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New Series of Exact Solutions for Gravitational Fields of Spinning Masses

Akira TOMIMATSU[†] and Humitaka SATO*

Department of Physics, Kyoto University, Kyoto
**Research Institute for Fundamental Physics*
Kyoto University, Kyoto

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New series of solutions for space-times which are regarded as representing the gravitational fields of spinning bound masses is derived from a series of Weyl metrics, following Ernst's formulation on axi-symmetric stationary fields. This series of solutions includes the Kerr metric as a member of the simplest one. Except in the case of the Kerr metric, the space-times have ring singularities outside event horizons, that is, there exist naked singularities. Therefore, these solutions seem to give a very different picture concerning the ultimate fate of gravitational collapse compared with a current picture such as the Israel-Carter conjecture.

The naked ring singularity may become an active energy source of gravitational wave, and yield many interesting phenomena in astrophysics.

§ 1. Introduction and summary

As the exact solutions for space-times which are regarded as representing the exterior gravitational fields of bound sources, there were three well-known metrics: the Schwarzshild metric, the Weyl metric[1] and the Kerr metric.[2] Besides these solutions, we recently found the fourth solution,[3] which is regarded as representing the field of spinning mass. In this paper we present other exact solutions of a similar kind and discuss their properties. These solutions form a series of solutions, which are called the T-S metrics hereafter. The Kerr metric can also be regarded as the simplest member of this series of solutions.

Concerning the ultimate fate of gravitational collapse, some authors[4),5)] have proposed the conjecture that the space-times around black-holes may always be represented by the Kerr metric. This conjecture comes from the hope that all singularities in space-time are hidden behind the non-singular event horizon in the course of collapse. If the above hope is correct, the space-time must be represented by the Kerr metric, following the Israel-Carter theorem.[6),7)] However, the T-S metrics do have ring singularities on the equatorial plane *outside* the non-singular event horizons, which do not exist in the case of the Weyl metrics. Therefore, we suppose that the above conjecture may be a hasty conclusion. The

[†] Present address: Research Institute for Theoretical Physics, Hiroshima University, Takehara, Hiroshima.

existence of the T-S metrics dose not contradict the Israel-Carter theorem because one of the premises of this theorem is not satisfied in the T-S metrics.

The four types of solutions representing the exterior gravitational fields of bound masses are classified as follows:

$$
\begin{array}{ccc}
\text{Schwarzschild metric} & \xrightarrow{\text{rotation}} & \text{Kerr metric} \\
(\rho\neq 0,\ \omega=\sigma=0) & & (\rho\neq 0,\ \omega\neq 0,\ \sigma=0) \\
\downarrow \text{deformation} & & \downarrow \text{deformation} \\
\text{Weyl metric} & \xrightarrow{\text{rotation}} & \text{T-S metrics} \\
(\rho\neq 0,\ \omega=0,\ \sigma\neq 0) & & (\rho\neq 0,\ \omega\neq 0,\ \sigma\neq 0).
\end{array}
$$

In the above diagram, ρ, ω and σ denote divergence, rotation and shear of bunch of null geodesics in the corresponding space-times.[8] As seen from the above classification, the Kerr metric was derived by generalizing the Schwarzshild metric to its stationary one. In the same way, the new solutions were derived by generalizing a series of the Weyl metrics to its stationary one. Some methods to generalize the Schwarzshild metric to the Kerr metric were proposed.[9),10] Among them, we adopted Ernst's formulation on the axi-symmetric stationary metrics.[10]

The T-S metrics obtained in this way contains three parameters, namely mass m, angular momentum about the z axis J and distortion parameter δ, δ being the parameter to classify a series of the Weyl metrics[11),12] and being related with the quadrupole moment[12] of the field as $Q/m^3=(\delta^2-1)/3\delta^2$ in the case $J=0$.*) Though δ can take any positive number in the Weyl metric, the T-S metric is restricted to the cases of integer δ. For slow rotation, however, the approximate solutions can be obtained for general δ as we have shown elsewhere.[14] In our series of metrics, the metric for $\delta=1$ is justly the Kerr metric and that for $\delta=2$ has already been published in a preceding paper.[8] In this paper, the metrics for $\delta=3$ and 4 will be given and, further, the general rules to compute the solutions for higher integer δ will be given.

If we know an axi-symmetric stationary solution, it is not so difficult to get the corresponding solution of the coupled Einstein-Maxwell equations which is regarded as representing the field of charged source.[15] Ernst has already derived such a solution corresponding to the T-S metric for $\delta=2$.[16]

In § 2, Ernst's complex functions ξ for $\delta=1\sim 4$ are presented explicitly and their general properties of the expressions are summarized. In § 3, the metric functions are given explicitly. In § 4, the properties of the space-times in the distant region and in the inner region are discussed, where the quadrupole moment of space-time and the space-time structures such as infinite redshift surface,

*) Recently, E. N. Glass[13] has verified the physical meaning of the parameters in the T-S metric for $\delta=2$.

event horizon and true singularity are discussed. In § 5, the geodesics in the T-S metrics are discussed especially for the case of null geodesics on the equatorial plane.

