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Riemann: Geometry and Physics

Figure 18.1 Riemann

18.1 Riemann

Riemann was the archetype of the shy mathematician, not much drawn to topics other than mathematics, physics and philosophy, devout in his religion, conventional in his tastes, close to his family and awkward outside them.[1] As a child, he was taught by his father, a pastor, and then for some years at school before going to Göttingen University. There he had initially intended to study theology, in accordance with his father's wishes – Göttingen was the only university in Riemann's native Hanover with strong links to the Hanover church – but his remarkable ability at mathematics led him to switch subjects. He was

[1] For a biography of Riemann, see Laugwitz, *Bernhard Riemann* [148].

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always inclined to the conceptual side of things, rather than the computational or algorithmic. His written German is scholarly and old-fashioned – Victorian, one might say – and his Latin (required for academic purposes) is no easier.

However, the level of mathematical education in Göttingen was not particularly high, and in accordance with the German freedom to study anywhere Riemann left Göttingen for Berlin in 1847, where he spent two years learning from Dirichlet about potential theory and partial differential equations, number theory and theory of integration. He also attended Jacobi's lectures on analytical mechanics and higher algebra. In particular, the influence of Jacobi's lectures might well have been to stimulate Riemann's tendency to think in an abstract and sophisticated way about the relation of mathematics to physics and the real world.

But Berlin was a bustling capital, and in other ways Göttingen was the ideal choice for Riemann. His brief involvement in the revolutionary events of 1848 in Berlin also seem to have been a factor in his decision to return, as the subsequent reactionary crackdown sought to punish those who had been involved.[2]

He returned home in 1848, then went back to Berlin, and then came back to Göttingen in 1849. Here he attended Wilhelm Weber's lectures on mathematical physics and for a while he devoted himself to studies in physics and Naturphilosophie. This influenced his way of dealing with geometry. There he quickly met Richard Dedekind, who was five years younger than him, and who was also a truly conceptual thinker, but unlike Riemann, one who was to move slowly and steadily throughout a long life. In 1855 matters improved greatly for Riemann. Gauss had for some years been a distant but real presence in Göttingen, but after his death in 1855 Dirichlet was called as the senior professor. Socially there was something of a strain between the sociable and highly musical Dedekind and Dirichlet (who was married to a sister of Mendelssohn-Bartholdy) on the one hand, and the gauche Riemann. But mathematically, Dirichlet, who may be regarded as the man who brought rigorous mathematics to Germany, was the ideal abstract thinker to guide Riemann, and the young man remained very grateful to him. His influence is visible in Riemann's work, not just the published papers but the lectures he began to give once he was working for his Habilitation.

However, Dirichlet died in 1859 and Riemann was very quickly appointed his successor (the more charismatic figure of Dedekind being by then a professor in Zürich). He still attracted only a few students, and one wonders how matters might have developed had he not contracted pleurisy in 1862. This led to a permanent weakening of his lungs, and he was advised to spend as much time as possible in the south. He accordingly spent as much time as he could in Italy,

[2] See the letters in Neuenschwander [175, pp. 85–131].

but his health deteriorated and on 20 July 1866 he died near Lake Maggiore, where he is buried. He left behind a number of published papers, several more in a good enough state to be published, and yet more that could be edited and printed in the first edition of his *Werke* (edited by Dedekind and Heinrich Weber in 1876).

It was not only the difficult and boldly original nature of Riemann's ideas that hindered their reception. He had had a few good students, but several of them, Hankel, Hattendorff and Roch, died young. After 1866 the Göttingen tradition then lapsed into the hands of Schering, who was not a profound mathematician. More might have come from Clebsch's discovery of Riemann's ideas, had he also not died, aged 39, in 1872. Thereafter, the German scene was increasingly dominated by the school around Weierstrass and Kronecker, which placed much more emphasis on algebra than geometry, let alone topology.

18.2 Riemann's publications

The first of Riemann's publications was the privately distributed doctoral thesis (1851) on the foundations of a theory of functions of a complex variable, where the idea of a "Riemann surface" was presented for the first time. The thesis was accepted and his defence successfully conducted in December 1851. In the German university system in the 19th century, and indeed until recently, there was a crucial post-doctoral qualification called, in untranslatable German, the Habilitation. In 1854 he successfully presented his Habilitationsschrift (essay – on the theory of functions) and gave his Habilitationsvortrag (lecture – on geometry).

