Math 475 **Text: Brualdi, Introductory Combinatorics 5th Ed.**
Prof: Paul Terwilliger
Selected solutions for Chapter 5

1. For an integer k and a real number n, we show

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

First assume $k \leq -1$. Then each side equals 0. Next assume $k = 0$. Then each side equals 1. Next assume $k \geq 1$. Recall

$$P(n,k) = n(n-1)(n-2)\cdots(n-k+1).$$

We have

$$\binom{n}{k} = \frac{P(n,k)}{k!} = \frac{nP(n-1,k-1)}{k!}.$$

$$\binom{n-1}{k-1} = \frac{P(n-1,k-1)}{(k-1)!} = \frac{kP(n-1,k-1)}{k!}.$$

$$\binom{n-1}{k} = \frac{P(n-1,k)}{k!} = \frac{(n-k)P(n-1,k-1)}{k!}.$$

The result follows.

2. Pascal's triangle begins

$$\begin{array}{ccccccccccccccccccccc}
&&&&&&&&&&1&&&&&&&&&&\\
&&&&&&&&&1&&1&&&&&&&&&\\
&&&&&&&&1&&2&&1&&&&&&&&\\
&&&&&&&1&&3&&3&&1&&&&&&&\\
&&&&&&1&&4&&6&&4&&1&&&&&&\\
&&&&&1&&5&&10&&10&&5&&1&&&&&\\
&&&&1&&6&&15&&20&&15&&6&&1&&&&\\
&&&1&&7&&21&&35&&35&&21&&7&&1&&&\\
&&1&&8&&28&&56&&70&&56&&28&&8&&1&&\\
&1&&9&&36&&84&&126&&126&&84&&36&&9&&1&\\
1&&10&&45&&120&&210&&252&&210&&120&&45&&10&&1\\
&&&&&&&&&\cdot&\cdot&\cdot&&&&&&&&&
\end{array}$$

3. Let $\mathbb{Z}$ denote the set of integers. For nonnegative $n \in \mathbb{Z}$ define $F(n) = \sum_{k\in\mathbb{Z}} \binom{n-k}{k}$. The sum is well defined since finitely many summands are nonzero. We have $F(0) = 1$ and $F(1) = 1$. We show $F(n) = F(n-1) + F(n-2)$ for $n \geq 2$. Let n be given. Using Pascal's formula and a change of variables $k = h + 1$,

$$\begin{aligned} F(n) &= \sum_{k\in\mathbb{Z}} \binom{n-k}{k} \\ &= \sum_{k\in\mathbb{Z}} \binom{n-k-1}{k} + \sum_{k\in\mathbb{Z}} \binom{n-k-1}{k-1} \\ &= \sum_{k\in\mathbb{Z}} \binom{n-k-1}{k} + \sum_{h\in\mathbb{Z}} \binom{n-h-2}{h} \\ &= F(n-1) + F(n-2). \end{aligned}$$

Thus $F(n)$ is the nth Fibonacci number.

4. We have

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

and

$$(x+y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6.$$

5. We have

$$(2x-y)^7 = \sum_{k=0}^{7} \binom{7}{k} 2^{7-k}(-1)^k x^{7-k} y^k.$$

6. The coefficient of x^5y^{13} is $3^5(-2)^{13}\binom{18}{5}$. The coefficient of x^8y^9 is 0 since $8 + 9 \neq 18$.

7. Using the binomial theorem,

$$3^n = (1+2)^n = \sum_{k=0}^{n} \binom{n}{k} 2^k.$$

Similarly, for any real number r,

$$(1+r)^n = \sum_{k=0}^{n} \binom{n}{k} r^k.$$

8. Using the binomial theorem,

$$2^n = (3-1)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} 3^{n-k}.$$

9. We have

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}10^k = (-1)^n\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}10^k = (-1)^n(10-1)^n = (-1)^n 9^n.$$

The sum is 9^n for n even and -9^n for n odd.

