(Un)countable and (non)effective versions of Ramsey's theorem

Dietrich Kuske

ABSTRACT. We review Ramsey's theorem and its extensions by Jockusch for computable partitions, by Sierpiński and by Erdős and Rado for uncountable homogeneous sets, by Rubin for automatic partitions, and by the author for ω-automatic (in particular uncountable) partitions.

Introduction

Every infinite graph has an infinite clique or an infinite anticlique – this is the paradigmatic formulation of Ramsey's theorem [**14**]. But this theorem is highly non-constructive since there are computable infinite graphs none of its infinite cliques and anticliques is computable (they are not even in Σ_2^0 [**7**]). On the positive side, Jockusch also showed that every infinite computable graph contains an infinite clique or anticlique from Π_2^0. Soon after Ramsey's paper from 1930, authors got interested in a quantitative analysis: how many nodes are necessary and sufficient to obtain a clique or anticlique of size $\aleph_1$. The answer was given by Sierpiński [**17**] $((2^{\aleph_0})^+$ nodes are necessary) and Erdős & Rado [**6**] $((2^{\aleph_0})^+$ nodes are sufficient). We unify the proofs of these positive results by the notion of a Ramsey tree. In doing so, we prove in particular that every infinite computable finitely branching tree contains an infinite path from Π_2^0.

Recall that a graph is computable if both its set of nodes and its set of edges can be decided by a Turing machine. Replacing these Turing machines by finite automata, one obtains the more restrictive notion of an *automatic graph*: the set of nodes is a regular set and whether a pair of nodes forms an edge can be decided by a synchronous two-tape automaton (this concept is known since the beginning of automata theory, a systematic study started with [**9, 3**], see [**15**] for a recent overview). Compared to computable graphs, here, the situation is much more favourable: every infinite automatic graph contains an infinite regular clique or an infinite regular anticlique [**15**].

Since automatic graphs contain at most $\aleph_0$ nodes, we need a more general notion for a recursion-theoretic analysis of uncountable graphs. For this, we use

---

Blumensath & Grädel's [**3**] ω-automatic graphs: the names of nodes form a regular ω-language and the edge relation (on names) as well as the relation "these two names denote the same node" can be decided by a synchronous 2-tape Büchi-automaton. We show that any such graph of size $2^{\aleph_0}$ has a clique or anticlique of size $2^{\aleph_0}$, but these (anti)cliques are not necessarily described by a Büchi-automaton.

While the results by Ramsey, by Jockusch, by Erdős & Rado, and by Rubin generalise to hypergraphs, this is not the case for our result on ω-automatic graphs: We present a ternary ω-automatic hypergraph of size $2^{\aleph_0}$ that does not contain any uncountable clique or anticlique.

Finally, it should be noted that all the positive results, except those by Rubin and by Kuske are shown by an analysis of the Ramsey tree of a hypergraph. This proof technique seems not to work in the (ω-)autmatic setting where other strategies are employed.

Acknowledgment Markus Lohrey's careful and sympathetic reading of the manuscript eliminated more mistakes than I thought possible. I am very grateful for the time and energy he devoted to this task.

1. Partitions and the Ramsey tree

For a set V and a natural number $k \geq 1$, let $[V]^k$ denote the set of k-element subsets of V. A (k,ℓ)-*partition* is a tuple $P = (V, E_1, \ldots, E_\ell)$ where V is a set and $(E_1, \ldots, E_\ell)$ is a partition of $[V]^k$ into (possibly empty) sets. For $1 \leq i \leq \ell$, a set $W \subseteq V$ is E_i-*homogeneous* if $[W]^k \subseteq E_i$; it is *homogeneous* if it is E_i-homogeneous for some $1 \leq i \leq \ell$. The case $k = \ell = 2$ is special: any $(2,2)$-partition $G = (V, E_1, E_2)$ can be considered as an (undirected loop-free) graph (V, E_1). Homogeneous sets in G are then complete or discrete induced subgraphs of (V, E_1).

This paper is concerned with the following question: Does every (k,ℓ)-partition $G = (V, E_1, \ldots, E_\ell)$ with $|V| = \kappa$ have a homogeneous set of size λ (where κ and λ are cardinal numbers and $k, \ell \geq 2$ are natural numbers). If this is the case, one writes

$$\kappa \to (\lambda)_\ell^k$$

(a notation due to Erdős & Rado [**5**]).

OBSERVATION 1.1. *Let $P = (V, E_1, \ldots, E_\ell)$ be a (k,ℓ)-partition, $\leq$ a well-order on V, and $v \in V$. Then there exists a unique ordinal sequence $(v_\alpha)_{\alpha \leq \beta}$ satisfying:*

(R1) for $\alpha \leq \beta$, the node v_α is the least element of $(V, \leq)$ such that
- *$v_\gamma < v_\alpha$ for all $\gamma < \alpha$ and*
- *$A \cup \{v_\alpha\} \in E_i$ iff $A \cup \{v\} \in E_i$ for all $A \in [\{v_\gamma \mid \gamma < \alpha\}]^{k-1}$ and $1 \leq i \leq \ell$*

(R2) $v_\beta = v$

EXAMPLE 1.2. We consider the Rado graph as a $(2,2)$-partition $(\mathbb{N}, E_1, E_2)$ where $\{m, n\} \in E_1$ with $m < n$ if, in the binary expansion of n, position m carries 1 (i.e., the unique presentation of n as sum of powers of 2 contains the summand 2^m).

For a word w over $\{0, 1\}$, let $[w]$ denote its value at base 2, i.e., $[\varepsilon] = 0$ and $[wx] = 2 \cdot w + x$ for $x \in \{0, 1\}$.

Then $0, 1, 5, 33, 37$ is the Ramsey sequence of 37: Clearly, every Ramsey sequence has to start with $v_0 = 0$. Then $[100101] = 37$, so $\{0, 37\} \in E_1$. Hence we determine the minimal number $v_1 > v_0$ whose binary expansion ends with 1 – this is obviously $v_1 = 1$. Since $\{1, 37\} \in E_2$, we now search for the minimal number

$v_2 > v_1 = 1$ whose binary expansion ends in 01 – hence $v_2 = [101] = 5$. Again, $\{5, 37\} \in E_1$, so v_3 is the minimal number above 5 whose binary expansion ends with $1abc01$ for some $a, b, c \in \{0, 1\}$ – hence $v_3 = [100001] = 33$. Now $\{33, 37\} \in E_2$, so v_4 is the minimal number whose binary expansion ends with $0w1abc01$ with $w \in \{0, 1\}^*$ of length 28 and $a, b, c \in \{0, 1\}$ – thus $v_4 = [100101] = 37$.

Let $v \in V$. Then one can construct, by transfinite induction, the sequence of nodes v_α satisfying (R1). In particular, $v_0 < v_1 < \cdots < v_{k-2}$ is the initial segment of $(V, \leq)$ of length $k - 1$ (if v is not among these elements). Eventually, v is the least element satisfying the two conditions in (R1) – at this stage β, the process terminates. We call the sequence $(v_\alpha)_{\alpha \leq \beta}$ the *Ramsey sequence of v* and denote it by $\bar{v}$; its *length* is β. Using these sequences, we can define a partial order $\preceq$ on V by setting $v \preceq w$ if and only if $\bar{v}$ is a prefix of $\bar{w}$ (which is the case if and only if v appears in the sequence $\bar{w}$). The structure $(V, \preceq)$ is the *Ramsey tree* of the (k, ℓ)-partition P with respect to the well-order $\leq$.