Riemann is a prime example, with Galois, of the turn towards conceptual thinking and away from prodigious calculation that characterises the 19th century, and it was his conceptual novelties that made his work hard to accept. He was deeply involved for a time in studying the philosophy of Herbart, and worked enthusiastically in experimental physics, where Wilhelm Weber exerted a life-long influence upon him, even though Riemann's ideas in physics on action at a distance became distinctly unWeberian. In particular, Riemann speculated on how distortions in space might enable forces such as electricity, magnetism and gravity to travel. His idea was that these distortions of space would somehow enable influence to travel from one object to another, thus explaining the otherwise mysterious action at a distance of these fundamental forces, and mathematically they would show up as variations in the underlying metrical relations, thus altering the geometry of space.[3]

[3] For more details, see Bottazzini and Tazzioli [25].

His involvement in both philosophy and physics enabled him to reformulate the concept of a mathematical quantity. He came to argue that mathematics is about "n-fold extended quantities" ("n-tuples of real numbers" is the equally unattractive modern term) to which is added some appropriate extra structure. In particular, complex numbers are pairs of real numbers; some extra structure (given by the Cauchy–Riemann equations) enables one to define analytic functions.

This new metaphysical basis for mathematics marks a break with the old theory of magnitudes that still preoccupied Gauss. In the context of differential geometry it freed the mind to propose many different descriptions of physical space (any set of n-fold extended quantities with a metric will do) and to regard Euclidean geometry as just one possibility among many. It also fits very well to a view of physics as being about variable quantities that one can measure and whose variations obey some physical "laws".

18.3 Riemann on geometry

In 1854 Riemann submitted three topics for his Habilitation, and contrary to his hopes, as he later wrote, Gauss chose the one "On the hypotheses which lie at the foundations of geometry" that therefore formed his lecture for the Habilitation (the *Habilitationsvortrag* (1854)).[4] The lecture, of which lengthy extracts will be found below, is not an easy read. It lacks the formulae which would help mathematicians understand it, because the lecture was given to the philosophy faculty, of which mathematics formed a department. Moreover, the philosopher Lotze was on the panel, and Riemann may have known what Klein was to find out to his pain some years later (1870) that Lotze was very dismissive of non-Euclidean geometry. Gauss, on the other hand, knew that his health was very poor and was eager to hear what this bright young man had to say on a subject that was among his lifelong interests. According to Dedekind, the lecture exceeded all his expectations, he sat through it in the greatest astonishment and spoke to his friend Wilhelm Weber afterwards in a rare state of excitement about the profundity of Riemann's ideas.[5]

In it Riemann did not explicitly mention non-Euclidean geometry, and he never mentioned the names Bolyai and Lobachevskii. He referred only to Legendre when describing the darkness which he said has covered the foundations since the time of Euclid, and later in [209, II, §5 and III, §1] he described two homogeneous geometries in which the angle sum of all triangles is deter-

[4] Riemann's *Lebenslauf* in Riemann, *Gesammelte Werke*, 3rd edn. [210, p. 579].
[5] Riemann's *Lebenslauf* in Riemann, *Gesammelte Werke*, 3rd edn. [210, p. 581].

mined once it is known for one triangle, as occurring on surfaces of constant non-zero curvature. This oblique glance at the characterisation of three homogeneous geometries is typical of the formulation of the "problem of parallels" since Saccheri; it can be found in many editions of Legendre's *Éléments de géométrie*, e.g. [149], and would have alerted any mathematician to the implications for non-Euclidean geometry without its being mentioned by name. On the other hand, not naming it explicitly would avoid philosophical misapprehensions about what he had to say, so one may perhaps ascribe the omission to a Gaussian prudence.

