussed. Students' computational and concrete percent knowledge was described, and the relationship of the development of such knowledge to the development of principled-conceptual percent knowledge was discussed. Students' internalisation of the proportional number line method was hypothesised through presentation of delayed posttest scores. Two models were proposed as a result of analysis of the change and growth, and permanence of percent knowledge for students in this study. A model of percent instruction suggested how the teaching program presented to students in this study promotes students' percent knowledge, and enhances prior knowledge of associated mathematical topics, and also lays the foundation for building and extending such knowledge through further instruction. A model of percent knowledge was suggested, consisting of four components, to be addressed further in section 6.3.4.

6.3 Metacognition

6.3.1 Overview

The focus of discussion in this section is metacognition. In section 6.3.2, the application of metacognitive skills for each group in the study is described, and related to successful implementation of the metacognitive training episode of the teaching program. In section 6.3.3, the interplay of metacognition and domain specific knowledge is discussed, particularly in how metacognitive awareness promoted greater application of metacognitive skills and successful task performance. In section 6.3.4, the models of percent instruction and percent knowledge, presented in section 6.2 are modified to include metacognitive training and metacognition respectively. A summary of key points is presented in section 6.3.5.

6.3.2 Metacognitive skills and successful task performance

Chan (1993) stated that successful learners are those with strongly developed metacognitive skills and knowledge, such as self-directed goal setting, planning of work tactics, self-monitoring of progress and evaluation for self-improvement. The development of metacognitive skills can be facilitated by provision of metacognitive training programs (Chan, 1993; Kirby & Williams, 1991). In this study, the Conceptual Mediation Program (CMP) (Lyndon, 1995), which is a collection of ideas, concepts, and learning strategies drawn from psychology literature, was incorporated into a metacognitive training episode. As discussed previously (see section 5.5.2) the metacognitive training episode (adapted from the CMP), was implemented entirely with Groups 3 and 4, but only partially with groups 1 and 2.

Throughout the study, there was evidence of much greater application of metacognitive skills by Groups 3 and 4 students than with Groups 1 and 2. Groups 3 and 4 students exhibited goal-directed behaviour, such as: paying attention during all teaching episodes; application of practice strategies to internalise the steps of the proportional number line method and Rule of Three procedure; spontaneous and selfdifferentiation of knowledge as an "old way" or a "new way"; willingness to take control of PI during the O/N trial on the language of percent increase. With Groups 1 and 2, less goal-directed behaviour was observed, contributing to difficulties in the teaching-learning situation. For example, students' lack of attention and lack of self control rather than paying attention led to the failure of many teaching episodes; students' prior knowledge of solution procedures for Type I percent problems interfered with students' paying attention to instruction on the number line model; students' reluctance to practise the steps in the proportional number line method led to errors in construction of proportion equations; students' failure to check solutions led to non-detection of unsuitable problem solutions in some cases; display of avoidance behaviours led to non- completion of set tasks.

The CMP program strongly emphasises the responsibility of the individual in the learning situation. Clearly, in this study, Groups 3 and 4 students demonstrated more control of the learning process than Groups 1 and 2 students. Group 3 students' diary entries suggest that the metacognitive training contributed to building students' awareness of "self" in the learning process, as exemplified in the following student written responses:

> *"I think I will soon be able to remember more than I used to remember because I know what it takes to remember."*
>
> *"I learnt that you could train your brain to learn things. I thought you were either smart of not smart. I think I am getting better at teaching my brain to take things in better."*
>
> *"This lesson was helpful to me because now I know how my brain works I will take over and teach my brain to recall better."*
>
> *"This lesson has actually taught me more about the brain and how it works. It has made me realise how my attention gets distracted very easy. I was pleased that I found out how my attention flies away."*

The above comments suggest that the metacognitive training was responsible for revealing that control of the brain could be managed by the individual, which was evident in student use of such phrases as "teaching my brain", "I will take over", "train my brain". For one student in particular, it appeared that the metacognitive training episode suggested to her that greater levels of performance could be achieved through application of CMP strategies, and that it was not the case that "you were either smart or not smart." In terms of academic performance, Groups 3 and 4 posttest results showed much greater improvement in performance from pretest results, compared with Groups 1 and 2. Groups 3 and 4 students were thus more successful learners than Groups 1 and 2 students. In this study, the metacognitive training component of the CMP appears to have contributed to this greater level of performance. Clearly, the metacognitive training component of the teaching program appeared to satisfy many requirements of instruction to assist students become successful learners, as they actively influenced their own learning (Weinstein & Mayer, 1986); they worked hard to become experts (Baird & White, 1982); they realised that they were in control of their own learning (Mercer & Miller, 1992); they were convinced that their own efforts, persistence and application of learning strategies would contribute to successful task performance (Chan, 1993).

6.3.3 The interplay of metacognition and domain specific knowledge

As stated in Chapter 3 (section 3.3.5), Cole and Chan (1990) suggested that explicit instruction in both cognitive and metacognitive strategies is the means of ensuring students' successful task performance. Cognitive strategies are generally defined as those strategies which assist students' performance in a specific domain; metacognitive strategies are those applicable over a range of situations (Cole & Chan, 1990). In this study, instruction was planned so that students would be presented with metacognitive awareness and strategies for successful learning in general, and the proportional number line method for successful operation in the specific domain of percent. The interaction of metacognition and percent knowledge was seen to contribute to successful task performance.

As presented in section 5.5.4 (see Figure 5.22), test scores in this study indicated that all groups displayed enhanced performance on percent problems after instruction, with Groups 3 and 4 students improving to a greater extent that Group 1 and 2 students. In the study, Groups 1 and 2 students were not presented with the metacognitive training episode in its entirety; Groups 3 and 4 students were. It seems a reasonable hypothesis that the interplay of the metacognitive training and proportional number line method contributed to Group 3 and 4 students' greater level of successful task performance over Groups 1 and 2 students, although it is acknowledged that Group 3 and 4 students may have greater capacity to apply metacognitive skills prior to instruction. However, when presented with the metacognitive training, Group 3 and 4 students were more willing to listen to the key notions presented, and their comments indicated that the material presented made sense (apparent from comments presented in previous section 6.3.2). The metacognitive training program appeared to crystalise metacognitive thinking, which led to an interesting interplay between metacognitive strategies and knowledge and application of the dual-scale number, percent knowledge and the part-whole-percent schema. Data gathered in this study was insufficient for direct analysis of such interplay, and this is taken up in chapter 7 as a limitation to this study.

Students' successful task performance, as a result of application of the proportional number line method, led to greater levels of confidence, as exemplified by the following Group 3 students' diary entries about the proportional number line method:

> *"I'm really confident with percent now so this has helped."*
> *"I was pleased that I knew what I was doing."*
> *"I feel that I have learnt heaps more from when I first began percentage. I feel 100% more confident in mathematics."*
> *"Percentage is so easy. I really enjoy it."*
> *"I'm really happy I'm getting much better at percent, and I feel much more confident."*

The above student responses indicate that students regarded percent as an easy mathematical topic, and therefore operating in the domain of percent was an enjoyable experience.

Results of this study highlight the interplay of domain specific knowledge and metacognition in successful task performance. It has been repeatedly suggested that mathematical task performance is dependent upon metacognition (Flavell, 1976; Garofalo & Lester, 1985; Prawatt, 1989; Silver, 1982). Further, as stated in Chapter 1 (see section 1.1.4), mathematical knowledge is influenced by three metacognitive categories of person, task and strategy knowledge (Garofalo & Lester, 1985). The instructional program presented in this study, promoted students' *person* knowledge, as they were confident in their own ability to operate in the domain of percent; students' *task* knowledge changed in that students came to believe that percent was an easy mathematical topic; and students' *strategy* knowledge increased through provision of instruction in the proportional number line method for percent problem solving. Similarly, instruction in this teaching program can be seen to enhance students' knowledge, strategic and metastrategic thinking, and disposition, which have been defined by Prawatt (1989) as factors influencing mathematical performance. In this study, it was seen that the instructional program contributed to the development of students' knowledge of percent; it provided students with skills for percent problem solving; and as a result, students developed a positive disposition and confidence in operating with percent.

6.3.4 Metacognitive knowledge and instruction

In light of results of this study, the model of percent instruction proposed in section 6.2.5, can be modified to include metacognitive training. A model of instruction which embodies the teaching sequence followed in this study is one which depicts instruction commencing with metacognitive training, to promote greater student focus on the proportional number line method for percent problem solving. In this model of instruction, it is proposed that through percent problem solving, conceptual percent knowledge is developed, which leads to enhanced knowledge of associated mathematics topics decimal-fractions, ratio, proportion), and also enhanced knowledge of the Rule of Three within problem solving contexts. The modified model of percent instruction is depicted in Figure 6.5.

Comparing the two proposed models of percent instruction (Figure 6.3 and Figure 6.5), it is proposed that the inclusion of metacognition takes knowledge of the Rule of Three from a procedural level, to a principled-conceptual level of knowledge. Metacognition enhances percent problem solving skill and understanding of the Rule of Three. In this model, inclusion of metacognitive training is suggested as the means to application of greater metacognitive skills which leads to greater enhancement of knowledge of other associated mathematics topics than instruction on percent alone.

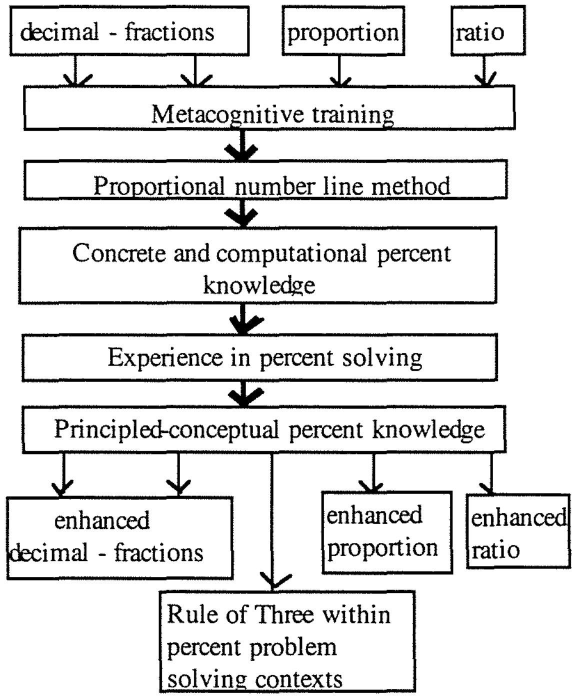


Figure 6.5. A model of percent instruction including metacognitive training.

Metacognitive training within an instructional sequence may lead to greater application of metacognitive skills. This has implications for the model of percent knowledge (proposed previously in section 6.2.6). This study gives rise to two models of percent knowledge: (i) percent knowledge without metacognition; and (ii) percent knowledge with metacognition. In this study, Groups 1 and 2 students did not receive complete metacognitive training as planned. As discussed in sections 5.1.11, 6.3.2 and 6.3.3, Group 1 and 2 students exhibited limited metacognitive control in task performance; - they were often unfocused; they did not enable themselves to engage in percent problem solving consistently; they frequently searched for short-cuts to percent problem solving. A model of percent knowledge without metacognition is suggested in light of results of teaching experiments with Groups 1 and 2. Groups 1 and 2 students were presented with models and strategies for percent problem solving; namely, the proportional number line method. The proportional number line method appeared to promote students' real-world experience with percent, but also may have restricted their understanding of real-world percent situations to only those presented in the classroom. The proportional number line method promoted students' understanding of the three types of percent problems and thus extends to topics related to percent, such as decimal-fractions, ratio, and proportion. Students' lack of focus upon the percent problems experienced, results in minimal growth of principled-conceptual percent knowledge. A model of percent knowledge without metacognition if presented in Figure 6.6.

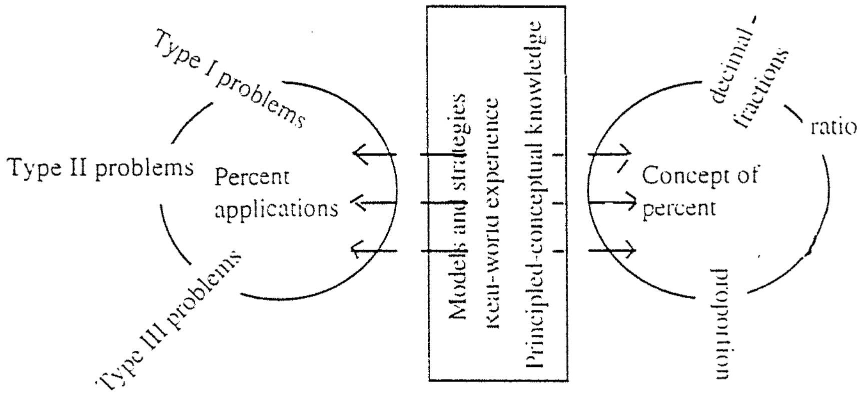


Figure 6.6. A model of percent knowledge without metacognition.

The model of percent knowledge without metacognition presented in Figure 6.6, suggests that models and strategies serve as a bridge to: (i) link the two percent knowledge nodes, (ii) to provide real-world experience with percent situations, and (iii) assist in the development of principled-conceptual percent knowledge.

In this study, Groups 3 and 4 students were presented with metacognitive training, and as stated in sections 5.3.10, 6.3.2, and 6.3.3, exhibited greater metacognitive control than Group 1 and 2 students, which led to enhanced task performance. Whether this was a result of the metacognitive training was not determined clearly through this study. A model of percent knowledge with metacognition is suggested in light of results for Groups 3 and 4 students. Like the model presented in Figure 6.6, a model of percent knowledge with metacognition depicts how models and strategies bridge the two nodes of percent knowledge, promoting the development of percent applications and percent concepts. With inclusion of metacognition, focus on percent in the real-world is accentuated, drawing understanding and experience of percent to a greater level than percent situations presented in the classroom. In this model, it is proposed that metacognition raises principled-conceptual knowledge of percent as students engage in percent problem solving. Diagrammatically, the model of percent knowledge with metacognition is presented in Figure 6.7.