Some properties of the Ramsey tree are obvious:

(1) If $v \in V$, then its predecessors in the Ramsey tree (i.e., the set $\{w \in V \mid w \preceq v\}$) are well-ordered. This justifies to call $(V, \preceq)$ an order tree.

(2) Let $v \in V$ and let B be the cardinality of its Ramsey sequence. For any two distinct brothers x and y of v, there exists a $(k - 1)$-elements subset A of $\bar{v}$ not containing v and $1 \leq i \leq \ell$ such that $A \cup \{x\} \in E_i$ and $A \cup \{y\} \notin E_i$. Hence the number of brothers is at most

$$\ell^{\binom{B}{k-1}} = \ell^{|[B]^{k-1}|}.$$

In particular, it is finite if $\bar{v}$ is finite, and it is at most $2^{\aleph_n}$ if $|B| = \aleph_n$.

Different versions of Ramsey's theorem will be derived from an analysis of the Ramsey tree. This is based on the following observation and result that allows to infer the existence of homogeneous sets in (k, ℓ)-partitions from their existence in $(k - 1, \ell)$-partitions.

OBSERVATION 1.3. *Let $(V, \preceq)$ be the Ramsey tree of the (k, ℓ)-partition $P = (V, E_1, \ldots, E_\ell)$ with respect to the well-order $\leq$ and let $Y \subseteq V$ be some chain in the Ramsey tree (i.e., linearly ordered subset) without maximal element.*

For $A \in [Y]^{k-1}$, set $A \in E_i^Y$ if and only if there exists $y \in Y$ with $A \prec y$ and $A \cup \{y\} \in E_i$. Then $P^Y = (Y, E_1^Y, \ldots, E_\ell^Y)$ is a $(k - 1, \ell)$-partition.

PROOF. Since Y has no maximal element with respect to $\preceq$, any set $A \in [Y]^{k-1}$ belongs to some E_i^Y. Suppose $A \in E_i^Y \cap E_j^Y$. Then there are $y_i, y_j \in Y$ with $A \prec y_i, y_j$, $A \cup \{y_i\} \in E_i$, and $A \cup \{y_j\} \in E_j$. For symmetry reasons, we can assume $y_i \preceq y_j$. But then $i = j$ by (R1) since y_i appears in the Ramsey sequence of y_j. □

PROPOSITION 1.4. *Let $k \geq 2$ and let $P = (V, E_1, \ldots, E_\ell)$ be a (k, ℓ)-partition and Y a chain without maximal element in the Ramsey tree $(V, \preceq)$. Any set homogeneous in the $(k - 1, \ell)$-partition P^Y induced by Y is homogeneous in P.*

PROOF. Let $W \subseteq Y \subseteq V$ be E_i^Y-homogeneous, i.e., $[W]^{k-1} \subseteq E_i^Y$. Let $B \in [W]^k \subseteq [Y]^k$ and let $v < w$ be the maximal elements of $(B, \leq)$. Then $A := B \backslash \{w\} \in [W]^{k-1} \subseteq E_i^Y$. By the definition of E_i^Y, there exists $y \in Y$ with $A \leq v < y$ and $A \cup \{y\} \in E_i$. Since $(Y, \preceq)$ is linearly ordered, y appears in the Ramsey sequence of w or vice versa. In any case, $B = A \cup \{w\} \in E_i$ follows from (R1). □

2. The countable case

2.1. Ramsey's theorem.

THEOREM 2.1 (Ramsey [**14**]). *If $k, \ell \geq 1$ and $\kappa \geq \aleph_0$, then $\kappa \to (\aleph_0)_\ell^k$.*

PROOF. It suffices to prove the theorem for $\kappa = \aleph_0$ in which case it is shown by induction on k. The base case ($k = 1$) is the pigeonhole principle. So suppose $\aleph_0 \to (\aleph_0)_\ell^{k-1}$ and let $P = (\mathbb{N}, E_1, \ldots, E_\ell)$ be a (k, ℓ)-partition. The well-order $\leq$ on the base set $\mathbb{N}$ of this partition is the natural order on the natural numbers. Then the Ramsey tree of P is infinite and, since all Ramsey sequences are finite, finitely branching. Hence, by König's lemma, it contains some infinite branch Y (that does not have a maximal element). Then P^Y is an infinite $(k-1, \ell)$-partition, so it contains by the induction hypothesis an infinite homogeneous set. But this set is, by Prop. 1.4 also homogeneous in P.   □

2.2. Jockusch's theorems. A partition $(\mathbb{N}, E_1, \ldots, E_\ell)$ is computable if all the sets E_i are decidable. For $k, \ell \geq 2$ and classes of sets $\mathcal{C} \subseteq 2^{[\mathbb{N}]^k}$ and $\mathcal{D} \subseteq 2^\mathbb{N}$, write

$$(\aleph_0, \mathcal{C}) \to (\aleph_0, \mathcal{D})_\ell^k$$

if every (k, ℓ)-partition $(\mathbb{N}, E_1, \ldots, E_\ell)$ with $E_1, \ldots, E_\ell \in \mathcal{C}$ contains some infinite homogeneous set $H \in \mathcal{D}$. In this section, the sets $\mathcal{C}$ and $\mathcal{D}$ will be classes from the arithmetical hierarchy: A set $A \subseteq \mathbb{N}$ belongs to Σ_n^0 if and only if there is a computable predicate $P \subseteq \mathbb{N}^{n+1}$ such that

$$A = \{x \in \mathbb{N} \mid \exists x_1 \forall x_2 \ldots \exists/\forall x_n : (x, x_1, \ldots, x_n) \in P\}.$$

The set A belongs to Π_n^0 if its complement belongs to Σ_n^0, it belongs to Δ_n^0 if it is both, Σ_n^0 and Π_n^0, and it belongs to the arithmetical hierarchy AH if it belongs to Σ_n^0 for some $n \in \mathbb{N}$. We also write REC for Δ_1^0 since these are precisely the computable sets. Identifying a finite set $A \in [\mathbb{N}]^k$ with its Gödel number, we can also say that a set $E \subseteq [\mathbb{N}]^k$ belongs to Σ_n^0 etc.

We now discuss the computational content of the above proof of Ramsey's theorem. First note that clearly

$$(\aleph_0, \mathrm{REC}) \to (\aleph_0, \mathrm{REC})_\ell^1$$

since, given an infinite computable $(1, \ell)$-partition, one of the classes $E_1, \ldots, E_\ell$ is infinite and they are all computable. Next, let $(\mathbb{N}, E_1, \ldots, E_\ell)$ be some computable (k, ℓ)-partition and let $\leq$ be the natural order on $\mathbb{N}$. Then the Ramsey tree of P wrt. $\leq$ has a computable copy: it consists of all Ramsey sequences (that are necessarily finite) ordered by the prefix order. An isomorphism is given by mapping v to its Ramsey sequence $\bar{v}$. In order to apply Prop. 1.4, we have to find an infinite branch in this computable copy. First note that also the successor relation in this computable copy is computable. We therefore now make a short excursion into the theory of computable successor trees.

