Math 475 **Text: Brualdi, Introductory Combinatorics 5th Ed.**
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Selected solutions for Chapter 2

1. Case $\emptyset$. Fill in the blanks:

$$\begin{array}{ccccc} & \rule{2em}{0.4pt} & \rule{2em}{0.4pt} & \rule{2em}{0.4pt} & \rule{2em}{0.4pt} \\ \#\text{ choices}: & 5 & 5 & 5 & 5 \end{array}$$

The answer is 5^4.
Case $\{a\}$. Fill in the blanks left to right:

$$\begin{array}{ccccc} & \rule{2em}{0.4pt} & \rule{2em}{0.4pt} & \rule{2em}{0.4pt} & \rule{2em}{0.4pt} \\ \#\text{ choices}: & 5 & 4 & 3 & 2 \end{array}$$

The answer is $5 \times 4 \times 3 \times 2 = P(5, 4) = 120$.
Case $\{b\}$. Fill in the blanks:

$$\begin{array}{ccccc} & \rule{2em}{0.4pt} & \rule{2em}{0.4pt} & \rule{2em}{0.4pt} & \rule{2em}{0.4pt} \\ \#\text{ choices}: & 5 & 5 & 5 & 2 \end{array}$$

The answer is $5 \times 5 \times 5 \times 2 = 250$.
Case $\{a, b\}$. Fill in the blanks right to left:

$$\begin{array}{ccccc} & \rule{2em}{0.4pt} & \rule{2em}{0.4pt} & \rule{2em}{0.4pt} & \rule{2em}{0.4pt} \\ \#\text{ choices}: & 2 & 3 & 4 & 2 \end{array}$$

The answer is $2 \times 3 \times 4 \times 2 = 48$.

2. We proceed in stages:

stage	to do	# choices
1	order the suits	4!
2	order the ranks for suit 1	13!
3	order the ranks for suit 2	13!
4	order the ranks for suit 3	13!
5	order the ranks for suit 4	13!

The answer is $4! \times (13!)^4$.

3. If we count the order in which the cards are dealt, the answer is $P(52, 5)$. The number of different poker hands is $\binom{52}{5}$.

4. (a) Each divisor has the form $3^r \times 5^s \times 7^t \times 11^u$ where

$$0 \leq r \leq 4, \qquad 0 \leq s \leq 2, \qquad 0 \leq t \leq 6, \qquad 0 \leq u \leq 1.$$

We proceed in stages:

stage	to do	# choices
1	choose r	5
2	choose s	3
3	choose t	7
4	choose u	2

The answer is $5 \times 3 \times 7 \times 2 = 210$.
(b) Consider the prime factorization $620 = 2^2 \times 5 \times 31$. Each divisor has the form $2^r \times 5^s \times 31^t$ where

$$0 \leq r \leq 2, \qquad 0 \leq s \leq 1, \qquad 0 \leq t \leq 1.$$

We proceed in stages:

stage	to do	# choices
1	choose r	3
2	choose s	2
3	choose t	2

The answer is $3 \times 2 \times 2 = 12$.
(c) Consider the prime factorization $10^{10} = 2^{10} \times 5^{10}$. Proceeding as above we find the answer is $11 \times 11 = 121$.

5. (a) Since $10 = 5 \times 2$ we consider the power of 5 and 2 in the prime factorization of 50!. Below we list the positive integers at most 50 that are divisible by 5, together with the power of 5 in their prime factorization:

integer	5	10	15	20	25	30	35	40	45	50
power of 5	1	1	1	1	2	1	1	1	1	2

The table shows that in the prime factorization of 50! the power of 5 is 12. In the prime factoriztion of 50! the power of 2 is clearly greater than 12, so the answer is 12.
(b) Note that $1000 = 5 \times 200$. Therefore there exist 200 positive integers at most 1000 that are divisible by 5. Note that $200 = 5 \times 40$. Therefore there exist 40 positive integers at most 1000 that are divisible by 5^2. Note that $40 = 5 \times 8$. Therefore there exist 8 positive integers at most 1000 that are divisible by 5^3. Also 625 is the unique positive integer at most 1000 that is divisible by 5^4. Therefore in the prime factorization of 1000! the power of 5 is $200 + 40 + 8 + 1 = 249$. In the prime factorization of 1000! the power of 2 is greater than 249. Therefore the answer is 249.

6. Consider the set S of integers that meet the requirements. We find $|S|$. The maximal integer in S is 98654310, and this has 8 digits. The minimal integer in S is 5401, with 4 digits. For $4 \leq n \leq 8$ let S_n denote the set of integers in S that have exactly n digits. By

construction $|S| = \sum_{n=4}^{8} |S_n|$. To find $|S_8|$, we count the number of elements in S_8 by filling in the blanks left to right:

$$\begin{array}{lcccccccc} & \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ \\ \#\text{ choices}: & 7 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{array}$$

So $|S_8| = 7 \times 7!$. More generally we find

n	8	7	6	5
$\lvert S_n\rvert$	$7 \times 7!$	$7 \times P(7,6)$	$7 \times P(7,5)$	$7 \times P(7,4)$

To find $|S_4|$ we partition S_4 into four subsets as follows:

type of element	# choices
has form $540*$	5
has form $54**$ but not $540*$	5×5
has form $5***$ but not $54**$	$3 \times 6 \times 5$
has form $****$ but not $5***$	$3 \times 7 \times 6 \times 5$

So $|S_4| = 5 + 5 \times 5 + 3 \times 6 \times 5 + 3 \times 7 \times 6 \times 5 = 5 + 25 + 90 + 630 = 750$. Therefore

$$|S| = 7 \times 7! + 7 \times P(7,6) + 7 \times P(7,5) + 7 \times P(7,4) + 750.$$

7. Pick a man and call him H. We proceed in stages:

stage	to do	# choices
1	order the remaining men clockwise from H	3!
2	order the women clockwise from H	8!

