Math 475 **Text: Brualdi, Introductory Combinatorics 5th Ed.**
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Selected solutions for Chapter 4

1. The permutation 31524 is followed by 35124 and preceded by 31254. To see this, recall the display on page 90. In that display the permutations of $(1, 2, 3, 4)$ are listed in 24 rows. According to the algorithm in Section 4.1, for each row $1, 2, \ldots, 24$ we insert a 5 in the five possible locations of the given permutation, working from right to left for odd numbered rows and left to right for even numbered rows. The permutation 3124 appears on row 9 which is odd, so for this row the 5's are inserted from right to left.

2. The mobile integers are 8, 3, 7. For these integers the arrow points to an adjacent smaller integer.

3. Working from left to right across each row,

$\overleftarrow{1}\,\overleftarrow{2}\,\overleftarrow{3}\,\overleftarrow{4}\,\overleftarrow{5}$	$\overleftarrow{1}\,\overleftarrow{2}\,\overleftarrow{3}\,\overleftarrow{5}\,\overleftarrow{4}$	$\overleftarrow{1}\,\overleftarrow{2}\,\overleftarrow{5}\,\overleftarrow{3}\,\overleftarrow{4}$	$\overleftarrow{1}\,\overleftarrow{5}\,\overleftarrow{2}\,\overleftarrow{3}\,\overleftarrow{4}$	$\overleftarrow{5}\,\overleftarrow{1}\,\overleftarrow{2}\,\overleftarrow{3}\,\overleftarrow{4}$
$\overrightarrow{5}\,\overleftarrow{1}\,\overleftarrow{2}\,\overleftarrow{4}\,\overleftarrow{3}$	$\overleftarrow{1}\,\overrightarrow{5}\,\overleftarrow{2}\,\overleftarrow{4}\,\overleftarrow{3}$	$\overleftarrow{1}\,\overleftarrow{2}\,\overrightarrow{5}\,\overleftarrow{4}\,\overleftarrow{3}$	$\overleftarrow{1}\,\overleftarrow{2}\,\overleftarrow{4}\,\overrightarrow{5}\,\overleftarrow{3}$	$\overleftarrow{1}\,\overleftarrow{2}\,\overleftarrow{4}\,\overleftarrow{3}\,\overrightarrow{5}$
$\overleftarrow{1}\,\overleftarrow{4}\,\overleftarrow{2}\,\overleftarrow{3}\,\overleftarrow{5}$	$\overleftarrow{1}\,\overleftarrow{4}\,\overleftarrow{2}\,\overleftarrow{5}\,\overleftarrow{3}$	$\overleftarrow{1}\,\overleftarrow{4}\,\overleftarrow{5}\,\overleftarrow{2}\,\overleftarrow{3}$	$\overleftarrow{1}\,\overleftarrow{5}\,\overleftarrow{4}\,\overleftarrow{2}\,\overleftarrow{3}$	$\overleftarrow{5}\,\overleftarrow{1}\,\overleftarrow{4}\,\overleftarrow{2}\,\overleftarrow{3}$
$\overrightarrow{5}\,\overleftarrow{4}\,\overleftarrow{1}\,\overleftarrow{2}\,\overleftarrow{3}$	$\overleftarrow{4}\,\overrightarrow{5}\,\overleftarrow{1}\,\overleftarrow{2}\,\overleftarrow{3}$	$\overleftarrow{4}\,\overleftarrow{1}\,\overrightarrow{5}\,\overleftarrow{2}\,\overleftarrow{3}$	$\overleftarrow{4}\,\overleftarrow{1}\,\overleftarrow{2}\,\overrightarrow{5}\,\overleftarrow{3}$	$\overleftarrow{4}\,\overleftarrow{1}\,\overleftarrow{2}\,\overleftarrow{3}\,\overrightarrow{5}$
$\overrightarrow{4}\,\overleftarrow{1}\,\overleftarrow{3}\,\overleftarrow{2}\,\overleftarrow{5}$	$\overrightarrow{4}\,\overleftarrow{1}\,\overleftarrow{3}\,\overleftarrow{5}\,\overleftarrow{2}$	$\overrightarrow{4}\,\overleftarrow{1}\,\overleftarrow{5}\,\overleftarrow{3}\,\overleftarrow{2}$	$\overrightarrow{4}\,\overleftarrow{5}\,\overleftarrow{1}\,\overleftarrow{3}\,\overleftarrow{2}$	$\overleftarrow{5}\,\overrightarrow{4}\,\overleftarrow{1}\,\overleftarrow{3}\,\overleftarrow{2}$
$\overrightarrow{5}\,\overleftarrow{1}\,\overrightarrow{4}\,\overleftarrow{3}\,\overleftarrow{2}$	$\overleftarrow{1}\,\overrightarrow{5}\,\overrightarrow{4}\,\overleftarrow{3}\,\overleftarrow{2}$	$\overleftarrow{1}\,\overrightarrow{4}\,\overrightarrow{5}\,\overleftarrow{3}\,\overleftarrow{2}$	$\overleftarrow{1}\,\overrightarrow{4}\,\overleftarrow{3}\,\overrightarrow{5}\,\overleftarrow{2}$	$\overleftarrow{1}\,\overrightarrow{4}\,\overleftarrow{3}\,\overleftarrow{2}\,\overrightarrow{5}$
$\overleftarrow{1}\,\overleftarrow{3}\,\overrightarrow{4}\,\overleftarrow{2}\,\overleftarrow{5}$	$\overleftarrow{1}\,\overleftarrow{3}\,\overrightarrow{4}\,\overleftarrow{5}\,\overleftarrow{2}$	$\overleftarrow{1}\,\overleftarrow{3}\,\overleftarrow{5}\,\overrightarrow{4}\,\overleftarrow{2}$	$\overleftarrow{1}\,\overleftarrow{5}\,\overleftarrow{3}\,\overrightarrow{4}\,\overleftarrow{2}$	$\overleftarrow{5}\,\overleftarrow{1}\,\overleftarrow{3}\,\overrightarrow{4}\,\overleftarrow{2}$
$\overrightarrow{5}\,\overleftarrow{1}\,\overleftarrow{3}\,\overleftarrow{2}\,\overrightarrow{4}$	$\overleftarrow{1}\,\overrightarrow{5}\,\overleftarrow{3}\,\overleftarrow{2}\,\overrightarrow{4}$	$\overleftarrow{1}\,\overleftarrow{3}\,\overrightarrow{5}\,\overleftarrow{2}\,\overrightarrow{4}$	$\overleftarrow{1}\,\overleftarrow{3}\,\overleftarrow{2}\,\overrightarrow{5}\,\overrightarrow{4}$	$\overleftarrow{1}\,\overleftarrow{3}\,\overleftarrow{2}\,\overrightarrow{4}\,\overrightarrow{5}$
$\overleftarrow{3}\,\overleftarrow{1}\,\overleftarrow{2}\,\overleftarrow{4}\,\overleftarrow{5}$	$\overleftarrow{3}\,\overleftarrow{1}\,\overleftarrow{2}\,\overleftarrow{5}\,\overleftarrow{4}$	$\overleftarrow{3}\,\overleftarrow{1}\,\overleftarrow{5}\,\overleftarrow{2}\,\overleftarrow{4}$	$\overleftarrow{3}\,\overleftarrow{5}\,\overleftarrow{1}\,\overleftarrow{2}\,\overleftarrow{4}$	$\overleftarrow{5}\,\overleftarrow{3}\,\overleftarrow{1}\,\overleftarrow{2}\,\overleftarrow{4}$
$\overrightarrow{5}\,\overleftarrow{3}\,\overleftarrow{1}\,\overleftarrow{4}\,\overleftarrow{2}$	$\overleftarrow{3}\,\overrightarrow{5}\,\overleftarrow{1}\,\overleftarrow{4}\,\overleftarrow{2}$	$\overleftarrow{3}\,\overleftarrow{1}\,\overrightarrow{5}\,\overleftarrow{4}\,\overleftarrow{2}$	$\overleftarrow{3}\,\overleftarrow{1}\,\overleftarrow{4}\,\overrightarrow{5}\,\overleftarrow{2}$	$\overleftarrow{3}\,\overleftarrow{1}\,\overleftarrow{4}\,\overleftarrow{2}\,\overrightarrow{5}$