THEOREM 2.2. *Let $T = (V, \mathrm{succ})$ with $V \subseteq \mathbb{N}$ be an infinite computable finitely branching successor tree. Then T contains an infinite branch from Π_2^0.*

PROOF. On V, we define the lexicographic order: $m \leq_{\mathrm{lex}} n$ if and only if $(m, n) \in \mathrm{succ}^*$ or there exist nodes $x, m', n' \in V$ with $(x, m'), (x, n') \in \mathrm{succ}$, $(m', m), (n', n) \in \mathrm{succ}^*$, and $m' < n'$. Note that the linear order $\leq_{\mathrm{lex}}$ is decidable. Furthermore, the root of T is its minimal element and every node $v \in V$ has

a successor in the linear order $(V, \leq_{\text{lex}})$ (if it is not maximal in this linear order). Hence $(V, \leq_{\text{lex}})$ is isomorphic to $\omega + (V', \leq_{\text{lex}})$ for some set $V' \subseteq V$. For $n \geq 1$ let v_n denote the nth element of $(V, \leq_{\text{lex}})$ and define

$$C = \{v \in V \mid \forall n : (v \leq_{\text{lex}} v_n \Rightarrow (v, v_n) \in \text{succ}^*) \wedge \exists m : v = v_m\}.$$

This set is a chain in the tree (V, succ^*): Let $v, w \in C$. Then there exist $m, n \in \mathbb{N}$ with $v = v_m$ and $w = v_n$, we assume $m \leq n$. By the definition, this implies $v_m \leq_{\text{lex}} v_n$ and therefore (since $v = v_m \in C$) also $(v_m, v_n) \in \text{succ}^*$, i.e., $(v, w) \in \text{succ}^*$.

Our next aim is to show that C is infinite. First note that the root v_1 belongs to C. Let $c \in C$. Then there are infinitely many $m \in \mathbb{N}$ with $c \leq_{\text{lex}} v_m$ and therefore with $(c, v_m) \in \text{succ}^*$. Let c' be the lexicographically largest son of c of the form v_n. Then, for all $s \geq n$, we have $c <_{\text{lex}} c' = v_n \leq_{\text{lex}} v_s$ implying $(c, v_s) \in \text{succ}^*$ and therefore $(c', v_s) \in \text{succ}^*$. Hence $c' \in C$, i.e., the chain C does not contain a maximal element. It follows that C is even an infinite branch (i.e., downwards closed).

To show that C is Π_2^0, it suffices to argue that the relation $R = \{(n, v_n) \mid n \in \mathbb{N}\}$ is Σ_2^0. But this is obvious since it is defined by

$$\exists v_1, v_2, \ldots, v_{n-1} \forall v : \quad \bigwedge_{1 \leq i < n-1} v_i <_{\text{lex}} v_{i+1}$$
$$\wedge \quad v \not\leq_{\text{lex}} v_1 \wedge \bigwedge_{1 \leq i < n-1} \neg(v_i <_{\text{lex}} v <_{\text{lex}} v_{i+1}).$$

$\square$

LEMMA 2.3. *Let $P = (\mathbb{N}, E_1, \ldots, E_\ell)$ be a computable and infinite $(k+1, \ell)$-partition and let $C \subseteq \mathbb{N}$ be an infinite branch in its Ramsey tree from Π_2^0. Then $E_i^C \subseteq \mathbb{N}$ is Π_2^0 for all $1 \leq i \leq \ell$.*

PROOF. Let $A \in [C]^k$ with maximal element a. Let furthermore $\mathbb{A}$ denote the finite set $\{x \in V \mid x \preceq a\}$. Then $A \in E_i^C$ if and only if there exists a function $f : [\mathbb{A}]^k \to \{1, \ldots, \ell\}$ with $f(A) = i$ such that

$$\forall y (a \prec\!\!\cdot\, y \wedge \bigwedge_{B \in [\mathbb{A}]^k} B \cup \{y\} \in E_{f(B)} \to y \in C)$$
$$\wedge \exists y (x \prec y \wedge \bigwedge_{B \in [\mathbb{A}]^k} B \cup \{y\} \in E_{f(B)})$$

where $\prec\!\!\cdot$ is the immediate successor relation of the tree T. The existentially quantified formula and the premise in the universally quantified formula are computable predicates. Since the quantification over the functions f is computably bounded, the whole expression belongs to Π_2^0.

$\square$

THEOREM 2.4. *Let $P = (V, E_1, \ldots, E_\ell)$ be some infinite computable $(2, \ell)$-partition. Then P contains an infinite homogeneous set from Π_2^0.*

PROOF. Since also the successor relation of the Ramsey tree of P is computable, we can apply Theorem 2.2, i.e., there exists an infinite chain C from Π_2^0 in the Ramsey tree of P. Since C is infinite, one of the sets E_i^C is infinite. By Lemma 2.3, this set E_i^C belongs to Π_2^0 and, by Prop. 1.4, it is homogeneous in P.

$\square$

THEOREM 2.5 (attributed to Manaster in [**7**, p. 276]). *For all $k, \ell \geq 1$, we have*

$$(\aleph_0, \text{AH}) \to (\aleph_0, \text{AH})_\ell^k.$$

PROOF. Note that Theorem 2.2 and Lemma 2.3 and therefore Theorem 2.4 also hold in their relativized form. Hence we can prove this theorem by induction in the same way that we proved Ramsey's theorem.   □

Jockusch [**7**] proved the more precise result

$$(\aleph_0, \Delta_n^0) \to (\aleph_0, \Pi_{n+k-1}^0)_\ell^k$$

for $k, \ell, n \geq 1$. In general, the proof proceeds by induction as before: from a (k, ℓ)-partition P whose partition classes all belong to Δ_n^0, one builds a $(k-1, \ell)$-partition whose homogeneous sets are also homogeneous in P. Once the inductive construction reaches a $(2, \ell)$-partition, the above Theorem 2.4 is invoked. While Manaster uses the Π_2^0-branch from Theorem 2.2 in the inductive step, Jockusch uses a branch C such that C' (the first jump of C) is Δ_3^0 – existence of such a branch is shown in [**8**].

Esthetically, this result is not as satisfactory as the previous ones since the levels of the arithmetical hierarchy on the left and on the right do not coincide. Jockusch [**7**] also showed that this cannot be avoided. His construction uses the following limit lemma twice.

LEMMA 2.6 (Shoenfield [**16**]). *Let $k \geq 2$. There is a total function $F : \mathbb{N}^3 \to \{0, 1\}$ computable in $\emptyset^{(k-2)}$ such that, for any $A \in \Delta_k^0$, there is $e \in \mathbb{N}$ satisfying*

(1) the sequence $(F(e, m, s))_{s \in \omega}$ is ultimately constant for all $m \in \mathbb{N}$, and
(2) $A = \{m \mid F(e, m, s) = 1 \text{ for almost all } s\}$.