10. Given integers $1 \le k \le n$ we show

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

Let S denote the set of ordered pairs (x, y) such that x is a k-subset of $\{1, 2, \ldots, n\}$ and y is an element of x. We compute $|S|$ in two ways. (i) To obtain an element (x, y) of S there are $\binom{n}{k}$ choices for x, and for each x there are k choices for y. Therefore $|S| = k\binom{n}{k}$. (ii) To obtain an element (x, y) of S there are n choices for y, and for each y there are $\binom{n-1}{k-1}$ choices for x. Therefore $|S| = n\binom{n-1}{k-1}$. The result follows.

11. Given integers $n \ge 3$ and $1 \le k \le n$. We show

$$\binom{n}{k} - \binom{n-3}{k} = \binom{n-1}{k-1} + \binom{n-2}{k-1} + \binom{n-3}{k-1}.$$

Let S denote the set of k-subsets of $\{1, 2, \ldots, n\}$. Let S_1 consist of the elements in S that contain 1. Let S_2 consist of the elements in S that contain 2 but not 1. Let S_3 consist of the elements in S that contain 3 but not 1 or 2. Let S_4 consist of the elements in S that do not contain 1 or 2 or 3. Note that $\{S_i\}_{i=1}^4$ partition S so $|S| = \sum_{i=1}^4 |S_i|$. We have

$$|S| = \binom{n}{k}, \quad |S_1| = \binom{n-1}{k-1}, \quad |S_2| = \binom{n-2}{k-1}, \quad |S_3| = \binom{n-3}{k-1}, \quad |S_4| = \binom{n-3}{k}.$$

The result follows.

12. We evaluate the sum

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2.$$

First assume that $n = 2m + 1$ is odd. Then for $0 \le k \le m$ the k-summand and the $(n-k)$-summand are opposite. Therefore the sum equals 0. Next assume that $n = 2m$ is even. To evaluate the sum in this case we compute in two ways the the coefficient of x^n in $(1-x^2)^n$. (i) By the binomial theorem this coefficient is $(-1)^m\binom{2m}{m}$. (ii) Observe $(1-x^2) = (1+x)(1-x)$. We have

$$(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k,$$
$$(1-x)^n = \sum_{k=0}^{n}\binom{n}{k}(-1)^k x^k.$$

By these comments the coefficient of x^n in $(1-x^2)^n$ is

$$\sum_{k=0}^{n} \binom{n}{n-k} (-1)^k \binom{n}{k} = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^2.$$

Therefore

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 = (-1)^m \binom{2m}{m}.$$

13. We show that the given sum is equal to

$$\binom{n+3}{k}.$$

The above binomial coefficient is in row $n+3$ of Pascal's triangle. Using Pascal's formula, write the above binomial coefficient as a sum of two binomial coeffients in row $n+2$ of Pascal's triangle. Write each of these as a sum of two binomial coeffients in row $n+1$ of Pascal's triangle. Write each of these as a sum of two binomial coeffients in row n of Pascal's triangle. The resulting sum is

$$\binom{n}{k} + 3\binom{n}{k-1} + 3\binom{n}{k-2} + \binom{n}{k-3}.$$

14. Given a real number r and integer k such that $r \neq k$. We show

$$\binom{r}{k} = \frac{r}{r-k} \binom{r-1}{k}.$$

First assume that $k \leq -1$. Then each side is 0. Next assume that $k = 0$. Then each side is 1. Next assume that $k \geq 1$. Observe

$$\binom{r}{k} = \frac{P(r,k)}{k!} = \frac{rP(r-1,k-1)}{k!},$$

and

$$\binom{r-1}{k} = \frac{P(r-1,k)}{k!} = \frac{(r-k)P(r-1,k-1)}{k!}.$$

The result follows.

15. For a variable x consider

$$(1-x)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k x^k.$$

Take the derivative with respect to x and obtain

$$-n(1-x)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k k x^{k-1}.$$

Now set $x = 1$ to get

$$0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k k.$$

The result follows.