In summary, the model of percent knowledge including metacognition proposes that percent knowledge consists of two knowledge nodes (percent applications and percent concepts), and these nodes are connected to each other through models and strategies. This model proposes that real-world experience with percent develops through percent problem solving, which develops knowledge of percent applications and percent concepts. With metacognition, it is proposed that the relationship between the nodes, and models and strategies, is revealed, and real-world experience now encompasses the nodes as well as the models and strategies with the development of principled-conceptual percent knowledge. In other words, through application of models and strategies to real situations, students are able to see/use the whole schema. They are not restricted to specific models and strategies in their thinking; their thinking is no longer prototypical or tied specifically to one given model and/or strategy.

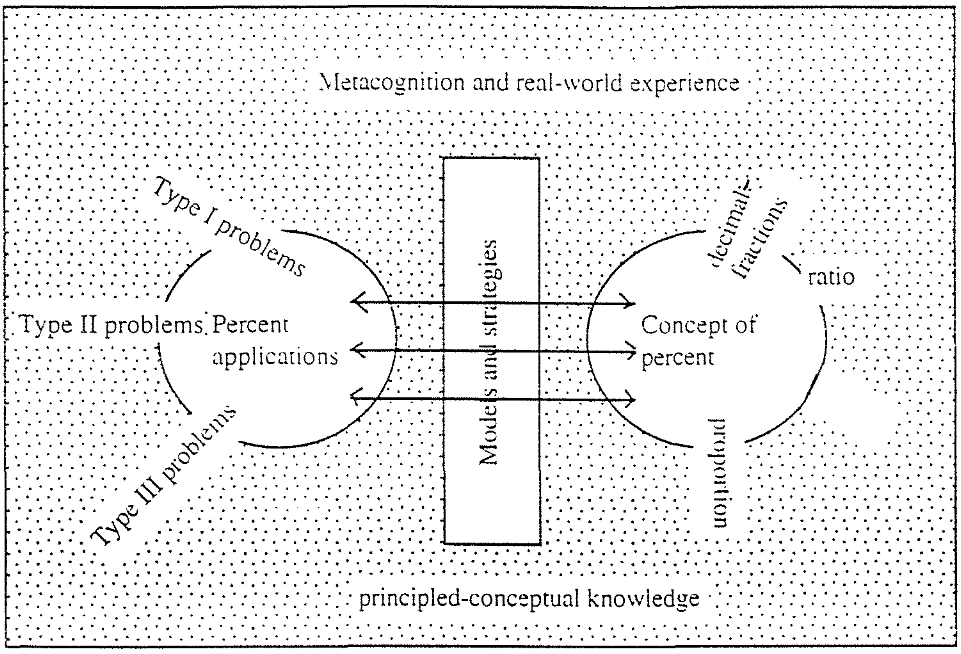


Figure 6.7. A model of percent knowledge including metacognition.

6.3.5 A developmental model of percent

In this study, it was seen that students' knowledge of percent developed through instruction. The Percent Knowledge Test revealed that initially, students in this study had some conceptual knowledge of percent, and could perform percent calculations, primarily of Type I. Through instruction, students developed knowledge of the proportional number line method. Student utilisation of the proportional number line method appeared to lead to the development of percent concepts and percent applications (the two percent knowledge nodes). Although not directly measured, it appeared that with application of greater metacognition as a result of the metacognitive training, connections between models and strategies and the percent knowledge nodes became explicit. Metacognition promoted real-world percent experience, which led to development of principled-conceptual percent knowledge, contributing to development of the full experiential schema for percent. As a result of this study, it is suggested that percent knowledge may develop for students over five phases, as summarised below:

1. Student possesses some ideas of percent;
2. Student possesses some ideas of percent and percent applications;
3. Student develops knowledge of models and strategies for percent applications, but knowledge of percent is prototypical in that it is restricted to one given model and/or strategy;
4. Student makes connections between the two knowledge nodes, made explicit through application of metacognition;
5. Student has developed full experiential percent schema and percent knowledge is not longer prototypical.

As a result of this study, there was some evidence of the growth of percent knowledge through the five phases. As stated in Chapter 5 (section 5.5.4) all students began from similar knowledge bases, as reflected by results of the Percent Knowledge Test. All four groups of students exhibited some knowledge of percent (phase 1), with some students exhibiting some knowledge of Type I percent applications (phase 2) and all three types of percent applications (phase 3). Through instruction, and thus provision of models and strategies, students developed knowledge of percent and knowledge of all three types of percent applications. In this study, students in Groups 1 and 2 appeared to reach phase 3 of the developmental model, and it is hypothesised that for the majority of Group 1 and 2 students, connection of knowledge of percent applications and percent concepts was in a transitional phase. For these students, it is suggested that their percent knowledge at this phase was prototypical. Groups 3 and 4 students exhibited greater metacognitive control, and this appeared to enable the connection between percent knowledge and percent applications to become explicit (phase 4). In this study, metacognition appeared to enable the schema for real-world experience to be extended to include the percent knowledge nodes and the development of principled-conceptual percent knowledge. In this developmental model, the highest level of percent knowledge is phase 5, where it is proposed that understanding from models and strategies is widened to encompass a full experiential schema; where principled-conceptual knowledge is linked to intuitive, concrete and computational knowledge, as suggested by Leinhardt's (1988) model of the development of mathematical knowledge. Some students in Group 4 in this study were seen to reach this stage of percent knowledge development (see section 6.1.3). The five phases of the developmental model of percent knowledge are depicted in Figure 6.8.

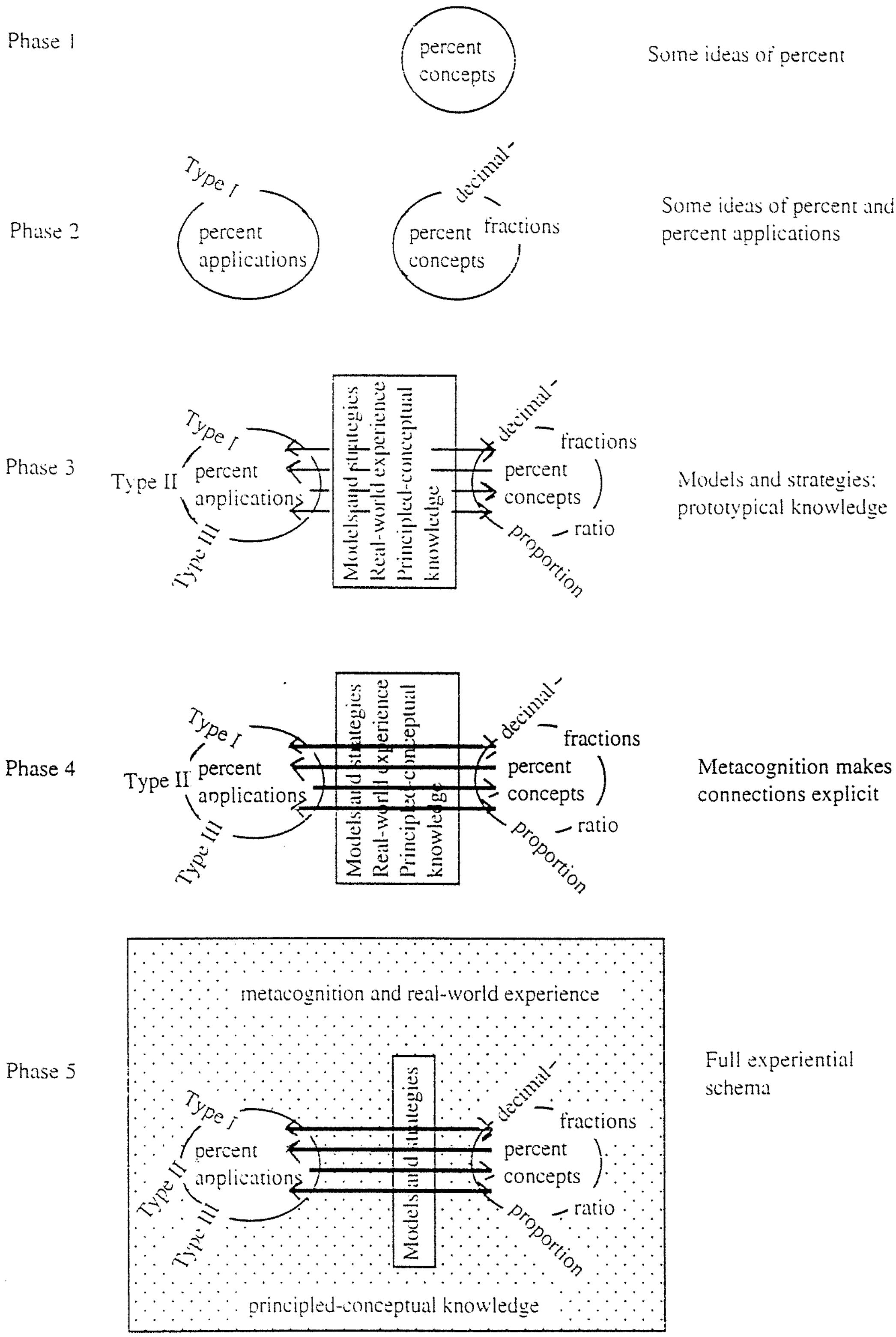


Figure 6.8. The five phases of the developmental model of percent knowledge

6.3.6 Summary of key points

In this section, metacognitive knowledge and skills and their relationship to successful task performance were discussed. The metacognitive training program based on CMP was seen to promote students' greater task performance in this study. Metacognition was included within the percent knowledge model to highlight the interplay of metacognitive with domain specific knowledge for greater task performance. Models of percent knowledge, with and without metacognition were presented. The development of percent knowledge, as dependent upon models and strategies was described. The development of percent knowledge is based on the model of percent instruction, which can be seen to draw on the principles of pedagogic mediation (Schmittau, 1993a; 1993b), and belief in teacher efficacy (Smith, 1996). In this study, it was seen that growth of percent knowledge was dependent upon instruction.

6.4 Diagnostic-prescriptive instruction

6.4.1 Overview

The focus of this section is on diagnostic-prescriptive instruction, and particularly the Old Way/New Way (O/N) strategy within a model of diagnostic-prescriptive instruction. In section 6.4.2, the O/N strategy applied to students' difficulties in interpreting the multiplicative language of percent increase situations is discussed. In section 6.4.3, incorporation of O/N in the classroom, as it occurred in this study is described. In section 6.4.4, the O/N as a teaching strategy, how it differs to good teaching and reteaching strategies, and how it was adopted by the teacher-researcher in this study, is discussed. O/N as an integral element within a diagnostic-prescriptive model of instruction; and the use of O/N within real classroom situations are discussed in sections 6.4.5. and 6.4.6 respectively. A summary of key points is presented in section 6.4.7.

6.4.2 Application of O/N for dealing with prior knowledge

The influence of prior, erroneous knowledge on acquisition of new knowledge in mathematics is well documented (e.g., Ashlock, 1994; Borassi, 1994; Confrey; 1990a; Connell & Peck, 1993; Lyndon, 1995). In this study, the O/N strategy for overcoming the interfering effects of prior knowledge was trialled with some Group 4 students who demonstrated difficulty interpreting the multiplicative language of percent increase. Results of this trial suggested that the O/N strategy assisted Group 4 students to interpret percent increase situations on the posttest better than students in Groups 1, 2 and 3, and that this knowledge change was permanent, as presented in section 6.2.4. As previously discussed in section 5.5.4, all students demonstrated difficulty in interpreting the additive and multiplicative language of percent increase on the pretest, and Groups 1, 2 and 3 students continued to exhibit such difficulty on the posttest to a greater degree than Group 4. All groups were exposed to direct teaching on the language of percent increase and decrease; Group 4 students engaged in an O/N trial to overcome the misinterpretation of the multiplicative nature of percent increase situations. On the delayed posttest for Groups 3 and 4, Group 4 students demonstrated greater facility in interpreting the additive and multiplicative nature of percent increase situations than did Group 3. The O/N trial, therefore, appears to have directly influenced knowledge change associated with the multiplicative language of percent increase over conventional instructional methods.

In terms of the theoretical background of O/N, knowledge is protected from change by proactive inhibition (PI), an unconscious brain mechanism which serves to prevent conflict of incoming information with prior knowledge (Baddeley, 1990). In this study, students' intuitive notions of percent increase appeared to interfere with correct interpretation of percent increase situations. As described in section 6.2.3, students' performance on tasks relating to interpreting an additive change in percent situations was always much greater than their performance in interpreting percent increase situations multiplicatively. Students' additive structures are naturally better developed than multiplicative structures, as students intuitively interpret proportion situations additively rather than multiplicatively (Hart, 1981). As seen in all groups, direct teaching and reteaching of interpretation of percent increase situations did not significantly develop students' understanding of the multiplicative language of percent increase. O/N however, did. As a result of this study, it appears that O/N can be implemented in large group situations to precipitate conceptual change, and not merely skill change, which supports conclusions of research elsewhere (Rowell, Dawson & Lyndon, 1990).

6.4.3 Incorporation of O/N within mathematics instruction

In this study, the O/N strategy was described to students as a strategy for taking control of accelerated forgetting, and presented within the context of the metacognitive training episode. The study, conducted in real classroom situations, highlighted the difficulty of using the O/N strategy by the classroom teacher, but also highlighted the potential of this strategy in mathematics instruction. In this study, the O/N strategy was not used with Groups 1 and 2 during instruction due to incomplete implementation of the metacognitive training episode. The O/N strategy was not used with Group 3 students due to classroom timetable constraints and the need to progress through the teaching program. Also, specific diagnosis of students' difficulties in mathematics had not been carried out to identify students who may have benefited from engaging in O/N on an individual basis. During instruction, few systematic errors surfaced during the course of the teaching program. With Group 4 students, the

O/N strategy was trialled in a large group situation, as discussed in Chapter 5 (see section 5.4.5).

The O/N trial occurred when the teacher-researcher could group the students for instruction, so that students not engaging in the O/N trial were occupied with other tasks not requiring teacher assistance. The teacher-researcher was thus freed to devote attention to the students engaged in the O/N trial. The O/N trial was planned in response to diagnosis of students' misinterpretation of the language of percent increase after students had received instruction on the language of percent increase. The pivotal role of diagnosis to determine applicability of utilisation of the O/N strategy was highlighted in this study. As a strategy for dealing with inappropriate knowledge and misconceptions, O/N may be seen as similar to the conflict teaching strategy (Bell, 1986-1987); the use of errors as springboards for enquiry (Borassi, 1985; 1994); and belief-based teaching (Rauff, 1994), where student errors are brought to the fore, and are the focus of instruction and/or discussion. The O/N strategy, however, differs from these strategies in that the O/N is a prescriptive series of steps where student errors and misconceptions are actively differentiated between by the students themselves with the help of the teacher. O/N is a dialogue where the teacher plays a direct mediating role between students' knowledge and the inappropriate knowledge, in a similar vein to the approach suggested in the cognitive apprenticeship model of Reid & Stone (1991). O/N offers the script for conceptual mediation which other approaches do not.

