Knots, braids and their invariants

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History

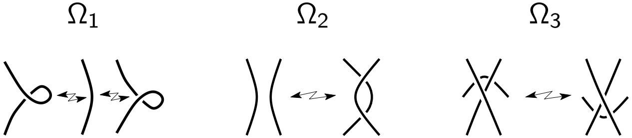
The first mathematician to study knots as mathematical objects was Carl Friedrich Gauss. While Gauss himself wrote little about knots, his student Johann Benedict Listing dedicated much of his monograph to the study of knots and links. By the late 19th and early 20th centuries, large tables of knots had been compiled, but it was not clear that knots within them were distinct. A revolution in knot theory occurred with the introduction of the Jones polynomial in 1985 [3]. This was quickly followed by the HOMFLY polynomial [2] and Kauffman polynomial [4].

Quantum invariants

These knot polynomials were later generalized by quantum knot invariants, one of which is studied at a relatively elementary level in this work. The concept of quantum invariants originates from quantum physics, first introduced in Witten's groundbreaking paper[9]. Although Witten's formulation was not mathematically rigorous, it was later formalized by Reshetikhin and Turaev[7, 6], who used quantum groups, which had been introduced shortly before this.

Reidemeister moves

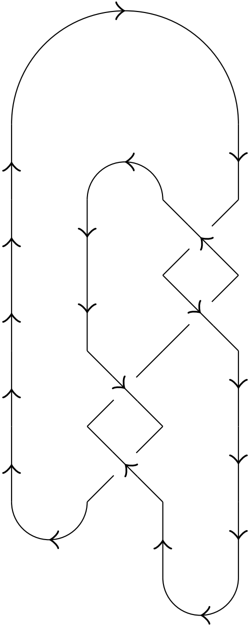
Knots are conveniently represented by knot diagrams. A knot diagram is a projection of a knot onto a plane, where the curve has only a finite number of self-intersections, and at each intersection, one branch of the curve passes over the other. The Reidemeister theorem [1] asserts that two knots are equivalent if and only if their diagrams can be transformed into each other through a sequence of three types of Reidemeister moves:



Reidemeister moves

In addition to knots, their close relatives, called *tangles*, are often studied. A tangle is both a generalization and a building block of knots. On one hand, knots are a special case of tangles, and on the other, any knot can be represented as a combination of elementary tangles.

Figure-eight knot



- A knot with a *framing*—a smooth family of perpendiculars to the knot—is called *framed*. Framed knots can be thought of as knots tied on a ribbon.
- The HOMFLY polynomial of a knot is a polynomial P in two variables, typically denoted by a and z. It was introduced in [2, 5], where it was defined by the skein relation:

$$aP\Big(L_+\Big) - a^{-1}P\Big(L_-\Big) = zP\Big(L_0\Big)$$

Result

In this work, I provide an explicit construction of the invariant $\theta_{\mathfrak{sl}_N}^{St}$, which is equivalent to the HOMFLY polynomial and a more general invariant $\theta_{\mathfrak{sl}_N}^{fr,St}$ defined for framed knots. Moreover, this construction also applies to tangles.

Operator R

To construct the invariant $\theta^{fr,St}_{\mathfrak{sl}_N}$, we introduce some notation. Let V be an N-dimensional vector space with basis $e_1, \dots, e_N$. Define the operator $R : V \otimes V \to V \otimes V$ on the basis as follows:

$$R(e_i \otimes e_j) = \begin{cases} q^{\frac{-1}{2N}} e_j \otimes e_i, & i < j; \\ q^{\frac{N-1}{2N}} e_i \otimes e_j, & i = j; \\ q^{\frac{-1}{2N}} e_j \otimes e_i + \left(q^{\frac{N-1}{2N}} - q^{\frac{-N-1}{2N}}\right) e_i \otimes e_j, & i > j. \end{cases}$$

This operator can be extended to $V \otimes V$ by linearity.

Inverse operator

The inverse operator $R^{-1} : V \otimes V \to V \otimes V$ is defined on the basis as follows:

$$R^{-1}(e_i \otimes e_j) = \begin{cases} q^{\frac{1}{2N}} e_j \otimes e_i + \left(-q^{\frac{N+1}{2N}} + q^{\frac{-N+1}{2N}}\right) e_i \otimes e_j, & i < j; \\ q^{\frac{-N+1}{2N}} e_i \otimes e_j, & i = j; \\ q^{\frac{1}{2N}} e_j \otimes e_i, & i > j. \end{cases}$$

Verifying $RR^{-1} = id$

WLOG assume $i < j$. Then:

$$R^{-1}(e_i \otimes e_j) = q^{\frac{1}{2N}} e_j \otimes e_i + \left(-q^{\frac{N+1}{2N}} + q\right.$$

$$R\left(q^{\frac{1}{2N}} e_j \otimes e_i + \left(-q^{\frac{N+1}{2N}} + q^{\frac{-N+1}{2N}}\right) e_i \otimes e_j\right) = e_i \otimes e_j.$$

Thus, the operator $R \circ R^{-1}$ coincides with the identity operator on $V \otimes V$, verifying that R^{-1} behaves as expected.