The answer is $3! \times 8!$.

8. Pick a man and call him H. We proceed in stages:

stage	to do	# choices
1	order the remaining men clockwise from H	5!
2	order the women clockwise from H	6!

The answer is $5! \times 6!$.

9. Declare A to be the head of the table. We proceed in stages:

stage	to do	# choices
1	pick the person to A's right	13
2	pick the person to A's left	12
3	order the remaining people clockwise in the remaining seats	12!

The answer is $13 \times 12 \times 12! = 12 \times 13!$.
Now suppose that B only refuses to sit on A's right. We proceed in stages:

stage	to do	# choices
1	pick the person to A's right	13
2	order the remaining people clockwise in the remaining seats	13!

The answer is $13 \times 13!$.

10. We partition the set of committees according to the number of women:

# women	# ways to pick men	# ways to pick women	# committees
2	$\binom{10}{3}$	$\binom{12}{2}$	$\binom{10}{3}\binom{12}{2}$
3	$\binom{10}{2}$	$\binom{12}{3}$	$\binom{10}{2}\binom{12}{3}$
4	$\binom{10}{1}$	$\binom{12}{4}$	$\binom{10}{1}\binom{12}{4}$
5	1	$\binom{12}{5}$	$\binom{12}{5}$

The answer is

$$\binom{10}{3}\binom{12}{2} + \binom{10}{2}\binom{12}{3} + \binom{10}{1}\binom{12}{4} + \binom{12}{5}.$$

Now assume that a particular man (Adam) and a particular woman (Eve) refuse to serve together on a committee. We partition the set of committees according to the participation of Adam and Eve:

Adam on committee?	Eve on Committee?	# committees
Y	N	$\binom{9}{2}\binom{11}{2} + \binom{9}{1}\binom{11}{3} + \binom{9}{0}\binom{11}{4}$
N	Y	$\binom{9}{3}\binom{11}{1} + \binom{9}{2}\binom{11}{2} + \binom{9}{1}\binom{11}{3} + \binom{9}{0}\binom{11}{4}$
N	N	$\binom{9}{3}\binom{11}{2} + \binom{9}{2}\binom{11}{3} + \binom{9}{1}\binom{11}{4} + \binom{9}{0}\binom{11}{5}$

The answer is the sum of the entries in the right-most column of the above table.

11. Let $a < b < c$ denote the elements of the 3-integer set. Define

$$x = a - 1, \qquad y = b - a - 2, \qquad z = c - b - 2, \qquad w = 20 - c.$$

Then each of x, y, z, w is nonnegative and $x + y + z + w = 15$. Thus the desired quantity is equal to the number of integral solutions to

$$x \geq 0, \quad y \geq 0, \quad z \geq 0, \quad w \geq 0, \qquad\qquad x + y + z + w = 15.$$

This number is $\binom{15+4-1}{4-1} = \binom{18}{3}$ by Theorem 2.5.1.

12. Let A and B denote the football players that can play in the backfield or on the line. We partition the set of football teams according to the employment of A and B, in the backfield, on the line, or on the bench.

location of A	location of B	# teams
bench	bench	$\binom{5}{4}\binom{8}{7}$
bench	backfield	$\binom{5}{3}\binom{8}{7}$
bench	line	$\binom{5}{4}\binom{8}{6}$
backfield	bench	$\binom{5}{3}\binom{8}{7}$
backfield	backfield	$\binom{5}{2}\binom{8}{7}$
backfield	line	$\binom{5}{3}\binom{8}{6}$
line	bench	$\binom{5}{4}\binom{8}{6}$
line	backfield	$\binom{5}{3}\binom{8}{6}$
line	line	$\binom{5}{4}\binom{8}{5}$

The answer is the sum of the entries in the right-most column of the above table.

13. (a) We proceed in stages:

stage	to do	# choices
1	fill dorm A	$\binom{100}{25}$
2	fill dorm B	$\binom{75}{35}$
3	fill dorm C	1

The answer is

$$\binom{100}{25}\binom{75}{35} = \frac{100!}{25! \times 35! \times 40!}.$$

(b) We proceed in stages:

stage	to do	# choices
1	fill dorm A	$\binom{50}{25}$
2	fill dorm B	$\binom{50}{35}$
3	fill dorm C	1

The answer is

$$\binom{50}{25}\binom{50}{35}.$$

14. We proceed in stages:

stage	to do	# choices
1	seat the 5 front-row students	$P(8,5)$
2	seat the 4 back-row students	$P(8,4)$
3	seat the remaining 5 students	$P(7,5)$

The answer is $P(8,5)P(8,4)P(7,5)$.

15. (a) Label the men $1, 2, \ldots, 15$. We proceed in stages:

stage	to do	# choices
1	pick the woman for man 1	20
2	pick the woman for man 2	19
.	...	.
.	...	.
15	pick the woman for man 15	6

The answer is $20 \times 19 \times 18 \times \cdots \times 6 = P(20, 15)$.

(b) We proceed in stages:

stage	to do	# choices
1	pick 10 men	$\binom{15}{10}$
2	pick 10 women	$\binom{20}{10}$
3	match the above men and women	10!

The answer is $\binom{15}{10} \times \binom{20}{10} \times 10!$.