16. For a variable x consider

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Integrate with respect to x and obtain

$$\frac{(1+x)^{n+1}}{n+1} = \sum_{k=0}^{n} \binom{n}{k} \frac{x^{k+1}}{k+1} + C$$

for a constant C. Set $x = 0$ to find $C = 1/(n+1)$. Thus

$$\frac{(1+x)^{n+1} - 1}{n+1} = \sum_{k=0}^{n} \binom{n}{k} \frac{x^{k+1}}{k+1}.$$

Now set $x = 1$ to get

$$\frac{2^{n+1} - 1}{n+1} = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1}.$$

17. Routine.

18. For a variable x consider

$$(x-1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} x^k.$$

Integrate with respect to x and obtain

$$\frac{(x-1)^{n+1}}{n+1} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \frac{x^{k+1}}{k+1} + C$$

for a constant C. Set $x = 0$ to find $C = (-1)^{n+1}/(n+1)$. Thus

$$\frac{(x-1)^{n+1} - (-1)^{n+1}}{n+1} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \frac{x^{k+1}}{k+1}.$$

Now set $x = 1$ to get

$$\frac{(-1)^n}{n+1} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \frac{1}{k+1}.$$

Therefore

$$\frac{1}{n+1} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{k} \frac{1}{k+1}.$$

19. One readily checks

$$2\binom{m}{2} + \binom{m}{1} = m(m-1) + m = m^2.$$

Therefore

$$\begin{aligned} \sum_{k=1}^{n} k^2 &= \sum_{k=0}^{n} k^2 \\ &= 2\sum_{k=0}^{n} \binom{k}{2} + \sum_{k=0}^{n} \binom{k}{1} \\ &= 2\binom{n+1}{3} + \binom{n+1}{2} \\ &= \frac{(n+1)n(2n+1)}{6}. \end{aligned}$$

20. One readily checks

$$m^3 = 6\binom{m}{3} + 6\binom{m}{2} + \binom{m}{1}.$$

Therefore

$$\begin{aligned} \sum_{k=1}^{n} k^3 &= \sum_{k=0}^{n} k^3 \\ &= 6\sum_{k=0}^{n} \binom{k}{3} + 6\sum_{k=0}^{n} \binom{k}{2} + \sum_{k=0}^{n} \binom{k}{1} \\ &= 6\binom{n+1}{4} + 6\binom{n+1}{3} + \binom{n+1}{2} \\ &= \frac{(n+1)^2 n^2}{4} \\ &= \binom{n+1}{2}^2. \end{aligned}$$

21. Given a real number r and an integer k. We show

$$\binom{-r}{k} = (-1)^k \binom{r+k-1}{k}.$$

First assume that $k < 0$. Then each side is zero. Next assume that $k \geq 0$. Observe

$$\begin{aligned} \binom{-r}{k} &= \frac{(-r)(-r-1)\cdots(-r-k+1)}{k!} \\ &= (-1)^k \frac{r(r+1)\cdots(r+k-1)}{k!} \\ &= (-1)^k \binom{r+k-1}{k}. \end{aligned}$$

22. Given a real number r and integers k, m. We show

$$\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k}.$$

First assume that $m < k$ or $k < 0$. Then each side is zero. Next assume that $0 \leq k \leq m$. Observe

$$\begin{aligned} \binom{r}{m}\binom{m}{k} &= \frac{r(r-1)\cdots(r-m+1)}{m!} \frac{m!}{k!(m-k)!} \\ &= \frac{r(r-1)\cdots(r-m+1)}{k!(m-k)!} \\ &= \frac{r(r-1)\cdots(r-k+1)}{k!} \frac{(r-k)(r-k-1)\cdots(r-m+1)}{(m-k)!} \\ &= \binom{r}{k}\binom{r-k}{m-k}. \end{aligned}$$

23. (a) $\binom{24}{10}$.

(b) $\binom{9}{4}\binom{15}{6}$.

(c) $\binom{9}{4}\binom{9}{3}\binom{6}{3}$.

(d) $\binom{9}{4}\binom{15}{6} - \binom{9}{4}\binom{9}{3}\binom{6}{3}$.