4. For $1 \le k \le n$ the direction of k changes immediately after we move a mobile integer m $(1 \le m < k)$. For $k = 1$ there is no such m. For $k = 2$ the only candidate for m is 1, but 1 is never mobile. Therefore the directions of 1 and 2 never change.

5. The integer k is equal to the total number of inversions for the given permutation. Any switch of adjacent terms $ab \to ba$ either decreases this total by one (if $a > b$) or increases this total by one (if $a < b$). Therefore we cannot bring the given permutation to $12 \cdots n$ by fewer than k successive switches of adjacent terms.

6. (a) We have

i	1	2	3	4	5	6	7	8
b_i	2	4	0	4	0	0	1	0

(b) We have

i	1	2	3	4	5	6	7	8
b_i	6	5	1	1	3	2	1	0

7. (a) Using algorithm 1,

8
87
867
8657
48657
486573
4865723
48165723

Using algorithm 2,

		1					
		1				2	
		1				2	3
4		1				2	3
4		1		5		2	3
4		1	6	5		2	3
4		1	6	5	7	2	3
4	8	1	6	5	7	2	3

(b) Using algorithm 1,

8
78
768
7658
76584
736584
7365842
73658412

Using algorithm 2,

$$
\begin{array}{cccccccc}
 & & & & & & 1 & \\
 & & & & & & 1 & 2 \\
 & 3 & & & & & 1 & 2 \\
 & 3 & & & & 4 & 1 & 2 \\
 & 3 & & 5 & & 4 & 1 & 2 \\
 & 3 & 6 & 5 & & 4 & 1 & 2 \\
7 & 3 & 6 & 5 & & 4 & 1 & 2 \\
7 & 3 & 6 & 5 & 8 & 4 & 1 & 2
\end{array}
$$

8. (a) For a permutation of $\{1, 2, 3, 4, 5, 6\}$ the corresponding inversion sequence $(b_1, b_2, b_3, b_4, b_5, b_6)$ satisfies $0 \le b_i \le 6 - i$ for $1 \le i \le 6$. The total number of inversions is $\sum_{i=1}^{6} b_i$. This total is at most $5 + 4 + 3 + 2 + 1 + 0 = 15$ with equality if and only if $b_i = 6 - i$ for $1 \le i \le 6$. Therefore there is just one permutation with 15 inversions.
(b) The number of permutations with 14 inversions is equal to the number of integral solutions to

$$0 \le b_i \le 6 - i \qquad (1 \le i \le 6), \qquad\qquad \sum_{i=1}^{6} b_i = 14.$$

Each solution $(b_1, b_2, b_3, b_4, b_5, b_6)$ is obtained from $(5, 4, 3, 2, 1, 0)$ by subtracting 1 from one of the first 5 coordinates. This can be done in 5 ways. Therefore there are 5 permutations with 14 inversions.
(c) The number of permutations with 13 inversions is equal to the number of integral solutions to

$$0 \le b_i \le 6 - i \qquad (1 \le i \le 6), \qquad\qquad \sum_{i=1}^{6} b_i = 13.$$

Each solution $(b_1, b_2, b_3, b_4, b_5, b_6)$ is obtained from $(5, 4, 3, 2, 1, 0)$ by subtracting 1 from two of the first 5 coordinates ($\binom{5}{2}$ ways) or subtracting 2 from one of the first four coordinates (4 ways). Therefore the number of solutions is $\binom{5}{2} + 4 = 14$. There are 14 permutations with 13 inversions.

9. For a permutation $i_1 i_2 \cdots i_n$ of $\{1, 2, \ldots, n\}$ an inversion is an ordered pair (i_k, i_ℓ) such that $k < \ell$ and $i_k > i_\ell$. There are $\binom{n}{2}$ possibilities for (i_k, i_ℓ). Therefore the number of inversions is at most $\binom{n}{2}$. The following are equivalent: (i) there are $\binom{n}{2}$ inversions; (ii) $i_k > i_\ell$ for $1 \le k < \ell \le n$; (iii) $i_k = n - k + 1$ for $1 \le k \le n$. Thus $n \cdots 321$ is the unique permutation with $\binom{n}{2}$ inversions. The permutations of $\{1, 2, \ldots, n\}$ that have exactly $\binom{n}{2} - 1$ inversions are obtained from $n \cdots 321$ by switching a single pair of adjacent terms. There are $n - 1$ such permutations.