§ 2. Solutions of Ernst's equation

In general, we can express the line element of stationary and axi-symmetric vacuum space in the form

$$ds^2=f^{-1}[e^{2\gamma}(d\rho^2+dz^2)+\rho^2 d\phi^2]-f(dt-\omega d\phi)^2, \tag{2·1}$$

where f, ω and γ are functions of ρ and z only. According to Ernst's method,[10] these functions can be obtained from a complex function ξ which satisfies the following differential equation,

$$(\xi\xi^*-1)\nabla^2\xi=2\xi^*\nabla\xi\cdot\nabla\xi. \tag{2·2}$$

In this equation ∇ denotes the three-dimensional divergence operator. The relations between the metric functions and ξ are given as follows,

$$f=\mathrm{Re}\frac{\xi-1}{\xi+1}, \tag{2·3}$$

$$\frac{\partial\omega}{\partial\rho}=-\frac{2\rho}{(\xi\xi^*-1)^2}\mathrm{Im}\left[(\xi^*+1)^2\frac{\partial\xi}{\partial z}\right], \tag{2·4}$$

$$\frac{\partial\omega}{\partial z}=\frac{2\rho}{(\xi\xi^*-1)^2}\mathrm{Im}\left[(\xi^*+1)^2\frac{\partial\xi}{\partial\rho}\right], \tag{2·5}$$

$$\frac{\partial\gamma}{\partial\rho}=\frac{\rho}{(\xi\xi^*-1)^2}\left[\frac{\partial\xi}{\partial\rho}\frac{\partial\xi^*}{\partial\rho}-\frac{\partial\xi}{\partial z}\frac{\partial\xi^*}{\partial z}\right] \tag{2·6}$$

and

$$\frac{\partial\gamma}{\partial z}=\frac{2\rho}{(\xi\xi^*-1)^2}\mathrm{Re}\left[\frac{\partial\xi}{\partial\rho}\frac{\partial\xi^*}{\partial z}\right]. \tag{2·7}$$

In the cases of no rotation, many solutions for ξ are easily obtainable, which derive the Weyl metrics. In these solutions there exists an interesting series of solutions[11),12)] with a positive parameter δ, expressed in the form

$$\xi=\frac{(x+1)^\delta+(x-1)^\delta}{(x+1)^\delta-(x-1)^\delta}, \tag{2·8}$$

where prolate spheroidal coordinates (x, y) are used in place of cylindrical coordinates (ρ, z), and the two coordinates are related as follows,

$$\rho=\kappa(x^2-1)^{1/2}(1-y^2)^{1/2}, \quad z=\kappa xy. \tag{2·9}$$

The metrics obtained from the solutions (2·8) are regarded as representing the gravitational fields of axi-symmetric masses, whose deformation is described by δ. In particular, in the case $\delta=1$ the Schwarzschild metric is obtained and it will be concluded that the solutions with $\delta>1\,(\delta<1)$ represent the exterior gravitational fields of the oblate (prolate) axi-symmetric masses in view of the quadrupole moment of space-time. We shall generalize the solution (2·8) to the stationary one in order to find out new solutions involving rotation.

Though we can obtain approximate solutions for any δ in the case of slow rotation, it is very difficult to obtain exact analytic solutions of Eq. (2·2), but it seems possible if we consider only the cases of integer δ. In this case, as anticipated from Eq. (2·8), the solution of ξ is expected to be written as a quotient of polynomials;

$$\xi=\frac{\alpha}{\beta}, \tag{2·10}$$

where α and β are complex polynomials of x and y. Substituting Eq. (2·10) into Eq. (2·2) and using the coordinates (x, y), we obtain the basic equation as

$$\Big[(x^2-1)(\alpha\alpha^*-\beta\beta^*)\Big(\frac{\partial^2\alpha}{\partial x^2}\beta-\alpha\frac{\partial^2\beta}{\partial x^2}\Big)+\Big\{2x(\alpha\alpha^*-\beta\beta^*)-2(x^2-1)$$
$$\times\Big(\alpha^*\frac{\partial\alpha}{\partial x}-\beta^*\frac{\partial\beta}{\partial x}\Big)\Big\}\Big(\frac{\partial\alpha}{\partial x}\beta-\alpha\frac{\partial\beta}{\partial x}\Big)\Big]$$
$$-[\text{the same expression replacing } x \text{ by } y]=0\,. \tag{2·11}$$

The solutions which have been obtained are as follows:

i) *Ernst's solution*[10] *or the T-S solution for* $\delta=1$

$$\alpha=px-iqy \quad \text{and} \quad \beta=1\,, \tag{2·12}$$

where p and q are parameters relating to each other as $p^2+q^2=1$. In the case of no rotation, $p=1$ and $q=0$. This solution gives the well-known Kerr metric.

ii) *the T-S solution for* $\delta=2$[8]

$$\alpha=p^2x^4+q^2y^4-1-2ipqxy(x^2-y^2)$$

and

$$\beta=2px(x^2-1)-2iqy(1-y^2). \tag{2·13}$$

From these solutions, it is verified that they satisfy the following rules:

a) The term $(\beta\partial\alpha/\partial x-\alpha\partial\beta/\partial x)$ is real and the term $(\beta\partial\alpha/\partial y-\alpha\partial\beta/\partial y)$ is pure-imaginary.

b) Terms involving y of even power index are real and those of odd power index are pure-imaginary. Furthermore, the coefficients of the former terms include q with even power index, and those of the latter include q with odd power index. This property is understood easily from the consideration of parity of rotation.

c) Apart from positive or negative sign of each term, both α and β are symmetrical with respect to x, p and y, iq, respectively. This will be expected from the symmetry of Eq. (2·11).

d) As will be understood afterwards, α and β are the polynomials with powers of δ^2 and δ^2-1, respectively.

e) The power indices of p and q are at most δ.

f) The numerical coefficients of the terms without y in α and β are identical with those of the respective terms of the polynomials

$$(x^2-1)^{\delta(\delta-1)/2}\{(x+1)^\delta \pm (x-1)^\delta\}, \tag{2·14}$$

where the plus sign is taken for α and minus sign for β.

g) In the limit $q \ll 1$, ξ becomes as follows:

$$\xi \simeq \frac{(x+1)^\delta + (x-1)^\delta}{(x+1)^\delta - (x-1)^\delta} + iq\xi_1(x, y) + \cdots \tag{2·15}$$

and

$$\xi_1(x, y) = \frac{1}{\{(x+1)^\delta - (x-1)^\delta\}^2} \sum_{l=1}^{\delta} a_{2l-1}(x) P_{2l-1}(y) ,$$

where $P_l(y)$ is Legendre's polynomial and $a_l(x)$ is a polynomial with the maximum power index of $2\delta - l - 1$, a_l satisfying the equation

$$(x^2-1)\frac{d^2 a_l}{dx^2} - (4\delta - 2)x\frac{da_l}{dx} + (4\delta^2 - 2\delta - l(l+1))a_l = 0 . \tag{2·16}$$

The fact that ξ_1 contains higher order terms of l may imply that the solutions for $\delta \geq 2$ correspond to the gravitational fields of differentially rotating sources.