16. Let X denote a set with $|X| = n$. For $0 \le r \le n$ let X_r denote the set of r-subsets of X. Recall $|X_r| = \binom{n}{r}$. The map $X_r \to X_{n-r}$, $S \mapsto \overline{S}$ is a bijection. Therefore $|X_r| = |X_{n-r}|$ so

$$\binom{n}{r} = \binom{n}{n-r}.$$

17. For indistinguishable rooks the answer is 6!. Now suppose that there are 2 red and 4 blue rooks. We proceed in stages:

stage	to do	# choices
1	select rook locations	6!
2	decide which rooks are red	$\binom{6}{2}$

The answer is

$$6! \times \binom{6}{2} = \frac{(6!)^2}{2! \times 4!}.$$

18. We proceed in stages:

stage	to do	# choices
1	select red rook locations	$\binom{8}{2}\binom{8}{2}2!$
2	select blue rook locations	$\binom{6}{4}\binom{6}{4}4!$

The answer is

$$\binom{8}{2}^2 \times \binom{6}{4}^2 \times 2! \times 4!.$$

19. (a) We proceed in stages:

stage	to do	# choices
1	select red rook locations	$\binom{8}{5}\binom{8}{5}5!$
2	select blue rook locations	$\binom{3}{3}\binom{3}{3}3!$

The answer is

$$\binom{8}{5}^2 \times 5! \times 3!.$$

(b) We proceed in stages:

stage	to do	# choices
1	select red rook locations	$\binom{12}{5}\binom{12}{5}5!$
2	select blue rook locations	$\binom{7}{3}\binom{7}{3}3!$

The answer is

$$\binom{12}{5}^2 \times \binom{7}{3}^2 \times 5! \times 3!.$$

20. The total number of circular permutations of $\{0, 1, 2, \ldots, 9\}$ is $9!$. We now compute the number of circular permutations of $\{0, 1, 2, \ldots, 9\}$ for which 0 and 9 are opposite. Place 0 at the head of a circular table. Seat 9 opposite 0. The number of ways to seat the remaining 8 is $8!$. Therefore the answer is $9! - 8! = 8 \times 8!$.

21. We proceed in stages:

stage	to do	# choices
1	select A location	9
2	select D locations	$\binom{8}{2}$
3	select E locations	$\binom{6}{2}$
4	select R location	4
5	select S locations	$\binom{3}{3}$

The answer is the product of the entries in the right-most column, which comes to

$$\frac{9!}{(2!)^2 \times 3!}.$$

Concerning the number of 8-permutations the answer is the same.

22. We partition the set of solutions according to the pattern of ties:

# in 1st place	# in 2nd place	# in 3d place	# in 4th place	# choices
1	1	1	1	$4!$
2	1	1	0	4×3
1	2	1	0	4×3
1	1	2	0	4×3
3	1	0	0	4
2	2	0	0	$\binom{4}{2}$
1	3	0	0	4
4	0	0	0	1

The answer is the sum of the entries in the right-most column, which comes to 75.

23. We proceed in stages:

stage	to do	# choices
1	choose the hand of player 1	$\binom{52}{13}$
2	choose the hand of player 2	$\binom{39}{13}$
3	choose the hand of player 3	$\binom{26}{13}$
4	choose the hand of player 4	1

The answer is the product of the entries in the right-most column, which comes to

$$\frac{52!}{(13!)^4}.$$

24. To clarify the problem, until further notice assume that both the left/right and front/back seating patterns are important. There are 20! different ways for the ride to begin. Now suppose that a certain two people want to sit in different cars. Call them A and B. We proceed in stages:

stage	to do	# choices
1	choose A's seat	20
2	fill A's car	$18 \times 17 \times 16$
3	fill the remaining seats	16!

The answer is the product of the entries in the right-most column, which comes to $20 \times 16 \times 18!$.

Now let us solve the problem again, this time assuming that the left/right seating pattern is unimportant. To answer the initial question we proceed in stages:

stage	to do	# choices
1	fill car 1, front seats	$\binom{20}{2}$
2	fill car 1, back seats	$\binom{18}{2}$
3	fill car 2, front seats	$\binom{16}{2}$
4	fill car 2, back seats	$\binom{14}{2}$
.	$\cdots$	.
.	$\cdots$	.
10	fill car 10, back seats	$\binom{2}{2}$

The answer is the product of the entries in the right-most column, which comes to

$$\frac{20!}{2^{10}}.$$

Now again suppose that persons A and B want to sit in different cars. We proceed in stages:

stage	to do	# choices
1	select A's car	5
2	Put A in front or back?	2
3	fill A's car	$18\binom{17}{2}$
4	select B's car	4
5	Put B in front or back?	2
6	fill B's car	$15\binom{14}{2}$
7	fill remaining seats	$12!/2^6$

The answer is the product of the entries in the right-most column, which comes to

$$\frac{5 \times 18!}{2^4}.$$

25. To clarify the problem, until further notice assume that both the left/right and circular seating patterns are important. There are $20!/5$ ways for the ride to begin. Suppose that a certain two people want to sit in different cars. Call them A and B. We proceed in stages:

stage	to do	# choices
1	choose A's seat	$20/5$
2	choose B's seat	16
3	fill remaining seats	$18!$

The answer is the product of the entries in the right-most column, which comes to $64 \times 18!$.