Square equation on R

The operator R satisfies the equation:

$$q^{\frac{1}{2N}} R - q^{-\frac{1}{2N}} R^{-1} = \left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right) id_{V \otimes V}.$$

For $i < j$:

$$\begin{aligned}
& q^{\frac{1}{2N}} R(e_i \otimes e_j) - q^{-\frac{1}{2N}} R^{-1}(e_i \otimes e_j) \\
& = q^{\frac{1}{2N}} q^{\frac{-1}{2N}} e_i \otimes e_j - q^{-\frac{1}{2N}} \left(q^{\frac{1}{2N}} e_j \otimes e_i + \left(-q^{\frac{N+1}{2N}} + q^{\frac{-N+1}{2N}}\right) e_i \otimes e_j\right) \\
& = \left(q^{\frac{N}{2N}} - q^{\frac{-N}{2N}}\right) e_i \otimes e_j.
\end{aligned}$$

Hence, all eigenvalues of R are $q^{\frac{N-1}{2N}}$ or $-q^{\frac{-N-1}{2N}}$.

Constructing the invariant $\theta^{fr,St}_{\mathfrak{sl}_N}$

To construct the invariant $\theta^{fr,St}_{\mathfrak{sl}_N}$, we introduce additional operators:

$$\underrightarrow{\min} : \mathbb{C} \to V^* \otimes V, \qquad \underrightarrow{\min}(1) := \sum_{k=1}^{N} q^{\frac{-N-1}{2}+k} e^k \otimes e_k,$$

$$\underleftarrow{\min} : \mathbb{C} \to V \otimes V^*, \qquad \underleftarrow{\min}(1) := \sum_{k=1}^{N} e_k \otimes e^k,$$

$$\overrightarrow{\max} : V \otimes V^* \to \mathbb{C}, \qquad \overrightarrow{\max}(e_i \otimes e^j) = \begin{cases} 0, & i \neq j, \\ q^{\frac{N+1}{2}-i}, & i = j, \end{cases}$$

$$\overleftarrow{\max} : V^* \otimes V \to \mathbb{C}, \qquad \overleftarrow{\max}(e^i \otimes e_j) = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases}$$

Constructing the invariant $\theta_{\mathfrak{sl}_N}^{fr,St}$

To build the invariant $\theta_{\mathfrak{sl}_N}^{fr,St}$, represent the tangle as a composition of elementary tangles. On the boundary between two elementary tangles, associate the space V to each thread going upwards, and V^* to each thread going downwards. The boundary itself is associated with the tensor product of these spaces.

The mappings for trivial tangles are defined as follows: a thread going from bottom to top without intersecting other threads corresponds to the identity map from V to V. Similarly, a thread going from top to bottom corresponds to the identity map from V^* to V^*.

Elementary tangles

Each "event" in a tangle corresponds to an operator. For example, a minimum going left is assigned to the linear transformation $\underleftarrow{\text{min}} : \mathbb{C} \to V \otimes V^*$:

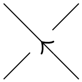
Similarly, a minimum going right corresponds to $\underrightarrow{\text{min}}$:

A maximum going right corresponds to $\overrightarrow{\text{max}}$:

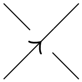
A maximum going left corresponds to $\overleftarrow{\text{max}}$:

Elementary tangles

If an event is a crossing, there are eight possible cases. If both lines are going up, then:



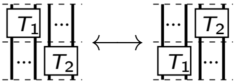
is assigned to the operator R, and:

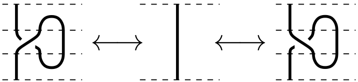
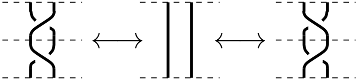
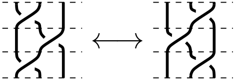
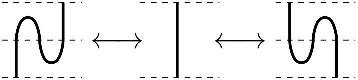
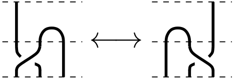
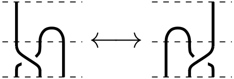


is assigned to the operator R^{-1}. All other cases can be expressed using previously defined operators.

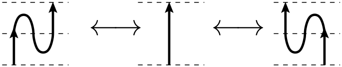
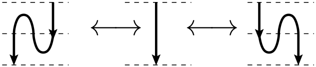
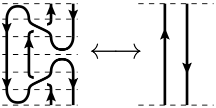
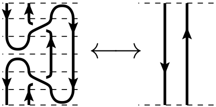
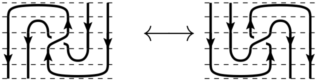
If a tangle represents a knot, its boundaries are empty, and it is assigned a linear transformation from $\mathbb{C}$ to $\mathbb{C}$. Denote the image of 1 by $\theta_{\mathfrak{sl}_N}^{fr,St}$.
For tangle diagrams, there exists an analogue of the Reidemeister moves called Turaev moves [8]. Tangle diagrams are equivalent if and only if they can be transformed into each other by a finite sequence of Turaev moves.