Referring to these rules for computation, we can determine explicit forms for α and β from Eq. (2·11). After the lengthy calculation, we obtain the solutions as follows:

iii) *the T-S solution for* $\delta = 3$

$$\begin{aligned} \alpha &= p(x^2-1)^3(x^3+3x) + iq(1-y^2)^3(y^3+3y) \\ &\quad - pq^2(x^2-y^2)^3(x^3+3xy^2) - ip^2q(x^2-y^2)^3(y^3+3x^2y) \end{aligned} \tag{2·17}$$

and

$$\beta = p^2(x^2-1)^3(3x^2+1) - q^2(1-y^2)^3(3y^2+1) - 12ipqxy(x^2-y^2)(x^2-1)(1-y^2),$$

iv) *the T-S solution for* $\delta = 4$

$$\begin{aligned} \alpha &= p^2(x^2-1)^6(x^4+6x^2+1) + q^2(1-y^2)^6(y^4+6y^2+1) - p^2q^2(x^2-y^2)^6(x^4+6x^2y^2+y^4) \\ &\quad - 4ip^3qxy(x^2-y^2)\{(x^2+1)(x^2-1)^5 + (4x^2+6)(x^2-1)^4(1-y^2) \\ &\quad + (5x^2+15)(x^2-1)^3(1-y^2)^2\} \\ &\quad - 4ipq^3xy(x^2-y^2)\{(y^2+1)(1-y^2)^5 + (4y^2+6)(1-y^2)^4(x^2-1) \\ &\quad + (5y^2+15)(1-y^2)^3(x^2-1)^2\} \end{aligned}$$

and

$$\begin{aligned}\beta=&p(x^2-1)^6(4x^3+4x)+iq(1-y^2)^6(4y^3+4y)\\&-4pq^2x(x^2-y^2)^3\{(x^2+1)(x^2-1)^3-(3x^2+3)(x^2-1)^2(1-y^2)\\&+(6x^2+10y^2-4)(x^2-1)(1-y^2)^2\}\\&-4ip^2qy(x^2-y^2)^3\{(y^2+1)(1-y^2)^3-(3y^2+3)(1-y^2)^2(x^2-1)\\&+(6y^2+10x^2-4)(1-y^2)(x^2-1)^2\}.\end{aligned} \tag{2·18}$$

These two solutions which satisfy Eq. (2·11) also satisfy the above rules for computation and it will be permissible to determine expressions of solutions from the above rules a)~g) without solving Eq. (2·11) directly. In this way we can also derive solutions for $\delta\geq 5$ in more lengthy expressions. It is hoped to find an expression for any integer δ as in the case of Legendre's polynomial but it has not been successful.

However, it seems sufficient to know the properties of the space-times corresponding to any integer δ from those for $\delta=1\sim 4$. Then, in what follows, we sometimes give expressions in terms of general integer δ, which are really verified for $\delta=1, 2, 3$ and 4 but have not been verified explicitly for larger δ.

§ 3. Metric functions for the T-S metrics

It is rather straightforward to calcualte the metric functions f, ω and γ from ξ. Using Eq. (2·3) and writing α and β as $u+iv$ and $m+in$ respectively, f becomes as follows,

$$f=\frac{A}{B}, \tag{3·1}$$

where $A=u^2+v^2-m^2-n^2$, $B=(u+m)^2+(v+n)^2$. Furthermore, rewriting Eqs. (2·4) and (2·5) by the coordinates (x, y), and using the rule a) for α and β, ω can be expressed in the form

$$\omega=\frac{2mq}{A}(1-y^2)C, \tag{3·2}$$

where C is a polynomial with the maximum power of $2\delta^2-1$ and satisfies the following equations:

$$A\frac{\partial C}{\partial x}-C\frac{\partial A}{\partial x}=\frac{\kappa}{mq}\left(\frac{\partial u}{\partial y}n+\frac{\partial v}{\partial y}m-u\frac{\partial n}{\partial y}-v\frac{\partial m}{\partial y}\right)\{(u+m)^2-(v+n)^2\} \tag{3·3}$$

and

$$A\frac{\partial C}{\partial y}-C\frac{\partial A}{\partial y}=\frac{-2\kappa}{mq}\left(\frac{\partial u}{\partial x}m-\frac{\partial v}{\partial x}n-u\frac{\partial m}{\partial x}+v\frac{\partial n}{\partial x}\right)(u+m)(v+n). \tag{3·4}$$

Similarly, we can write the equation for γ from Eqs. (2·6) and (2·7) as follows,

$$\frac{\partial\gamma}{\partial x}=\frac{x(1-y^2)}{A^2(x^2-y^2)}\left\{(x^2-1)\left(\frac{\partial u}{\partial x}m-\frac{\partial v}{\partial x}n-u\frac{\partial m}{\partial x}+v\frac{\partial n}{\partial x}\right)^2\right.$$
$$\left.-(1-y^2)\left(\frac{\partial u}{\partial y}n+\frac{\partial v}{\partial y}m-u\frac{\partial n}{\partial y}-v\frac{\partial m}{\partial y}\right)^2\right\} \quad (3\cdot5)$$

and

$$\frac{\partial\gamma}{\partial y}=\frac{y(x^2-1)}{A^2(x^2-y^2)}\left\{(x^2-1)\left(\frac{\partial u}{\partial x}m-\frac{\partial v}{\partial x}n-u\frac{\partial m}{\partial x}+v\frac{\partial n}{\partial x}\right)^2\right.$$
$$\left.-(1-y^2)\left(\frac{\partial u}{\partial y}n+\frac{\partial v}{\partial y}m-u\frac{\partial n}{\partial y}-v\frac{\partial m}{\partial y}\right)^2\right\}. \quad (3\cdot6)$$

The solution is found to be expressed as

$$2\gamma=\ln\frac{A}{p^{2\delta}(x^2-y^2)^{\delta^2}}, \quad (3\cdot7)$$

after the lengthy calculations. As $e^{2\gamma}$ should tend to unity for $x\to\infty$, the maximum power of the polynomial A must be $2\delta^2$ and, therefore, that of α must be δ^2.