Now let us solve the problem again, this time assuming that the left/right seating pattern is unimportant. To answer the initial question, we pick a person A among the 20 and proceed in stages:

stage	to do	# choices
1	choose A's 3 companions	$\binom{19}{3}$
2	fill the four seats behind A	$\binom{16}{4}$
3	fill the next four seats	$\binom{12}{4}$
4	fill the next four seats	$\binom{8}{4}$
5	fill the last four seats	$\binom{4}{4}$

The answer is the product of the entries in the right-most column, which comes to

$$\frac{19!}{3! \times (4!)^4}.$$

Now again suppose that persons A and B want to sit in different cars. We proceed in stages:

stage	to do	# choices
1	choose A's 3 companions	$\binom{18}{3}$
2	select B's car	4
3	choose B's 3 companions	$\binom{15}{3}$
4	fill remaining seats	$12!/(4!)^3$

The answer is the product of the entries in the right-most colum, which comes to

$$\frac{18!}{(3!)^3 \times (4!)^2}.$$

26. (a) Name the teams $1, 2, \ldots, m$. We proceed in stages:

stage	to do	# choices
1	select team 1	$\binom{mn}{n}$
2	select team 2	$\binom{mn-n}{n}$
3	select team 3	$\binom{mn-2n}{n}$
.	...	.
.	...	.
m	select team m	$\binom{n}{n}$

The answer is the product of the entries in the right-most colum, which comes to

$$\frac{(mn)!}{(n!)^m}.$$

(b) To get the answer, divide the answer to (a) by $m!$. The answer is

$$\frac{(mn)!}{m!(n!)^m}.$$

27. By assumption there is a rook in the first row; suppose it is in column r. By assumption there is a rook in the first column; suppose it is in row s. Note that either (i) $r = s = 1$;

or (ii) $r \neq 1$, $s \neq 1$. The number of solutions for case (i) is equal to the number of ways to place 4 nonattacking rooks on a 7×7 chessboard. Concerning (ii), there are 7 choices for r and 7 choices for s. For each r, s the number of solutions is equal to the number of ways to place 3 nonattacking rooks on a 6×6 chessboard. Therefore the answer is

$$4! \times \binom{7}{4}^2 + 7^2 \times 3! \times \binom{6}{3}^2.$$

28. (a) Represent each route by a sequence of length 17 consisting of 9 E's and 8 N's. The number of these sequences is $\binom{17}{9}$.

(b) Denote the intersections by (i, j) with $0 \leq i \leq 9$ and $0 \leq j \leq 8$. Thus a route starts at intersection $(0, 0)$ and ends at intersection $(9, 8)$. Note that the underwater block has corners $(4, 3)$, $(4, 4)$, $(5, 4)$, $(5, 3)$. Consider the set U of routes that use the underwater block. We partition U as follows:

route passes through	# routes
$(4,3),(4,4),(5,4)$	$\binom{7}{3}\binom{8}{4}$
$(4,3),(5,3),(5,4)$	$\binom{7}{3}\binom{8}{4}$
$(4,3),(4,4),(4,5)$	$\binom{7}{3}\binom{8}{3}$
$(3,4),(4,4),(5,4)$	$\binom{7}{3}\binom{8}{4}$
$(5,2),(5,3),(5,4)$	$\binom{7}{2}\binom{8}{4}$
$(4,3),(5,3),(6,3)$	$\binom{7}{3}\binom{8}{3}$
$(3,4),(4,4),(4,5)$	$\binom{7}{3}\binom{8}{3}$
$(5,2),(5,3),(6,3)$	$\binom{7}{2}\binom{8}{3}$

Thus $|U|$ is the sum of the entries in the right-most column. Our answer is $\binom{17}{9} - |U|$ which comes to

$$\binom{17}{9} - 3\binom{7}{3}\binom{8}{3} - 3\binom{7}{3}\binom{8}{4} - \binom{7}{2}\binom{8}{3} - \binom{7}{2}\binom{8}{4}.$$

29. Routine.

30. We proceed in stages:

stage	to do	# choices
1	pick gender to the parent's right	2
2	order the girls clockwise	5!
3	order the boys clockwise	5!

The answer is $2 \times (5!)^2$.

Now assume that there are two parents, labelled P and Q. Suppose we move clockwise around the table from P to Q. Let n denote the number of seats between the two. Thus $n = 0$ (resp. $n = 10$) if Q sits next to P at P's left (resp. right). We now partition the set of solutions according to the value of n.

n	# seatings
0	$2(5!)^2$
1	$2(5!)^2$
2	$4(5!)^2$
3	$2(5!)^2$
4	$4(5!)^2$
5	$2(5!)^2$
6	$4(5!)^2$
7	$2(5!)^2$
8	$4(5!)^2$
9	$2(5!)^2$
10	$2(5!)^2$

The answer is the sum of the entries in the right-most column, which comes to $30 \times (5!)^2$.

31. We proceed in stages:

stage	to do	# choices
1	award the gold	15
2	award the silver	14
3	award the bronze	13
4	choose the 3 losers	$\binom{12}{3}$

The answer is the product of the entries in the right-most column, which comes to

$$\frac{15!}{3!}.$$

32. Since the multiset S has 12 elements, the number of 11-permutations of S is equal to the number of permutations of S, which is

$$\frac{12!}{3! \times 4! \times 5!}.$$

33. Each 10-permutation of S involves all but two elements of S. We partition the set of solutions according to the missing two elements:

missing elements	# solutions
a, a	$\frac{10!}{1!4!5!}$
b, b	$\frac{10!}{3!2!5!}$
c, c	$\frac{10!}{3!4!3!}$
a, b	$\frac{10!}{2!3!5!}$
a, c	$\frac{10!}{2!4!4!}$
b, c	$\frac{10!}{3!3!4!}$

The answer is the sum of the entries in the right-most column, which comes to

$$\frac{85 \times 10!}{3! \times 4! \times 5!}$$

34. Since the multiset S has 12 elements, the number of 11-permutations of S is equal to the number of permutations of S, which is

$$\frac{12!}{(3!)^4}.$$

35. We list the 3-combinations:

example	# a's	# b's	# c's
1	2	1	0
2	2	0	1
3	1	1	1
4	1	0	2
5	0	1	2
6	0	0	3

We now list the 4-combinations:

example	# a's	# b's	# c's
1	2	1	1
2	2	0	2
3	1	1	2
4	1	0	3
5	0	1	3

36. The combinations correspond to the sequences $(x_1, x_2, \ldots, x_k)$ such that $0 \le x_i \le n_i$ for $1 \le i \le k$. We proceed in stages:

stage	to do	# choices
1	pick x_1	n_1+1
2	pick x_2	n_2+1
.	$\cdots$	.
.	$\cdots$	.
k	pick x_k	n_k+1

The answer is $(n_1+1)(n_2+1)\cdots(n_k+1)$.