6.4.4 The O/N strategy as a teaching strategy

In the O/N trial, the teacher-researcher's reflections suggested that the students actively engaged in the O/N trial, yet the teacher-researcher was "uncomfortable" with O/N as a style of teaching. In this study, it was seen that the teacher-researcher had to consciously differentiate between two styles of teaching so that the O/N trial did not revert to a reteaching episode. As indicated by the report on trialling O/N in large group situations (see Chapter 5, section 5.4.5), O/N was a very different teaching strategy which, in this study, appeared to be in conflict with the teacher-researcher's usual teaching style. The O/N was implemented successfully, and the teacher-researcher adhered to the prescribed steps of the strategy. The O/N strategy, therefore, can be seen to contrast teaching strategies suggested for teaching the same content in a different way, which can thus be labelled as reteaching strategies. O/N in this study, was seen as different to a reteaching strategy, and thus appears to be a strategy in a category of its own.

As a result of using the O/N strategy in this study as a distinct strategy for mediation, the true definition of the term remediation becomes apparent. Teaching strategies which are different ways of presenting the same content can be categorised as reteaching strategies, and are the types of strategies traditionally suggested in programs of mathematics remediation (see, for example, Ashlock, 1994; Booker et al., 1980; Resnick, 1982; Wilson, 1976a). These have been shown to be good teaching strategies (e.g., Ashlock, 1994; Wilson, 1976a) and will be successful for "filling gaps" in students' knowledge. These strategies are not so successful when prior knowledge exists (see, for example, Connell & Peck, 1993).

In this study it was seen that, prior to implementation of the teaching program, students misinterpreted the multiplicative language of percent increase. Instruction was implemented to overcome this misinterpretation. It was found that, after instruction, students continued to misinterpret the language of percent increase. Application of the O/N strategy assisted students to overcome this misinterpretation. The difference between O/N as a teaching or reteaching strategy and a definitive strategy for conceptual mediation is highlighted in this study. Thus, there is a difference between good teaching and remedial teaching. The effectiveness of the use of the O/N strategy in this study over good teaching for dealing with inappropriate conceptualisations indicates that good teaching strategies fall into a category distinct from true remediation or mediation strategies. Only strategies which attack prior knowledge and thus overcome the power of PI can be classed as mediation or remediation strategies.

6.4.5 O/N and diagnostic-prescriptive mathematics instruction

The theoretical basis of O/N is described in detail in Chapter 3 (see section 3.5). In this study, the theoretical underpinnings of the O/N strategy served as a blue-print for a diagnostic-prescriptive approach to instruction (as described in section 3.6.3). In this study, it was seen that instruction stemmed from diagnosis. Prior to instruction, the students in this study were presented with a pen and paper test, which served as a relatively simple way to determine students' percent knowledge (in relation to the categories of intuitive, concrete, computational, and principled-conceptual percent knowledge), and to diagnose, in general, students' strengths and weaknesses in this domain. Upon diagnosis of students' knowledge of this topic, in this study, instruction was seen to occur as in the diagnostic-prescriptive model of teaching presented in Chapter 3, section 3.6.3 (see Figure 3.1). In that model, diagnosis is the essential element, and instruction flows from diagnosis for the purposes of teaching (and/or reteaching) to build students' knowledge of a topic, or for unteaching to overcome interfering prior knowledge.

The model of diagnostic-prescriptive teaching in this study proceeded in the following two steps:

1. Diagnosis of students' knowledge of the topic, including identification of

systematic errors and/or misconceptions.

2. Instruction for (i) teaching (and/or reteaching), or for
(ii) unteaching.

These two steps are seen to be incorporated into the updated model of diagnostic-prescriptive mathematics teaching and unteaching presented in Figure 6.9.

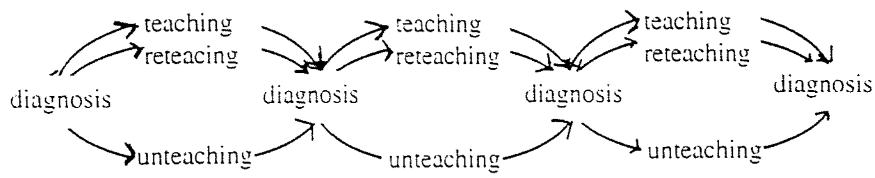


Figure 6.9. A model of diagnostic-prescriptive teaching and unteaching.

In this model, teaching and reteaching occurs via implementation of good teaching strategies for building students' knowledge of the topic, and following Leinhardt's (1988) model of mathematical knowledge, this would be instruction to build students' intuitive, concrete, computational, and principled-conceptual knowledge. For unteaching, the O/N strategy is utilised.

This diagnostic-prescriptive model of instruction shares features of other such models. It is based on the premise that, to assist students who are experiencing difficulty with the study of mathematics, determining the nature of the difficulty is required (e.g., Ashlock, 1994; Underhill et al., 1980). It is also based on the premise that errors indicate knowledge (Ashlock, 1994; Ashlock et al., 1983); that errors are constructed knowledge (Confrey, 1990a). This model also assumes that careful lesson planning and sequencing of instruction will help students who are experiencing difficulties in mathematics (Ashlock, 1994; Ashlock et al., 1983; Connell & Peck, 1993, Mercer & Miller, 1992). This model of diagnostic-prescriptive instruction differs in that it differentiates between the need for good teaching strategies to help children overcome difficulties in mathematics, and the need to employ strategies for mediation and remediation. In this model, the O/N is incorporated as a strategy for remediation.

6.4.6 Diagnostic-prescriptive teaching in real classrooms

As stated in Chapter 3 (see section 3.1.2) it is when a student begins to "fail" that those concerned with the student's mathematical progress are alerted to the fact that the student is experiencing difficulty with the study of mathematics. Traditionally, the blame for the student's learning difficulty was attributed to the student (e.g., Kephart, 1960; Valett, 1978). However, this perception has altered to focus on instructional methods (MacDonald, 1972; Shapiro, 1989; Woodward & Howard, 1994). A diagnostic-prescriptive approach to instruction has been suggested as a means through which students' difficulties in mathematics can be minimised (Ashlock et al., 1983; see also Chapter 3, section 3.3.2) as diagnosis and careful sequencing of instruction are the key components of such a model.

In this study, a diagnostic-prescriptive approach to instruction was implemented in real classroom situations. It was found that diagnosis assisted in developing instruction, and that instruction incorporating good teaching strategies was relatively simple to implement in the classroom. In this study, implementation of the O/N strategy as a conceptual mediation strategy, was difficult in terms of preparing students for use of the strategy (that is, in providing metacognitive training) and in grouping students for instruction. However, it was accomplished, thus suggesting that the style of diagnostic-prescriptive model of teaching can be implemented in whole class situations. Assisting students to overcome difficulties with the study of mathematics can be achieved in typical classroom situations by the classroom teacher.

6.4.7 Summary of key points

In this section, the Old Way/New Way strategy was discussed in relation to a diagnostic-prescriptive model of instruction. Within the teaching program presented in this study, the O/N strategy was seen to be a useful strategy for overcoming students' misinterpretation of the multiplicative language of percent increase. Trialling the use of the O/N strategy within this study has served to illuminate the classroom constraints which may hinder successful application of such a strategy, including the need to acquaint students with the CMP, the requirement of careful diagnosis of students' specific difficulties, and the necessity to group students for application of the strategy. Trialling of the O/N strategy in this study also highlighted how it is different to "good teaching" strategies, and therefore may not easily be adopted within a teacher's usual teaching style. In this section, it was seen that the O/N strategy, or other strategies which deal directly with prior knowledge and proactive inhibition, should be an integral element of diagnostic-prescriptive instruction.

CHAPTER SUMMARY

discussed. The focus of the chapter was on knowledge and instruction. In this chapter, percent knowledge and metacognitive knowledge were seen to result in successful task performance. The proportional number line method was proposed as the means to promote percent knowledge; metacognitive training incorporating the CMP was proposed as the means to promote metacognitive knowledge. A model of percent instruction was presented, to show the pathway to creating percent knowledge for Year 8 students in this study. A model of percent knowledge was created through analysis of students' percent knowledge growth as a result of instruction. In the next chapter, the models of percent knowledge and percent instruction as presented in chapter 6, are briefly discussed, and conclusions to this study are presented.

CHAPTER 7
CONCLUSIONS

CHAPTER OVERVIEW

In this chapter, conclusions, limitations and implications of the research conducted in this study are presented. There are four sections. In sections 7.1 and 7.2 respectively, the two aims of the study are discussed, and the extent to which they were achieved through this research addressed. In section 7.3, the limitations of this study are outlined. In section 7.4, the implications of this study for both future research and classroom instruction are overviewed.

7.1 The first aim

The first aim of this study, presented in Chapter 1, was to develop a program for effectively teaching percent applications in real classrooms which:

(a) is diagnostic-prescriptive in that it caters for all types of learners, ranging from learners who have experienced limited formal instruction in percent, to learners who have received considerable formal instruction in percent;

(b) is *effective* (in terms of student outcome) in relation to promoting students' (i) intuitive, concrete, computational, and principled-conceptual percent knowledge; (ii) percent application problem solving skill; and (iii) permanence of knowledge over time; and

(c) is *efficient* (in terms of teacher input) of (i) teacher preparation requirements; (ii) time requirements to implement in the real world (i.e. in the school situation); (iii) resources; and (iv) cost.

As a result of this study, a teaching program for percent, which caters to all learners; which is effective in terms of student outcome; and which is efficient in terms of teacher input, is a teaching program which has the following components:

1. Metacognitive training;
2. Models and strategies for percent problem solving;
3. Models which exemplify the additive and multiplicative relationships inherent in percent increase situations;
4. Provision of practice in percent problem solving to develop real-world experience of percent, and thus principled-conceptual knowledge of percent.

The teaching program implemented in this study was diagnostic-prescriptive, catering to all types of learners. The students in this study came from varying schools and, therefore, varying backgrounds in relation to percent experience. Following syllabus guidelines, it was expected that the students would have received some prior, direct instructional experiences to develop their conceptual understanding of percent, and that they possibly would be able to perform Type I percent calculations. Taking account of expected prior experiences with the topic of percent, it was assumed that students in this study would have a degree of conceptual knowledge of percent, but the extent to which they could solve the three types of percent problems would vary.

Pretest results confirmed this view, as pretest results indicated that students' intuitive and principled-conceptual percent knowledge was relatively high prior to instruction, but percent calculations and problem solving skills were very low. From the pretest results, students in this study were considered "midstream" learners, who had developed basic percent concepts through experience and/or instruction, and were "midstream" in their exposure to formal instruction on the topic of percent. The teaching program developed for this study focussed on instruction in the proportional number line method. The teaching program, with its focus on the proportional number line method, was presented to all students in the study. The proportional number line method was readily adopted by the majority of students in this study, and students' proficiency in percent problem solving increased. The teaching program, therefore, appeared to be well-suited to the Year 8 students in this study, as it catered to all students with their varying knowledge of percent prior to instruction.

The teaching program implemented in this study was effective in promoting students' percent knowledge and percent problem solving skill, and contributed to permanence of such knowledge over time. In this study, it was seen that students developed concrete and computational percent knowledge, and rapidly became proficient percent problem solvers through application of the proportional number line method. As a result of experience in solving a variety of percent problems using this method, students developed principled-conceptual percent knowledge embedded within the proportional number line method. The proportional number line method was seen to be internalised by students in that they could still apply the method for solving percent problems some weeks after instruction was completed.

The teaching program implemented was seen to be an efficient program on several fronts. It was efficient in terms of teacher preparation requirements: once the instructional program had been planned, and appropriate resources constructed (i.e., worksheets), implementation of the program occurred with the teacher-researcher presenting herself to the designated classroom at the designated time, without prior organisation of specialist materials, equipment, or a particular teaching environment. It was efficient in terms of time requirements: implementation of the teaching program spanned the allocated time for percent instruction in real classrooms imposed by the school timetable. It was efficient as it required no specialist resources for implementation: resources used (for example, worksheets, newspapers, cardboard) were those typically found and used in real classroom situations, and thus the program was also cost-effective.

In summary, the teaching program developed for assisting students become proficient percent problem solvers, was seen to successfully cater to all types of learners, ranging from those who had limited percent knowledge to those who had received extensive instruction in percent; it was an effective program as students became proficient problem solvers and also developed principled-conceptual percent knowledge; and it was an efficient program in terms of preparation requirements, time requirements for implementation, costs incurred, and resources required. This study has thus achieved the first aim.

7.2 The second aim

The second aim of this study, presented in Chapter 1, was to draw implications for percent knowledge, percent instruction, and mathematical teaching in general, specifically, in terms of:

(a) Students' knowledge of percent;

(b) A model for teaching percent problem solving; and

(c) A model for diagnostic-prescriptive teaching of mathematics.

As a result of this study, a model of percent knowledge was developed. The model suggested that percent knowledge consists of two knowledge nodes, these being percent concepts and percent applications. Models and strategies serve to link these two knowledge nodes, enabling students to solve percent problems and thus build their real-world percent experience.

Results of this study suggested a five phase developmental model of students' percent knowledge (as described in section 6.3.5). In this study, Year 8 students' percent knowledge was seen to consist of some percent ideas (phase 1), and/or some knowledge of percent applications (phase 2). With models and strategies, students' knowledge of percent concepts and percent applications developed, with models and strategies serving as a bridge between percent concepts and percent applications. Students entered phase 3 of percent knowledge development. With application of metacognition, connections between percent applications and percent concepts became explicit. Principled-conceptual knowledge of percent was developed, encompassing knowledge of percent concepts and percent knowledge. Students entered phase 4 of percent knowledge development. In the last phase (phase 5) a full experiential schema was developed, where all knowledge was integrated and connections between percent concepts and percent applications led to transfer of knowledge to other related mathematical topics.