PROOF. For $s \in \mathbb{N}$ and $X \subseteq \mathbb{N}$, let K_s^X denote the set of all $e \leq s$ such that the oracle Turing machine with index e stops, when started with input e and oracle X, after at most s steps.

Next, let $F : \mathbb{N}^3 \to \{0, 1\}$ be defined by $F(e, m, s) = 1$ if and only if the oracle Turing machine with index e stops successfully after at most s steps, when started with input m and oracle $K_s^{\emptyset^{(n-2)}}$. Then F is computable in $\emptyset^{(k-2)}$ and total. Let $A \in \Delta_k^0$. Then A is computable in $\emptyset^{(k-1)}$, i.e., there exists $e \in \mathbb{N}$ such that the oracle Turing machine M_e with index e computes the characteristic function χ_A using the oracle $\emptyset^{(k-1)}$. Now let $m \in \mathbb{N}$. Then M_e has a unique computation C with input m and oracle $\emptyset^{(k-1)}$. Let $t \in \mathbb{N}$ be at least the length of C such that, for any oracle access o in C, we have

$$o \in \emptyset^{(k-1)} = K^{\emptyset^{(k-2)}} \iff o \in K_t^{\emptyset^{(k-2)}}.$$

Then, for any $s \geq t$, C is the computation of the oracle Turing machine M_e with oracle $K_s^{\emptyset^{(k-2)}}$ and input m. In other words, $F(e, m, s) = \chi_A(m)$ for any $s \geq t$. This proves (1) and (2).   □

Jockusch's first use of the limit lemma is in the base case:

LEMMA 2.7. *For any $n \geq 2$, we have $(\aleph_0, \Delta_{n-1}^0) \not\to (\aleph_0, \Delta_n^0)_2^2$*

PROOF. Let F be the function from Lemma 2.6 that is computable in $\emptyset^{(n-2)}$. We set $\{m, s\} \in E_1$ (with $m < s$) if and only if $m \in H(s)$ where $H(s)$ is defined by the procedure from Fig. 1; $E_2 = [\mathbb{N}]^2 \setminus E_1$ completes the definition of the $(2, 2)$-partition $(\mathbb{N}, E_1, E_2)$. Note that E_1 and E_2 are both computable in $\emptyset^{(n-2)}$ and therefore belong to Δ_{n-1}^0.

We now show by contradiction that this partition does not have any infinite homogeneous set in Δ_n^0. Let $A \in \Delta_n^0$ be infinite. Hence there is $e \in \mathbb{N}$ satisfying (1)

```
0    function H(s ∈ ℕ): subset of {1, 2, ..., s}
1      H₁ := ∅; H₂ := ∅;
2      for e := 0 to s − 1 do
3        if |{n ≤ s : F(e, n, s) = 0} \ (H₁ ∪ H₂)| ≥ 2 then
4          let a < b be minimal in {n ≤ s | F(e, n, s) = 0} \ (H₁ ∪ H₂);
5          set H₁ := H₁ ∪ {a} and H₂ := H₂ ∪ {b};
6      return H₁;
```

FIGURE 1. Computation of $H(s)$

and (2) in Lemma 2.6. Let $a_1 < a_2 < \cdots < a_{2e+2}$ be the $2e + 2$ minimal elements of A. By Lemma 2.6, there is $t \in \mathbb{N}$ such that

$$(2.1) \qquad \forall a \leq a_{2e+2} \forall s \geq t : a \in A \iff F(e, a, s) = 0.$$

Since A is infinite, there exists $s \in A$ with $s \geq a_{2e+2}$ and $s \geq t$. We analyse the set $H(s)$: Note that in any run through the loop in Fig. 1, $|H_1 \cup H_2|$ increases by at most 2. Hence, whenever we execute line 3, we get $|H_1 \cup H_2| \leq 2e$. With e satisfying (1) and (2) in Lemma 2.6, the test in line 3 succeeds since $\{n \leq s : F(e, n, s) = 0\} = \{a_1, \ldots, a_{2e+2}\}$ by (2.1). For the same reason, the elements a and b chosen in line 4 both belong to $\{a_1, \ldots, a_{2e+2}\} \subseteq A$. Since no element is ever deleted from H_1, we obtain $a \in H(s)$. On the other hand, no element from H_2 will ever be put into H_1, so $b \notin H(s)$. Hence, we found $a, b \in A$ with $\{a, s\} \in E_1$ and $\{b, s\} \in E_2$, i.e., the set A is not homogeneous. Since all we assumed about the set A was $A \in \Delta_n^0$ and A infinite, we proved that no infinite Δ_n^0-set is homogeneous in the $(2, 2)$-partition $(\mathbb{N}, E_1, E_2)$. □

By induction, Jockusch then increases the gap between Δ_{n-1}^0 and Δ_n^0 in the above lemma at the expense of partitioning $[\mathbb{N}]^k$ for $k > 2$. This inductive argument makes use of the limit lemma, again.

LEMMA 2.8. *Let $P = (\mathbb{N}, A, B)$ be a $(k, 2)$-partition with $A, B \in \Delta_n^0$. Then there exists a $(k + 1, 2)$-partition $P' = (\mathbb{N}, A', B')$ with $A', B' \in \Delta_{n-1}^0$ such that any infinite set homogeneous in P' is also homogeneous in P.*

PROOF. Let F be the function from Lemma 2.6 that is computable in $\emptyset^{(n-2)}$. Hence there is $e \in \mathbb{N}$ satisfying (1) and (2) in Lemma 2.6 (where we identify a finite set with its Gödel number). For $a_1 < a_2 < \cdots < a_k < s$, set $\{a_1, \ldots, a_k, s\} \in A'$ if and only if $F(e, \{a_1, a_2, \ldots, a_k\}, s) = 0$ and define $B' = [\mathbb{N}]^{k+1} \setminus A'$. Clearly, A' is computable in $\emptyset^{(n-2)}$ implying $A', B' \in \Delta_{n-1}^0$.

Now let $H \subseteq \mathbb{N}$ be homogeneous in P'. It suffices to consider the case $[H]^{k+1} \subseteq A'$ since the other one is analogous. To prove $[H]^k \subseteq A$, let $h_1 < h_2 < \cdots < h_k$ be elements of H. Then, for any $t > h_k$ from H, we get $F(e, \{h_1, \ldots, h_k\}, t) = 0$. Since the sequence $(F(e, \{h_1, \ldots, h_k\}, s))_{s \in \omega}$ is ultimately constant, this implies $F(e, \{h_1, \ldots, h_k\}, s) = 0$ for almost all s, i.e., $\{h_1, \ldots, h_k\} \in A$. □

THEOREM 2.9 (Jockusch [7]). *Let $k, \ell, n \geq 2$. Then $(\aleph_0, \Delta_n^0) \nrightarrow (\aleph_0, \Sigma_{n+k-1}^0)_\ell^k$.*

PROOF. We first prove $(\aleph_0, \Delta_n^0) \nrightarrow (\aleph_0, \Delta_{n+k-1}^0)_2^k$:

By Lemma 2.7, we find a $(2, 2)$-partition $P_2 = (\mathbb{N}, E_1^2, E_2^2)$ with $E_1^2, E_2^2 \in \Delta_{n+k-2}^0$ without any homogeneous set in Δ_{n+k-1}^0. Applying Lemma 2.8 several

times, we find $(m, 2)$-partitions $P_m = (\mathbb{N}, E_1^m, E_2^m)$ with $E_1^m, E_2^m \in \Delta_{n+k-m}^0$ whose infinite homogeneous sets are also homogeneous in P_2 (and therefore do not belong to Δ_{n+k-1}^0). With $m = k$, the claim follows.