24. The number of walks of length 45 is equal to the number of words of length 45 involving 10 x's, 15 y's, and 20 z's. This number is

$$\frac{45!}{10! \times 15! \times 20!}.$$

25. Given integers $m_1, m_2, n \geq 0$. Show

$$\sum_{k=0}^{n} \binom{m_1}{k}\binom{m_2}{n-k} = \binom{m_1+m_2}{n}.$$

Let A denote a set with cardinality m_1+m_2. Partition A into subsets A_1, A_2 with cardinalities m_1 and m_2 respectively. Let S denote the set of n-subsets of A. We compute $|S|$ in two ways. (i) By construction

$$|S| = \binom{m_1+m_2}{n}.$$

(ii) For $0 \leq k \leq n$ let the set S_k consist of the elements in S whose intersection with A_1 has cardinality k. The sets $\{S_k\}_{k=0}^{n}$ partition S, so $|S| = \sum_{k=0}^{n} |S_k|$. For $0 \leq k \leq n$ we now compute $|S_k|$. To do this we construct an element $x \in S_k$ via the following 2-stage procedure:

stage	to do	# choices
1	pick $x \cap A_1$	$\binom{m_1}{k}$
2	pick $x \cap A_2$	$\binom{m_2}{n-k}$

The number $|S_k|$ is the product of the entries in the right-most column above, which comes to $\binom{m_1}{k}\binom{m_2}{n-k}$. By these comments

$$|S| = \sum_{k=0}^{n} \binom{m_1}{k}\binom{m_2}{n-k}.$$

The result follows.

26. For an integer $n \geq 1$ show

$$\sum_{k=1}^{n} \binom{n}{k}\binom{n}{k-1} = \frac{1}{2}\binom{2n+2}{n+1} - \binom{2n}{n}.$$

Using Problem 25,

$$\begin{aligned}
\sum_{k=1}^{n} \binom{n}{k}\binom{n}{k-1} &= \sum_{k=0}^{n} \binom{n}{k}\binom{n}{k-1} \\
&= \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n+1-k} \\
&= \binom{2n}{n+1} \\
&= \frac{1}{2}\binom{2n}{n-1} + \frac{1}{2}\binom{2n}{n+1}.
\end{aligned}$$

It remains to show

$$\frac{1}{2}\binom{2n}{n-1} + \frac{1}{2}\binom{2n}{n+1} = \frac{1}{2}\binom{2n+2}{n+1} - \binom{2n}{n}.$$

This holds since

$$\begin{aligned}
\binom{2n}{n-1} + 2\binom{2n}{n} + \binom{2n}{n+1} &= \binom{2n+1}{n} + \binom{2n+1}{n+1} \\
&= \binom{2n+2}{n+1}.
\end{aligned}$$

27. Given an integer $n \geq 1$. We show

$$n(n+1)2^{n-2} = \sum_{k=1}^{n} k^2 \binom{n}{k}.$$

Let S denote the set of 3-tuples (s, x, y) such that s is a nonempty subset of $\{1, 2, \ldots, n\}$ and x, y are elements (not necessarily distinct) in s. We compute $|S|$ in two ways. (i) Call an element (s, x, y) of S *degenerate* whenever $x = y$. Partition S into subsets S^+, S^- with S^+ (resp. S^-) consisting of the degenerate (resp. nondegenerate) elements of S. So $|S| = |S^+| + |S^-|$. We compute $|S^+|$. To obtain an element (s, x, x) of S^+ there are n choices for x, and given x there are 2^{n-1} choices for s. Therefore $|S^+| = n2^{n-1}$. We compute $|S^-|$. To obtain an element (s, x, y) of S^- there are n choices for x, and given x there are $n-1$ choices for y, and given x, y there are 2^{n-2} choices for s. Therefore $|S^-| = n(n-1)2^{n-2}$. By these comments

$$|S| = n2^{n-1} + n(n-1)2^{n-2} = n(n+1)2^{n-2}.$$

(ii) For $1 \leq k \leq n$ let S_k denote the set of elements (s, x, y) in S such that $|s| = k$. The sets $\{S_k\}_{k=1}^n$ give a partition of S, so $|S| = \sum_{k=1}^n |S_k|$. For $1 \leq k \leq n$ we compute $|S_k|$. To obtain an element (s, x, y) of S_k there are $\binom{n}{k}$ choices for s, and given s there are k^2 ways to choose the pair x, y. Therefore $|S_k| = k^2\binom{n}{k}$. By these comments

$$|S| = \sum_{k=1}^{n} k^2 \binom{n}{k}.$$

The result follows.