In non-oriented case

- 

- 
- 
- 
- 
- 
- 

In oriented case

- The same as in non-oriented case with arbitrary orientation of threads.
- The same as in non-oriented case with threads going from the bottom to the top
- 
- 
- 
- 
- 

Yang-Baxter equation

The operator R satisfies the Yang-Baxter equation. Let
$R_{12} : V \otimes V \otimes V \to V \otimes V \otimes V = R \otimes id_V$, and
$R_{23} : V \otimes V \otimes V \to V \otimes V \otimes V = id_V \otimes R$. Then:

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}. \tag{1}$$

This can be verified by checking the actions of both sides of (1) on the basis elements $e_i \otimes e_j \otimes e_k$ of $V \otimes V \otimes V$. If their actions coincide on the basis, they will coincide on the entire space.

Skein-relation

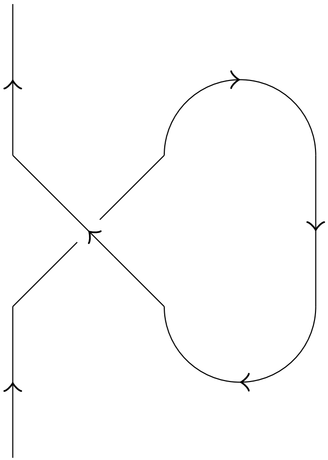
Let us prove that for $\theta^{fr,St}_{\mathfrak{sl}_N}$, the following skein-relation holds:

$$q^{\frac{1}{2N}}\theta^{fr,St}_{\mathfrak{sl}_N}(L_+) - q^{-\frac{1}{2N}}\theta^{fr,St}_{\mathfrak{sl}_N}(L_-) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\theta^{fr,St}_{\mathfrak{sl}_N}(L_0).$$

Consider a knot diagram where the knot is decomposed into elementary tangles, with L_-, L_0, and L_+ oriented upwards. They correspond to R, $id_V \otimes id_V$, and R^{-1}, respectively. By linearity, it suffices to check that $q^{\frac{1}{2N}}R - q^{-\frac{1}{2N}}R^{-1} = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})id_{V\otimes V}$, which was previously verified.

First Turaev move

First Turaev move replaces the loop with one line, but $\theta^{fr,St}_{\mathfrak{sl}_N}$ is not preserved. Let us look what is exactly happening to $\theta^{fr,St}_{\mathfrak{sl}_N}$.



First Turaev move

First Turaev move replaces the loop with one line, but $\theta^{fr,St}_{\mathfrak{sl}_N}$ is not preserved. Let us look what is exactly happening to $\theta^{fr,St}_{\mathfrak{sl}_N}$.

Loop is oriented upside down and gives us an operator $V \to V$ by formula $(id_V \otimes \overrightarrow{\mathrm{max}})(R \otimes id_{V^*})(id_V \otimes \underleftarrow{\mathrm{min}})$.

One can check that changing writhe by ± 1 corresponds to multiplying $\theta^{fr,St}_{\mathfrak{sl}_N}$ by $q^{\pm \frac{N-1/N}{2}}$.

Deframing

Define the invariant $\theta^{St}_{\mathfrak{sl}_N}$ of a knot by the relation:

$$\theta^{St}_{\mathfrak{sl}_N} := q^{-\frac{N-1/N}{2}w}\theta^{fr,St}_{\mathfrak{sl}_N},$$

where w is the writhe, the difference between the number of positive crossings (associated with R, $R_{\searrow\nearrow}$, $R_{\nwarrow\swarrow}$, and $R_{\searrow\swarrow}$) and negative crossings (associated with R^{-1}, $R^{-1}_{\searrow\nearrow}$, $R^{-1}_{\nwarrow\swarrow}$, and $R^{-1}_{\searrow\swarrow}$). Both $\theta^{fr,St}_{\mathfrak{sl}_N}$ and the writhe must be calculated for the same diagram. The skein-relation for $\theta^{St}_{\mathfrak{sl}_N}$ is:

$$q^{\frac{1}{2N}}q^{\frac{N-1/N}{2}}\theta^{fr,St}_{\mathfrak{sl}_N}(L_+) - q^{-\frac{1}{2N}}q^{-\frac{N-1/N}{2}}\theta^{fr,St}_{\mathfrak{sl}_N}(L_-) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\theta^{fr,St}_{\mathfrak{sl}_N}(L_0).$$

The invariant $\theta^{St}_{\mathfrak{sl}_N}$ can be obtained from the HOMFLY polynomial $P(Knot)$ by substituting $q^{\frac{N}{2}}$ for a and $q^{\frac{1}{2}} - q^{-\frac{1}{2}}$ for z, up to the factor $\frac{q^{\frac{N}{2}} - q^{\frac{-N}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}$. Thus, $\theta^{St}_{\mathfrak{sl}_N}$ is equivalent to the HOMFLY polynomial.

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