37. Suppose that for $1 \le i \le 6$ we choose n_i pastries of the ith kind. Each of $\{n_i\}_{i=1}^6$ is nonnegative and $\sum_{i=1}^6 n_i = 12$. The number of ways to pick the $\{n_i\}_{i=1}^6$ is

$$\binom{12+6-1}{6-1} = \binom{17}{5}.$$

Now suppose that we choose at least one pastry of each type. In other words $n_i \ge 1$ for $1 \le i \le 6$. Define $m_i = n_i - 1$ for $1 \le i \le 6$. Then each of $\{m_i\}_{i=1}^6$ is nonnegative and $\sum_{i=1}^6 m_i = 6$. The answer is equal to the number of ways to pick the $\{m_i\}_{i=1}^6$, which is

$$\binom{6+6-1}{6-1} = \binom{11}{5}.$$

38. We make a change of variables. Define

$$y_1 = x_1 - 2, \qquad y_2 = x_2, \qquad y_3 = x_3 + 5, \qquad y_4 = x_4 - 8.$$

Note that $\{x_i\}_{i=1}^4$ is a solution to the original problem if and only if

$$y_1 \ge 0, \quad y_2 \ge 0, \quad y_3 \ge 0, \quad y_4 \ge 0, \qquad y_1 + y_2 + y_3 + y_4 = 25.$$

Therefore the number of solutions is

$$\binom{25+4-1}{4-1} = \binom{28}{3}.$$

39. (a) There are $\binom{20}{6}$ ways to choose six sticks from the twenty available sticks.

(b) Label the sticks $1, 2, \ldots, 20$. Suppose we choose six sticks labelled $\{x_i\}_{i=1}^6$ with $x_1 < x_2 < \cdots < x_6$. Define

$$y_1 = x_1 - 1, \qquad y_2 = x_2 - x_1 - 2, \qquad y_3 = x_3 - x_2 - 2, \qquad y_4 = x_4 - x_3 - 2,$$
$$y_5 = x_5 - x_4 - 2, \qquad y_6 = x_6 - x_5 - 2, \qquad y_7 = 20 - x_6.$$

Observe that the solutions $\{x_i\}_{i=1}^6$ to the original problem correspond to the integral solutions $\{y_i\}_{i=1}^7$ for

$$y_i \ge 0 \quad (1 \le i \le 7), \qquad \sum_{i=1}^{7} y_i = 9.$$

Therefore the number of solutions $\{x_i\}_{i=1}^{6}$ to the original problem is

$$\binom{9+7-1}{7-1} = \binom{15}{6}.$$

(c) We proceed as in (b) with the modification

$$\begin{aligned} &y_1 = x_1 - 1, \qquad y_2 = x_2 - x_1 - 3, \qquad y_3 = x_3 - x_2 - 3, \qquad y_4 = x_4 - x_3 - 3, \\ &y_5 = x_5 - x_4 - 3, \qquad y_6 = x_6 - x_5 - 3, \qquad y_7 = 20 - x_6. \end{aligned}$$

The solutions $\{x_i\}_{i=1}^{6}$ to the original problem correspond to the integral solutions $\{y_i\}_{i=1}^{7}$ for

$$y_i \geq 0 \quad (1 \leq i \leq 7), \qquad \sum_{i=1}^{7} y_i = 4.$$

Therefore the number of solutions $\{x_i\}_{i=1}^{6}$ to the original problem is

$$\binom{4+7-1}{7-1} = \binom{10}{6}.$$

40. (a) There are $\binom{n}{k}$ ways to choose k sticks from the n available sticks.

(b) Label the sticks $1, 2, \ldots, n$. Suppose we choose k sticks labelled $\{x_i\}_{i=1}^{k}$ with $x_1 < x_2 < \cdots < x_k$. Define

$$y_1 = x_1 - 1, \qquad y_i = x_i - x_{i-1} - 2 \qquad (2 \leq i \leq k), \qquad y_{k+1} = n - x_k.$$

Observe that the solutions $\{x_i\}_{i=1}^{k}$ to the original problem correspond to the integral solutions $\{y_i\}_{i=1}^{k+1}$ for

$$y_i \geq 0 \quad (1 \leq i \leq k+1), \qquad \sum_{i=1}^{k+1} y_i = n - 2k + 1.$$

Therefore the number of solutions $\{x_i\}_{i=1}^{k}$ to the original problem is

$$\binom{n-2k+1+k}{k} = \binom{n-k+1}{k}.$$

(c) We proceed as in (b) with the modification

$$y_1 = x_1 - 1, \qquad y_i = x_i - x_{i-1} - \ell - 1 \quad (2 \leq i \leq k), \qquad y_{k+1} = n - x_k.$$

The solutions $\{x_i\}_{i=1}^{k}$ to the original problem correspond to the integral solutions $\{y_i\}_{i=1}^{k+1}$ for

$$y_i \geq 0 \quad (1 \leq i \leq k+1), \qquad \sum_{i=1}^{k+1} y_i = n - k - \ell k + \ell.$$

Therefore the number of solutions $\{x_i\}_{i=1}^k$ to the original problem is

$$\binom{n-k-\ell k+\ell+k}{k} = \binom{n-\ell k+\ell}{k}.$$

41. We proceed in stages:

stage	to do	# choices
1	hand out the orange	3
2	give one apple to each of the other children	1
3	distribute remaining 10 apples to 3 children	$\binom{12}{2}$

The answer is $3 \times \binom{12}{2}$.