10. We have

$$\begin{array}{c}256143\\251643\\215643\\125643\\125634\\125364\\123564\\123546\\123456\end{array}$$

and

$$\begin{array}{c}436251\\436215\\436125\\431625\\413625\\143625\\143265\\142365\\124365\\123465\\123456\end{array}$$

11. (a) The set $\{x_5, x_4, x_3\}$ corresponds to

coordinate	7	6	5	4	3	2	1	0
entry	0	0	1	1	1	0	0	0

(b) The set $\{x_7, x_5, x_3, x_1\}$ corresponds to

coordinate	7	6	5	4	3	2	1	0
entry	1	0	1	0	1	0	1	0

(c) The set $\{x_6\}$ corresponds to

coordinate	7	6	5	4	3	2	1	0
entry	0	1	0	0	0	0	0	0

12. (a) $\{x_4, x_3, x_1, x_0\}$; (b) $\{x_6, x_4, x_2, x_0\}$; (c) $\{x_3, x_2, x_1, x_0\}$.

13. The subsets of $\{x_4, x_3, x_2, x_1, x_0\}$ are listed in the table below. Each row describes a subset. For $0 \leq i \leq 4$ we write 1 (resp. 0) below x_i whenever the subset contains (resp. does not contain) x_i.

rank	x_4	x_3	x_2	x_1	x_0
0	0	0	0	0	0
1	0	0	0	0	1
2	0	0	0	1	0
3	0	0	0	1	1
4	0	0	1	0	0
5	0	0	1	0	1
6	0	0	1	1	0
7	0	0	1	1	1
8	0	1	0	0	0
9	0	1	0	0	1
10	0	1	0	1	0
11	0	1	0	1	1
12	0	1	1	0	0
13	0	1	1	0	1
14	0	1	1	1	0
15	0	1	1	1	1
16	1	0	0	0	0
17	1	0	0	0	1
18	1	0	0	1	0
19	1	0	0	1	1
20	1	0	1	0	0
21	1	0	1	0	1
22	1	0	1	1	0
23	1	0	1	1	1
24	1	1	0	0	0
25	1	1	0	0	1
26	1	1	0	1	0
27	1	1	0	1	1
28	1	1	1	0	0
29	1	1	1	0	1
30	1	1	1	1	0
31	1	1	1	1	1

14. The subsets of $\{x_5, x_4, x_3, x_2, x_1, x_0\}$ are listed in the table below. Each row describes a subset. For $0 \le i \le 5$ we write 1 (resp. 0) below x_i whenever the subset contains (resp. does not contain) x_i.

rank	x_5	x_4	x_3	x_2	x_1	x_0
0	0	0	0	0	0	0
1	0	0	0	0	0	1
2	0	0	0	0	1	0
3	0	0	0	0	1	1
4	0	0	0	1	0	0
5	0	0	0	1	0	1
6	0	0	0	1	1	0
7	0	0	0	1	1	1
8	0	0	1	0	0	0
9	0	0	1	0	0	1
10	0	0	1	0	1	0
11	0	0	1	0	1	1
12	0	0	1	1	0	0
13	0	0	1	1	0	1
14	0	0	1	1	1	0
15	0	0	1	1	1	1
16	0	1	0	0	0	0
17	0	1	0	0	0	1
18	0	1	0	0	1	0
19	0	1	0	0	1	1
20	0	1	0	1	0	0
21	0	1	0	1	0	1
22	0	1	0	1	1	0
23	0	1	0	1	1	1
24	0	1	1	0	0	0
25	0	1	1	0	0	1
26	0	1	1	0	1	0
27	0	1	1	0	1	1
28	0	1	1	1	0	0
29	0	1	1	1	0	1
30	0	1	1	1	1	0
31	0	1	1	1	1	1

rank	x_5	x_4	x_3	x_2	x_1	x_0
32	1	0	0	0	0	0
33	1	0	0	0	0	1
34	1	0	0	0	1	0
35	1	0	0	0	1	1
36	1	0	0	1	0	0
37	1	0	0	1	0	1
38	1	0	0	1	1	0
39	1	0	0	1	1	1
40	1	0	1	0	0	0
41	1	0	1	0	0	1
42	1	0	1	0	1	0
43	1	0	1	0	1	1
44	1	0	1	1	0	0
45	1	0	1	1	0	1
46	1	0	1	1	1	0
47	1	0	1	1	1	1
48	1	1	0	0	0	0
49	1	1	0	0	0	1
50	1	1	0	0	1	0
51	1	1	0	0	1	1
52	1	1	0	1	0	0
53	1	1	0	1	0	1
54	1	1	0	1	1	0
55	1	1	0	1	1	1
56	1	1	1	0	0	0
57	1	1	1	0	0	1
58	1	1	1	0	1	0
59	1	1	1	0	1	1
60	1	1	1	1	0	0
61	1	1	1	1	0	1
62	1	1	1	1	1	0
63	1	1	1	1	1	1

15. We convert to binary, add 1, and then convert back:

subset	$\{x_4, x_1, x_0\}$	$\{x_7, x_5, x_3\}$	$\{x_7, x_5, x_4, x_3, x_2, x_1, x_0\}$	$\{x_0\}$
binary rep	00010011	10101000	10111111	00000001
binary rep + 1	00010100	10101001	11000000	00000010
next subset	$\{x_4, x_2\}$	$\{x_7, x_5, x_3, x_0\}$	$\{x_7, x_6\}$	$\{x_1\}$

16. We convert to binary, subtract 1, and then convert back:

subset	$\{x_4, x_1, x_0\}$	$\{x_7, x_5, x_3\}$	$\{x_7, x_5, x_4, x_3, x_2, x_1, x_0\}$	$\{x_0\}$
binary rep	00010011	10101000	10111111	00000001
binary rep - 1	00010010	10100111	10111110	00000000
prec. subset	$\{x_4, x_1\}$	$\{x_7, x_5, x_2, x_1, x_0\}$	$\{x_7, x_5, x_4, x_3, x_2, x_1\}$	$\emptyset$

17. $150 = 128 + 16 + 4 + 2$ has the binary representation 10010110 so the 150th subset is $\{x_7, x_4, x_2, x_1\}$. $200 = 128 + 64 + 8$ has binary representation 11001000 so the 200th subset is $\{x_7, x_6, x_3\}$. $250 = 128 + 64 + 32 + 16 + 8 + 2$ has binary representation 11111010 so the 250th subset is $\{x_7, x_6, x_5, x_4, x_3, x_1\}$.