In terms of a model for teaching percent problem solving, results of this study revealed how a teaching program, focusing on models and strategies for percent problem solving, is powerful in enabling students to become proficient percent problem solvers, and in promoting principled-conceptual percent knowledge. In this study, instruction commenced with metacognitive training, followed by direct instruction on the proportional number line method. Through application of the proportional number line method to percent problem situations, concrete and computational percent knowledge was developed. This knowledge provided students with experience in percent problem solving, and it is suggested that this led to the development of principled-conceptual knowledge of percent associated with the proportional number line method. Metacognitive training ensured greater application of metacognitive control, which led to greater student focus on percent problems and percent situations in the real-world. Development of principled-conceptual knowledge associated with the proportional number line method potentially may enhance knowledge of related mathematical topics (such as fractions, decimal, ratio, proportion). This instructional sequence, which presented students with a proportional method for percent problem solving, took account of students' prior knowledge in percent and other related mathematical topics (decimals, fractions, ratio, proportion) but did not require students to possess a specified level of knowledge of those topics. After instruction in percent problem solving, it appeared that the foundation had been laid and/or cemented for further instruction in topics related to percent, particularly the topic of proportion. Results of this study, therefore, indicated that instruction for promoting percent problem solving led to significantly increased student proficiency in percent problem solving. Such instruction may also have promoted principled-conceptual percent knowledge, and led to enhanced development of knowledge of related mathematical topics.

In terms of a model for diagnostic-prescriptive teaching of mathematics, results of this study suggested that effective mathematics teaching is dependent upon diagnosing whether instruction should proceed for teaching, reteaching or unteaching. As a result of this study, a model of diagnostic-prescriptive teaching, which differentiates between strategies for teaching and reteaching, and strategies for unteaching was proposed. Strategies for teaching and/or reteaching are those strategies for promoting students' knowledge of a topic where such knowledge does not yet exist. Strategies for unteaching are those which specifically focus on prior inappropriate knowledge and break down the psychological barrier which protects such knowledge from change. To determine whether to use teaching/reteaching strategies, or to use unteaching strategies in teaching situations, is through diagnosis to identify a student's learning difficulty as stemming from a lack of knowledge of the topic, or from the interfering effects of existing inappropriate knowledge. Therefore, the model of diagnostic-prescriptive teaching (proposed in section 6.4.5) has a focus on diagnosis, and pathways of instruction are either for the application of good teaching/reteaching, or through the application of unteaching strategies.

As a result of this study, implications for percent knowledge, percent instruction, and mathematics teaching in general, were drawn, specifically in terms of students' knowledge of percent; a model for teaching percent problem solving; and a model for diagnostic-prescriptive teaching of mathematics. This was the second aim of this study; therefore this study has achieved it second stated aim.

7.3 Limitations

One of the limitations of the research conducted in this study is the fact that students' cognitions about the proportion number line method were not delved into to truly provide a detailed picture of students' percent knowledge structure. The data collection methods of the study (pre-, post- and delayed posttests, field notes, students' diaries, anecdotal records, students' work samples) were suitable to match the design of the study where the emphasis was on attaining naturalism within the classroom environment. Test results and field notes assisted the teacher-researcher to be as unobtrusive as possible to the participants' usual school routine, but also resulted in under-utilisation of such data collecting methods as clinical interviews, (e.g., Ginsburg, 1981; Swanson, Schwartz, Ginsburg & Kossan, 1981) and video records and protocol methods (e.g., Ginsburg, Kossan, Schwartz & Swanson, 1983).

The focus of the research on students' percent problem solving efficacy in this study can also be regarded as a limitation to the study. The Percent Knowledge Test as a data gathering instrument for this study was seen as useful in determining the extent to which students could solve percent problems of the three types as well as perform percent calculations. Other items on the test were constructed to determine students' intuitive, principled-conceptual, concrete and other computational knowledge as per Leinhardt's (1988) mathematical knowledge model. Assessment of students' percent knowledge may have been more precisely determined through construction of test items similar to those for Grade 5 students developed through the Math in Context project at the University of Wisconsin, Madison (see Van den Heuvel-Panhuizen, 1994; Van den Heuvel-Panhuizen, Middleton & Streefland, 1995). Through the Math in Context project, researchers developed test items to cover key concepts and key abilities in Grade 5 percent, generated through students posing their own "easy" and "difficult" percent problems.

Through research conducted in the study reported here, the teaching program with its inclusion of a metacognitive training program and the proportional number line method for percent problem solving, was seen to be powerful in promoting students' percent problem solving skills, particularly for the third and fourth groups of students who participated in this study. The students in Groups 3 and 4 attended a private girls' college, which is a contrasting school environment to that attended by Group 1 and 2 students. Two more limitations of this study, therefore, are that the teaching program was only trialled in two school environments and the student responses were contrasted across two environments. Linked to this is the fact that the teaching program was only trialled in four classroom situations; the number of teaching experiments conducted, therefore, limits the generalisability of the results.

The nature of the study was exploratory, where a problem was identified, and through analysis of ideas on teaching percent in particular and mathematics in general, a teaching program was created and trialled. The research is not conclusive, but can be seen to be more generative, where results unearthed many other ideas and suggestions for further research. This is another limitation of the study.

7.4 Implications

The identified limitations of this study suggest the need for further research in this field. Primarily, results of this study have suggested the power of instruction which focuses on models and strategies for problem solving together with metacognitive training to promote knowledge. This finding leads to formulation of several research questions, including:

- Does metacognitive training promote principled-conceptual knowledge of all mathematical topics?
- Will provision of models and strategies for problem solving in the absence of metacognitive training promote principled-conceptual mathematical knowledge to the same degree as with metacognitive training?
- Is instruction which focuses on methods and strategies for problem solving superior to a learning environment which encourages students to develop their intuitive strategies for problem solving?

Further research is required to pursue such questions.

The teaching program created and implemented in this study was successful for developing students' percent problem solving skills. Further research is required to explore implementation of the teaching program in other school environments. To effectively conduct such research, ideally the teaching program should be presented to teachers from a range of school environments as part of a professional development program, with teachers' implementation of the program studied. This would lead to interesting comparative studies. Similarly, trialling this program with Year 9 and Year 10 students would also lead to interesting results, and insightful suggestions for curriculum change. Students posing their own "hard" and "easy" percent questions could assist the modification of the Percent Knowledge Test as a data gathering instrument in a similar vein to the approach taken in the Math in Context research (Van den Heuvel-Panhuizen, 1994; Van den Heuvel-Panhuizen et al., 1995). Such an inclusion into the teaching program would also provide interesting insights into what particular percent problems students find difficult before and after instruction on the proportion number line method.

Results of this study also suggest further research which looks inwards upon this study. Exploring students' cognitions about percent were not the specific focus of this study, but would potentially excite future research. Probing students' cognitions about the proportional number line method, and attempting to pin-point when students make cognitive leaps and the factors which precipitated such cognitive change and growth, would require a methodology different to the one chosen here. Clinical methods or teaching experiments conducted with a team for observation and interviewing purposes, would be suitable research methodologies for such studies.

Another element of this study, which suggests further research, is the metacognitive training program based on CMP. Of particular interest would be research exploring the extent to which the CMP could assist students become more metacognitive in other mathematical topics (as well as other curriculum topics), and also specifically delving into students' minds to assess their level of metacognitive development and the influence of CMP upon such development and change. The need for two research studies with contrasting methodologies becomes apparent in order to pursue such lines of inquiry. One study would be on providing inservice training to teachers in the CMP and the exploring of students' application of the CMP to mathematics learning via a teaching experiment methodology. Another study would be a clinical study with CMP and the percent teaching program presented to a small group of students.

Apart from the stated implications for further research, the results of this study have implications for instruction. The study clearly showed how the proportional number line method can be implemented in real classroom situations to promote percent problem solving. In this study, students were presented with all three types of percent problems, but not overtly taught to differentiate between the three different types of percent problems. Provision of such holistic percent instruction is at odds with the current approach suggested for students in Queensland schools. As described in section 1.2.2, instruction in percent is staggered with Type I problems introduced in Year 7, Type II problems in Year 8, and Type III problems in Year 9. In such an approach students are presented with each Type of percent problem separately, and various procedures taught and practiced to enable solution of the three types of percent problems. In this study, through the teaching program based on the proportional number line method, students became very competent in solving all three types of percent problems and in applying this knowledge to multi-step problems. Thus, this method of instruction appears to support the presentation of the whole conceptual field (Resnick, 1992) for percent and to dispense with staggering of instruction across three year levels. This study has shown that a holistic approach to percent instruction appears to demystify percent situations, and enables the instructional process to appear more streamlined. For application to the classroom, this study suggests the need to reconsider the structure of the curriculum pertaining to the mathematical topic of percent.

The proportional number line method is powerful in its ability to promote students' knowledge and problem solving skill. The dual-scale number line within the proportional number line method organises elements of percent situations and embodies percent situations into statements of proportion. The structure of the number line mirrors the structure for organising multiplication and division problems as suggested by Vernguard (1983; 1988). Some examples, presented by Greer (1992) summarising Vernguard's work, are in Figure 7.1.

The division and multiplication examples presented in Figure 7.1 can be represented in a similar fashion to percent problems represented on the dual-scale number line. In this manner, percent problems can be seen to belong to the multiplicative conceptual field, proposed by Vernguard (1983; 1988). The proportional number line method, therefore, has implications for instruction in terms of providing a link to other topics and building students understanding of the elements within the multiplicative conceptual field, as well as proportional reasoning. The proportional number line method has applications beyond percent.

3 children each have 4 oranges. How many oranges do they have altogether?

children	oranges
1	4
3	?

A boat moves 13.9m in 3.3 seconds. What is its average speed in metres per second?

seconds	metres
1	?
3.3	13.9

An inch is about 2.54cm. About how long in inches is 7.84cm?

inches	cm
1	2.54
?	7.84

Figure 7.1. Multiplication and division problems as proportions (Greer, 1992, p. 282, 283).

Inclusion of the metacognitive training in the teaching program has implications for instruction. In this study, the metacognitive training based on CMP appeared to enhance students' mathematical performance. Metacognitive training should possibly be an integral part of mathematics instruction. As such, this would require teachers to attend professional development workshops on metacognitive training programs. Linked to metacognitive training based on CMP is the application of the Old Way/New Way (O/N) strategy for overcoming the interfering effects of prior erroneous knowledge. The theoretical background from which O/N draws suggests the need for effective teaching strategies together with effective remediation strategies for effective teaching. Application of the effective remediation strategies in classroom situations suggests the use of diagnostic-prescriptive teaching approaches to mathematics instruction. This study has shown how this can be achieved.

CHAPTER SUMMARY

This final chapter of the report has served to overview what was achieved through research conducted in this study . In this chapter, the aims of the study were discussed, the limitations of the study outlined, and implications of this study for both future research and for instruction were described. This concluding chapter has summarised the relationship between instruction, learning and unlearning in actual classrooms for the purpose of developing instruction to facilitate Year 8 students' access to percent knowledge for solving common percent problems.

BIBLIOGRAPHY

Allinger, G. D. (1985). Percent, calculators and general mathematics. *School Science and Mathematics, 85* (7), 567-573.

Anderson, J. R. (1985). *Cognitive psychology and its implications*. New York: W. H. Freeman and Company.

Ashlock, R. B. (1994). *Error patterns in computation - a semi-programmed approach* (6th ed.). New York: MacMillan.

Ashlock, R. B., Johnson, M. L., Wilson, J. W., & Jones, W. L. (1983). *Guiding each child's learning of mathematics: A diagnostic approach to instruction.* Columbus: Charles E. Merrill.

Ausubel, D. (1968). *Educational psychology: A cognitive view*. New York: Holt, Rinehart & Winston.

Ausubel, D. (1985). Learning as constructing meaning. In N. Entwistle (Ed.), *New Directions in Educational Psychology 1: Learning and Teaching* (pp. 71-82). London: The Falmer Press.

Baddeley, A. (1990). *Human memory: Theory and practice.* Hillsdale, NJ: Lawrence Erlbaum.

Baird, J. R., & White, R. T. (1982). Promoting self-control of learning. *Instructional Science, 11*, 227-247.

Baxter, P., & Dole, S. (1990). Working with the brain, not against it: Correction of systematic errors in subtraction. *British Journal of Special Education, 17*(1) Research Supplement, 19-22.

Baxter, E. P., & Lyndon, E.H. (1987). *Old Way/New Way handbook, 1987, revised edition for teachers.* For distribution to persons who have completed an accredited program of Old Way/New Way training. Unpublished manuscript.

Behr, M., & Harel, G. (1990). Students' errors, misconceptions, and cognitive conflict in application of procedures. *Focus on Learning Problems in Mathematics, 12* (3&4), 75-84.

Behr, M., Harel, G., Post, T., & Lesh, R. (1992). Rational number, ratio and proportion. In D. Grouws (Ed.), *Handbook on research of teaching and learning* (pp. 296-333). New York: MacMillan.

Bell, A. (1986-87). Diagnostic teaching 1-3. *Mathematics Teaching*, Nos. 115, 116, 118.

Bell, A., Swan, M., Onslow, B., Pratt, K., & Purdy, D. (1985). *Diagnostic teaching: Teaching for long term learning (Report of an ESRC Project).* Shell Centre for Mathematical Education, University of Nottingham.

Bennett, A. B., & Nelson, L. T. (1994). A conceptual model for solving percent problems. *Mathematics teaching in the middle school, 1* (1), 20-24.

Boling, B. A. (1985). Sharing teaching ideas: A different method for solving percentage problems. *Mathematics Teacher, 78*, 523-524.

Booker, G., Briggs, J., Davey, G., and Nisbet, S. (1992). *Teaching primary mathematics.* Melbourne: Longman Cheshire.

Booker, G., Irons, C., & Jones, G. (1980). *Fostering arithmetic in the primary school.* Canberra: Curriculum Development Centre.

Borassi, R. (1994). Capitalising on errors as "springboards for inquiry": A teaching experiment. *Journal for Research in Mathematics Education, 25* (2), 166-208.

Borassi, R. (1985). Using errors as springboards for the learning of mathematics: An introduction. *Focus on Learning Problems in Mathematics, 7* (3&4), 1-14.