For $\delta=1\sim3$, the expressions of A, B and C are shown as follows:

i) $\delta=1$

$$A=p^2(x^2-1)-q^2(1-y^2), \quad (3\cdot8)$$

$$B=(px+1)^2+q^2y^2, \quad (3\cdot9)$$

$$C=-px-1. \quad (3\cdot10)$$

ii) $\delta=2$

$$A=p^4(x^2-1)^4+q^4(1-y^2)^4$$
$$-2p^2q^2(x^2-1)(1-y^2)\{2(x^2-1)^2+2(1-y^2)^2+3(x^2-1)(1-y^2)\}, \quad (3\cdot11)$$

$$B=\{p^2(x^2+1)(x^2-1)-q^2(y^2+1)(1-y^2)+2px(x^2-1)\}^2$$
$$+4q^2y^2\{px(x^2-1)+(px+1)(1-y^2)\}^2, \quad (3\cdot12)$$

$$C=-p^3x(x^2-1)\{2(x^2+1)(x^2-1)+(x^2+3)(1-y^2)\}$$
$$-p^2(x^2-1)\{4x^2(x^2-1)+(3x^2+1)(1-y^2)\}+q^2(px+1)(1-y^2)^3. \quad (3\cdot13)$$

iii) $\delta=3$

$$A=p^6(x^2-1)^9-3p^4q^2(x^2-1)^4(1-y^2)[3(x^2-1)^4+12(x^2-1)^3(1-y^2)$$
$$+28(x^2-1)^2(1-y^2)^2+30(x^2-1)(1-y^2)^3+12(1-y^2)^4]$$
$$+3p^2q^4(x^2-1)(1-y^2)^4[12(x^2-1)^4+30(x^2-1)^3(1-y^2)$$
$$+28(x^2-1)^2(1-y^2)^2+12(x^2-1)(1-y^2)^3+3(1-y^2)^4]-q^6(1-y^2)^9, \quad (3\cdot14)$$

$$B=\{p(x^3+3x)(x^2-1)^3-pq^2(x^3+3xy^2)(x^2-y^2)^3+p^2(3x^2+1)(x^2-1)^3$$
$$-q^2(3y^2+1)(1-y^2)^3\}^2+q^2y^2\{p^2(x^2-y^2)^3(3x^2+y^2)$$
$$-(1-y^2)^3(y^2+3)+12px(x^2-y^2)(x^2-1)(1-y^2)\}^2, \tag{3·15}$$

$$C=-p^5x(x^2-1)^4[(x^2-1)^2(3x^2+1)(x^2+3)+4(x^2-1)(1-y^2)(x^4+8x^2+3)$$
$$+2(1-y^2)^2(x^4+10x^2+5)]+p^3q^2x(x^2-1)(1-y^2)^3[8(x^2-1)^2(3x^4-4x^2-3)$$
$$+4(x^2-1)(1-y^2)(13x^4-10x^2-15)+4(1-y^2)^2(9x^4-13)+6(1-y^2)^3$$
$$\times(x^2+3)]-p^4(x^2-1)^4[(x^2-1)^2(3x^2+1)^2+2(x^2-1)(1-y^2)$$
$$\times(9x^4+14x^2+1)+2(1-y^2)^2(5x^4+10x^2+1)]+p^2q^2(x^2-1)$$
$$\times(1-y^2)^3[54(x^2-1)^3(1-y^2)+4(x^2-1)^2(9x^4-18x^2+1)+4(x^2-1)$$
$$\times(1-y^2)(9x^4-8x^2-13)+6(1-y^2)^3(3x^2+1)+2(1-y^2)^2$$
$$\times(37x^4-42x^2-3)]-q^4(px+1)(1-y^2)^8. \tag{3·16}$$

Furthermore it should be noted that our metrics can be extended for the case $q>1$. When $q>1$, we must use new coordinate variables and parameters connected by the transformation $x\to ix$, $y\to y$, $p\to -ip$, and $q\to q$, which are related with the cylindrical coordinates (ρ, z) as

$$\rho=\frac{mp}{\delta}(x^2+1)^{1/2}(1-y^2)^{1/2},\quad z=\frac{mp}{\delta}xy. \tag{3·17}$$

Thus it is easy to get the metric functions for $q>1$ by the above transformation.

§ 4. Properties of the T-S metrics

a) *Properties in the distant region*

For $px\gg1$, f and ω behave as follows,

$$f\sim1-\frac{2\delta}{px}+\frac{2\delta^2}{p^2x^2}+\frac{2\delta^3}{p^3x^3}\left(q^2y^2-1+\frac{\delta^2-1}{3\delta^2}p^2\right) \tag{4·1}$$

and

$$\omega\sim2mq(y^2-1)\frac{\delta}{px}. \tag{4·2}$$

Making two successive coordinate transformations, i.e., $(x, y)\to(r, \theta)\to(R, \theta)$,

$$\rho=\frac{pm}{\delta}(x^2-1)^{1/2}(1-y^2)^{1/2}=(r^2-2mr+m^2q^2)^{1/2}\sin\theta, \tag{4·3}$$
$$z=\frac{pm}{\delta}xy=(r-m)\cos\theta,$$

and

$$r^2 \simeq R^2 - m^2 q^2 \sin^2\theta\,, \tag{4·4}$$

f and ω become

$$f \sim 1 - \frac{2m}{R} + \frac{2m^3}{R^3}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)\left(\frac{\delta^2-1}{3\delta^2}p^2 + q^2\right) \tag{4·5}$$

and $\omega \sim -2m^2 q \sin^2\theta/R$. This shows that the T-S metrics represent the gravitational fields of rotating masses with the angular momentum about the z axis $J \equiv m^2 q$ and the quadrupole moments of the fields are

$$Q = m^3\left(\frac{\delta^2-1}{3\delta^2}p^2 + q^2\right). \tag{4·6}$$

It should be noted that the quadrupole moment for $\delta \geq 2$ is larger than that for $\delta = 1$, i.e., $Q(\delta \geq 2) > Q_{\text{Kerr}} \equiv m^3 q^2$, in the case $q < 1$. Of course, for $q = 0$, Q reduces to the expression for the Weyl metrics.