18. Consider the reflected Gray code of order 4. For $0 \leq m \leq 15$ the mth term is $g_3 g_2 g_1 g_0$ where

m	g_3	g_2	g_1	g_0
0	0	0	0	0
1	0	0	0	1
2	0	0	1	1
3	0	0	1	0
4	0	1	1	0
5	0	1	1	1
6	0	1	0	1
7	0	1	0	0
8	1	1	0	0
9	1	1	0	1
10	1	1	1	1
11	1	1	1	0
12	1	0	1	0
13	1	0	1	1
14	1	0	0	1
15	1	0	0	0

See the course notes for a 4-cube representation.

19. The following is a noncyclic Gray code of order 3:

rank	codeword
0	000
1	001
2	011
3	010
4	110
5	100
6	101
7	111

20. The following is a cyclic Gray code of order 3 that is not the reflected Gray code:

rank	codeword
0	000
1	001
2	011
3	111
4	101
5	100
6	110
7	010

21. Consider the reflected Gray code of order 5. For $0 \le m \le 31$ the mth term is $g_4 g_3 g_2 g_1 g_0$ where

m	g_4	g_3	g_2	g_1	g_0
0	0	0	0	0	0
1	0	0	0	0	1
2	0	0	0	1	1
3	0	0	0	1	0
4	0	0	1	1	0
5	0	0	1	1	1
6	0	0	1	0	1
7	0	0	1	0	0
8	0	1	1	0	0
9	0	1	1	0	1
10	0	1	1	1	1
11	0	1	1	1	0
12	0	1	0	1	0
13	0	1	0	1	1
14	0	1	0	0	1
15	0	1	0	0	0
16	1	1	0	0	0
17	1	1	0	0	1
18	1	1	0	1	1
19	1	1	0	1	0
20	1	1	1	1	0
21	1	1	1	1	1
22	1	1	1	0	1
23	1	1	1	0	0
24	1	0	1	0	0
25	1	0	1	0	1
26	1	0	1	1	1
27	1	0	1	1	0
28	1	0	0	1	0
29	1	0	0	1	1
30	1	0	0	0	1
31	1	0	0	0	0

22. Consider the reflected Gray code of order 6. For $0 \leq m \leq 63$ the mth term is $g_5 g_4 g_3 g_2 g_1 g_0$ where

m	g_5	g_4	g_3	g_2	g_1	g_0
0	0	0	0	0	0	0
1	0	0	0	0	0	1
2	0	0	0	0	1	1
3	0	0	0	0	1	0
4	0	0	0	1	1	0
5	0	0	0	1	1	1
6	0	0	0	1	0	1
7	0	0	0	1	0	0
8	0	0	1	1	0	0
9	0	0	1	1	0	1
10	0	0	1	1	1	1
11	0	0	1	1	1	0
12	0	0	1	0	1	0
13	0	0	1	0	1	1
14	0	0	1	0	0	1
15	0	0	1	0	0	0
16	0	1	1	0	0	0
17	0	1	1	0	0	1
18	0	1	1	0	1	1
19	0	1	1	0	1	0
20	0	1	1	1	1	0
21	0	1	1	1	1	1
22	0	1	1	1	0	1
23	0	1	1	1	0	0
24	0	1	0	1	0	0
25	0	1	0	1	0	1
26	0	1	0	1	1	1
27	0	1	0	1	1	0
28	0	1	0	0	1	0
29	0	1	0	0	1	1
30	0	1	0	0	0	1
31	0	1	0	0	0	0

m	g_5	g_4	g_3	g_2	g_1	g_0
32	1	1	0	0	0	0
33	1	1	0	0	0	1
34	1	1	0	0	1	1
35	1	1	0	0	1	0
36	1	1	0	1	1	0
37	1	1	0	1	1	1
38	1	1	0	1	0	1
39	1	1	0	1	0	0
40	1	1	1	1	0	0
41	1	1	1	1	0	1
42	1	1	1	1	1	1
43	1	1	1	1	1	0
44	1	1	1	0	1	0
45	1	1	1	0	1	1
46	1	1	1	0	0	1
47	1	1	1	0	0	0
48	1	0	1	0	0	0
49	1	0	1	0	0	1
50	1	0	1	0	1	1
51	1	0	1	0	1	0
52	1	0	1	1	1	0
53	1	0	1	1	1	1
54	1	0	1	1	0	1
55	1	0	1	1	0	0
56	1	0	0	1	0	0
57	1	0	0	1	0	1
58	1	0	0	1	1	1
59	1	0	0	1	1	0
60	1	0	0	0	1	0
61	1	0	0	0	1	1
62	1	0	0	0	0	1
63	1	0	0	0	0	0

23. (a) The sum of the entries is 4, which is even, so we change coordinate zero. The successor is 010100111.
(b) The sum of the entries is 4, which is even, so we change coordinate zero. The successor is 110001101.
(c) the sum of the entries is 9, which is odd. Coordinate zero is 1 so we change coordinate one. The successor is 111111101.

24. (a) The sum of the entries is 4, which is even. The sequence ends with 10 so we change coordinate two. The predecessor is 010100010.
(b) The sum of the entries is 4, which is even. The sequence ends with 100 so we change coordinate three. The predecessor is 110000100
(c) The sum of the entries is 9, which is odd. So we change coordinate zero. The predecessor is 111111110.

25.

26. The lexicographic ordering is 12, 13, 14, 15, 23, 24, 25, 34, 35, 45.

27. The lexicographic ordering is 123, 124, 125, 126, 134, 135, 136, 145, 146, 156, 234, 235, 236, 245, 246, 256, 345, 346, 356, 456.

28. In the lexicographic order the 6-subset that follows $2, 3, 4, 6, 9, 10$ is $2, 3, 4, 7, 8, 9$ and the 6-subset that precedes $2, 3, 4, 6, 9, 10$ is $2, 3, 4, 6, 8, 10$.

29. In the lexicographic order the 7-subset that follows $1, 2, 4, 6, 8, 14, 15$ is $1, 2, 4, 6, 9, 10, 11$ and the 7-subset that precedes $1, 2, 4, 6, 8, 14, 15$ is $1, 2, 4, 6, 8, 13, 15$.