It is less certain, however, that Riemann had read Lobachevskii, and very unlikely he had read Bolyai. Only Gauss appreciated them in Göttingen at that time, and it is not known if he discussed these matters with Riemann. There is, however, the tantalising fact that a major paper of Dirichlet's on the theory of Fourier series, which we can be certain Riemann read, was published in the same volume of Crelle's *Journal* as Lobachevskii's "Géométrie imaginaire" of 1837 [152, vol. 17]. It is hard to believe that Riemann's eyes would not have caught that title and, thus drawn, read the article itself.[6] That said, the crucial idea in Riemann's paper is his presentation of geometry as intrinsic, grounded in the free mobility of infinitesimal measuring rods, and to be expressed mathematically in terms of curvature. This is an immense generalisation of Gauss's idea of the intrinsic curvature of a surface [84], itself a profound novelty.

18.3.1 Surfaces

Applied to surfaces, Riemann's proposals went like this. A surface is a two-dimensional set of points (something that is swept out by a moving curve, just as a curve is swept out by a moving point). Given a path in the surface, its length is measured approximately by using a finite ruler, and smaller and smaller rulers give better and better approximations. The length is measured correctly only in the limit and, by means of an infinitesimal ruler, Riemann gave a precise description in formulae. Accordingly, computing length is done by integration, and the concept of length is infinitesimal and calculus based, hence the name differential geometry for this branch of mathematics. Once you have a concept of length you can find the geodesic between two points and compute the curvature of the surface as Gauss had shown. If the surface is a plane, a geodesic is a straight line, and on a sphere it is a great circle.

[6] The page numbers are 35–56 for Dirichlet and 295–320 for Lobachevskii [152, vol. 17].

But almost all Riemann said about non-Euclidean geometry was that the angle sum of a triangle on a surface of constant negative curvature is always less than π. He never mentioned the subject by name, or any investigator of it except Legendre. (See the extract below, taken from the first English translation of the hypotheses, produced, oddly unclearly, by his leading English contemporary, W. K. Clifford [39].) He did, however, drop one remark, one formula, which shows just how deep his insight was into the topic.

Riemann wished to discuss surfaces without reference to any ambient space. His inspiration here was Gauss's discovery of curvature, which Gauss showed was something that could be determined from quantities measured in the surface alone. Indeed, Riemann's whole insight into geometry may be summarised by saying that geometry is first of all the study of the intrinsic properties of n-dimensional manifolds, and that the study of how a manifold inherits properties from a larger ambient space should be reformulated accordingly. Now, when one studies a surface intrinsically, one has only coordinates. For example, the familiar latitude and longitude coordinates on the earth (regarded as a perfect sphere) may be thought of as specifying points in a plane, and indeed this is exactly how a map of the surface may be constructed and depicted in the pages of an atlas. In fact, maps of the earth may be constructed in many ways. Stereographic projection, for example, maps the sphere of radius r onto the tangent plane to the sphere at the north pole by joining a point on the sphere with a straight line from the south pole to a point P in the plane. If the point P has coordinates (x, y) and distance is measured according to the formula

$$ds = \frac{\sqrt{dx^2 + dy^2}}{1 + (1/4r^2)\,(x^2 + y^2)}$$

the intrinsic geometry of the plane is identical with spherical geometry, because this formula is what the usual metric on the sphere looks like after stereographic projection. A rather complicated calculation done once and for all in courses on differential geometry allows mathematicians to compute the curvature of a surface from its metric, and if one starts with the plane with the metric just given, then, happily, the space being described is indeed a space of constant curvature r^{-2}.

This little formula conceals a remarkable statement. If one sets r^{-2} negative, say, for simplicity, $r^{-2} = -4$, the formula for distance becomes $ds = \frac{\sqrt{dx^2+dy^2}}{1-(x^2+y^2)}$, which makes sense only when $x^2 + y^2 < 1$. But inside this region, which is the interior of the circle of radius 1, the formula for distance describes a two-dimensional space of constant negative curvature (-4, to be precise). There can be no doubt that Riemann knew this perfectly well, even though he did not draw attention to it, because he had been talking about negative curvature just a page before.

18.4 Riemannian geometry

The vital point about Riemann's lecture is that Riemann said that all geometry is based on intrinsic measurement. Gauss had made only the much more limited claim that it is possible to do geometry on a surface without referring to the surrounding Euclidean (three-dimensional) space. Following Gauss, Riemann showed that any two surfaces have different geometries, that is, different mathematical theorems are true for them if they have different curvatures anywhere; but he did much more. Riemann gave a wholly novel answer to the question: what is geometry? To him, geometry was to do with concepts like length and angle which could be intrinsically defined on a surface or space of some sort. It follows that there are many geometries, one for each kind of surface and each definition of distance: a geometry arises from anything in which it makes sense to talk of a distance between two points, and this geometry will have a set of theorems associated with it.