In the limit $q = 1$, all of the T-S metrics coincide with the Kerr metric of $q = 1$ or $J = m^2$. It must be noticed, however, that this limit is taken by retaining a value such as px to be finite as anticipated from the coordinate transformation (4·3), where ρ and z represent the physical length. Concerning the recent numerical analysis for the gravitational fields of rotating sources,[17),18)] the above-mentioned conclusions are very interesting.

b) *Properties in the inner region*

Next we investigate the properties of the T-S metrics in the inner region near the surfaces where $x^2 = 1$. For this purpose, it is convenient to rewrite the metric (2·1) as

$$ds^2 = m^2[e^{2\mu_2}dx^2 + e^{2\mu_3}dy^2 + e^{2\psi}(d\phi - \Omega dt)^2] - e^{2\nu}dt^2 \tag{4·7}$$

where

$$\Omega = -\frac{2\delta^2 qC}{mD}, \quad e^{2\nu} = \frac{p^2(x^2-1)B}{D}, \quad e^{2\psi} = \frac{(1-y^2)D}{\delta^2 B}, \tag{4·8}$$

$$e^{2\mu_2} = \frac{B}{\delta^2 p^{2\delta-2}(x^2-1)(x^2-y^2)^{\delta^2-1}}$$

and

$$e^{2\mu_3} = \frac{B}{\delta^2 p^{2\delta-2}(1-y^2)(x^2-y^2)^{\delta^2-1}},$$

and D is a polynomial determined by the equation

$$DA = p^2(x^2-1)B^2 - 4\delta^2 q^2(1-y^2)C^2\,. \tag{4·9}$$

For $\delta = 1$,

$$D = \{(px+1)^2 + q^2\}^2 - p^2 q^2 (x^2-1)(1-y^2), \tag{4·10}$$

and for $\delta = 2$,

$$
\begin{aligned}
D=&p^6(x^2-1)(x^8+28x^6+70x^4+28x^2+1)-16q^6(1-y^2)^3\\
&+p^4q^2[(x^2-1)\{32x^2(x^4+4x^2+1)-4(1-y^2)(x^2-1)^3\\
&+(-6x^4+12x^2+10)(1-y^2)^3\}-4(1-y^2)^3(x^4+6x^2+1)]\\
&+p^2q^4[(x^2-1)\{64x^4+(1-y^2)^2(y^4+14y^2+1)\}-16(1-y^2)^3(x^2+2)]\\
&+8p^5x(x^4-1)(x^4+6x^2+1)-32pq^4x(1-y^2)^3\\
&+8p^3q^2x[(x^2-1)\{8x^2(x^2+1)+(1-y^2)^2(2y^2-x^2+1)\}-4(1-y^2)^3]. \quad (4\cdot11)
\end{aligned}
$$

Here Ω is the angular velocity of rotation of the local inertial system, and the surfaces where $A=0$ are the infinite red-shift surfaces because $g_{00}\equiv f=0$ at $A=0$. The number of the infinite red-shift surfaces is 2δ.

Let λ be an affine parameter of geodesics. As will be discussed in § 5 (see Eq. (5·3)), except the case $q=0$ and the poles where $y^2=1$, along the null geodesics $dt/d\lambda$ behaves as $(x^2-1)^{-1}$ near the surfaces of $x^2=1$, but $dx/d\lambda$ remains finite, as in the case of the Schwarzshild space-time. Therefore, the surfaces where $x^2=1$ will be the event horizons except a peculiarity at the poles.

In order to investigate the singularity of the space-time, we consider the physical components of the Riemann tensor in the locally nonrotating frame.[19] For example, $R_{(t)(\phi)(t)(\phi)}$ is given as

$$
R_{(t)(\phi)(t)(\phi)}=\frac{\partial\nu}{\partial x}\frac{\partial\psi}{\partial x}e^{-2\mu_2}+\frac{\partial\nu}{\partial y}\frac{\partial\psi}{\partial y}e^{-2\mu_3}+\frac{1}{4}\left\{\left(\frac{\partial\Omega}{\partial x}\right)^2e^{-2\mu_2}+\left(\frac{\partial\Omega}{\partial y}\right)^2e^{-2\mu_3}\right\}e^{2\psi-2\nu}. \quad (4\cdot12)
$$

As will be understood by substituting the expression (4·8) into $R_{(t)(\phi)(t)(\phi)}$, the surfaces where $x^2=1$ are shown to be non-singular except for the case $q=0$, because B, C and D are non-zero at $x^2=1$ for $q\neq0$. However only at the poles where $y^2=1$ and $x^2=1$, the space-time seems to be singular, except for the case $\delta=1$. Writing the third term of $R_{(t)(\phi)(t)(\phi)}$ as

$$
\frac{1}{4}\left(\frac{\partial\Omega}{\partial x}\right)^2e^{2\psi-2\nu-2\mu_2}=\frac{\delta^4q^2}{m^2}(1-y^2)(x^2-y^2)^{\delta^2-1}\frac{D^2}{B^3}\left(\frac{\partial}{\partial x}\frac{C}{D}\right)^2, \quad (4\cdot13)
$$

it is seen, for the case $\delta\geq2$, that this term becomes infinite when y^2 approaches one in the surface where $x^2=1$, but zero when x^2 approaches unity on the axis where $y^2=1$. This is the directional singularity, also existing in the Weyl space-times, and seems to cause many difficult problems on the treatment of the T-S space-time near the poles where $x^2=1$ and $y^2=1$.