The general result now follows since:

- Any infinite set H from Σ_{n+k-1}^0 contains an infinite set from Δ_{n+k-1}^0 that is homogeneous whenever H is.
- If $\ell > 2$, extend the $(n, 2)$-partition by sufficiently many empty classes.

$\square$

Let $P = (\mathbb{N}, E_1, E_2)$ be some infinite computable $(2, 2)$-partition. Let $R = (\mathbb{N}, \preceq)$ be its Ramsey tree (with respect to the natural order $\le$ on $\mathbb{N}$) and suppose $C \subseteq \mathbb{N}$ is an infinite chain in R from Σ_2^0. Then the sets E_i^C of natural numbers $c \in C$ with

$$\exists d : d \in C \wedge c < d \wedge \{c, d\} \in E_i$$

belong to Σ_2^0 and are homogeneous in P^C and therefore in P. Furthermore, one of E_1^C and E_2^C is infinite. But this contradicts $(\aleph_0, \Delta_1^0) \not\rightarrow (\aleph_0, \Sigma_2^0)_2^2$. Hence the Ramsey tree of P does not necessarily contain an infinite chain from Σ_2^0, i.e., we proved

COROLLARY 2.10. *There exists a computable finitely branching successor tree without any infinite chain from Σ_2^0.*

2.3. Rubin's theorem. Rubin [15] considered Ramsey's theorem in the context of automatic structures [9, 3]. While a computable structure is given by a tuple of Turing machines, an automatic structure is given by a tuple of finite automata. We therefore first sketch some basic notions from the theory of finite automata.

2.3.1. *Languages of finite words.* Let Γ be some finite alphabet. The set of all (finite) words over Γ is denoted Γ^*, the empty word is ε, and Γ^+ is the set of non-empty words. By $\le_{\mathrm{pref}}$, we denote the prefix order on Γ^*.

A *finite automaton* M is a tuple $M = (Q, \Gamma, \delta, I, F)$ where Q is a finite set of states, $I \subseteq Q$ is the set of initial states, $F \subseteq Q$ is the set of final states, and $\delta \subseteq Q \times \Gamma \times Q$ is the transition relation. A *run* of M on a word $x = a_1 a_2 \cdots a_n$ with $n \ge 0$ and $a_i \in \Gamma$ is a word $q_0 q_1 \ldots q_n \in Q^+$ with $q_i \in Q$ such that $(q_i, a_{i+1}, q_{i+1}) \in \delta$ for all $0 \le i < n$. The run is *successful* if $q_0 \in I$ and $q_n \in F$. The language $L(M) \subseteq \Gamma^*$ defined by M is the set of all words that admit a successful run. A language $L \subseteq \Gamma^*$ is *regular* if there exists a finite automaton M with $L(M) = L$. By REG, we denote the class of all regular languages.

For words $x_i = a_i^0 a_i^1 a_i^2 \ldots a_i^{k_i} \in \Gamma^*$, the *convolution* $(x_1, x_2, \ldots, x_n)^\otimes \in ((\Gamma \uplus \{\#\})^n)^*$ is defined by

$$(x_1, \ldots, x_n)^\otimes = (b_1^0, \ldots, b_n^0)(b_1^1, \ldots, b_n^1)(b_1^2, \ldots, b_n^2) \cdots (b_1^k, \ldots, b_n^k)$$

where $k = \max\{k_1, \ldots, k_n\}$ and

$$b_i^\ell = \begin{cases} a_i^\ell & \text{if } \ell \le k_i \\ \# & \text{otherwise.} \end{cases}$$

In other words, the convolution is obtained by first adding some occurrences of the new letter $\#$ to each word to make them the same length and then "glueing" the prolonged words together. An n-ary relation $R \subseteq (\Gamma^*)^n$ is called *automatic* if the language $\{(x_1, \ldots, x_n)^\otimes \mid (x_1, \ldots, x_n) \in R\}$ is regular. In order to also capture

partition classes of a (k, ℓ)-partition, we will call a set $E \subseteq [\Gamma^*]^k$ automatic if the relation

$$\{(u_1, \ldots, u_n) \mid \{u_1, \ldots, u_n\} \in E\}$$

is automatic, i.e., if the language

$$\{(u_1, \ldots, u_n)^\otimes \mid \{u_1, \ldots, u_n\} \in E\}$$

is regular. We write A for the class of automatic subsets of $[\Gamma^*]^k$ for any $k \in \mathbb{N}$ and alphabet Γ.

2.3.2. *Ramsey's theorem for automatic partitions.* Note that any partition class from A is decidable. Hence, any (k, ℓ)-partition with automatic partition classes is computable and therefore has an infinite homogeneous set in the arithmetical hierarchy. Rubin showed that in this case, much simpler infinite homogeneous sets can be found. Namely, he proved

PROPOSITION 2.11 (Rubin [**15**]). *Let $(V, E_1, \ldots, E_\ell)$ be some (k, ℓ)-partition with $E_i \in \mathsf{A}$ for all $1 \leq i \leq \ell$ that has some infinite E_1-homogeneous set. Then there is some infinite regular E_1-homogeneous set.*

Since every infinite partition has, by Ramsey's theorem, some homogeneous set, we get immediately:

THEOREM 2.12 (Rubin [**15**, Prop. 3.21]). *For all $k, \ell \geq 1$, we have*

$$(\aleph_0, \mathsf{A}) \to (\aleph_0, \mathsf{A})_\ell^k .$$

Since $L \subseteq \Gamma^*$ is automatic if and only if it is regular, we can alternatively express this as

$$(\aleph_0, \mathsf{A}) \to (\aleph_0, \mathrm{REG})_\ell^k .$$

The proof of Prop. 2.11 does not use the Ramsey tree of the partition for two reasons. First, the proof is simpler since it constructs the regular homogeneous set directly. Secondly, it is not clear whether the Ramsey tree of an automatic partition is automatic, i.e., can be described by finite automata. Hence, we do not know how to perform the induction in the automatic setting.

The direct construction of the regular homogeneous set in Prop. 2.11 uses the concept of a comb: a *comb* is a set of words $C = \{s_0 s_1 s_2 \ldots s_{i-1} t_i \mid 0 \leq i < \alpha\}$ for some $\alpha \leq \omega$ where $s_i, t_i \in \Gamma^+$ and $|t_i| = |s_i|$ for all $0 \leq i < \alpha$ (note that, given C, these conditions determine s_i and t_i completely). A comb C is visualized in Fig. 2: it consists of the paths from the empty node to the filled nodes since these paths are labeled t_0, $s_0 t_1$, $s_0 s_1 t_2$, $\ldots$, $s_0 s_1 s_2 s_3 t_4$.