Bourke, S. F. (1980). *Numeracy in school: An integrated approach to community expectation, student assessment, and error analysis.* Unpublished Master's thesis, Monash University, Melbourne, Vic.

Brown, A., & DeLoache, J. S. (1983). Metacognitive skills. In M. Donaldson, R. Grieve & C. Pratt (Eds.), *Early Childhood Development and Education*, (pp. 280-289). Oxford: Basil Blackwell.

Brown, A., & Palinscar, A. (1982). Inducing strategy learning from text by means of informed, self-control training. *Topics in learning and learning disabilities, 2* (1), 1-17.

Brown, G. W., & Kinney, L. B. (1973). Let's teaching them about ratio. *Mathematics Teacher, 66*, 352-355.

Brown, J. S., & Van Lehn, K. (1982). Towards a generative theory of "bugs". In T. P. Carpenter, J. M. Moser, and T. A. Romberg (Eds.), *Addition and subtraction - A cognitive perspective* (pp. 117-135). Hillsdale, NJ: Lawrence Erlbaum Associates.

Brownell, W. A. (1956/1987). AT Classic: Meaning and skill - Maintaining the balance. *Arithmetic Teacher, 34* (8), 18-25.

Brueckner L. J., & Grossnickle, F. E. (1953). *Making arithmetic meaningful.* Philadelphia: John C. Winston.

Brumfield, R., & Moore, B. (1985). Problems with the basic facts may not be the problem. *Arithmetic Teacher, 33* (3), 17-18.

Castro, C. S. (1998). Teaching probability for conceptual change. *Educational Studies in Mathematics, 35*, 233-258.

Chan, L. (1993). Remedial education: Research and practice. *Australian Journal of Remedial Education, 25* (4), 22-29.

Chan, L. (1991). Metacognition and remedial education. *Australian Journal of Remedial Education, 23* (1), 4-10.

Clark, F. B., & Kamii, C. (1996). Identification of multiplicative thinking in children in Grades 1-5. *Journal for Research in Mathematics Education, 27* (1), 41-51.

Cole, P., & Chan, L. (1990). *Methods and strategies for special education.* New York: Prentice Hall.

Cole, B. L., & Weissenfluh, H. S. (1974). An analysis of teaching percentages. *Arithmetic Teacher, 21*, 226-228.

Confrey, J. (1990a). What constructivism implies for teaching. In R. B. Davis, C. A. Maher, & N. Noddings (Eds.), *Constructivist views on teaching and learning of mathematics* (pp. 107 - 122). Reston, VA: NCTM.

Confrey, J. (1990b). Inconsistencies and cognitive conflict: A constructivist's view. *Focus on Learning Problems in Mathematics, 12* (3&4), 99-109.

Confrey, J., & Harel, G. (1994). Introduction. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. vii-xxviii). New York: SUNY Press.

Connell, M., & Peck, D. (1993). Report of a conceptual change intervention in elementary mathematics. *Journal of Mathematical Behaviour, 12*, 329-350.

Cooper, T., & Irons, C. (1987). *Ratio, proportion and percentage in the primary school: Trial materials.* Brisbane, Queensland: Brisbane College of Advanced Education.

Cox, L. S. (1975). Diagnosing and remediating systematic errors in addition and subtraction computations. *Arithmetic Teacher, 22* (4), 151-157.

Cramer, K., Post, T., & Currier, S. (1992). Learning and teaching ratio and proportion: Research implications. In D. T. Owens, (Ed.), *Research ideas for the classroom: Middle grade mathematics* (pp. 159-178). New York: MacMillan.

Department of Education, Queensland. (1989a). *Years 1 to 10 mathematics sourcebook: Activities for teaching mathematics in Year 6.* Brisbane: Curriculum Services Branch.

Department of Education, Queensland. (1989b). *Years 1 to 10 mathematics sourcebook: Activities for teaching mathematics in Year 8.* Brisbane: Curriculum Services Branch.

Derry, S. J. (1990). Remediating academic difficulties through strategy training: The acquisition of useful knowledge. *Remedial and Special Education, 11* (6), 19-31.

Dewar, J. M. (1984). Another look at the teaching of percent. *Arithmetic Teacher, 31*, 48-49.

Dole, S. (1993). Error patterns and subtraction knowledge development - A comparison of methods. In J. Novak (Ed.), *Proceedings of the Third International Seminar on Misconceptions and Educational Strategies in Science and Mathematics*, Cornell University, Ithaca, New York. Electronic publishing of seminar proceedings.

Dole, S. (1995). Gaining access to mathematical literacy: Application of Old Way/New Way to mathematics instruction for adult learners. *Proceedings of the 3rd Annual International Conference on Post-compulsory Education and Training*, Brisbane.

Dole, S., Cooper, T. J., Baturo, A., & Conoplia, Z. (1997). Year 8, 9 and 10 students' understanding and access of percent knowledge. In F. Biddulph and K. Carr (Eds.), *People in Mathematics Education, Proceedings of the twentieth annual conference of the Mathematics Education Research Group of Australasia.* University of Waikato Printery, New Zealand.

Drake, B. M., & Amspaugh, L. B. (1994). What writing reveals in mathematics. *Focus on Learning Problems in Mathematics, 16* (3), 43-50.

Dubinsky, E., Dautermann, J., Leron, U., & Zazkis, R. (1994). On learning fundamental concepts of group theory. *Educational Studies in Mathematics, 27,* 267-305.

Ellerton, N. (1989). The interface between mathematics and language. *Australian Journal of Reading, 12* (2), 92-103.

Engelhardt, J. M. (1977). Focus on teacher education in diagnostic and prescriptive mathematics: Diagnostic/prescriptive mathematics as content for teacher education. *Focus on Learning Problems in Mathematics, 10* (1), 47-53.

English, L. D., & Halford, G. (1995). *Mathematics education - Models and processes.* Mahwah, NJ: Lawrence Erlbaum.

Erlwanger, S. H. (1975). Case studies of children's conceptions of mathematics. *Journal of Mathematical Behaviour, 1* (3), 157-283.

Eysenck, M. W. (1977). *Human memory: Theory, research and individual differences.* Oxford: Pergamon Press.

Fernald, G. M. (1971). *Remedial techniques and basic school subjects (Edited by Lorna Idol).* Austin TX: ProEd.

Fennell, F. M. (1981). *Elementary mathematics diagnosis and correction kit.* New York: Centre for Applied Research in Education.

Fischbein, E. (1987). *Intuition in science and mathematics: An educational approach.* AH Dordrecht, Holland: D. Reidel.

Fischbein, E., & Schnarch, D. (1997). The evolution with age of probabilistic intuitively based misconceptions. *Journal for Research in Mathematics Education, 28* (1), 96-105.

Fischer, F. E. (1989). *Diagnostic-prescriptive teaching of mathematics: A historical overview.* Department of Mathematics, Towson State University. ED341552.

Fisher, L. C. (1988). Strategies used by secondary mathematics teachers to solve proportion problems. *Journal for Research in Mathematics Education, 19* (2), 157-168.

Flavell, J. (1976). Metacognitive aspects of problem solving. In L. Resnick (Ed.), *The Nature of Intelligence* (pp. 231 - 236). Hillsdale, NJ: Lawrence Erlbaum.

Fong, H. (1995). Schematic model for categorising children's errors in solving a ratio and proportion problem. *Hiroshima Journal of Mathematics Education, 3*, 15-29.

Gable, R. A., Enright, B. E., & Hendrickson, J. M. (1991). A practical model for curriculum-based assessment and instruction in arithmetic. *Teaching Exceptional Children, 24* (1), 6-9.

Gagne, R. (1983). Some issues in the psychology of mathematics instruction. *Journal for Research in Mathematics Education, 14* (1), 7-18.

Garofalo, J., & Lester, F. K. Jr. (1985). Metacognition, cognitive monitoring, and mathematics performance. *Journal of Research in Mathematics Education, 16* (3), 163-176.

Ginsburg, H. (1981). The clinical interview in psychological research on mathematical thinking: Aims, rationales, and techniques. *For the Learning of Mathematics, 1* (3), 4-11.

Ginsburg, H. (1977). *Children's arithmetic: The learning process.* New York: Van Nostrond.

Ginsburg, H. P., Kossan, N. E., Schwartz, R., & Swanson, D. (1983). Protocol methods in research on mathematical thinking. In H. P. Ginsburg (Ed.), *The development of mathematical thinking* (pp. 8-47). New York: Academic Press.

Glatzer, D. J. (1984). Teaching percentage: Ideas and suggestions. *Arithmetic Teacher, 31*, 24-26.

Glaser, R., & Bassock, M. (1989). Learning theory and the study of instruction. *Annual Review of Psychology, 40*, 631-66.

Graeber, A., & Baker, K. M. (1991). Curriculum materials and misconceptions concerning multiplication and division. *Focus on Learning Problems in Mathematics, 13* (3), 25-38.

Greer, B. (1992). Multiplication and division models of situations. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 276-295). New York: MacMillan Publishing Co.

Guba, E. G., & Lincoln, Y. S. (1989). *Fourth generation evaluation.* Newbury Park, CA: Sage Publications.

Hart, K. (1981). Ratio and proportion. In K. M. Hart, M. L. Brown, D. Kuchemann, D. Kerslake, G. Ruddock, & M. McCartney (Eds.), *Children's understanding of mathematics: 11-16* (pp. 88-101). London: John Murray.

Haubner, M. A. (1992). Percents: Developing meaning through models. *Arithmetic Teacher, 40*, 232-234.

Hauck, E. (1954). Concrete materials for teaching percentage. *Arithmetic Teacher, 1*, 9-12.

Hawkins, J., & Collins, A. (1992). Design-experiments for infusing technology into learning. *Educational Technology, 32* (9), 63-67.

Hennessy, S. (1993). The stability of children's mathematical behaviour: When is a bug really a bug? *Learning and Instruction, 3* (4), 315-338.

Hiebert, J., & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 65-97). New York: MacMillan.

Hiebert, J., & Wearne, D. (1991). Methodologies for studying learning to inform teaching. In E. Fennema, T. P. Carpenter, & S. J. Lamon (Eds.), *Integrating research on teaching and learning mathematics* (pp. 153- 176). Albany, NY: SUNY Press.

Houston, J. P. (1991). *Fundamentals of learning and memory* (4th ed.). Orlando, Fl: Harcourt Brace Jovanovich.

Hunting, R. P. (1980). *Emerging methodologies for understanding internal processes governing children's mathematical behaviour.* Paper presented at the Fourth Annual Conference of Mathematics Education Research Group in Australasia. Hobart, May, 1980.

James, W. (1890). *Principles of psychology.* New York: Holt, Rinehart & Winston.

Jones, K., & Charlton, T. (1992). *Learning difficulties in primary classrooms: Delivering the whole curriculum.* London: Routledge.

Kamii, C. (1985). *Young children reinvent arithmetic: Implications of Piaget's theory.* New York: Teachers College Press.

Kantowski, M. G. (1978). The teaching experiment and Soviet studies of problem solving. In L. L. Hatfield (Ed.) & D. A. Bradbard (Tech. Ed.), *Mathematical problem solving: Papers from a research workshop.* Ohio State University: ERIC/SMEAC.

Kemmis, S. (1982). Action research. Prepared for Husen and Postlethwaite (Eds.) *International encyclopaedia of education: Research and studies.* Oxford: Pergaman.

Kemmis, S., & McTaggart, L. (1990). *The action research planner* (3rd ed.). Melbourne: Deakin University.

Kephart, N. C. (1960). *The slow learner in the classroom.* Columbus, Ohio: Charles E. Merrill.

Kirby, J. R., & Williams, N. H. (1991). *Learning problems: A cognitive approach.* Toronto: Kagan & Woo.

Kouba, V. L., Brown, C. A., Carpenter, T. P., Lindquist, M. M., Silver, E. A., & Swafford, J. O. (1988). Results of the fourth NAEP assessment of mathematics: Number, operations and word problems. *Arithmetic Teacher, 35*, 14-19.

Lampert, M. (1992). Teaching and learning long division for understanding in school. In G. Leinhardt, R. Putnam, & R. Hattrup (Eds.), *Analysis of arithmetic for mathematics teaching* (pp. 221-282). Hillsdale, NJ: Lawrence Erlbaum.

Lampert, M. (1991). Connecting mathematical teaching and learning. In E. Fennema, T. P. Carpenter, & S. J. Lamon (Eds.), *Integrating research on teaching and learning mathematics* (pp. 121 - 152). Albany, NY: SUNY Press.

Lampert, M. (1986). Knowing, doing and teaching multiplication. *Cognition and Instruction, 3* (4), 305-342.

Leinhardt, G. (1988). Getting to know: Tracing student's mathematical knowledge from intuition to competence. *Educational Psychologist, 23* (2), 119-144.

Lembke, L. O., & Reys, B. J. (1994). The development of, and interaction between, intuitive and school-taught ideas about percent. *Journal for Research in Mathematics Education, 25* (3), 237-259.

Lerman, S. (1990). The role of research in the practice of mathematics education. *For the Learning of Mathematics, 10* (2), 25-28.

Lesh, R., Post, T., & Behr, M. (1988). Proportional reasoning. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 93 - 118). Hillsdale, NJ: Lawrence Erlbaum Associates, Reston VA: NCTM.

Lo, J-J., & Watanabe, T. (1997). Developing ratio and proportion schemes: A story of a fifth grader. *Journal for Research in Mathematics Education, 28* (2), 216-236.

Lyndon, H. (1995). Conceptual mediation - a new approach to an old problem. *Voice - A Journal for Educators, 3* (2), 12-18. Department for Education and Children's Services, South Australia.

Lyndon, H. (1989). I did it my way! An Introduction to the Old Way/New Way methodology. *Australasian Journal of Special Education, 13* (1), 32-37.

MacDonald, T. H. (1972). *Basic mathematics and remedial instruction.* Sydney: Angus & Robertson.

Mahlios, J. (1988). Word problems: Do I add or subtract? *Arithmetic Teacher, 36* (3), 48-52.

Mansfield, H., & Happs, J. (1992). Using grade-eight students existing knowledge to teach about parallel lines. *School Science and Mathematics, 92* (8), 450-454.

Mason, R. K. (1975). A time-function model: A new approach to the study of per cent concepts with relevant statistical and verbal comparisons (Doctoral dissertation, West Virginia University, 1975). *Dissertation Abstracts International, 37*, 4231A.