This is a radical step: we go from having only one true geometry, to having infinitely many different geometries, none of which has any special status. There is not even a universal ambient space which endows all subspaces with their metrics. To see how radical this is, if we have a three-dimensional set of points which possess some sense of distance, then it carries a geometry. As an example, consider physical space. Do Riemann's ideas mean that space is Euclidean? Not at all, since we have not mentioned Euclidean geometry. Far from being the origin of geometrical properties, Euclidean space becomes just one candidate for physical space. To discover whether it is the correct one it would be necessary to make measurements and calculate the three-dimensional analogue of the curvature.

A point upon which Riemann insisted, indeed he opened the lecture with it, was that Euclid's postulates are completely subverted: no longer can they be regarded as unproblematically true assumptions about physical space. Instead, Riemann argued, all geometry is based on specific metrical considerations, and Euclid's geometry cannot occupy a paramount position as the geometry of space and the source of geometrical concepts which are induced onto embedded surfaces. Riemann's starting point was so radical that discussion of any axiom system for Euclidean geometry would miss the point, and this also contributes to what can seem to be the "case of the missing geometry". Non-Euclidean geometry is not centre stage in Riemann's lecture because axioms for (Euclidean) geometry are not fundamental, or even interesting, for Riemann.

Riemann made some other remarkable observations. He distinguished between a space (spaces?) which is unbounded and that which is infinite. He pointed out that there was no evidence that physical space was bounded, in the sense that it had a boundary beyond which one could not go. But this did not mean that it was infinite. It could simply be a (three-dimensional) sphere, for example. This idea has become commonplace since the days of Einstein and the modern theories of cosmology which go from the Big Bang to a finite universe, but it is hard to underestimate how shocking it must have seemed to everyone accustomed to thinking of space as infinite (whether for physical or theological reasons). He also had provocative, and remarkably modern, ideas about how geometry might break down in very very small regions.

Riemann did not seek to publish these ideas. They were somewhat further developed in his Paris prize entry of 1861, also unpublished until 1876, and first appeared in 1867, after his death. By then Beltrami had independently discovered the import of Gauss's ideas for non-Euclidean geometry.

18.5 From Riemann's *Habilitationsvortrag*

The extracts below are taken from Clifford's translation of Riemann's *Habilitationsvortrag* [209], originally published in *Nature* in 1873. William Kingdon Clifford was by common consent the best English mathematician of his generation, with a particular ability in geometry, and by this translation he signalled his appreciation of Riemann's ideas. He went on to do work on the difficult theory of Riemann surfaces before succumbing to tuberculosis. He died in 1879 at the age of 33.

The translation [39] has more than its share of obscure Victorian phraseology, which may be attributed to Riemann's dense German style and the fact that mathematical terminology was being created and complexities uncovered as Riemann and Clifford were writing. Clifford usually wrote more clearly than this, and was regarded as a gifted lecturer. Some of these are easy enough to penetrate. Riemann's German word "Mannigfaltigkeit" was translated by Clifford as "manifoldness" and sometimes as "extent" and can be taken to mean "manifold" (with some of the same lofty overtones) provided it is remembered that here it is an informal concept and today it is a precise one.

ON THE HYPOTHESES WHICH LIE AT THE BASES OF GEOMETRY

Plan of the Investigation.

It is known that geometry assumes, as things given, both the notion of space and the first principles of constructions in space. She gives definitions of them which are merely nominal, while the true determinations appear in the form of axioms. The relation of these assumptions remains consequently in darkness; we neither perceive whether and how far their connection is necessary, nor, *a priori*, whether it is possible.