30. In the table below we list the permutations of $\{1, 2, 3\}$ according to the lexicographic order of their inversion sequence.

permutation	123	132	213	312	231	321
inversion sequence	000	010	100	110	200	210

Below we similarly list the permutations of $\{1, 2, 3, 4\}$.

1234	1243	1324	1423	1342	1432
0000	0010	0100	0110	0200	0210

2134	2143	3124	4123	3142	4132
1000	1010	1100	1110	1200	1210

2314	2413	3214	4213	3412	4312
2000	2010	2100	2110	2200	2210

2341	2431	3241	4231	3421	4321
3000	3010	3100	3110	3200	3210

31. We first generate the 3-subsets of $\{1,2,3,4,5\}$ in lexicographic order: 123, 124, 125, 134, 135, 145. For each subset we order the elements in all possible ways, using the algorithm from Section 4.1. This yields the following ordering of the 3-permutations of $\{1,2,3,4,5\}$. Reading left to right,

123	132	312	321	231	213
124	142	412	421	241	214
125	152	512	521	251	215
134	143	413	431	341	314
135	153	513	531	351	315
145	154	514	541	451	415

32. We first generate the 4-subsets of $\{1,2,3,4,5,6\}$ in lexicographic order. Reading left to right,

1234	1235	1236	1245	1246
1256	1345	1346	1356	1456
2345	2346	2356	2456	3456

For each subset we order the elements in all possible ways, using the algorithm from Section 4.1. This yields an ordering of the 4-permutations of $\{1,2,3,4,5,6\}$. We omit the details due to length.

33. By convention the first term in the lexicographical order is position 1. The total number of 4-subsets of $\{1,2,3,4,5,6,7,8,9\}$ is $\binom{9}{4}$. We now count the number of 4-subsets that come after 2489 in the lexicographical order. A 4-subset comes after 2489 if and only if it has the form (i) $abcd$ $(3 \le a < b < c < d \le 9)$ or (ii) $2bcd$ $(5 \le b < c < d \le 9)$. The number of 4-subsets of type (i) is $\binom{7}{4}$. The number of 4-subsets of type (ii) is $\binom{5}{3}$. Therefore 2489 is in position

$$\binom{9}{4} - \binom{7}{4} - \binom{5}{3}.$$

34. Consider the lexicographical order of r-subsets for $\{1,2,\ldots,n\}$.
(a) The first $n-r+1$ terms are

$$\begin{gathered} 1,2,\ldots,r-1,r, \\ 1,2,\ldots,r-1,r+1, \\ \cdots \\ 1,2,\ldots,r-1,n. \end{gathered}$$

(b) The last $r+1$ terms are

$$\begin{gathered} n-r,n-r+1,\ldots,n-1, \\ n-r,n-r+1,\ldots,n-2,n, \\ \cdots \\ n-r+1,n-r+2,\ldots,n-1,n. \end{gathered}$$

35. Let A and B denote distinct r-subsets of $\{1, 2, \ldots, n\}$. We show that with respect to lexicographic order, $A < B$ if and only $\overline{B} < \overline{A}$. Recall that $A < B$ whenever A contains the minimum element in $(A \cup B) \backslash (A \cap B)$. Equivalently $A < B$ whenever the minimum element of $A \cap \overline{B}$ is smaller than the minimum element of $B \cap \overline{A}$. From this version we see that $A < B$ if and only $\overline{B} < \overline{A}$. The result follows.

36. View $X = \{1, 2, \ldots, n\}$. (i) There are $2^{(n^2)}$ different relations on X. Reason: To construct a relation R on X, for $1 \leq x, y \leq n$ we must decide whether or not xRy. There are n^2 choices for (x, y). The result follows by the multiplication principle.
(ii) There are $2^{n(n-1)}$ reflexive relations on X. Reason: To construct a reflexive relation R on X, for distinct $1 \leq x, y \leq n$ we must decide whether or not xRy. There are $n(n-1)$ choices for (x, y). The result follows by the multiplication principle.
(iii) Abbreviate $N = \binom{n+1}{2}$. There are 2^N symmetric (resp. antisymmetric) relations on X. Reason: To construct a symmetric (resp. antisymmetric) relation R on X, for $1 \leq x \leq y \leq n$ we must decide whether or not xRy. There are N choices for (x, y). The result follows by the multiplication principle.
(iv) Abbreviate $M = \binom{n}{2}$. There are 2^M reflexive symmetric (resp. antisymmetric) relations on X. Reason: To construct a reflexive symmetric (resp. antisymmetric) relation R on X, for $1 \leq x < y \leq n$ we must decide whether or not xRy. There are M choices for (x, y). The result follows by the multiplication principle.

37. Recall that a relation on X is a partial order if and only if it is reflexive, antisymmetric, and transitive. We now show that the relation R has these features.
R is reflexive: For $x \in X$ we show xRx. The relations R' and R'' are reflexive so $xR'x$ and $xR''x$. Therefore xRx.
R is antisymmetric: For distinct $x, y \in X$ such that xRy, we show that yRx fails. We assume xRy so $xR'y$. The relation R' is antisymmetric so $yR'x$ fails. Therefore yRx fails.
R is transitive: For $x, y, z \in X$ such that xRy and yRz we show xRz. By construction $xR'y$ and $yR'z$. The relation R' is transitive so $xR'z$. Similarly $xR''z$. Therefore xRz.

38. Recall that a relation on a set is a partial order if and only if it is reflexive, antisymmetric, and transitive. We show that the relation T on the set $X_1 \times X_2$ has these features.
T is reflexive: For $x \in X_1 \times X_2$ we show xTx. Write $x = (x_1, x_2)$. The relation $\leq_1$ is reflexive so $x_1 \leq_1 x_1$. Similarly $x_2 \leq_2 x_2$. Therefore xTx.
T is antisymmetric: For distinct $x, y \in X_1 \times X_2$ such that xTy, we show that yTx fails. Write $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Since x, y are distinct, there exists $i \in \{1, 2\}$ such that $x_i \neq y_i$. Since xTy we have $x_i \leq_i y_i$. The relation $\leq_i$ is antisymmetric so $y_i \not\leq_i x_i$. Therefore yTx fails.
T is transitive: For $x, y, z \in X_1 \times X_2$ such that xTy and yTz, we show xTz. Write $x = (x_1, x_2)$ and $y = (y_1, y_2)$ and $z = (z_1, z_2)$. Pick $i \in \{1, 2\}$. Since xTy we have $x_i \leq_i y_i$, and since yTz we have $y_i \leq_i z_i$. By this and since $\leq_i$ is transitive we find $x_i \leq_i z_i$. Therefore xTz.