28. Given an integer $n \geq 1$. We show

$$\sum_{k=1}^{n} k \binom{n}{k}^2 = n\binom{2n-1}{n-1}.$$

Let S denote the set of ordered pairs (s, x) such that s is a subset of $\{\pm 1, \pm 2, \ldots, \pm n\}$ and x is a positive element of s. We compute $|S|$ in two ways. (i) To obtain an element (s, x) of S There are n choices for x, and given x there are $\binom{2n-1}{n-1}$ choices for s. Therefore

$$|S| = n\binom{2n-1}{n-1}.$$

(ii) For $1 \leq k \leq n$ let S_k denote the set of elements (s, x) in S such that s contains exactly k positive elements. The sets $\{S_k\}_{k=1}^n$ partition S, so $|S| = \sum_{k=1}^n |S_k|$. For $1 \leq k \leq n$ we compute $|S_k|$. To obtain an element (s, x) of S_k there are $\binom{n}{k}$ ways to pick the positive elements of s and $\binom{n}{n-k} = \binom{n}{k}$ ways to pick the negative elements of s. Given s there are k ways to pick x. Therefore $|S_k| = k\binom{n}{k}^2$. By these comments

$$|S| = \sum_{k=1}^{n} k\binom{n}{k}^2.$$

The result follows.

29. The given sum is equal to

$$\binom{m_2 + m_2 + m_3}{n}.$$

To see this, compute the coefficient of x^n in each side of

$$(1+x)^{m_1}(1+x)^{m_2}(1+x)^{m_3} = (1+x)^{m_1+m_2+m_3}.$$

In this computation use the binomial theorem.

30, 31, 32. We refer to the proof of Theorem 5.3.3 in the text. Let $\mathcal{A}$ denote an antichain such that

$$|\mathcal{A}| = \binom{n}{\lfloor n/2 \rfloor}.$$

For $0 \leq k \leq n$ let α_k denote the number of elements in $\mathcal{A}$ that have size k. So

$$\sum_{k=0}^{n} \alpha_k = |\mathcal{A}| = \binom{n}{\lfloor n/2 \rfloor}.$$

As shown in the proof of Theorem 5.3.3,

$$\sum_{k=0}^{n} \frac{\alpha_k}{\binom{n}{k}} \leq 1,$$

with equality if and only if each maximal chain contains an element of $\mathcal{A}$. By the above comments

$$\sum_{k=0}^{n} \alpha_k \frac{\binom{n}{\lfloor n/2 \rfloor} - \binom{n}{k}}{\binom{n}{k}} \leq 0,$$

with equality if and only if each maximal chain contains an element of $\mathcal{A}$. The above sum is nonpositive but each summand is nonnegative. Therefore each summand is zero and the sum is zero. Consequently (a) each maximal chain contains an element of $\mathcal{A}$; (b) for $0 \leq k \leq n$ either α_k is zero or its coefficient is zero. We now consider two cases.

Case: n is even. We show that for $0 \le k \le n$, $\alpha_k = 0$ if $k \ne n/2$. Observe that for $0 \le k \le n$, if $k \ne n/2$ then the coefficient of α_k is nonzero, so $\alpha_k = 0$.
Case: n is odd. We show that for $0 \le k \le n$, either $\alpha_k = 0$ if $k \ne (n-1)/2$ or $\alpha_k = 0$ if $k \ne (n+1)/2$. Observe that for $0 \le k \le n$, if $k \ne (n \pm 1)/2$ then the coefficient of α_k is nonzero, so $\alpha_k = 0$. We now show that $\alpha_k = 0$ for $k = (n-1)/2$ or $k = (n+1)/2$. To do this, we assume that $\alpha_k \ne 0$ for both $k = (n \pm 1)/2$ and get a contradiction. By assumption $\mathcal{A}$ contains an element x of size $(n+1)/2$ and an element y of size $(n-1)/2$. Define $s = |x \cap y|$. Choose x, y such that s is maximal. By construction $0 \le s \le (n-1)/2$. Suppose $s = (n-1)/2$. Then $y = x \cap y \subseteq x$, contradicting the fact that x, y are incomparable. So $s \le (n-3)/2$. Let y' denote a subset of x that contains $x \cap y$ and has size $(n-1)/2$. Let x' denote a subset of $y' \cup y$ that contains y' and has size $(n+1)/2$. By construction $|x' \cap y| = s+1$. Observe y' is not in $\mathcal{A}$ since x, y' are comparable. Also x' is not in $\mathcal{A}$ by the maximality of s. By construction x' covers y' so they are together contained in a maximal chain. This chain does not contain an element of $\mathcal{A}$, for a contradiction.