Besides the above directional singularity, there exists the ordinary and important singularity at the place of $B=0$, that is, the ring singularity on the equatorial plane, whose location is determined by the condition

$$
u+m=0 \qquad \text{and} \qquad y=0. \quad (4\cdot14)
$$

The number of the ring singularity is δ and, on this ring, all of A, B, C and D become zero.

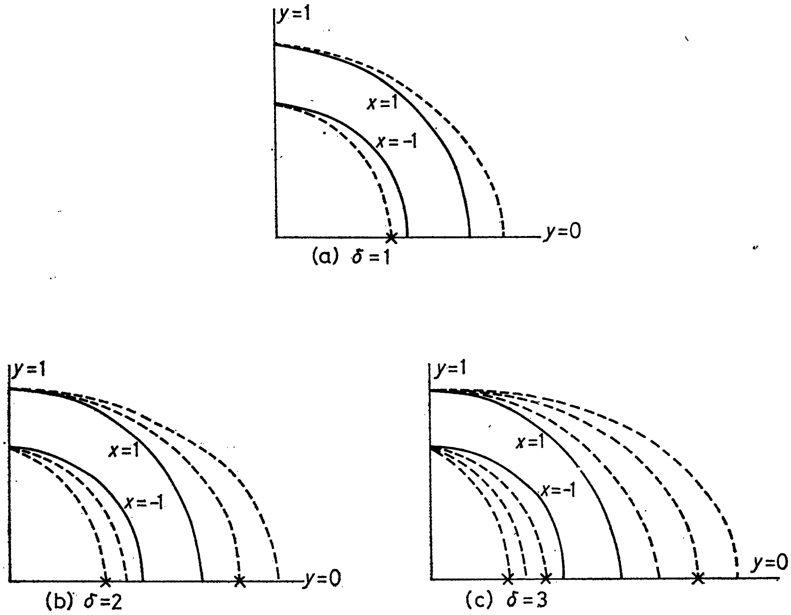

Fig. 1. The structure of the space-time. x and y should be regarded as a radial coordinate and $\cos\theta$, respectively. The dotted-lines and the solid-lines denote the infinite redshift surfaces and the event horizons respectively. The crossed sign denotes the ring singularity. About the peculiarity of space-time at the pole, see the text.

The structure of the space-time is schematically written in Fig. 1 for $\delta=1, 2$ and 3. The structure near $x^2=1$ will become more complex as δ becomes larger, which may be due to the more complex differential rotation as noted in § 2. It should be noted that, for $\delta\geq 2$, the ring singularities always lie outside the outer event horizon where $x=1$. This result is the most remarkable difference from the Kerr metric, i.e., the T-S metric for $\delta=1$. Physical implication of the existence of naked singularities has been mentioned in § 1.

§ 5. Geodesics in the T-S space-time

The geodesics in the metric of Eq. (4·7) are derived from the following Lagrangian:[20),21)]

$$2L=[e^{2\mu_2}\dot{x}^2+e^{2\mu_3}\dot{y}^2+e^{2\psi}(\dot{\phi}-\Omega\dot{t})^2]-e^{2\nu}\dot{t}^2, \tag{5·1}$$

where we have taken the unit of time as $m=1$, $\dot{x}^\mu=dx^\mu/d\lambda$ and λ is an affine parameter. Writing the momentum conjugate to x^μ as p_μ, the Hamiltonian H is given as

$$H=e^{-2\mu_2}p_x{}^2+e^{-2\mu_3}p_y{}^2+e^{-2\psi}p_\phi{}^2-e^{-2\nu}(p_\phi\Omega+p_t)^2\,. \tag{5·2}$$

Here we have used relations such as

$$\dot{t}=-e^{-2\nu}(p_\phi\Omega+p_t),\quad \dot{\phi}=e^{-2\psi}p_\phi-e^{-2\nu}\Omega(p_\phi\Omega+p_t),$$

$$\dot{x}=e^{-2\mu_2}p_x \quad \text{and} \quad \dot{y}=e^{-2\mu_3}p_y\,. \tag{5·3}$$

As the coordinates λ, t and ϕ are cyclic ones in this Hamiltonian, we can express the Hamilton-Jacobi functions S as

$$S=-\mu^2\lambda+\varepsilon t+l\phi+S(x,y), \tag{5·4}$$

where the constants of motion μ, ε and l denote the rest mass of a test particle ($\mu^2\equiv -g^{\mu\nu}p_\mu p_\nu$), the energy ($\varepsilon=p_t$) and the z component of the angular momentum ($l=p_\phi$), respectively. Using Eqs. (5·2) and (5·4), the Hamilton-Jacobi equation, that is, $\partial S/\partial\lambda=H(\partial S/\partial x^\mu, x^\mu)$ becomes

$$e^{-2\mu_2}\left[\frac{\partial S}{\partial x}\right]^2+e^{-2\mu_3}\left[\frac{\partial S}{\partial y}\right]^2+\mu^2+e^{-2\psi}l^2-e^{-2\nu}(\varepsilon+l\Omega)^2=0\,. \tag{5·5}$$