From Euclid to Legendre (to name the most famous of modern reforming geometers) this darkness was cleared up neither by mathematicians nor by such philosophers as concerned themselves with it. The reason of this is doubtless that the general notion of multiply extended magnitudes (in which space-magnitudes are included) remained entirely unworked. I have in the first place, therefore, set myself the task of constructing the notion of a multiply extended magnitude out of general notions of magnitude. It will follow from this that a multiply extended magnitude is capable of different measure-relations, and consequently that space is only a particular case of a triply extended magnitude. But hence flows as a necessary consequence that the propositions of geometry cannot be derived from general notions of magnitude, but that the properties which distinguish space from other conceivable triply extended magnitudes are only to be deduced from experience. Thus arises the problem, to discover the simplest matters of fact from which the measure-relations of space may be determined; a problem which from the nature of the case is not completely determinate, since there may be several systems of matters of fact which suffice to determine the measure-relations of space – the most important system for our present purpose being that which Euclid has laid down as a foundation. These matters of fact are – like all matters of fact – not necessary, but only of empirical certainty; they are hypotheses. We may therefore investigate their probability, which within the limits of observation is of course very great, and inquire about the justice of their extension beyond the limits of observation, on the side both of the infinitely great and of the infinitely small.
[...]

II. *Measure-relations ...*

§1. Measure-determinations require that quantity should be independent of position, which may happen in various ways. The hypothesis which first presents itself, and which I shall here develop, is that according to which the length of lines is independent of their position, and consequently every line is measurable by means of every other. Position-fixing being reduced to quantity-fixings, and the position of a point in the n-dimensioned manifoldness being consequently expressed by means of n variables x_1, x_2, $x_3, \ldots, x_n$ the determination of a line comes to the giving of these quantities as functions of one variable. The problem consists then in establishing a mathematical expression for the length of a line, and to this end we must consider the quantities x as expressible in terms of certain units. I shall treat this problem only under certain restrictions, and I shall confine myself in the first place to lines in which the ratios of the increments dx of the respective variables vary continuously. We may then conceive these lines broken up into elements, within which the ratios of the quantities dx may be regarded as constant; and the problem is then reduced to establishing for each point a general expression for the linear element ds starting from that point, an expression which will thus contain the quantities x and the quantities dx. I shall suppose, secondly, that the length of the linear element, to the first order, is unaltered when all the points of this element undergo the same infinitesimal displacement, which implies at the same time that if all the quantities dx are increased in the same ratio, the linear element will vary also in the same ratio.

On these suppositions, the linear element may be any homogeneous function of the first degree of the quantities dx, which is unchanged when we change the signs of all the dx, and in which the arbitrary constants are continuous functions of the quantities x. To find the simplest cases, I shall seek first an expression for manifoldnesses of $n-1$ dimensions which are everywhere equidistant from the origin of the linear element; that is, I shall seek a continuous function of position whose values distinguish them from one another. In going outwards from the origin, this must either increase in all directions or decrease in all directions; I assume that it increases in all directions, and therefore has a minimum at that point. If, then, the first and second differential coefficients of this function are finite, its first differential must vanish, and the second differential cannot become negative; I assume that it is always positive. This differential expression, then, of the second order remains constant when ds remains constant, and increases in the duplicate ratio[7] when the dx, and therefore also ds, increase in the same ratio; it must therefore be ds^2 multiplied by a constant, and consequently ds is the square root of an always positive integral homogeneous function of the second order of the quantities dx, in which the coefficients are continuous functions of the quantities x. For space, when the position of points is expressed by rectilinear co-ordinates, $ds = \sqrt{\Sigma(dx)^2}$; space is therefore included in this simplest case. The next case in simplicity includes those manifoldnesses in which the line-element may be expressed as the fourth root of a quartic differential expression. The investigation of this more general kind would require no really different principles, but would take considerable time and throw little new light on the theory of space, especially as the results cannot be geometrically

[7] That is, as the square. This terminology was obsolete when Clifford used it; even Riemann's German is more direct.

expressed; I restrict myself, therefore, to those manifoldnesses in which the line-element is expressed as the square root of a quadric differential expression. Such an expression we can transform into another similar one if we substitute for the n independent variables functions of n new independent variables. In this way, however, we cannot transform any expression into any other; since the expression contains $n(n+1)$ coefficients which are arbitrary functions of the independent variables; now by the introduction of new variables we can only satisfy n conditions, and therefore make no more than n of the coefficients equal to given quantities. The remaining $n(n-1)$ are then entirely determined by the nature of the continuum to be represented, and consequently $n(n-1)$ functions of positions are required for the determination of its measure-relations. Manifoldnesses in which, as in the Plane and in space, the line-element may be reduced to the form $\sqrt{\Sigma dx^2}$ are therefore only a particular case of the manifoldnesses to be here investigated; they require a special name, and therefore these manifoldnesses in which the square of the line-element may be expressed as the sum of the squares of complete differentials I will call flat. In order now to review the true varieties of all the continua which may be represented in the assumed form, it is necessary to get rid of difficulties arising from the mode of representation, which is accomplished by choosing the variables in accordance with a certain principle.