Given a (k, ℓ)-partition $P = (V, E_1, \ldots, E_\ell)$ with E_i automatic, our first lemma ensures that any infinite homogeneous set contains some infinite comb:

LEMMA 2.13. *Let $X \subseteq \Gamma^*$ be infinite. Then there exists an infinite comb $C \subseteq X$.*

PROOF. Let $t_0 \in X \setminus \{\varepsilon\}$ be arbitrary. Now suppose we defined $s_0, \ldots, s_{j-1}$ and $t_0, \ldots, t_j$ such that

(1) $s_0 s_1 \ldots s_{i-1} t_i \in X$ for all $0 \leq i \leq j$
(2) $|s_i| = |t_i|$ for all $0 \leq i < j$
(3) $X \cap s_0 s_1 \ldots s_{j-1} \Gamma^+$ is infinite.

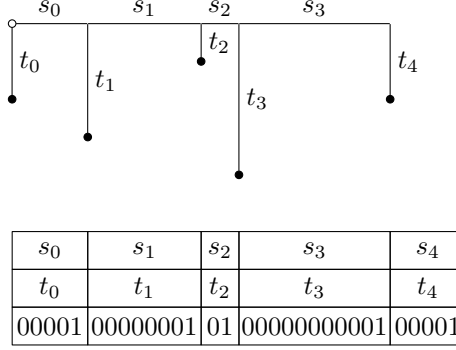| s_0	s_1	s_2	s_3	s_4
t_0	t_1	t_2	t_3	t_4
00001	00000001	01	00000000001	00001

FIGURE 2. A comb and its coding

Since there are only finitely many words of length $|t_j|$, (3) implies the existence of a word $s_j \in \Gamma^+$ with $|s_j| = |t_j|$ such that $X \cap s_0 s_1 \dots s_j \Gamma^+$ is infinite ensuring (2) and (3) for $j+1$. Choose $t_{j+1} \in \Gamma^+$ with $s_0 s_1 \dots s_j t_{j+1} \in X$ arbitrary. Since this ensures (1), we can proceed by induction. Then the set of words $s_0 s_1 \dots s_{i-1} t_i$ is an infinite comb $C \subseteq X$.   □

Words from $(\Sigma^2 \times \{0,1\})^*$ encode finite combs as follows:

DEFINITION 2.14. Let $n_0, n_1, \dots, n_\alpha \in \mathbb{N}$ and $w = 0^{n_0} 1 0^{n_1} 1 \dots 0^{n_{\alpha-1}} 1 0^{n_\alpha}$. Furthermore, let $s_i, t_i \in \Sigma^*$ with $|s_i| = |t_i| = n_i + 1$ for $0 \leq i < \alpha$ and $|s_\alpha| = |t_\alpha| = n_\alpha$. The comb encoded by $(s_0 s_1 \dots s_\alpha, t_0 t_1 \dots t_\alpha, w)^\otimes$ is the set of words $\{s_0 s_1 \dots s_{i-1} t_i \mid 0 \leq i \leq \alpha\}$.

Note that any word from $(\Sigma^2 \times \{0,1\})^*$ determines α, n_i, s_i, and t_i and therefore encodes a unique comb. Conversely, every finite comb can be encoded by some worf from $(\Sigma^2 \times \{0,1\})^*$ (which is not unique since s_α is arbitrary).

LEMMA 2.15. *Let $E \subseteq [\Sigma^*]^k$ be automatic. Then the set of encodings of finite E-homogeneous combs is regular.*

PROOF. Since the class of regular languages is closed under complementation, it suffices to construct a finite automaton that accepts a word from $(\Sigma^2 \times \{0,1\})^*$ if and only if it does *not* encode an E-homogeneous comb.

So let $M = (Q, (\Sigma \cup \{\#\})^k, \delta, I, F)$ be a finite automaton that accepts the set $E^\otimes = \{(u_1, \dots, u_k)^\otimes \mid \{u_1, \dots, u_k\} \in E\}$. We can assume that M is deterministic and complete, i.e., $|I| = 1$ and for any $p \in Q$ and $a \in (\Sigma \cup \{\#\})^k$, there exists a unique state $q \in Q$ with $(p, a, q) \in \delta$. We build a finite automaton M' as follows:

- states of M' are pairs (q, f) with $q \in Q$, $f : \{1, \dots, k\} \to \{1, 2, 3\}$, and $|f^{-1}(2)| \leq 1$,
- a state (q, f) is initial if and only if $q \in I$ and $f(i) \leq 2$ for all $1 \leq i \leq k$,
- a state (q, f) is accepting (i.e., belongs to F') if and only if $q \notin F$ and $f(i) \geq 2$ for all $1 \leq i \leq k$, and
- a triple $((q, f), (a, b, x), (q', f')) \in Q' \times (\Sigma^2 \times \{0,1\}) \times Q'$ is a transition from δ' if and only if
    - $f(i) \leq f'(i) \leq f(i) + x$ for all $1 \leq i \leq k$ and

$-$ $(q, (c_i)_{1 \leq i \leq k}, q') \in \delta$ where

$$c_i = \begin{cases} a & \text{if } f(i) = 1 \\ b & \text{if } f(i) = 2 \\ \# & \text{otherwise.} \end{cases}$$

Let $(s, t, w)^{\otimes} \in (\Sigma^* \times \{0, 1\})^*$. We use α, n_i, s_i, and t_i as in Definition 2.14. Then the automaton M' has a run from some initial state to (q, f) labeled $(s, t, w)^{\otimes}$ if and only if there are pairwise distinct numbers $0 \leq j_i < \alpha$ for $f(i) > 1$ and a run of M from $\iota \in I$ to q labeled $(u_1, \ldots, u_k)^{\otimes}$ where

(1) $u_i = \begin{cases} s & \text{if } f(i) = 1 \\ s_0 s_1 \ldots s_{j_i} t_{j_i+1} & \text{if } f(i) > 1, \end{cases}$

(2) $j_i = \alpha - 1$ implies $f(i) = 2$ or $n_\alpha = 0$, and

(3) $f(i) = 2$ implies $j_i = \alpha - 1$.

Now suppose $(s, t, w)^{\otimes} \in (\Sigma^2 \times \{0, 1\})^*$. Then $(s, t, w)^{\otimes}$ is accepted by M' if and only if there is some $q \in Q \setminus F$ such that M' has a run from some initial state to (q, f) labeled $(s, t, w)^{\otimes}$ where $f(i) = 3$ for all i. This is, by the above observation, equivalent to the existence of pairwise distinct $0 \leq j_i < \alpha$ for $1 \leq i \leq k$ (where $j_i = \alpha - 1$ is only allowed if $n_\alpha = 0$) and a run of M from $\iota \in I$ to q labeled $(u_1, \ldots, u_k)^{\otimes}$ where $u_i = s_0 s_1 \ldots s_{j_i} t_{j_i+1}$ for $1 \leq i \leq k$. Since M is deterministic and complete, such a run exists if and only if $\{u_1, \ldots, u_k\} \notin E$. Hence, indeed, the finite automaton M' accepts a word $(s, t, w)^{\otimes}$ from $(\Sigma^2 \times \{0, 1\})^*$ if and only if it encodes a comb that is not E-homogeneous. $\square$

PROOF OF PROP. 2.11. Let $(V, E_1, \ldots, E_\ell)$ be an infinite (k, ℓ)-partition with E_i automatic for all $1 \leq i \leq \ell$ and let $X \subseteq V$ by E_1-homogeneous. By Lemma 2.13, there exists an infinite comb $C \subseteq X$ (that is also E_1-homogeneous). Let $C = \{s_0 s_1 s_2 \ldots s_{i-1} t_i \mid 0 \leq i < \omega\}$ with $s_i, t_i \in \Gamma^+$ and $|t_i| = |s_i|$ for all $0 \leq i < \omega$. For $0 < \alpha < \omega$, let W_α encode the finite comb $\{s_0 s_1 s_2 \ldots s_{i-1} t_i \mid 0 \leq i < \alpha\}$. We can assume $W_1 <_{\text{pref}} W_2 <_{\text{pref}} W_3 \ldots$.