33. Define a poset $(X, \le)$ as follows. The set X consists of the subsets of $\{1, 2, \ldots, n\}$. For $x, y \in X$ define $x \le y$ whenever $x \subseteq y$. For $n = 3, 4, 5$ we display a symmetric chain decomposition of this poset. We use the inductive procedure from the text.
For $n = 3$,

$$\begin{aligned} &\emptyset, 1, 12, 123 \\ &2, 23 \\ &3, 13. \end{aligned}$$

For $n = 4$,

$$\begin{aligned} &\emptyset, 1, 12, 123, 1234 \\ &4, 14, 124 \\ &2, 23, 234 \\ &24, \\ &3, 13, 134 \\ &34. \end{aligned}$$

For $n = 5$,

$$\begin{aligned} &\emptyset, 1, 12, 123, 1234, 12345 \\ &5, 15, 125, 1235 \\ &4, 14, 124, 1245 \\ &45, 145 \\ &2, 23, 234, 2345 \\ &25, 235 \\ &24, 245 \\ &3, 13, 134, 1345 \\ &35, 135 \\ &34, 345. \end{aligned}$$

34. For $0 \le k \le \lfloor n/2 \rfloor$ there are exactly $\binom{n}{k} - \binom{n}{k-1}$ symmetric chains of length $n - 2k + 1$.

35. Let S denote the set of 10 jokes. Each night the talk show host picks a subset of S for his repertoire. It is required that these subsets form an antichain. By Corollary 5.3.2 each antichain has size at most $\binom{10}{5}$, which is equal to 252. Therefore the talk show host can continue for 252 nights.

36. Compute the coefficient of x^n in either side of

$$(1+x)^{m_1}(1+x)^{m_2} = (1+x)^{m_1+m_2},$$

In this computation use the binomial theorem.

37. In the multinomial theorem (Theorem 5.4.1) set $x_i = 1$ for $1 \le i \le t$.

38. $(x_1 + x_2 + x_3)^4$ is equal to

$$\begin{aligned} &x_1^4 + x_2^4 + x_3^4 + 4(x_1^3 x_2 + x_1^3 x_3 + x_1 x_2^3 + x_2^3 x_3 + x_1 x_3^3 + x_2 x_3^3) \\ &+ 6(x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2) + 12(x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2). \end{aligned}$$

39. The coefficient is

$$\frac{10!}{3! \times 1! \times 4! \times 0! \times 2!}$$

which comes to 12600.

40. The coefficient is

$$\frac{9!}{3! \times 3! \times 1! \times 2!} \times 1^3 \times (-1)^3 \times 2 \times (-2)^2$$

which comes to -40320.

41. One routinely obtains the multinomial theorem (Theorem 5.4.1) with $t = 3$.

42. Given an integer $t \ge 2$ and positive integers $n_1, n_2, \ldots, n_t$. Define $n = \sum_{i=1}^t n_i$. We show

$$\binom{n}{n_1\ n_2\ \cdots\ n_t} = \sum_{k=1}^t \binom{n-1}{n_1\ \cdots\ n_{k-1}\ n_k - 1\ n_{k+1}\ \cdots\ n_t}.$$