42. We proceed in stages:

stage	to do	# choices
1	hand out lemon drink	4
2	hand out lime drink	3
3	give one orange drink to each of the remaining two students	1
4	distribute remaining 8 orange drinks to 4 students	$\binom{11}{3}$

The answer is the product of the entries in the right-most column, which comes to $12 \times \binom{11}{3}$.

43. For an r-combination in question, either it contains a_1 or it does not. The number of r-combinations that contain a_1 is equal to the number of $(r-1)$-combinations of

$$\{\infty \cdot a_2, \ldots, \infty \cdot a_k\},$$

which comes to $\binom{r+k-3}{k-2}$. The number of r-combinations that do not contain a_1 is equal to the number of r-combinations of

$$\{\infty \cdot a_2, \ldots, \infty \cdot a_k\},$$

which comes to $\binom{r+k-2}{k-2}$. The answer is

$$\binom{r+k-2}{k-2} + \binom{r+k-3}{k-2}.$$

44. Label the objects $1, 2, \ldots, n$. We proceed in stages:

stage	to do	# choices
1	distribute object 1	k
2	distribute object 2	k
.	$\cdots$	.
.	$\cdots$	.
n	distribute object n	k

The answer is the product of the entries in the right-most column, which comes to k^n.

45. (a) For $1 \leq i \leq 5$ let x_i denote the number of books on shelf i. We seek the number of integral solutions to

$$x_i \geq 0 \quad (1 \leq i \leq 5), \qquad \sum_{i=1}^{5} x_i = 20.$$

The answer is

$$\binom{20+5-1}{5-1} = \binom{24}{4}.$$

(b) We proceed in stages:

stage	to do	# choices
1	put book 1 on a shelf	5
2	put book 2 on a shelf	5
.	...	.
.	...	.
20	put book 20 on a shelf	5

The answer is the product of the entries in the right-most column, which comes to 5^{20}.

(c) We proceed in stages:

stage	to do	# choices
1	order the books	20!
2	pick a solution $\{x_i\}_{i=1}^5$ to part (a)	$\binom{24}{4}$
3	put the first x_1 books on shelf 1	1
4	put the next x_2 books on shelf 2	1
5	put the next x_3 books on shelf 3	1
6	put the next x_4 books on shelf 4	1
7	put the last x_5 books on shelf 5	1

The answer is the product of the entries in the right-most column, which comes to

$$20! \times \binom{24}{4}.$$

46. (a) Call this number M_n. To find M_n, we pick a person at the party and proceed in stages:

stage	to do	# choices
1	choose the person's partner	$2n-1$
2	match the remaining $2n-2$ people	M_{n-1}

This shows that $M_n = (2n-1)M_{n-1}$. We also have $M_1 = 1$. Therefore

$$M_n = 1 \times 3 \times 5 \times \cdots \times (2n-1).$$

(b) We proceed in stages:

stage	to do	# choices
1	choose the loner	$2n+1$
2	match the remaining $2n$ people	$1 \times 3 \times \cdots \times (2n-1)$

The answer is the product of the entries in the right-most column, which comes to

$$1 \times 3 \times 5 \times \cdots \times (2n+1).$$

47. Observe that no shelf can hold more than n books. Imagine that we start with $3n$ books, n books per shelf. Remove a total of $n-1$ books to leave $2n+1$ books. For $i = 1, 2, 3$ let b_i denote the number of books removed from shelf i. We seek the number of integral solutions to

$$b_1 \geq 0, \quad b_2 \geq 0, \quad b_3 \geq 0, \qquad\qquad b_1 + b_2 + b_3 = n-1.$$

The answer is $\binom{n+1}{2}$.

48. Let P denote the set of permutations of m A's and at most n B's. We show

$$|P| = \binom{m+n+1}{m+1}.$$

Let Q denote the set of permutations of $m+1$ A's and n B's. We have

$$|Q| = \binom{m+n+1}{m+1}.$$

We show $|P| = |Q|$. To do this we display a bijection $Q \to P$. For $x \in Q$ obtain x' from x by deleting from x the right-most A along with the B's to its right. Observe that the map $Q \to P$, $x \mapsto x'$ is a bijection. The result follows.

49. Let R denote the set of permutations of at most m A's and at most n B's. We show

$$|R| = \binom{m+n+2}{m+1} - 1.$$

Let S denote the set of permutations of $m+1$ A's and $n+1$ B's. We have

$$|S| = \binom{m+n+2}{m+1}.$$

Obtain $S^\vee$ from S by deleting the permutation $AA\cdots ABB\cdots B$. Of course $|S^\vee| = |S| - 1$. We show $|R| = |S^\vee|$. To do this we display a bijection $S^\vee \to R$. For $x \in S^\vee$ obtain x' from

x by (i) deleting from x the left-most B along with all A's to its left; (ii) deleting from x the right-most A along with all B's to its right. Observe that the map $S^\vee \to R$, $x \mapsto x'$ is a bijection. The result follows.

50. We proceed in stages:

stage	to do	# choices
1	choose 2 rows for horiz. sides of rectangle	$\binom{8}{2}$
2	choose 2 columns for vert. sides of rectangle	$\binom{8}{2}$
3	place the 5th rook	$64 - 4$

The answer is the product of the entries in the right-most column, which comes to

$$\binom{8}{2}^2 \times 60.$$

51. We construct the desired n-combinations in stages:

stage	to do	# choices
1	choose a subset x of $\{1, 2, \ldots, n\}$	2^n
2	add $n - \|x\|$ a's to x	1

The number of n-combinations is 2^n.