Let L consist of all words from $(\Sigma^2 \times \{0, 1\})^* (\Sigma^2 \times \{1\})$ that encode some E_1-homogeneous comb. Since L is regular by Lemma 2.15, it can be accepted by a deterministic and complete finite automaton. Hence there are $m < n$ and a word W with $W_n = W_m W$ such that $W_m W^* \subseteq L$ (we write V for W_m such that $V W^* \subseteq L$).

There are words $s_V, t_V, s_W, t_W \in \Sigma^+$ and $w_V, w_W \in \{0, 1\}^+$ such that $V = (s_V, t_V, w_V)^{\otimes}$ and $W = (s_W, t_W, w_W)^{\otimes}$. Let n be such that $0^n 1$ is a prefix of w_W and let t be the prefix of t_W of length $n + 1$.

Since $V \in L$, the word w_V ends with 1. For $1 \leq i \leq k$, let $u_i = s_V s_W^{n_i} t$ for pairwise distinct numbers n_i and let $\alpha < \omega$ be some upper bound for the numbers n_i. Then all the words u_i belong to the comb encoded by $V W^\alpha \in L$. Since this comb is E_1-homogeneous, we get $[s_V s_W^+ t]^k \subseteq E_1$. Hence, indeed, there is some infinite regular homogeneous set. $\square$

The concept of a comb was first used by Khoussainov et al. in [10] where they show that every automatic order tree with at least one infinite branch has a regular infinite branch. Büchi exploited the relation between automata and logic and showed, in this context, that the first-order theory of every automatic structure is decidable [9]. Rubin [15] proved, using combs again, that this holds even if first-order logic is extended by Ramsey quantifiers. This result was then extended by

Kuske & Lohrey [**12**] to a weak form of second-order quantification, namely to quantifications of the form

$$\exists R \text{ infinite} : \varphi$$

where the relational variable R occurs only negatively in φ. This decidability implies that certain graph problems that are Σ_1^1-complete for computable graphs are decidable for automatic graphs [**12**].

The original proofs of all these results used infinite combs and therefore Büchi-automata that accept infinite words. Differently, the above proof works in the realm of finite automata and finite words.

3. The uncountable case

We now deal with the existence of uncountable homogeneous sets. The first observation in this direction is due to Sierpiński who showed that the direct analogue for Ramsey's theorem $\aleph_0 \to (\aleph_0)_\ell^k$ does not hold:

THEOREM 3.1 (Sierpiński [**17**]). *If $k, \ell \geq 2$, then $2^{\aleph_0} \not\to (\aleph_1)_\ell^k$ and therefore in particular $2^{\aleph_0} \not\to (2^{\aleph_0})_\ell^k$.*

PROOF. We first sketch the proof of $2^{\aleph_0} \not\to (\aleph_1)_2^2$: Let $\sqsubseteq$ be some well-order on $\mathbb{R}$ and let $E_1 = \{\{x, y\} \in [\mathbb{R}]^2 \mid x \leq y \iff x \sqsubseteq y\}$ and $E_2 = [\mathbb{R}]^2 \setminus E_1$. If $H \subseteq \mathbb{R}$ is E_1-homogeneous, then $(H, \sqsubseteq) = (H, \leq) \hookrightarrow (\mathbb{R}, \leq)$, i.e., the well-order $(H, \sqsubseteq)$ embeds into the linear order $(\mathbb{R}, \leq)$. Hence H is countable. If H is E_2-homogeneous, then $(H, \sqsubseteq) = (H, \geq) \hookrightarrow (\mathbb{R}, \geq) \cong (\mathbb{R}, \leq)$, so again H is countable. Hence the $(2, 2)$-partition $(\mathbb{R}, E_1, E_2)$ does not contain any uncountable homogeneous set.

To also show $2^{\aleph_0} \not\to (\aleph_1)_\ell^k$ for $k > 2$, define $A \in E_i'$ if and only if $\{x_1, x_2\} \in E_i$ where $x_1 < x_2$ are the two minimal elements of A. Then $(\mathbb{R}, E_1', E_2', \emptyset, \ldots, \emptyset)$ is a (k, ℓ)-partition of size $2^{\aleph_0}$ without uncountable homogeneous sets. $\square$

3.1. Erdős & Rado's theorem. This theorem is concerned with homogeneous sets of size $\aleph_1$. We will assume the generalized continuum hypothesis $2^{\aleph_k} = \aleph_{k+1}$ for all $k \in \omega$.

THEOREM 3.2 (Erdős & Rado [**6**]). *For all $k, \ell \geq 1$, we have*

$$\aleph_k \to (\aleph_1)_\ell^k.$$

PROOF. The proof is, as the proof of Ramsey's theorem, by induction on k where the base case $k = 1$ is trivial.

So let $P = (V, E_1, \ldots, E_\ell)$ be some $(k+1, \ell)$-partition of size $\aleph_{k+1}$ and consider the Ramsey tree of P with respect to some well-order on V. Recall that every node at level $\alpha < \omega_k$ has at most $2^{\aleph_{k-1}} = \aleph_k$ brothers. Hence, by induction, there are at most $\aleph_k$ nodes at level α. In total, there are only $\aleph_k$ nodes on levels $< \omega_k$. Since the size of the Ramsey tree equals that of the $(k+1, \ell)$-partition, there is at least one node at level ω_k. So the Ramsey tree contains a chain Y of size at least $\aleph_k$. Let P^Y be the (k, ℓ)-partition induced by Y. Since, by induction, we have $\aleph_k \to (\aleph_1)_\ell^k$, the (k, ℓ)-partition P^Y contains some homogeneous set of size $\aleph_1$. But this set is also homogeneous in P by Prop. 1.4. $\square$

3.2. Ramsey's theorem for ω-automatic partitions. We next want to find an effective analogue of the theorem of Erdős and Rado. For this, we use the framework of ω-automatic structures as defined formally in [**2, 3**].

3.2.1. *Languages of infinite words.* An ω-*word* over the alphabet Γ is an infinite ω-sequence $x = a_0 a_1 a_2 \cdots$ with $a_i \in \Gamma$. The set of all ω-words over Γ is denoted by Γ^ω. For a set $V \subseteq \Gamma^+$ of finite words let $V^\omega \subseteq \Gamma^\omega$ be the set of all ω-words of the form $v_0 v_1 v_2 \cdots$ with $v_i \in V$.