Consider the multiset

$$\{n_1 \cdot x_1, n_2 \cdot x_2, \ldots, n_t \cdot x_t\}.$$

Let P denote the set of permutations of this multiset. We compute $|P|$ in two ways.
(i) We saw earlier that

$$|P| = \frac{n!}{n_1! \times n_2! \times \cdots \times n_t!} = \binom{n}{n_1\ n_2\ \cdots\ n_t}.$$

(ii) For $1 \le k \le t$ let P_k denote the set of elements in P that have first coordinate x_k. The sets $\{P_k\}_{k=1}^t$ partition P, so $|P| = \sum_{k=1}^t |P_k|$. For $1 \le k \le t$ we compute $|P_k|$. Observe that $|P_k|$ is the number of permutations of the multiset

$$\{n_1 \cdot x_1, \ldots, n_{k-1} \cdot x_{k-1}, (n_k - 1) \cdot x_k, n_{k+1} \cdot x_{k+1}, \ldots, n_t \cdot x_t\}.$$

Therefore

$$|P_k| = \binom{n-1}{n_1 \ \cdots \ n_{k-1} \ n_k - 1 \ n_{k+1} \ \cdots \ n_t}.$$

By these comments

$$|P| = \sum_{k=1}^{t} \binom{n-1}{n_1 \ \cdots \ n_{k-1} \ n_k - 1 \ n_{k+1} \ \cdots \ n_t}.$$

The result follows.

43. Given an integer $n \ge 1$. Show by induction on n that

$$\frac{1}{(1-z)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} z^k, \qquad |z| < 1.$$

The base case $n = 1$ is assumed to hold. We show that the above identity holds with n replaced by $n+1$, provided that it holds for n. Thus we show

$$\frac{1}{(1-z)^{n+1}} = \sum_{\ell=0}^{\infty} \binom{n+\ell}{\ell} z^\ell, \qquad |z| < 1.$$

Observe

$$\begin{aligned} \frac{1}{(1-z)^{n+1}} &= \frac{1}{(1-z)^n} \frac{1}{1-z} \\ &= \left(\sum_{k=0}^{\infty} \binom{n+k-1}{k} z^k \right) \left(\sum_{h=0}^{\infty} z^h \right) \\ &= \sum_{\ell=0}^{\infty} c_\ell z^\ell \end{aligned}$$

where

$$\begin{aligned} c_\ell &= \binom{n-1}{0} + \binom{n}{1} + \binom{n+1}{2} + \cdots + \binom{n+\ell-1}{\ell} \\ &= \binom{n+\ell}{\ell}. \end{aligned}$$

The result follows.

44. (Problem statement contains typo) The given sum is equal to $(-3)^n$. Observe

$$\begin{aligned}(-3)^n &= (-1-1-1)^n \\ &= \sum_{n_1+n_2+n_3=n} \binom{n}{n_1\ n_2\ n_3} (-1)^{n_1+n_2+n_3} \\ &= \sum_{n_1+n_2+n_3=n} \binom{n}{n_1\ n_2\ n_3} (-1)^{n_1-n_2+n_3}.\end{aligned}$$

Also

$$\begin{aligned}1 &= (1-1+1)^n \\ &= \sum_{n_1+n_2+n_3=n} \binom{n}{n_1\ n_2\ n_3} (-1)^{n_2}.\end{aligned}$$

45. (Problem statement contains typo) The given sum is equal to $(-4)^n$. Observe

$$\begin{aligned}(-4)^n &= (-1-1-1-1)^n \\ &= \sum_{n_1+n_2+n_3+n_4=n} \binom{n}{n_1\ n_2\ n_3\ n_4} (-1)^{n_1+n_2+n_3+n_4} \\ &= \sum_{n_1+n_2+n_3+n_4=n} \binom{n}{n_1\ n_2\ n_3\ n_4} (-1)^{n_1-n_2+n_3-n_4}.\end{aligned}$$

Also

$$\begin{aligned}0 &= (1-1+1-1)^n \\ &= \sum_{n_1+n_2+n_3+n_4=n} \binom{n}{n_1\ n_2\ n_3\ n_4} (-1)^{n_2+n_4}.\end{aligned}$$