[...]

§3. In the idea of surfaces, together with the intrinsic measure-relations in which only the length of lines on the surfaces is considered, there is always mixed up the position of points lying out of the surface. We may, however, abstract from external relations if we consider such deformations as leave unaltered the length of lines – i.e., if we regard the surface as bent in any way without stretching, and treat all surfaces so related to each other as equivalent. Thus, for example, any cylindrical or conical surface counts as equivalent to a plane, since it may be made out of one by mere bending, in which the intrinsic measure-relations remain, and all theorems about a plane – therefore the whole of planimetry – retain their validity. On the other hand they count as essentially different from the sphere, which cannot be changed into a plane without stretching. According to our previous investigation the intrinsic measure-relations of a two-fold extent in which the line-element may be expressed as the square root of a quadric differential, which is the case with surfaces, are characterised by the total curvature. Now this quantity in the case of surfaces is capable of a visible interpretation, viz., it is the product of the two curvatures of the surface, or multiplied by the area of a small geodesic triangle, it is equal to the spherical excess of the same. The first definition assumes the proposition that the product of the two radii of curvature is unaltered by mere bending; the second, that in the same place the area of a small triangle is proportional to its spherical excess. To give an intelligible meaning to the curvature of an n-fold extent at a given point and in a given surface-direction through it, we must start from the fact that a geodesic proceeding from a point is entirely determined when its initial direction is given. According to this we obtain a determinate surface if we prolong all the geodesics proceeding from the given point and lying initially in the given surface-direction; this surface has at the given point a definite curvature, which is also the curvature of the n-fold continuum at the given point in the given surface-direction.

§4. Before we make the application to space, some considerations about flat manifoldnesses in general are necessary; i.e., about those in which the square of the line-element is expressible as a sum of squares of complete differentials. In a flat n-fold extent the total curvature is zero at all points in every direction; it is sufficient, however (according to the preceding investigation), for the determination of measure-relations, to know that at each point the curvature is zero in $\frac{1}{2}n(n-1)$ independent surface-directions. Manifoldnesses whose curvature is constantly zero may be treated as a special case of those whose curvature is constant. The common character of these continua whose curvature is constant may be also expressed thus, that figures may be moved in them without stretching. For clearly figures could not be arbitrarily shifted and turned round in them if the curvature at each point were not the same in all directions. On the other hand, however, the measure-relations of the manifoldness are entirely determined by the curvature; they are therefore exactly the same in all directions at one point as at another, and consequently the same constructions can be made from it: whence it follows that in aggregates with constant curvature figures may have an arbitrary position given them. The measure-relations of these manifoldnesses depend only on the value of the curvature, and in relation to the analytic expression it may be remarked that if this value is denoted by α, the expression for the line-element may be written

$$\frac{1}{1+\frac{1}{4}\alpha\Sigma x^2}\sqrt{\Sigma dx^2}.$$

[...]

III. Application to Space

§1. By means of these inquiries into the determination of measure-relations of an n-fold extent the conditions may be declared which are necessary and sufficient to determine the metric properties of space, if we assume the independence of line-length from position and expressibility of the line-element as the square root of a quadric differential, that is to say, flatness in the smallest parts.

First, they may be expressed thus: that the curvature at each point is zero in three surface-directions; and thence the metric properties of space are determined if the sum of the angles of a triangle is always equal to two right angles.

Secondly, if we assume with Euclid not merely an existence of lines independent of position, but of bodies also, it follows that the curvature is everywhere constant; and then the sum of the angles is determined in all triangles when it is known in one.

Thirdly, one might, instead of taking the length of lines to be independent of position and direction, assume also an independence of their length and direction from position. According to this conception changes or differences of position are complex magnitudes expressible in three independent units.