52. For each of the n-combinations in question let the *index* be the number of elements used among $1, 2, \ldots, n+1$. For $0 \le i \le n$ the number of n-combinations with index i is $\binom{n+1}{i}(n+1-i)$. Therefore the number of n-combinations is

$$\sum_{i=0}^{n} \binom{n+1}{i}(n+1-i).$$

We evaluate the sum as follows. By the binomial theorem

$$(x+1)^{n+1} = \sum_{i=0}^{n+1} \binom{n+1}{i} x^{n+1-i}.$$

In this equation take the derivative of each side with respect to x:

$$(n+1)(x+1)^n = \sum_{i=0}^{n} \binom{n+1}{i}(n+1-i)x^{n-i}.$$

In this equation set $x = 1$ to get

$$(n+1)2^n = \sum_{i=0}^{n} \binom{n+1}{i}(n+1-i).$$

The answer is $(n+1)2^n$.

53. Let $x_1 x_2 \cdots x_n$ denote a permutation of $\{1, 2, \ldots, n\}$. For $0 \le k \le n$ define $A_k = \{x_1, x_2, \ldots, x_k\}$. The desired one-to-one correspondence sends $x_1 x_2 \cdots x_n$ to the tower

$$A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_n.$$

54. For $0 \le k \le n$ there are $\binom{n}{k}$ subsets B of $\{1, 2, \ldots, n\}$ such that $|B| = k$. For each such B there are 2^k choices for A. Therefore the number of solutions is

$$\sum_{k=0}^{n} \binom{n}{k} 2^k.$$

We evaluate this sum. By the binomial theorem

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Setting $x = 2$ we obtain

$$3^n = \sum_{k=0}^{n} \binom{n}{k} 2^k.$$

The answer is 3^n.

55. (a) The word has 17 letters with repetitions

letter	A	B	D	E	H	I	K	O	P	R	S	T
mult	3	1	1	1	1	3	2	1	1	1	1	1

The number of permutations is

$$\frac{17!}{2! \times (3!)^2}.$$

(b) The word has 29 letters with repetitions

letter	A	C	F	H	I	L	N	O	P	T	U
mult	2	4	2	1	9	3	3	2	1	1	1

The number of permutations is

$$\frac{29!}{(2!)^3 \times (3!)^2 \times 4! \times 9!}.$$

(c) The word has 45 letters with repetitions

letter	A	C	E	I	L	M	N	O	P	R	S	T	U	V
mult	2	6	1	6	3	2	4	9	2	2	4	1	2	1

The number of permutations is

$$\frac{45!}{(2!)^5 \times 3! \times (4!)^2 \times (6!)^2 \times 9!}.$$

(d) The number of permutations is 15!.

56. The number of poker hands is $\binom{52}{5}$. This is the denominator. To obtain the numerator we proceed in stages:

stage	to do	# choices
1	pick the suit	4
2	pick 5 cards of that suit	$\binom{13}{5}$

The numerator is $4 \times \binom{13}{5}$. The desired probability is

$$\frac{4 \times \binom{13}{5}}{\binom{52}{5}}.$$

57. The number of poker hands is $\binom{52}{5}$. This is the denominator. To compute the numerator we proceed in stages:

stage	to do	# choices
1	pick the rank of the pair	13
2	pick 2 suits for the pair	$\binom{4}{2}$
3	pick the ranks for remaining 3 cards	$\binom{12}{3}$
4	pick the suits for remaining 3 cards	4^3

The numerator is the product of the entries in the right-most column. The desired probability is

$$\frac{13 \times \binom{4}{2} \times \binom{12}{3} \times 4^3}{\binom{52}{5}}.$$

58. The number of poker hands is $\binom{52}{5}$. This is the denominator. We now compute the numerator. The number of straights is 10×4^5. The number of flushes is $\binom{13}{5} \times 4$. The number of straight flushes is 10×4. The numerator is

$$\binom{13}{5} \times 4^5 - 10 \times 4^5 - \binom{13}{5} \times 4 + 40.$$

The desired probability is

$$\frac{\binom{13}{5} \times 4^5 - 10 \times 4^5 - \binom{13}{5} \times 4 + 40}{\binom{52}{5}}.$$

59. For the reduced deck the number of poker hands is $\binom{40}{5}$. This is the denominator for each of (1)–(5) below.
(1) To compute the numerator we proceed in stages:

stage	to do	# choices
1	pick the three-card rank	10
2	pick the missing suit among previous three cards	4
3	pick the two-card rank	9
4	pick the 2 suits for the previous two cards	$\binom{4}{2}$

The numerator is the product of the entries in the right-most column. The desired probability is

$$\frac{10 \times 4 \times 9 \times \binom{4}{2}}{\binom{40}{5}}.$$

(2) To compute the numerator we proceed in stages:

stage	to do	# choices
1	pick the lowest rank in the straight	7
2	pick the suits	4^5

The numerator is 7×4^5. The desired probability is

$$\frac{7 \times 4^5}{\binom{40}{5}}.$$

(3) To compute the numerator we proceed in stages:

stage	to do	# choices
1	pick the lowest rank in the straight flush	7
2	pick the suit	4

The numerator is 7×4. The desired probability is

$$\frac{7 \times 4}{\binom{40}{5}}.$$

(4) To compute the numerator we proceed in stages:

stage	to do	# choices
1	pick the ranks for the 2 pairs	$\binom{10}{2}$
2	pick the suits for the previous cards	$\binom{4}{2}^2$
3	pick the rank of the remaining card	8
4	pick the suit of the previous card	4