Substituting Eq. (4·7) into Eq. (5·5), we obtain the relation

$$p^{2\delta-2}\delta^2(x^2-y^2)^{\delta^2-1}\left\{(x^2-1)\left[\frac{\partial S}{\partial x}\right]^2+(1-y^2)\left[\frac{\partial S}{\partial y}\right]^2\right\}+\mu^2B=\frac{\varepsilon^2K(x,y;\,l,q)}{p^2(x^2-1)}$$

and

$$K(x,y;\,l,q)=-\frac{\delta^2A}{1-y^2}l^2-4\delta^2qCl+D\,, \tag{5·6}$$

where l/ε has been replaced by l. This relation shows that the separation of variable such as $S(x,y)=S_1(x)+S_2(y)$ is impossible except in the case $\delta=1$, i.e., the case of Kerr metric. From Eq. (5·3) this relation is also expressed as

$$\frac{\dot{x}^2}{x^2-1}+\frac{\dot{y}^2}{1-y^2}+\frac{p^{2(\delta-1)}\delta^2\mu^2(x^2-y^2)^{\delta^2-1}}{B}=\frac{\delta^2(x^2-y^2)^{\delta^2-1}\varepsilon^2}{p^{2(2-\delta)}(x^2-1)B^2}K(x,y;\,l,q)\,. \tag{5·7}$$

If we consider a null geodesic on the equatorial plane, Eq. (5·7) becomes by putting $\mu=0$ and $y=0$ as

$$\dot{x}^2=\frac{\delta^2x^{2(\delta^2-1)}\varepsilon^2}{p^{2(2-\delta)}B^2}K(x,0;\,l,q)\,. \tag{5·8}$$

It is evident that the geodesic cannot extend through the singularity of space-time where $B=0$. Further, the geodesic motion is allowed only in the region satisfying the condition $K(x,0;\,l,q)>0$. This condition is usually drawn on the l-x plane. We shall now draw the forbidden region ($K<0$) of null geodesics on the l-x plane for the T-S metrics. The roots of the equation $K(x,0;\,l,q)=0$ for l are easily given as

$$-l_r=2q\frac{C\pm\sqrt{C^2+AD/4\delta^2q^2}}{A}=2q\frac{C\pm Bp\sqrt{(x^2-1)}/2\delta q}{A}\,. \tag{5·9}$$

In general, the above relation becomes

$$-l_r\to\pm\frac{p}{\delta}x \quad \text{for } |x|\gg 1 \tag{5·10}$$

and, if $q\neq 0$,

$$-l_r = 2\frac{1+p}{q} \quad \text{for } x=1\,,$$

$$= 2\frac{1-p}{q} \quad \text{for } x=-1\,. \tag{5·11}$$

It is interesting to note that these values are independent of δ.

(a) *Weyl metrics*

For a comparison with the T-S metrics, we consider at first the case of the Weyl metrics. In the Weyl metrics of Eq. (2·8), the metric functions are given as

$$A=(x^2-1)^{\delta^2},\quad B=\left[\frac{x+1}{x-1}\right]^{\delta}(x^2-1)^{\delta^2} \tag{5·12}$$

and

$$D=\left[\frac{x+1}{x-1}\right]^{2\delta}(x^2-1)^{\delta^2+1}\,.$$

Then, the roots l_r becomes

$$l_r=\pm\frac{1}{\delta}\left[\frac{x+1}{x-1}\right]^{\delta}\sqrt{x^2-1}\,. \tag{5·13}$$

The schematic figure of the forbidden region are given for $\delta=1, 2$ and 3 in Figs. 2(a), 3(a) and 4(a), respectively. Except in the case $\delta=1$, i.e., the Schwarzshild metric, the space-time is singular on the surface of $x=1$.

For the infalling photon of $l=0$, the equations of motion become

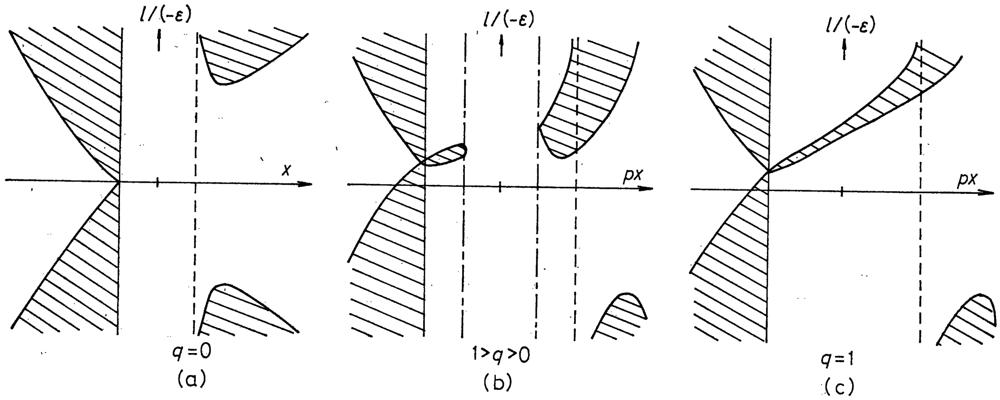

Fig. 2. The forbidden region of motion for a photon, moving on the equatorial plane in the Kerr space-time (i.e. the T-S space-time for $\delta=1$) and with the angular momentum about the z axis, l. The forbidden region is denoted on the l-x plane by the shaded region. The dotted-lines, the solid-lines and the chain-lines denote the places of the infinite redshift surfaces, the ring singularities and the event horizons, respectively.

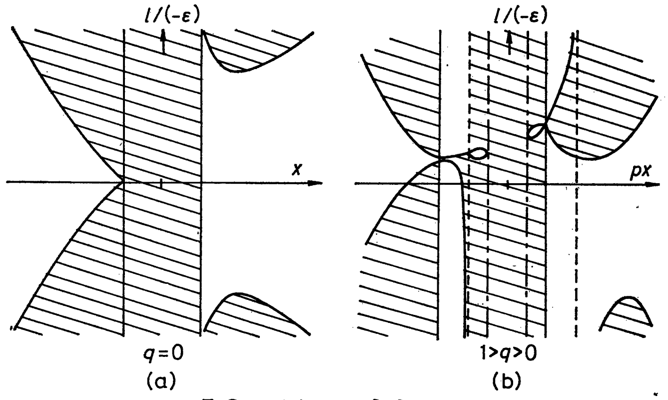

Fig. 3. The same figure with Fig. 2 but for the case of the T-S metric for $\delta=2$.