Maurer, S. B. (1987). New knowledge about errors and new ideas about learners: What they mean to educators and more educators would like to know. In A. H Schoenfeld (Ed.), *Cognitive Science and Mathematics Education* (pp. 165-187). Hillsdale, NJ: Lawrence Erlbaum.

Maxim, B. R. (1982). An evaluation of two sequences of percent. *Dissertation Abstracts International, 43* (10), 3249A.

McGivney, R. J., & Nitschke, J. (1988). Is-of, a mnemonic for percentage problems. *Mathematics Teacher, 81*, 455-456.

Mercer, C., & Miller, S. (1992). Teaching students with learning problems in math to acquire, understand, and apply basic math facts. *Remedial and Special Education, 13* (3), 19-35, 61.

Middleton, J. A., & Van den Heuvel-Panhuizen, M. (1995). The ratio table. *Mathematics Teaching in the Middle School, 1* (4), 282-288.

Noddings, N. (1990). Constructivism in mathematics education. In R. B. Davis, C. A. Maher, & N. Noddings (Eds.), *Constructivist views on teaching and learning of mathematics* (pp. 1 - 6). Reston, VA: NCTM.

Norman, D. A. (1980). Cognitive engineering and education. In D. T. Tuma & F. Reif (Eds.), *Problem solving and education* (pp. 97-107). Hillsdale, NJ: Lawrence Erlbaum.

Orey, M. A., & Burton, J. K. (1992). The trouble with error patterns. *Journal of Research on Computing in Education, 25* (1), 1-15.

Osiecki, B. A. (1988). Using percent problems to promote critical thinking. *Mathematics Teacher, 81*, 31-34.

Parker, M., & Leinhardt, G. (1995). Percent: A privileged proportion. *Review of Educational Research, 65* (4), 421-481.

Parker, M., & Leinhardt, G. (1993). *Percent: A privileged proportion.* Unpublished manuscript.

Posner, G., Strike, K., Hewson, P., & Gertzog, W. (1982). Accommodation of a scientific conception: Toward a theory of conceptual change. *Science Education, 66*, 211-227.

Post, T. R., Behr, M. J., & Lesh, R. (1988). Proportionality and the development of prealgebra understandings. In A. F. Coxford & A. P. Shulte (Eds.), *The ideas of algebra, K-12* (pp. 78-90). Reston, VA: NCTM.

Post, T. R., & Cramer, K. (1989). Knowledge representation and quantitative reasoning. In M. Reynolds (Ed.), *Knowledge base for beginning teachers* (pp. 221-232). Elmsford, NY: Pergamon Press.

Post, T. R., Harel, G., Behr, M. J., & Lesh, R. (1991). Intermediate teachers' knowledge of rational number concepts. In E. Fennema, T. P Carpenter, & S. J. Lamon (Eds.), *Integrating research on teaching and learning mathematics* (pp. 177-189). Albany, NY: State University of New York Press.

Prawatt, R. S. (1989). Promoting access to knowledge, strategy and disposition in students: A research synthesis. *Review of Educational Research, 59* (1), 1-41.

Putnam, R. T., Lampert, M., & Peterson, P. C. (1990). Alternative perspectives on knowing mathematics in elementary schools. *Review of Research in Education, 16*, 57-149.

Rachlin, S. L., Matsumoto, A., & Wada, L. A. (1987). The teaching experiment: A model for the marriage of teachers and researchers. *Focus on Learning Problems in Mathematics, 9* (3), 21-29.

Rauff, J. V. (1994). Constructivism, factoring and beliefs. *School Science and Mathematics, 94* (8), 421-426.

Reid, D, K., & Stone, C. A. (1991). Why is cognitive instruction effective? *Remedial and Special Education, 12* (3), 8-19.

Reisman, F. K. (1982). *A guide to the diagnostic teaching of arithmetic* (3rd ed.). Columbus, OH: Charles E. Merrill.

Resnick, L. B. (1992). From protoquantities to operators: Building mathematical competence on a foundation of everyday knowledge. In G. Leinhardt, R. Putnam & R. A. Hattrup (Eds.), *Analysis of arithmetic for mathematics teaching* (pp. 373-429). Hillsdale, NJ: Lawrence Erlbaum.

Resnick, L. B. (1982). Syntax and semantics in learning to subtract. In T. P. Carpenter, J. M. Moser & T. A. Romberg (Eds.), *Addition and subtraction: A cognitive perspective* (pp. 136-155). Hillsdale, NJ: Erlbaum.

Resnick, L. B., & Ford, W. W. (1984). *The psychology of mathematics for instruction*. Hillsdale, NJ: Lawrence Erlbaum.

Resnick, L., Nesher, P., Leonard, F., Magone, M., Omanson, S., & Peled, R. (1989). Conceptual bases of arithmetic errors: The case of decimal fractions. *Journal for Research in Mathematics Education, 20* (1), 8-27.

Resnick, L. B., & Omanson, S. F. (1987). Learning to understand arithmetic. In R. Glaser (Ed.), *Advances in instructional psychology (Vol. 3)* (pp. 41-95). Hillsdale, NJ: Lawrence Erlbaum.

Reys, R. E., Suydam, M. N., & Lindquist, M. M. (1992). *Helping children learn mathematics* (3rd ed.). Boston: Allyn & Bacon.

Robinson, F. G. (1981). *Rate and ratio: Classroom-tested curriculum materials for teachers at elementary level.* Ontario: OISE Press/The Ontario Institute for Studies in Education.

Rojewski, J. W., & Schell, J. W. (1994). Cognitive apprenticeship for learners with special needs: An alternative framework for teaching and learning. *Remedial and Special Education, 15* (4), 234-243.

Romberg, T. A. (1992). Perspectives on scholarship and research methods. In D. A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 49 - 64). New York: MacMillan Publishing Company.

Rowell, J. A., Dawson, C. J., & Lyndon, H. (1990). Changing misconceptions: A challenge to science educators. *International Journal of Science Education, 12* (2), 167-175.

Schmalz, R. (1977). The teaching of percent. *Mathematics Teacher, 70*, 340-343.

Schmittau, J. (1993a). Vygotskian scientific concepts: Implications for mathematics education. *Focus on Learning Problems in Mathematics, 15* (2 & 3), 29-39.

Schmittau, J. (1993b). Vygotskian psychology and dialectical logic: A psychological-epistemological foundation for contemporary education. *The Review of Education, 15*, 13-20.

Schulman, L. S. (1990). *Research in teaching and learning: Vol. 1, Paradigms and Programs.* New York: MacMillan.

Shapiro, E. S. (1989). *Academic Skills Problems: Direct Assessment and Intervention.* New York: The Guilford Press.

Silver, E. A. (1982). *Thinking about problem solving: Toward an understanding of metacognitive aspects of mathematical problem solving.* Paper presented at the Conference on Thinking, Suva, Fiji, January.

Skemp, R. R. (1978). Relational understanding and instrumental understanding. *Arithmetic Teacher, 16* (3), 9-15.

Smart, J. R. (1980). The teaching of percent problems. *School Science and Mathematics, 80*, 187-192.

Smith III, J. P. (1996). Efficacy and teaching mathematics by telling: A challenge to reform. *Journal for Research in Mathematics Education, 27* (4), 387-402.

Strauss, A., & Corbin, J. (1990). *Basics of qualitative research: Grounded theory procedures and techniques.* Newbury Park, CA: Sage Publications.

Stefanich, G. P., & Rokusek, T. (1992). An analysis of computational errors in the use of division algorithms by fourth-grade students. *School Science and Mathematics, 92* (4), 201-205.

Streefland, L. (1985). Search for the roots of ratio: some thoughts on the long term learning process (towards ... a theory). *Educational Studies in Mathematics, 16*, 75-94.

Stroop, J. R. (1935). Studies of interference in serial verbal reactives. *Journal of Experimental Psychology. 18*, 643-662.

Swanson, D., Schwartz, R., Ginsburg, H., & Kossan, N. (1981). The clinical interview: Validity, reliability, and diagnosis. *For the Learning of Mathematics, 2* (2), 31-38.

Sweller, J. (1992). Cognitive theories and their implications for mathematics instruction. In G. Leder (Ed.), *Assessment and learning of mathematics*, (pp. 46-62). Hawthorn, Vic: ACER.

Sweller, J. (1989). Cognitive technology: Some procedures for facilitating learning and problem solving in mathematics and science. *Journal of Educational Psychology, 81*, 457-466.

Sweller, J. (1988). Cognitive load during problem solving: Effects on learning. *Cognitive Science, 12*, 257-285.

Sweller, J., & Low, R. (1992). Some cognitive factors relevant to mathematics instruction. *Mathematics Education Research Journal, 4* (1), 83-94.

Swetz, F. (1992). Fifteenth and sixteenth century arithmetic texts: What can we learn from them? *Science and Education, 1*, 365-378.

Teahan, T. (1979). How I learned to do percents. *Arithmetic Teacher, 26*, 16-17.

Thipkong, S., & Davis, E. J. (1991). Preservice elementary teachers' misconceptions in interpreting and applying decimals. *School Science and Mathematics, 91* (3), 93-99.

Thompson, P. W. (1994). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 179-234). Albany, NY: SUNY Press.

Thorndike, E. L. (1922). *The psychology of arithmetic.* New York: MacMillan.

Thorndike, E. L. (1913). *Educational psychology, Vol. II. The Psychology of Learning*. New York: Teachers College, Columbia University.

Tirosh, D. (1990). Inconsistencies in students' mathematical constructs. *Focus on Learning Problems in Mathematics, 12* (3&4), 111-129.

Tirosh, D., & Graeber, A. (1990). Evoking cognitive conflict to explore preservice teachers' thinking about division. *Journal for Research in Mathematics Education, 21* (2), 98-108.

Tourniaire, F., & Pulos, S. (1985). Proportional reasoning: A review of the literature. *Educational Studies in Mathematics, 16*, 181-204.

Tredway, D. C. , & Hollister, G. E. (1963). An experimental study of two approaches to teaching percentage. *Arithmetic Teacher, 10*, 491-495.

Underhill, R. (1988). Focus on research into practice in diagnostic and prescriptive mathematics: Mathematics learners' beliefs: A review. *Focus on Learning Problems in Mathematics, 10* (1), 55-69.

Underhill, R., Uprichard, A. E., & Heddens, J. W. (1980). *Diagnosing mathematical difficulties.* Columbus, OH: Charles E. Merrill.

Underwood, B. J. (1966). *Experimental Psychology (*2nd ed.). New York: Appleton-Century Crofts.

Uprichard, A. E., & Engelhardt, J. (1986). A research context for diagnostic-prescriptive mathematics. *Focus on Learning Problems in Mathematics, 8* (1), 19-38.

Valett, R. E. (1976). *The remediation of learning disabilities: A handbook of psycho-educational resource programs* (2nd ed.). Belmont, Cal: Fearon.

Vance, J. H. (1992). Understanding equivalence: A number by any other name. *School Science and Mathematics, 92* (5), 263-266.

Van den Heuvel-Panhuizen, M. (1994). Improvement of (didactical) assessment by improvement of problems: An attempt with respect to percentage. *Educational studies in Mathematics, 27* (4), 341-372.

Van den Heuvel-Panhuizen, M., Middleton, J., & Streefland, L. (1995). Student-generated problems: Easy and difficult problems on percentage. *For the Learning of Mathematics, 15* (3), 21-27.

Van Lehn, K. (1983). On the representation of procedures in repair theory. In H. Ginsburg (Ed.), *The development of mathematical thinking* (pp. 201-252). Hillsdale, NJ: Academic Press.

Vernguard, G. (1988). Multiplicative structures. In J. Hiebert, & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 141-161). Reston, VA: NCTM; Hillsdale, NJ: Lawrence Erlbaum.

Vernguard, G. (1983). Multiplicative structures. In R. Lesh, & M. Landau (Eds.), *Acquisition of mathematical concepts and processes* (pp. 127-174). New York: Academic Press.

Vinner, S. (1990). Inconsistencies: Their causes and function in learning mathematics. *Focus on Learning Problems in Mathematics, 12* (3&4), 85-99.

Watson, J. (1994). The bu$ine$$ of decimals. *The Australian Mathematics Teacher, 50* (1), 4-6.

Wardrop, H. (1993). Mathematics language problems. *The Australian Mathematics Teacher, 49* (1), 10-13.

Wearne, D., & Hiebert, J. (1988). A cognitive approach to meaningful mathematics instruction: Testing a local theory using decimal numbers. *Journal for Research in Mathematics Education, 19* (5), 371-384.

Weibe, J. (1986). Manipulating percentages. *Mathematics Teacher, 79* (1), 23-26.

Weinstein, C. E., & Mayer, R. E. (1986). The teaching of learning strategies. In M. C. Wittrock (Ed.), *Handbook of research on teaching* (3rd ed.) (pp. 315-327). New York: Macmillan.

Wells, D. (1982) The effects of using a manipulative model for initiating regrouping. In M. L. Johnson, L. A. Raivek, J. Pitt, N. Wilson, & D. Wells. *Systematic approaches to remediation of mathematical difficulties.* Paper for the Ninth

Annual Conference of the Research Council for Diagnostic Prescriptive Mathematics.

Wendt, A. (1959). Per cent without cases. *Arithmetic Teacher, 6*, 209-214.

Wilson, J. W. (1976a). *Diagnosis and treatment in arithmetic: Beliefs, guiding models, and procedures.* Maryland: University of Maryland.

Wilson, J. W. (1976b). An epistemologically based system for classifying research and the role of the Clinical Intervention research within that system. In J. W. Hedden & F. D. Aquila (Eds.), *Proceedings of the Third National Conference on Remedial Mathematics.* Kent, Ohio: Kent State University.

Wilson, N. (1982) A remedial procedure for a specific multiplication error. In M. L. Johnson, L. A. Raivek, J. Pitt, N. Wilson, & D. Wells. *Systematic approaches to remediation of mathematical difficulties.* Paper for the Ninth Annual Conference on the Research Council for Diagnostic Prescriptive Mathematics.

Woodward, J., & Howard, J. (1994). The misconceptions of youth: Errors and their mathematical meaning. *Exceptional Children, 61* (2), 126-136.