46. Observe

$$\begin{aligned}\sqrt{30} &= 5\sqrt{\frac{30}{25}} \\ &= 5(1+z)^{1/2} \qquad\qquad z = 1/5, \\ &= 5\sum_{k=0}^{\infty} \binom{1/2}{k} z^k.\end{aligned}$$

For $n = 0, 1, 2, \dots$ the nth approximation to $\sqrt{30}$ is

$$a_n = 5\sum_{k=0}^{n} \binom{1/2}{k} 5^{-k}.$$

We have

n	a_n
0	5
1	5.5
2	5.475
3	5.4775
4	5.4771875
5	5.47723125
6	5.477224688
7	5.477225719
8	5.477225551
9	5.477225579

47. Observe

$$\begin{aligned} 10^{1/3} &= 2\left(\frac{10}{8}\right)^{1/3} \\ &= 2(1+z)^{1/3} \qquad\qquad z = 1/4, \\ &= 2\sum_{k=0}^{\infty}\binom{1/3}{k} z^k. \end{aligned}$$

For $n = 0, 1, 2, \ldots$ the nth approximation to $10^{1/3}$ is

$$a_n = 2\sum_{k=0}^{n}\binom{1/3}{k} 4^{-k}.$$

We have

n	a_n
0	2
1	2.166666667
2	2.152777778
3	2.154706790
4	2.154385288
5	2.154444230
6	2.154432769
7	2.154435089
8	2.154434605
9	2.154434708

48. We show that a poset with $mn + 1$ elements has a chain of size $m + 1$ or an antichain of size $n + 1$. Our strategy is to assume the result is false, and get a contradiction. By assumption each chain has size at most m and each antichain has size at most n. Let r denote the size of the longest chain. So $r \le m$. By Theorem 5.6.1 the elements of the poset can be partitioned into r antichains $\{A_i\}_{i=1}^{r}$. We have $|A_i| \le n$ for $1 \le i \le r$. Therefore

$$mn + 1 = \sum_{i=1}^{r} |A_i| \le rn \le mn,$$

for a contradiction. Therefore, the poset has a chain of size $m+1$ or an antichain of size $n+1$.

49. We are given a sequence of $mn+1$ real numbers, denoted $\{a_i\}_{i=0}^{mn}$. Let X denote the set of ordered pairs $\{(i, a_i) | 0 \leq i \leq mn\}$. Observe $|X| = mn+1$. Define a partial order $\leq$ on X as follows: for distinct $x = (i, a_i)$ and $y = (j, a_j)$ in X, declare $x < y$ whenever $i < j$ and $a_i \leq a_j$. For the poset $(X, \leq)$ the chains correspond to the (weakly) increasing subsequences of $\{a_i\}_{i=0}^{mn}$, and the antichains correspond to the (strictly) decreasing subsequences of $\{a_i\}_{i=0}^{mn}$. By Problem 48, there exists a chain of size $m+1$ or an antichain of size $n+1$. Therefore the sequence $\{a_i\}_{i=0}^{mn}$ has a (weakly) increasing subsequence of size $m+1$ or a (strictly) decreasing subsequence of size $n+1$.

50. (i) Here is a chain of size four: $1, 2, 4, 8$. Here is a partition of X into four antichains:

$$8, 12$$
$$4, 6, 9, 10$$
$$2, 3, 5, 7, 11$$
$$1$$

Therefore four is both the largest size of a chain, and the smallest number of antichains that partition X.

(ii) Here is an antichain of size six: $7, 8, 9, 10, 11, 12$. Here is a partition of X into six chains:

$$1, 2, 4, 8$$
$$3, 6, 12$$
$$9$$
$$5, 10$$
$$7$$
$$11$$

Therefore six is both the largest size of an antichain, and the smallest number of chains that partition X.

51. There exists a chain $x_1 < x_2 < \cdots < x_t$ of size $t \geq 2$ in the poset S such that $x_1 \not< x_t$ in the poset R. Indeed we could take $t = 2$ and let x_1, x_2 be elements of X such that $x_1 < x_2$ in S but $x_1 \not< x_2$ in R. Pick a chain as above with t maximal. Define $p = x_1$ and $q = x_t$. Then the pair (p, q) meets the requirements of the problem.