39. Define the set $J^n = J \times J \times \cdots \times J$ (n factors). Thus J^n consists of the n-tuples of zeros and ones. For $x \in J^n$ and $1 \leq i \leq n$ let x_i denote the entry in coordinate i of x, so that $x = (x_1, x_2, \ldots, x_n)$. Define a partial order $\leq$ on J^n such that for $x, y \in J^n$, $x \leq y$ whenever $x_i \leq y_i$ for $1 \leq i \leq n$. By construction the poset $(J^n, \leq)$ is the direct product of n

copies of the poset $(J, \leq)$. We show that the poset $(J^n, \leq)$ can be identified with the poset $(\mathcal{P}(X), \subseteq)$. View $X = \{1, 2, \ldots, n\}$. Define a function $f : J^n \to \mathcal{P}(X)$ by

$$f(x) = \{i \in X | x_i = 1\}, \qquad x \in J^n.$$

The function f is a bijection. We show that for $x, y \in J^n$, $x \leq y$ if and only if $f(x) \subseteq f(y)$. Let x, y be given and note that the following assertions are equivalent:

(i) $x \leq y$;

(ii) $x_i \leq y_i$ for $1 \leq i \leq n$;

(iii) $x_i = 1$ implies $y_i = 1$ for $1 \leq i \leq n$;

(iv) $i \in f(x)$ implies $i \in f(y)$ for $1 \leq i \leq n$;

(v) $f(x) \subseteq f(y)$.

We have shown that the poset $(J^n, \leq)$ can be identified with the poset $(\mathcal{P}(X), \subseteq)$.

40. For each integer $r \geq 0$ define a poset $[r]$ as follows. The poset consists of the set $\{0, 1, \ldots, r\}$ together with the total order $0 < 1 < \cdots < r$. We will be discussing the direct product $[n_1] \times [n_2] \times \cdots \times [n_m]$. Consider the multiset $X = \{n_1 \cdot a_1, n_2 \cdot a_2, \ldots, n_m \cdot a_m\}$. Let $\mathcal{P}(X)$ denote the set of all multisubsets of X. An element $x \in \mathcal{P}(X)$ has the form

$$x = \{x_1 \cdot a_1, x_2 \cdot a_2, \ldots, x_m \cdot a_m\}, \qquad 0 \leq x_j \leq n_j \quad (1 \leq j \leq m).$$

For $x, y \in \mathcal{P}(X)$ the following are equivalent:

(i) $x \subseteq y$;

(ii) $x_j \leq y_j$ for $1 \leq j \leq m$;

(iii) $(x_1, x_2, \ldots, x_m) \leq (y_1, y_2, \ldots, y_m)$ in the poset $[n_1] \times [n_2] \times \cdots \times [n_m]$.

Therefore the poset $[n_1] \times [n_2] \times \cdots \times [n_m]$ can be identified with the poset $(\mathcal{P}(X), \subseteq)$.

41. We show that a partial order $\leq$ on a finite set X is uniquely determined by its cover relation. This is a consequence of the following lemma.

Lemma The following are equivalent for all distinct $x, y \in X$:

(i) $x < y$;

(ii) there exists an integer $r \geq 2$ and a sequence $(x_1, x_2, \ldots, x_r)$ of elements in X such that $x = x_1$ and $x_r = y$ and x_i covers x_{i-1} for $2 \leq i \leq r$.

Proof: (i) ⇒ (ii) Consider the set S consisting of the finite sequences $(x_1, x_2, \dots, x_r)$ such that $x_1 = x$ and $x_r = y$ and $x_{i-1} < x_i$ for $2 \le i \le r$. The set S is finite since X is finite. The set S is nonempty since $(x, y) \in S$. Pick an element $(x_1, x_2, \dots, x_r)$ in S with r maximal. By construction $r \ge 2$. By the maximality of r we see that x_i covers x_{i-1} for $2 \le i \le r$.
(ii) ⇒ (i) The relation $\le$ is transitive.

42. The diagram of the cover relation is essentially the n-cube, where $n = |X|$.

43. The linear extensions are

$$abecfd, \qquad abefcd, \qquad aebcfd, \qquad aebfcd, \qquad abcefd, \qquad aefbcd.$$

44. Recall that a relation is an equivalence relation whenever it is reflexive and symmetric and transitive. We show that R has these features.
R is reflexive: For $x \in X$, x and x are in the same part of the partition.
R is symmetric: For $x, y \in X$, x and y are in the same part of the partition if and only if y and x are in the same part of the partition.
R is transitive: For $x, y, z \in X$, if x, y are in the same part of the partition, and y, z are in the same part of the partition, then x, z are in the same part of the partition.

45. The relation R is an equivalence relation. To verify this, one checks that R is reflexive, symmetric, and transitive. For the equivalence relation R one equivalence class consists of 0. Every other equivalence class consists of a positive integer and its opposite.

46. We show that R is an equivalence relation.
R is reflexive: For an integer a, certainly a and a have the same remainder when divided by m.
R is symmetric: For integers a, b suppose a and b have the same remainder when divided by m. Then b and a have the same remainder when divided by m.
R is transitive: For integers a, b, c suppose a, b have the same remainder when divided by m, and b, c have the same remainder when divided by m. Then a, c have the same remainder when divided by m. We have shown that R is an equivalence relation. The equivalence classes are $[0], [1], \dots, [m-1]$ where $[r] = \{r + im | i \in \mathbb{Z}\}$ for $0 \le r \le m-1$. The relation R has m equivalence classes.

47. (a) It is routinely checked that $\le$ is a partial order on Π_n.
(b) Given equivalence relations R and S on $\{1, 2, \dots, n\}$ we have $R \le S$ whenever xRy implies xSy for all $x, y \in \{1, 2, \dots, n\}$.