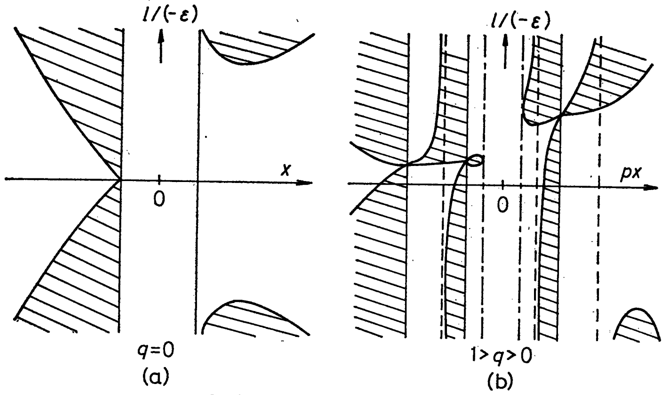

Fig. 4. The same figure with Fig. 2 but for the case of the T-S metric for $\delta=3$.

$$\frac{dx}{dt}=-\delta x^{\delta^2-1}\frac{1}{(x+1)^{(\delta^2+2\delta-1)/2}(x-1)^{(\delta^2-2\delta-1)/2}} \quad \text{for } y=0,$$
$$=-\delta\left[\frac{x+1}{x-1}\right]^{\delta} \quad \text{for } y^2=1. \qquad (5\cdot14)$$

For $x-1\ll1$, the solutions become

$$x-1\propto(t_0-t)^{2/(\delta-1)^2} \quad \delta\neq1,$$
$$\propto e^{-t/2} \quad \delta=1 \qquad (5\cdot15)$$

for $y=0$, and

$$x-1\propto t^{-1/(\delta-1)} \quad \delta>1,$$
$$\propto e^{-t/2} \quad \delta=1,$$
$$\propto(t_0-t)^{1/(1-\delta)} \quad \delta<1 \qquad (5\cdot16)$$

for $y^2=1$. It is noticeable that for $\delta\neq1$ the infalling photon reaches the singular surface of $x=1$ within finite interval of the proper time at distant observer. Therefore the singular surface where $x=1$ is not the event horizon in the Weyl metrics.

(b) *T-S metrics*

For the Kerr metric, the forbidden region on the l-x plane has been calculated as shown in Fig. 2.[20)~22)] For the T-S metrics for $\delta=2$ and 3, this forbidden regions are also obtained and shown in Figs. 3(b) and 4(b). These figures are not drawn in real scale but drawn schematically, since we are interested only in a gross behaviour of this relation. It is remarkable that the region inside the event horizon is the forbidden region for even δ, since A is positive in this region.

As will be seen from the cases $\delta=1\sim4$, on the equatorial plane the numbers of real roots of the equations $A=0, B=0$ and $C=0$ are equal to $2\delta, \delta$ and $2\delta-1$, respectively. Because A is an even function of x, we can denote the roots of $A=0$ as $\pm x_\delta, \pm x_{\delta-1}, \cdots, \pm x_1$, where $x_\delta>x_{\delta-1}>\cdots>x_1>0$. For $\delta=2$, the roots of $A=0$ are given as

$$px=\pm\tfrac{1}{2}[-1-\sqrt{t+1}\pm\sqrt{2-t+2\sqrt{t^2-t+1}}] \quad \text{for } p^2\leq\tfrac{1}{2},$$
$$=\pm\tfrac{1}{2}[-1+\sqrt{t+1}\pm\sqrt{2-t+2\sqrt{t^2-t+1}}] \quad \text{for } p^2>\tfrac{1}{2}, \qquad (5\cdot17)$$

where $t=(-4p^2q^2)^{1/3}$. Variation with q of these roots is shown in Fig. 5, and, for comparison, the same relation for $\delta=1$ is also shown. The roots of $B=0$ are $-x_\delta, -x_{\delta-2}, \cdots, x_{\delta-3}, x_{\delta-1}$. In the limit $q\to1, px_\delta/\delta$ tends to unity and all the other values such as $px_{\delta-1}/\delta, \cdots, px_1/\delta$ tend to zero. This result is consistent with the result mentioned in § 4 that, in the limit $q\to1$, all of the T-S metrics tend to the same one, that is, the Kerr metric with $q=1$.

The infalling photon on the equatorial plane is stopped at the true singularity where $x=x_{\delta-1}$ and cannot reach the inner region of this true singularity. However, if the photon departs from the equatorial plane, it will be possible to fall into the inner region

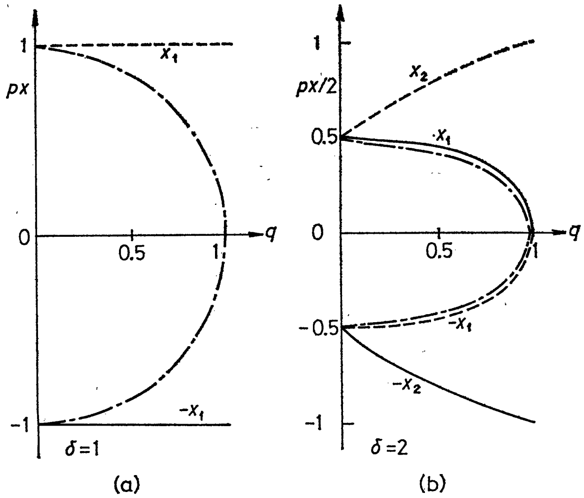

Fig. 5. The variations of the characteristic values of px/δ with q. The dotted-lines, the solid-lines and the chain-lines denote the values of px/δ where $A=0$ but $B\neq0$, those where $A=B=0$ and those where $x=\pm1$, respectively.

$x < x_{\delta-1}$. This problem will be investigated elsewhere.

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