The numerator is the product of the entries in the right-most column. The desired probability is

$$\frac{\binom{10}{2} \times \binom{4}{2}^2 \times 8 \times 4}{\binom{40}{5}}.$$

(5) The number of hands that contain no ace is $\binom{36}{5}$. The desired probability is

$$\frac{\binom{40}{5} - \binom{36}{5}}{\binom{40}{5}}.$$

60. The number of ways to pick the 15 bagels is equal to the number of integral solutions for

$$x_i \geq 0 \quad (1 \leq i \leq 6), \qquad \sum_{i=1}^{6} x_i = 15$$

which comes to $\binom{15+6-1}{6-1} = \binom{20}{5}$. This is the denominator. We now compute the numerator for the first probability. The number of ways to pick the 15 bagels so that you get at least one bagel of each kind is equal to the number of integral solutions for

$$y_i \geq 0 \quad (1 \leq i \leq 6), \qquad \sum_{i=1}^{6} y_i = 9$$

which is $\binom{9+6-1}{6-1} = \binom{14}{5}$. The first desired probability is

$$\frac{\binom{14}{5}}{\binom{20}{5}}.$$

We now compute the numerator for the second probability. The number of ways to pick the 15 bagels so that you get at least three sesame bagels is equal to the number of integral solutions for

$$z_i \geq 0 \quad (1 \leq i \leq 6), \qquad \sum_{i=1}^{6} z_i = 12$$

which is $\binom{12+6-1}{6-1} = \binom{17}{5}$. The second desired probability is

$$\frac{\binom{17}{5}}{\binom{20}{5}}.$$

61. The sample space S satisfies

$$|S| = 9! \times \binom{9}{4}.$$

The first event E satisfies

$$|E| = 9!.$$

The first desired probability is $|E|/|S|$ which comes to $\binom{9}{4}^{-1}$. The second event F satisfies

$$|F| = 5! \times 4!.$$

The second desired probability is $|F|/|S|$ which comes to

$$\frac{5! \times 4!}{9! \times \binom{9}{4}}.$$

62. The sample space has cardinality $\binom{52}{7}$. This will be the denominator for each of (a)–(f).

(a) To compute the numerator we proceed in stages:

stage	to do	# choices
1	pick the lowest rank in the straight	8
2	pick the suits	4^7

The answer is

$$\frac{8 \times 4^7}{\binom{52}{7}}.$$

(b) To compute the numerator we proceed in stages:

stage	to do	# choices
1	pick the four-card rank	13
2	pick the three-card rank	12
3	pick the missing suit among the previous three cards	4

The answer is

$$\frac{13 \times 12 \times 4}{\binom{52}{7}}.$$

(c) To compute the numerator we proceed in stages:

stage	to do	# choices
1	pick the three-card rank	13
2	pick the missing suit among previous three cards	4
3	pick the 2 two-card ranks	$\binom{12}{2}$
4	pick the suits for previous four cards	$\binom{4}{2}^2$

The answer is

$$\frac{13 \times 4 \times \binom{12}{2} \times \binom{4}{2}^2}{\binom{52}{7}}.$$

(d) To compute the numerator we proceed in stages:

stage	to do	# choices
1	pick the 3 two-card ranks	$\binom{13}{3}$
2	pick the suits for previous six cards	$\binom{4}{2}^3$
3	pick the one-card rank	10
4	pick the suit for previous card	4

The answer is

$$\frac{\binom{13}{3} \times \binom{4}{2}^3 \times 10 \times 4}{\binom{52}{7}}.$$

(e) To compute the numerator we proceed in stages:

stage	to do	# choices
1	pick the three-card rank	13
2	pick the missing suit among previous three cards	4
3	pick the 4 one-card ranks	$\binom{12}{4}$
4	pick the suits for previous four cards	4^4

The answer is

$$\frac{13 \times 4 \times \binom{12}{4} \times 4^4}{\binom{52}{7}}.$$

(f) To compute the numerator we proceed in stages:

stage	to do	# choices
1	pick the 7 ranks	$\binom{13}{7}$
2	pick the suits	4^7

The answer is

$$\frac{\binom{13}{7} \times 4^7}{\binom{52}{7}}.$$

63. The size of the sample space is 6^4. This is the denominator for (a)–(e) below.

(a) The die numbers must be some permutation of $3, 1, 1, 1$ (4 ways) or $2, 2, 1, 1$ ($\binom{4}{2}$ ways). The numerator is $4 + 6 = 10$. The desired probability is $10/6^4$.

(b) One dot occurs either once (4×5^3 ways) or twice ($\binom{4}{2} \times 5^2$ ways) or not at all (5^4 ways). The desired probability is

$$\frac{5^4 + 4 \times 5^3 + \binom{4}{2} \times 5^2}{6^4}.$$

(c) The desired probability is $5^4/6^4$.

(d) The desired probability is $P(6,4)/6^4$.

(e) The die numbers must be some permutation of i, i, i, j ($4 \times 6 \times 5$ ways) or i, i, j, j ($\binom{6}{2} \times \binom{4}{2}$ ways). Here we mean $i \neq j$. The desired probability is

$$\frac{4 \times 6 \times 5 + \binom{6}{2} \times \binom{4}{2}}{6^4}.$$

64. (a) The desired probability is

$$\binom{n}{2} \times \frac{\binom{n-2}{2}\binom{n}{2}\binom{n-2}{2}(n-4)! + (n-2)\binom{n}{3}(n-3)!}{n^n}.$$

(b) The desired probability is

$$\binom{n}{3} \times \frac{(n-3)\binom{n}{4}(n-4)! + (n-3)(n-4)\binom{n}{3}\binom{n-3}{2}(n-5)! + \binom{n-3}{3}\binom{n}{2}\binom{n-2}{2}\binom{n-4}{2}(n-6)!}{n^n}.$$