48. Consider the prime factorizations for a and b:

$$a = 2^{a_1} 3^{a_2} 5^{a_3} \cdots \qquad\qquad b = 2^{b_1} 3^{b_2} 5^{b_3} \cdots$$

Then

$$c = 2^{c_1} 3^{c_2} 5^{c_3} \cdots \qquad\qquad d = 2^{d_1} 3^{d_2} 5^{d_3} \cdots$$

where $c_i = \min\{a_i, b_i\}$ for $i \geq 1$ and $d_i = \max\{a_i, b_i\}$ for $i \geq 1$. The integer c (resp. d) is the *greatest common factor* (resp. *least common multiple*) of a, b.

49. It is routinely checked that $R \cap S$ is an equivalence relation. In general $R \cup S$ is not an equivalence relation.

50. There are 48 linear extensions.

51. For a permutation π of $\{1, 2, \ldots, n\}$ let $\mathrm{Inv}(\pi)$ denote the set of inversions of π. Let σ denote a permutation of $\{1, 2, \ldots, n\}$. By definition $\pi \leq \sigma$ whenever $\mathrm{Inv}(\pi) \subseteq \mathrm{Inv}(\sigma)$. We show that for the partial order $\leq$ the following are equivalent:

(i) σ covers π;

(ii) σ is obtained from π by applying a transposition $ab \to ba$ with $a < b$.

Proof: (i) $\Rightarrow$ (ii) By assumption $\pi < \sigma$ so $\mathrm{Inv}(\pi) \subseteq \mathrm{Inv}(\sigma)$. The containment is proper since a permutation is determined by its inversions. Pick an inversion ba ($a < b$) that is contained in $\mathrm{Inv}(\sigma)$ but not $\mathrm{Inv}(\pi)$. Then π and σ have the form

$$\pi = \cdots a \cdots b \cdots \qquad\qquad \sigma = \cdots b \cdots a \cdots$$

Of all the inversions ba that meet the above requirement, pick one for which the distance betwen the symbols b, a in σ is minimal. We show that b, a are adjacent in σ. Suppose that this is not the case. Then σ has the form

$$\sigma = \cdots b \cdots c \cdots a \cdots$$

By the minimality condition neither of bc and ca is an inversion that is contained in $\mathrm{Inv}(\sigma)$ but not in $\mathrm{Inv}(\pi)$. Consequently in π the c lies to the left of a and to the right of b. This is a contradiction so c does not exist. We have shown that b, a are adjacent in σ. In other words σ has the form

$$\sigma = \cdots ba \cdots$$

Apply the transposition $ba \to ab$ to σ and let p denote the resulting permutation. Thus

$$p = \cdots ab \cdots$$

The set $\mathrm{Inv}(p)$ is obtained from $\mathrm{Inv}(\sigma)$ by removing the inversion ba. Consequently $\pi \leq p < \sigma$. By assumption σ covers π so $\pi = p$. Therefore π is obtained from σ by applying the transposition $ba \to ab$. Consequently σ is obtained from π by applying the transposition $ab \to ba$.
(ii) $\Rightarrow$ (i) The permutations σ and π have the form

$$\pi = \cdots ab \cdots \qquad\qquad \sigma = \cdots ba \cdots$$

with agreement in all coordinates except the two shown. The set $\mathrm{Inv}(\sigma)$ is obtained from $\mathrm{Inv}(\pi)$ by adding the inversion ba. Thus $\mathrm{Inv}(\pi) \subseteq \mathrm{Inv}(\sigma)$ so $\pi \leq \sigma$. Also $|\mathrm{Inv}(\sigma)| =$

$|\text{Inv}(\pi)|+1$ so there does not exist a permutation τ such that $\pi < \tau < \sigma$. Therefore σ covers π.

Before solving problems 52 and 53 we recall a few points from the text.

Problem B For an integer $m \geq 0$ consider the base 2 representation of m as a sequence of zeros and ones. (i) Describe how to adjust this sequence to get the corresponding representation of $m+1$. (ii) For $m \geq 1$, describe how to adjust this sequence to get the corresponding representation of $m-1$.

Sol Write the base 2 representation of m as $\cdots b_2 b_1 b_0$, so that $m = \sum_{i=0}^{\infty} b_i 2^i$ with $b_i \in \{0,1\}$ for $i \geq 0$.
(i) To get the corresponding representation of $m+1$ we specify which coordinates to change. For $i \geq 0$ change b_i if and only if each of $b_0, b_1, \ldots, b_{i-1}$ is 1. Thus b_0 always gets changed; b_1 gets changed if and only if b_0 is 1; b_2 gets changed if and only if each of b_0, b_1 is 1, and so on.
(ii) Assume $m \geq 1$. To get the corresponding representation of $m-1$ we specify which coordinates to change. For $i \geq 0$ change b_i if and only if each of $b_0, b_1, \ldots, b_{i-1}$ is 0 . Thus b_0 always gets changed; b_1 gets changed if and only if b_0 is 0; b_2 gets changed if and only if each of b_0, b_1 is 0, and so on.

Problem G Let $\cdots g_2 g_1 g_0$ denote a term in the reflected Gray code. (i) Describe the next term in the code. (ii) Assume that there exists an integer $i \geq 0$ such that $g_i = 1$. Describe the preceding term in the code.

Sol (i) We specify which coordinate to change. First assume that $\sum_{i=0}^{\infty} g_i$ is even. Then change g_0. Next assume that $\sum_{i=0}^{\infty} g_i$ is odd. Change g_s for the unique integer $s \geq 1$ such that $g_{s-1} = 1$ and each of $g_0, g_1, \ldots, g_{s-2}$ is 0.
(ii) First assume that $\sum_{i=0}^{\infty} g_i$ is odd. Then change g_0. Next assume that $\sum_{i=0}^{\infty} g_i$ is even. Change g_s for the unique integer $s \geq 1$ such that $g_{s-1} = 1$ and each of $g_0, g_1, \ldots, g_{s-2}$ is 0.