Wolters, M. A. D. (1983). The part-whole schema and arithmetical problems. *Educational Studies in Mathematics, 14*, 127-138.

Appendix A

Percent Knowledge Test taxomony

SECTION I: Intuitive, Principled-Conceptual Percent Knowledge

Real world language - real world transactions

1. a) % discount as less than whole
   b) % profit as more than whole
   c) % loss as less than whole
   d) % interest on borrowings as repaying more than the whole
   e) % interest on savings as receiving more than the whole

Benchmarks

2. a) 50% is half
   b) 25% is one quarter
   c) 10% is one in ten

Concept

3. a) Whole is 100%
   b) Complement: part+part=whole
   c) % as part/whole - relationship
   d) Fraction as % as part/whole relationship
   e) Increase as more than 100%

Percent - decimal equivalence principle

4. Decimal hundredths expressed as percent

Percent - fraction equivalence principle

5. Fraction expressed as percent

Additive/subtractive language of decrease/increase

6. a) Discount as whole less % discount (complement)
   b) Increase as whole plus % increase (easy - <100%)
   c) Increase as whole plus % increase (hard - >100%)

Multiplicative language of increase/decrease

7. a) Multiplicative decrease/discount
   b) Multiplicative increase (easy - <100%)
   c) Multiplicative increase (hard - >100%)

Posing real world percent situation problems

8. a) Type 1
   b) Type 2
   c) Type 3

SECTION II: Conversions and benchmarking

Percent to fraction equivalence

1. a) 2 digit b) 1 digit c) 3 digit

Percent to decimal equivalence

2. a) 2 digit b) 1 digit c) 3 digit

Appendix A (Cont.)

Percent Knowledge Test taxomony

Decimal to percent equivalence

3. a) hundredths b) tenths c) ones & tenths

Fraction to percent equivalence

4. a) denominator 10 b) denominator 5 c) denominator 20

Benchmarks (mental computation)

5.	a)	25%	f)	150%
	b)	50%	g)	15% (10%+5%)
	c)	10%	h)	30% (3 x 10%)
	d)	100%	i)	60% (6 x 10%)
	e)	75%	j)	$33^{1}/_{3}$% ($^{1}/_{3}$)

SECTION III: Percent calculations and problem solving

Percent calculations

1. Type I percent calculations
2. Type II percent calculations
3. Type III percent calculations

Percent problem analysis and problem solving

4. Type I
5. Type II
6. Type III
   a) matching word problem to numerical expression
   b) diagrammatic representation of problem
   c) solving of problem
   d) expression of solution in words

Multi-step problem solving

7. Additive increase (NAEP item)
8. Subtractive decrease (NAEP item)
9. Increase (hard - > 100%)
10. Large numbers (type I)

Appendix B

Delayed posttest taxonomy

Intuitive, principled-conceptual percent knowledge

1. Interpreting additive language of percent increase/decrease (discount)
2. Interpreting multiplicative language of percent increase/decrease (increase>100%)

Concrete/computational percent knowledge

3. Solving percent equations
    a) Type I
    b) Type II
    c) Type III

Interpreting and solving percent problems

4. a) Type I
    b) Type II
    c) Type III

Solving two-step problems

5. a) additive/subtractive
    b) multiplicative

Appendix C

Type I, II and III percent problems for overhead transparencies

INTERPRETING % PROBLEMS

(i) Solve the following percent problems.

1. There were 240 people in the hall. 186 people were locals. What percent were local people?

2. Of the 368 people at the school, only 97 were girls. What percent were girls?

3. At the concert, 587 of the 975 people were under the age of 16. What percent were under the age of 16?

4. 358 of the 700 bricks had faults in them. What percent of bricks were faulty?

5. Of the 1075 people interviewed, 972 said they exercised regularly. What percent was that?

(ii) Write "story" problems to match the following equations.

1. Δ % of 50 = 18

2. Δ% of 98 = 72

3. Δ % of 12 = 10

Type I, II and III percent problems for overhead transparencies

INTERPRETING % PROBLEMS

(i) Solve the following percent problems.

1. Of the 195 people at the local football match, 20% were children. How many were children?

2. 83% of the cans were recycled. If there were 2547 cans, how many were recycled?

3. Sally earns $275 a week and has to spend 15% on sport. How much does she spend on sport?

4. In a school of 875 students, the measles epidemic infected 72% How many students were infected?

5. 23% of the 900 watches were diamond encrusted. How many was this?

(ii) Write "story" problems to match the following equations.

1. 25% of 80 = Δ

2. 16% of 900 = Δ

3. 52% of 45 = Δ

Type I, II and III percent problems for overhead transparencies

INTERPRETING % PROBLEMS

(i) Solve the following percent problems.

1. In a school, 243 of the students competed in the zone sports, which was 27% of the whole school. How many students attend the school?

2. In the carton of apples, 256 were bruised, which was 75% of the apples. How many apples were in the carton?

3. At a sale, I bought a shirt for $19.50, which was 35% of the original price. What was the original price?

4. 72% of the tadpoles grew into frogs, which was 198 tadpoles. How many tadpoles were there originally?

5. The school has 453 girls, which is 62% of the total school population. How many students are at the school?

(ii) Write "story" problems to match the following equations.

1. 65% of $\Delta = 29$

2. 35% of $\Delta = 86$

3. 16% of $\Delta = 216$

Appendix D

Structured worksheet to develop the proportional number line method

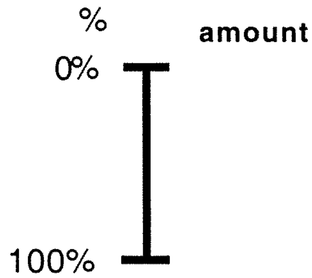
PERCENT PROBLEMS WORKSHEET 1

For each of the following, **identify**:
- the PART
- the WHOLE
- the PERCENT

Draw a diagram to show this information. **Write** a percent equation to show this information.

1. Of the 950 people at the basketball match, 490 were Raiders fans. What percent were Raiders fans?

DIAGRAM



PART: ______

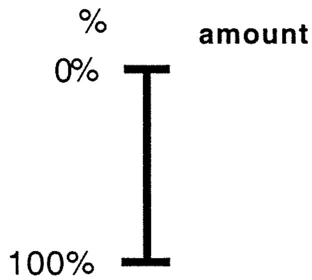
WHOLE: ______

PERCENT: ______

EQUATION: ___ % of ___ = ___

2. For my Science test, I got 58 out of 80. What was my percent for the test?

DIAGRAM



PART: ______

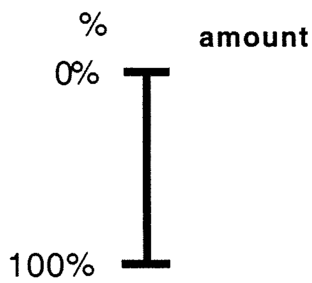
WHOLE: ______

PERCENT: ______

EQUATION: ___ % of ___ = ___

3. 25% of the 7018 teenagers interviewed said they were smokers. How many were smokers?

DIAGRAM



PART: ______

WHOLE: ______

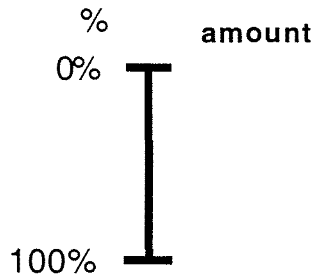
PERCENT: ______

EQUATION: ___ % of ___ = ___

Structured worksheet to develop the proportional number line method

4. In a school of 853 students, 60% travel to school by bus. How many is this?

DIAGRAM



PART: ______

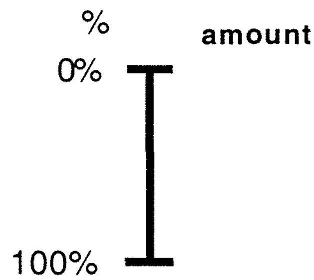
WHOLE: ______

PERCENT: ______

EQUATION: ____ % of ____ = ____

5. In the hospital building, there were 430 nurses in duty, which was 37% of the nursing staff. How many nurses are on staff?

DIAGRAM



PART: ______

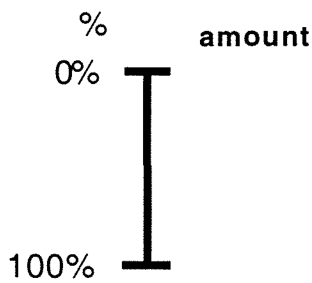
WHOLE: ______

PERCENT: ______

EQUATION: ____ % of ____ = ____

6. At a sale, I paid $350 for a jacket after it had been reduced 45%. What was the original price of the jacket?

DIAGRAM



PART: ______

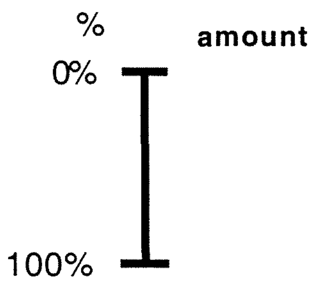
WHOLE: ______

PERCENT: ______

EQUATION: ____ % of ____ = ____

7. The survey reported that 35% of teenagers had experimented with alcohol before they were 16. If 9854 teenagers under sixteen were interviewed, how many had experimented with alcohol?

DIAGRAM



PART: ______

WHOLE: ______

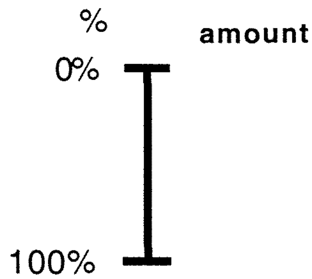
PERCENT: ______

EQUATION: ____ % of ____ = ____

Structured worksheet to develop the proportional number line method

8. The Corbett family spent $1148 for clothes last year. Their income was $12 600. What percentage of their income was spent on clothes?

DIAGRAM



PART: ______

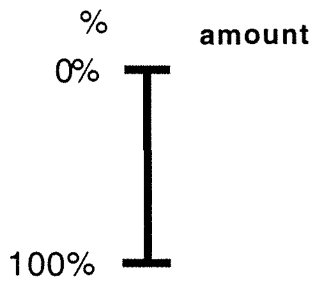
WHOLE: ______

PERCENT: ______

EQUATION: ____ % of ____ = ____

9. A family which has an income of $1923 per month anticipates that they will spend about 18% on their mortgage payment. What will their monthly mortgage payment be?

DIAGRAM



PART: ______

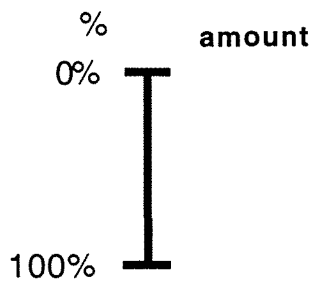
WHOLE: ______

PERCENT: ______

EQUATION: ____ % of ____ = ____

10. Mary Anne needed a score of 80% or better on a test containing 34 questions. How many could she miss and still the score she needed?

DIAGRAM



PART: ______

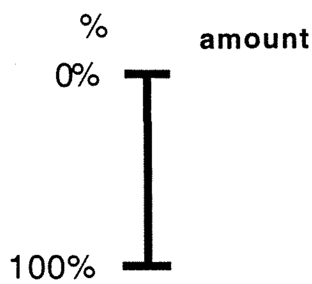
WHOLE: ______

PERCENT: ______

EQUATION: ____ % of ____ = ____

11. One school year, Bill was absent 12 days and present 168 days. What percent of the time was he absent?

DIAGRAM



PART: ______

WHOLE: ______

PERCENT: ______

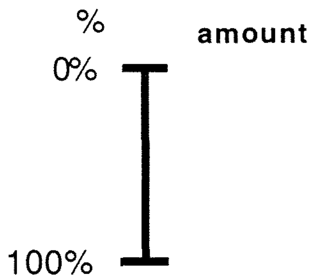
EQUATION: ____ % of ____ = ____

Appendix D (Cont.)

Structured worksheet to develop the proportional number line method

12. When buying a washing machine costing $400, the Bensons paid $32 deposit. What percentage of the purchase price was the deposit?

DIAGRAM



PART: ______

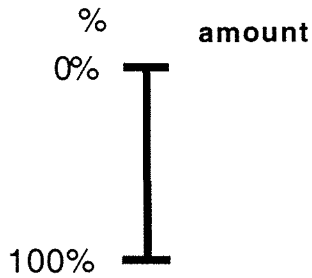
WHOLE: ______

PERCENT: ______

EQUATION: ____ % of ____ = ____

13. What percentage of 83 is 25?

DIAGRAM



PART: ______

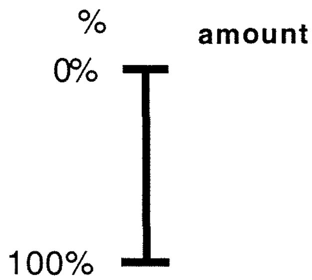
WHOLE: ______

PERCENT: ______

EQUATION: ____ % of ____ = ____

14. 12 is 35% of what number?

DIAGRAM



PART: ______

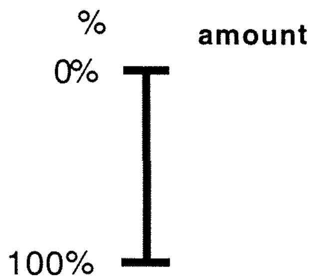
WHOLE: ______

PERCENT: ______

EQUATION: ____ % of ____ = ____

15. 90% of what number is 60?

DIAGRAM



PART: ______

WHOLE: ______

PERCENT: ______

EQUATION: ____ % of ____ = ____

Appendix E

Percent problems worksheet 2

PERCENT PROBLEMS WORKSHEET 2
Solve the following problems. Draw a diagram, and show your working.

PROBLEM	DIAGRAM AND WORKING
1. There were 240 people in the hall. 186 were locals. What percent were local people?	
2. Of the 368 people at the school, only 97 were girls. What percent were girls?	
3. Of the 195 people at the football match, 20% were children. How many were children?	
4. 85% of the cans were recycled. If there were 2547 cans, how many were recycled?	
5. In a school, 243 of the students competed in the zone sports, which was 27% of the whole school. How many students attend the school?	
6. In the carton of apples, 265 were bruised, which was 75% of the apples. How many apples were in the carton?	
7. At the concert, 587 of the 975 people were under the age of 16. What percent were under the age of 16?	