§2. In the course of our previous inquiries, we first distinguished between the relations of extension or partition and the relations of measure, and found that with the same extensive properties, different measure-relations were conceivable; we then investigated the system of simple size-fixings by which the measure-relations of space are completely determined, and of which all propositions about them are a necessary consequence; it remains to discuss the question how, in what degree, and to what extent these assumptions are borne out by experience. In this respect there is a real distinction between mere extensive relations and measure-relations; in so far as in the former, where the possible cases form a discrete manifoldness, the declarations of experience are indeed not quite certain, but still not inaccurate; while in the latter, where the possible cases form a continuous manifoldness, every determination from experience remains always inaccurate: be the probability ever so great that it is nearly exact. This consideration becomes important in the extensions of these empirical determinations beyond the limits of observation to the infinitely great and infinitely small; a latter may clearly become more inaccurate beyond the limits of observation, but not the former.

In the extension of space-construction to the infinitely great, we must distinguish between *unboundedness* and *infinite extent*, the former belongs to the extent relations, the latter to the measure-relations. That space is an unbounded three-fold manifoldness, is an assumption which is developed by every conception of the outer world according to which every instant the region of real perception is completed and the possible positions of a sought object are constructed, and which by these applications is for ever confirming itself. The unboundedness of space possesses in this way a greater empirical certainty than any external experience. But its infinite extent by no means follows from this; on the other hand if we assume independence of bodies from position, and therefore ascribe to space constant curvature, it must necessarily be finite provided this curvature has ever so small a positive value. If we prolong all the geodesics starting in a given surface-element, we should obtain an unbounded surface of constant curvature, i.e., a surface which in a flat manifoldness of three dimensions would take the form of a sphere, and consequently be finite.

[...]

§3. The questions about the infinitely great are for the interpretation of nature useless questions. But this is not the case with the questions about the infinitely small. It is upon the exactness with which we follow phenomena into the infinitely small that our knowledge of their causal relations essentially depends. The progress of recent centuries in the knowledge of mechanics depends almost entirely on the exactness of the construction which has become possible through the invention of the infinitesimal calculus, and through the simple principles discovered by Archimedes, Galileo, and Newton, and used by modern physics. But in the natural sciences which are still in want of simple principles for such constructions, we seek to discover the causal relations by following the phenomena into great minuteness, so far as the microscope permits. Questions about the measure-relations of space in the infinitely small are not therefore superfluous questions.

If we suppose that bodies exist independently of position, the curvature is everywhere constant, and it then results from astronomical measurements that it cannot be different from zero; or at any rate its reciprocal must be an area in comparison with which the range of our telescopes may be neglected. But if this independence of bodies from position does not exist, we cannot draw conclusions from metric relations of the great, to those of the infinitely small; in that case the curvature at each point may have an arbitrary value in three directions, provided that the total curvature of every measurable portion of space does not differ sensibly from zero. Still more complicated relations may exist if we no longer suppose the linear element expressible as the square root of a quadric differential.

Now it seems that the empirical notions on which the metrical determinations of space are founded, the notion of a solid body and of a ray of light, cease to be valid for the infinitely small. We are therefore quite at liberty to suppose that the metric relations of space in the infinitely small do not conform to the hypotheses of geometry; and we ought in fact to suppose it, if we can thereby obtain a simpler explanation of phenomena.

The question of the validity of the hypotheses of geometry in the infinitely small is bound up with the question of the ground of the metric relations of space. In this last question, which we may still regard as belonging to the doctrine of space, is found the application of the remark made above; that in a discrete manifoldness, the ground of its metric relations is given in the notion of it, while in a continuous manifoldness, this ground must come from outside. Either therefore the reality which underlies space must form a discrete manifoldness, or we must seek the ground of its metric relations outside it, in binding forces which act upon it.

The answer to these questions can only be got by starting from the conception of phenomena which has hitherto been justified by experience, and which Newton assumed as a foundation, and by making in this conception the successive changes required by facts which it cannot explain. Researches starting from general notions, like the investigation we have just made, can only be useful in preventing this work from being hampered in too narrow views, and progress in knowledge of the interdependence of things from being checked by traditional prejudices. This leads us into the domain of another science, of physics, into which the object of this work does not allow us to go to-day.