52. For an integer $m \geq 0$ let $\cdots b_2 b_1 b_0$ denote the base 2 representation of m and let $\cdots g_2 g_1 g_0$ denote term m in the reflected Gray code. We show that for $i \geq 0$, $b_i = 0$ if $g_i + g_{i+1} + \cdots$ is even, and $b_i = 1$ if $g_i + g_{i+1} + \cdots$ is odd. For $i \geq 0$ define $B_i = 0$ if $g_i + g_{i+1} + \cdots$ is even, and $B_i = 1$ if $g_i + g_{i+1} + \cdots$ is odd. We show $B_i = b_i$. Call the sequence $\cdots B_2 B_1 B_0$ the *B-sequence* for m. For the purpose of this proof call the sequence $\cdots b_2 b_1 b_0$ the *b-sequence* for m. Note that for $m = 0$ the B-sequence and b-sequence are both equal to $\cdots 000$. To finish the proof, it suffices to show that for $m \geq 0$ the B-sequence of $m+1$ is related to the B-sequence of m in the same way that the b-sequence of $m+1$ is related to the b-sequence of m. It is explained in Problem B(i) how the b-sequence of $m+1$ is related to the b-sequence of m. Now consider the B-sequences. First assume $\sum_{i=0}^{\infty} g_i$ is even, so that $B_0 = 0$. By Problem G, term $m+1$ in the reflected Gray code is obtained from $\cdots g_2 g_1 g_0$ by changing g_0. Therefore the B-sequence for $m+1$ is obtained from $\cdots B_2 B_1 B_0$ by changing B_0 to 1. Next assume $\sum_{i=0}^{\infty} g_i$ is odd, so that $B_0 = 1$. Consider the unique integer $s \geq 1$ such that $g_{s-1} = 1$ and each of $g_0, g_1, \ldots, g_{s-2}$ is 0. By construction $B_s = 0$ and $B_i = 1$ for $0 \leq i \leq s-1$. By Problem G, term $m+1$ in the reflected Gray code is obtained from $\cdots g_2 g_1 g_0$ by changing g_s. Therefore the B-sequence for $m+1$ is obtained from $\cdots B_2 B_1 B_0$ by changing B_s to 1 and B_i to 0 for $0 \leq i \leq s-1$. We have shown that the B-sequence of $m+1$ is related to the

B-sequence of m in the same way that the b-sequence of $m+1$ is related to the b-sequence of m. The result follows.

53. For an integer $m \geq 0$ let $\cdots b_2 b_1 b_0$ denote the base 2 representation of m and let $\cdots g_2 g_1 g_0$ denote term m in the reflected Gray code. We show that for $i \geq 0$, $g_i = 0$ if $b_i + b_{i+1}$ is even and $g_i = 1$ if $b_i + b_{i+1}$ is odd. For $i \geq 0$ define $G_i = 0$ if $b_i + b_{i+1}$ is even and $G_i = 1$ if $b_i + b_{i+1}$ is odd. We show $G_i = g_i$. Call the sequence $\cdots G_2 G_1 G_0$ the *G-sequence* for m. For the purpose of this proof call the sequence $\cdots g_2 g_1 g_0$ the *g-sequence* for m. Note that for $m = 0$ the G-sequence and g-sequence are both equal to $\cdots 000$. To finish the proof, it suffices to show that for $m \geq 0$ the G-sequence of $m+1$ is related to the G-sequence of m in the same way that the g-sequence of $m+1$ is related to the g-sequence of m. It is explained in Problem G(i) how the g-sequence of $m+1$ is related to the g-sequence of m. Now consider the G-sequences. First assume $b_0 = 0$, so that $\sum_{i=0}^{\infty} G_i$ is even. The base 2 representation of $m+1$ is obtained from $\cdots b_2 b_1 b_0$ by changing b_0 to 1. Therefore the G-sequence of $m+1$ is obtained from $\cdots G_2 G_1 G_0$ by changing G_0. Next assume that $b_0 = 1$, so that $\sum_{i=0}^{\infty} G_i$ is odd. There exists a unique integer $s \geq 1$ such that $b_s = 0$ and $b_i = 1$ for $0 \leq i \leq s-1$. By construction $G_{s-1} = 1$ and each of $G_0, G_1, \ldots, G_{s-2}$ is 0. The base 2 representation of $m+1$ is obtained from $\cdots b_2 b_1 b_0$ by changing b_s to 1 and b_i to 0 for $0 \leq i \leq s-1$. Therefore the G-sequence of $m+1$ is obtained from $\cdots G_2 G_1 G_0$ by changing G_s. We have shown that the G-sequence of $m+1$ is related to the G-sequence of m in the same way that the g-sequence of $m+1$ is related to the g-sequence of m. The result follows.

54. We augment the covering relation for $\leq$ by declaring that b covers a. View the augmented covering relation as the covering relation of a new partial order $\leq'$. The partial order $\leq'$ has a linear extension by Theorem 4.5.2. This linear extension has the desired features.

55. This is a routine consequence of 54.

56. It is routine to check that R is a partial order. Assume that $i_1 i_2 \cdots i_n$ is not the identity permutation. We show that R has dimension 2. Define a partial order $\leq_1$ on X such that $(a, b) \leq_1 (c, d)$ whenever $a \leq c$. Observe that $\leq_1$ is a linear extension of R. Define a partial order $\leq_2$ on X such that $(a, b) \leq_2 (c, d)$ whenever $b \leq d$. Observe that $\leq_2$ is a linear extension of R. The relation R is the intersection of $\leq_1$ and $\leq_2$. Therefore R has dimension at most 2. The partial orders $\leq_1$ and $\leq_2$ are not identical so R is not a total order; consequently R has dimension at least 2. By these comments R has dimension 2.

57.

58. The following are equivalent: (i) the relation is an equivalence relation; (ii) K_n is partitioned into complete graphs such that any two distinct vertices of K_n are connected by a red edge if and only if they are in the same complete subgraph.

59. Let T denote the total number of inversions for all $n!$ permutations of $\{1, 2, \ldots, n\}$. Thus T is the number of triples $(r, s; a_1 a_2 \cdots a_n)$ such that $a_1 a_2 \cdots a_n$ is a permutation of $\{1, 2, \ldots, n\}$ and (r, s) is an inversion of $a_1 a_2 \cdots a_n$. To count these triples we proceed in stages:

stage	to do	# choices
1	select the values of r, s	$\binom{n}{2}$
2	select the location of r, s among $a_1 a_2 \cdots a_n$	$\binom{n}{2}$
3	choose the remaining $n-2$ terms among $a_1 a_2 \cdots a_n$	$(n-2)!$

T is the product of the entries in the right-most column above, which comes to $n! n(n-1)/4$.