Appendix E (Cont.)

Percent problems worksheet 2

8. Sally earns $275 a week and has to spend 15% on sport. How much does she spend on sport?	
9. 72% of the tadpoles grew into frogs, which was 198 tadpoles. How many tadpoles were there originally?	
10. 358 of the 700 bricks had faults in them. What percent of bricks were faulty?	
11. Of the 1075 people interviewed, 972 said they exercised regularly. What percent was that?	
12. At a sale, I bought a shirt for $19.50. which had been marked down by 35%. What was the original cost of the shirt?	
13. 73% of what numbers is 450?	
14. What percentage of 75 is 43?	

Appendix E (Cont.)

Percent problems worksheet 2

Write "story problems" to match the following equations.

1. $\Delta\%$ of 50 = 18

____

____

2. $\Delta\%$ of 98 = 72

____

____

3. 25% of 80 = Δ

____

____

4. 16% of 900 =Δ

____

____

5. 65% of Δ = 29

____

____

6. 35% of Δ = 86

____

____

Appendix F

Multiple-choice worksheet for interpreting percent increase and decrease situations

INTERPRETING % INCREASE AND DECREASE PROBLEMS

Group name: ____

Group members: ____

____

____

SELECT the % **INCREASE** sentences which match the following situations.

NOTE: There may be more than one correct sentence for each situation.

1. A baby weighs 2kg at birth. Two months later, its weight has increased 25%. The baby's weight now is
   a) 25% of its birthweight
   b) 25% more than its birthweight
   c) 75% more than its birthweight
   d) 125% more than its birthweight
   e) 125% of its birthweight (2)

2. At 6:00am, there were 100 people lined up to buy State of Origin rugby tickets. At 9:00am, the crowd had increased 400%. The crowd size now is
   a) 3 times the original size (i.e. 300)
   b) 4 times the original size (i.e. 400)
   c) 5 times the original size (i.e. 500)
   d) 400% of the original size
   e) 500% of the original size
   f) 400% bigger than the original size
   g) 500% bigger than the original size (3)

3. The Mars Bar company decided to make Mars Bars 25% longer than before. Which of the following statements correctly describes the size of the new Mars Bar?
   a) The size of the new Mars Bar is 125% of the old Mars Bar.
   b) The new Mars Bar is 25% of the old Mars Bar.
   c) The size of the new Mars Bar increased by 125%.
   d) The new Mars Bar is 25% bigger than the old Mars Bar. (2)

Appendix F (Cont.)

Multiple-choice worksheet for interpreting percent increase and decrease situations

4. The family size pizza is 50% bigger than the large size pizza. Which of the following statements correctly describes the size of the family pizza to the large pizza?

   a) The family pizza is 150% bigger than the large pizza
   b) The family pizza is 50% bigger than the large pizza
   c) In the family pizza, you get 50% more pizza than in the large pizza.
   d) The family pizza is 50% greater than the large pizza.
   e) The family pizza is 150% greater than the large pizza. (3)

5. I am now 400% heavier than I was when I was 5.
   Which of the following statements correctly describes my mass now with what I weighed when I was five?

   a) I am 4 times heavier than I was at 5.
   b) I am 5 times heavier than I was at 5.
   c) My weight is 400% of my weight when I was 5.
   d) My weight is 500% of my weight when I was 5.
   e) My weight has increased 400% since I was 5.
   f) My weight has increased 500% since I was 5. (3)

6. The population of a town increased 200% following a minor baby boom. The new population is now

   a) 200 times larger than the original population
   b) 200% of the old population
   c) 300% of the old population
   d) 200% larger than the old population
   e) 300% larger than the old population (2)

Multiple-choice worksheet for interpreting percent increase and decrease situations

DECREASE % PROBLEMS

Name: ____

SELECT the % **DECREASE** sentences which match the following situations.
NOTE: There may be more than one correct sentence for each situation.

1. Registration for the SLAM DUNK coaching clinic is $400. For early enrolment there is a 35% reduction in the fee cost. Early registration fees are
   a) 35% of the normal registration fee of $400
   b) 35% less than the normal registration fee of $400
   c) 65% less than the normal registration fee of $400
   d) 65% of the normal registration fee of $400 (2)

2. At Target Michael Jackson's new CD is selling for $44.95. At sale time, this CD is to be discounted 25%. The sale price you will pay is
   a) 75% of the original price of $44.95
   b) 25% of the original price of $44.95
   c) 125% of the original price of $44.95
   d) 75% less than the original price of $44.95
   a) 25% less than the original price of $44.95 (2)

3. Last year, I weighed 60kg. Since I have taken up playing Squash 3 times a week, I have reduced my weight by 10%. Which of the following statements correctly compares my weight last year to my weight now?
   a) I weigh 10% of what I weighed last year.
   b) I weigh 90% of what I weighed last year.
   c) I weigh 110% of what I weighed last year.
   d) I have reduced my weight 90%.
   e) I have reduced my weight 10%.
   f) My weight has decreased by 10%.
   g) My weight has decreased by 90%. (3)

Multiple-choice worksheet for interpreting percent increase and decrease situations

4. Bronco football club jumpers were marked $89. At sale time, they were discounted 40%. To buy a club jumper, you would pay

   a) 40% of the marked price of $89
   b) 40% less than the marked price of $89
   c) 60% of the marked price of $89
   d) 60% less than the marked price of $89 (2)

5. The town of Gorgonville has a population of 28,000. After an epidemic virus swept through the town, 15% of people died. The population of Gorgonville is now

   a) 15% less than the original population of 28,000
   b) 85% less than the original population of 28,000
   c) 15% of the original population of 28,000
   d) 85% of the original population of 28,000
   e) reduced by 15%
   f) reduced by 85% (3)

Appendix G

Worksheet for percent, common and decimal fraction conversions

Fraction - decimal - percent equivalence

1. Express the following as percents

a)	0.29 = ________	f)	0.5 = ________
b)	0.58 = ________	g)	0.75 = ________
c)	0.01 = ________	h)	2.4 = ________
d)	0.1 = ________	i)	1 = ________
e)	1.25 = ________	j)	0.25 = ________

2. Express the following as fractions

a)	52% = ________	f)	75% = ________
b)	21% = ________	g)	10% = ________
c)	11% = ________	h)	25% = ________
d)	8% = ________	i)	$33^1/_3$% = ________
e)	50% = ________	j)	125% = ________

3. Express the following as decimals

a)	26% = ________	f)	25% = ________
b)	17% = ________	g)	10% = ________
c)	8% = ________	h)	110% = ________
d)	80% = ________	i)	75% = ________
e)	50% = ________	j)	100% = ________

Appendix G (Cont.)

Worksheet for percent, common and decimal fraction conversions

4. Express the following as fractions

a)	$^{1}/_{2}$= __________	i)	1 = __________
b)	$^{1}/_{4}$ = __________	j)	$^{47}/_{50}$ = __________
c)	$^{10}/_{100}$ = __________	k)	$^{17}/_{20}$ = __________
d)	$^{1}/_{10}$ = __________	l)	2 $^{97}/_{100}$ =__________
e)	$^{19}/_{25}$ = __________	m)	$^{4}/_{5}$ = __________
f)	$^{11}/_{20}$ =__________	n)	$^{3}/_{4}$ = __________
g)	$^{4}/_{5}$ = __________	o)	$^{22}/_{25}$ = __________
h)	$^{3}/_{10}$ = __________		

5. Complete the table below:

Percent	Fraction	Decimal
50%		
25%		
75%		
10%		
100%		
20%		
5%		
4%		
*33$^{1}/_{3}$%		
*66$^{2}/_{3}$%		

NOTE: Memorise the first 5 rows of this table

Worksheet for percent, common and decimal fraction conversions

6. Use the table to help you complete the following:

a)	50% of 40 = __________	i)	50% of 1100 = __________
b)	25% of 40 = __________	j)	25% of 100 = __________
c)	10% of 40 = __________	k)	100% of 60 = __________
d)	50% of 90 = __________	l)	150% of 60 = __________
e)	50% of 7806 = __________	m)	125% of 400 = __________
f)	100% of 500 = __________	n)	$33^1/_3$% of 900 = __________
g)	75% of 400 = __________	o)	15% of 60 = __________
h)	10% of 8520 = __________	p)	10% of 49,000 = __________

7. Use your knowledge of *fraction - decimal - percent* equivalence to help you calculate percent equations quickly. Try these:

a) In a bucket of 40 golf balls, 50% were white and the rest were coloured. How many golf balls were white? __________

b) In a class of 40 students, 25% of the students had part-time jobs. How many students had part-time jobs? __________

c) Sally earns $450 a week. She spends 10% of her wage on entertainment. How much is this? __________

d) Of the 32 children in the room, 100% had been immunised against measles. How many is this? __________

e) Of the 1200 people at the concert, 75% were under the age of 15. How many was this? __________

f) 20% of the golf balls hit landed in the pond. If 40 golf balls were hit, how many landed in the pond? __________

Worksheet for percent, common and decimal fraction conversions

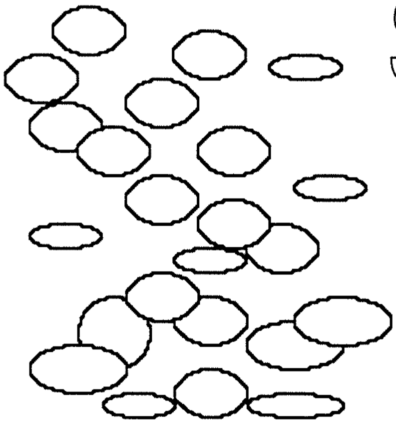
g) When the students voted, 50% said that they wanted to study Japanese instead of Indonesian. If 560 students voted, how many wanted to study Japanese?

h) Of the 780 tadpoles, only 25% matured to frogs. How many was this? __________

i) The Medicare levy requires wage earners to pay 1.25% of their income to Medicare. If a person earns $40 000 per year, approximately how much of their earnings is paid to Medicare? __________

Appendix H

Stimulus worksheet for analysing a collection



SMARTIE MATHS

Investigate the contents of your Smarties box.

Describe your findings in as many mathematical ways as possible.

fractions

decimals

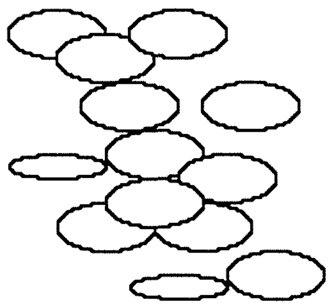
percents

ratio

graphs

proportion

equations



Appendix I

Summary of the historical development of percent

HISTORICAL DEVELOPMENT OF PERCENT

Year	Place	Development
300 BC	India	Interest rates given in panas per month per hundred.
200 - 100 BC	China	K'iu-ch'ang Suan-shu (Nine chapters on the mathematical art) The Rule of Three is used to solve problems.
499	India	Trairasiak - The Rule of Three terms: "*Multiply the first by the desire and divide by the measure. The result is the fruit of the desire.*" Compound interest.
628	India	Brahmagapta - Mercantile Rule of Three "*In the Rule of Three, Argument, Fruit, and Requisition are the names of the terms. The first and last terms must be similar. Requisition multiplied by Fruit and divided by Argument, is the Produce.*"
1202	Italy	Fibonacci - after travelling through Egypt, Syria, Greece and Sicily, writers *Liber Abaci* - an arithmetic of wide scope, including prices of goods, barter and partnership, utilising the Rule of Three in these areas.
1481	Italy	Earliest record available on the appearance of the word "*perceto*".
1545	Italy	Italian manuscripts use percent symbol and Rule of Three in commercial problems.
1650	Italy	*perceto*, *p.c.* etc begins to change to º, the precursor to the modern %.
1801	France	Playfair publishes the first pie chart in his *Statistical Breviary*.

Appendix J

Stimulus for discussion of percent language

The language of percent

Can we have:
50% class attendance?
90% class attendance?
100% class attendance?
110% class attendance?

What do the following mean:
75% full
25% full
100% full
25% empty

Can you be:
50% certain?
100% certain?
110% certain?

How much effort is:
50% effort?
10% effort?
100% effort?
110% effort?

Appendix K

A collection of percent problems for analysis in terms of part, whole and percent

INTERPRETING % PROBLEMS

1. There were 240 people in the hall. 186 people were locals. What percent were local people?

2. Of the 368 people at the school, only 97 were girls. What percent were girls?

3. Of the 195 people at the local football match, 20% were children. How many were children?

4. 83% of the cans were recycled. If there were 2547 cans, how many were recycled?

5. In a school, 243 of the students competed in the zone sports, which was 27% of the whole school. How many students attend the school?

6. In the carton of apples, 256 were bruised, which was 75% of the apples. How many apples were in the carton?

7. At the concert, 587 of the 975 people were under the age of 16. What percent were under the age of 16?

8. In a school of 875 students, the measles epidemic infected 72% How many students were infected?

9. At a sale, I bought a shirt for $19.50, which was 35% of the original price. What was the original price?

10. 358 of the 700 bricks had faults in them. What percent of bricks were faulty?

A collection of percent problems for analysis in terms of part, whole and percent

11. Of the 1075 people interviewed, 972 said they exercised regularly. What percent was that?

12. Sally earns $275 a week and has to spend 15% on sport. How much does she spend on sport?

13. 23% of the 900 watches were diamond encrusted. How many was this?

14. 72% of the tadpoles grew into frogs, which was 198 tadpoles. How many tadpoles were there originally?

15. The school has 453 girls, which is 62% of the total school population. How many students are at the school?

A collection of percent problems for analysis in terms of part, whole and percent

PART WHOLE % WORKSHEET

Identify the PART, the WHOLE, and the % in each of the problems.

1.	PART______	WHOLE______	%______
2.	PART______	WHOLE______	%______
3.	PART______	WHOLE______	%______
4.	PART______	WHOLE______	%______
5.	PART______	WHOLE______	%______
6.	PART______	WHOLE______	%______
7.	PART______	WHOLE______	%______
8.	PART______	WHOLE______	%______
9.	PART______	WHOLE______	%______
10.	PART______	WHOLE______	%______
11.	PART______	WHOLE______	%______
12.	PART______	WHOLE______	%______
13.	PART______	WHOLE______	%______
14.	PART______	WHOLE______	%______
15.	PART______	WHOLE______	%______