Two ω-words x and y are *eventually equal* (denoted $x \sim_e y$) if, from some position on, the two words coincide. For $\Sigma = \{0, 1\}$, the support $\mathrm{supp}(x) \subseteq \mathbb{N}$ is the set of positions of the letter 1 in the word $x \in \Sigma^\omega$.

A *run* of a finite automaton $M = (Q, \Gamma, \delta, I, F)$ on an ω-word $x = a_0 a_1 a_2 \cdots$ is an ω-word $r = p_0 p_1 p_2 \cdots$ over the set of states Q such that $(p_i, a_i, p_{i+1}) \in \delta$ for all $i \geq 0$. The run r is *successful* if $p_0 \in I$ and there exists a final state from F that occurs infinitely often in r. The ω-language $L^\omega(M) \subseteq \Gamma^\omega$ defined by M is the set of all ω-words that admit a successful run. Note that a finite automaton M accepts a language of finite words $L(M)$ and an ω-language $L^\omega(M)$. Whenever we are more interested in the latter, we will speak of M as a Büchi-automaton. An ω-language $L \subseteq \Gamma^\omega$ is *regular* if there exists a Büchi-automaton M with $L^\omega(M) = L$.

Alternatively, regular ω-languages can be represented algebraically. To this end, one defines ω-*semigroups* to be two-sorted algebras $S = (S_+, S_\omega; \cdot, *, \pi)$ where $\cdot : S_+ \times S_+ \to S_+$ and $* : S_+ \times S_\omega \to S_\omega$ are binary operations and $\pi : (S_+)^\omega \to S_\omega$ is an ω-ary operation such that the following hold:

- $(S_+, \cdot)$ is a semigroup,
- $s * (t * u) = (s \cdot t) * u$,
- $s_0 \cdot \pi((s_i)_{i \geq 1}) = \pi((s_i)_{i \geq 0})$,
- $\pi((s_i^1 \cdot s_i^2 \cdots s_i^{k_i})_{i \geq 0}) = \pi((t_j)_{j \geq 0})$ whenever

$$(t_j)_{j \geq 0} = (s_0^1, s_0^2, \ldots, s_0^{k_0}, s_1^1, \ldots, s_1^{k_1}, \ldots) .$$

The ω-semigroup S is *finite* if both, S_+ and S_ω are finite. The free ω-semigroup generated by Γ is

$$\Gamma^\infty = (\Gamma^+, \Gamma^\omega; \cdot, *, \pi)$$

where $u \cdot v$ and $u * x$ are the natural operations of prefixing a word by the finite word u, and $\pi((u_i)_{i \geq 0})$ is the ω-word $u_0 u_1 u_2 \ldots$. A homomorphism $h : \Gamma^\infty \to S$ of ω-semigroups maps finite words to elements of S_+ and ω-words to elements of S_ω and commutes with the operations $\cdot$, $*$, and π. The algebraic characterisation of regular ω-languages then reads as follows.

PROPOSITION 3.3. *An ω-language $L \subseteq \Gamma^\omega$ is regular if and only if there exists a finite ω-semigroup S and a homomorphism $\eta : \Gamma^\infty \to S$ such that $L = \eta^{-1}(\eta(L))$.*

Hence, every Büchi-automaton M is "equivalent" to a homomorphism into some finite ω-semigroup together with a distinguished set $\eta(L^\omega(M))$ (and vice versa). Even more, this translation is effective – see [13] for this and more results on regular ω-languages. Note that the equation $L = \eta^{-1}(\eta(L))$ can also be read as

$$\eta(x) = \eta(y) \implies (x \in L \iff y \in L)$$

for all ω-words x and y.

For ω-words $x_i = a_i^0 a_i^1 a_i^2 \cdots \in \Gamma^\omega$, the *convolution* $x_1 \otimes x_2 \otimes \cdots \otimes x_n \in (\Gamma^n)^\omega$ is defined by

$$(x_1, \ldots, x_n)^\otimes = (a_1^0, \ldots, x_n^0)(a_1^1, \ldots, a_n^1)(a_1^2, \ldots, a_n^2) \cdots .$$

An n-ary relation $R \subseteq (\Gamma^\omega)^n$ is called ω-*automatic* if the ω-language $\{(x_1, \ldots, x_n)^\otimes \mid (x_1, \ldots, x_n) \in R\}$ is regular, a partition class $E \subseteq [\Gamma^\omega]^k$ is ω-*automatic* if the relation $\{(x_1, \ldots, x_k) \mid \{x_1, \ldots, x_k\} \in E\}$ is.

To describe the complexity of ω-languages, we will use language-theoretic terms. Let ωLANG denote the class of all ω-languages (i.e., sets of ω-words over some finite set of symbols). By ωREG, we denote the regular ω-languages. A language is *context-free* if it can be accepted by a pushdown-automaton. An ω-language is *eventually regular context-free* if it is a finite union of ω-languages UV with U context-free and $V \in \omega$REG. Let co-ωerCF denote the set of complements of eventually regular context-free ω-languages.

A final, rather peculiar class of ω-languages is $\mathbf{\Lambda}$: it is the class of ω-languages L such that $(\mathbb{R}, \leq)$ embeds into $(L, \leq_{\mathrm{lex}})$ (the name derives from the notation $\boldsymbol{\lambda}$ for the order type of $(\mathbb{R}, \leq)$).

An ω-*automatic presentation of a (k, ℓ)-partition* $(V, E_1, \ldots, E_\ell)$ is a pair (L, h) consisting of an ω-language L and a surjection $h : L \to V$ such that

$$E_i^\otimes = \{(x_1, x_2, \ldots, x_k) \in L^k \mid \{h(x_1), h(x_2), \ldots, h(x_k)\} \in E_i\} \text{ for } 1 \leq i \leq k \text{ and}$$
$$R_\approx = \{(x_1, x_2) \in L^2 \mid h(x_1) = h(x_2)\}$$

are ω-automatic. An ω-automatic presentation is *injective* if h is a bijection. A (k, ℓ)-partition is *(injectively) ω-automatic* if it has an (injective) ω-automatic presentation. From [1], it follows that an uncountable ω-automatic (k, ℓ)-partition has $2^{\aleph_0}$ elements.

Extending notions we used before, we write

$$(\kappa, \omega\mathsf{A}) \to (\lambda, \mathcal{C})_\ell^k$$

if the following partition property holds: for every ω-automatic presentation (L, h) of a (k, ℓ)-partition P of size κ, there exists $H \subseteq L$ in $\mathcal{C}$ such that $h(H)$ is homogeneous in P and of size λ;

$$(\kappa, \omega\mathsf{iA}) \to (\lambda, \mathcal{C})_\ell^k$$

is to be understood similarly where we only consider injective ω-automatic presentations.

REMARK 3.4. Let $P = (V, E_1, \ldots, E_\ell)$ be some (k, ℓ)-partition with ω-automatic presentation (L, h). Then the partition property above requires that there is a "large" homogeneous set $X \subseteq V$ and an ω-language $H \in \mathcal{C}$ such that $h(H) = X$, in particular, every element of X has at least one representative in H. Alternatively, one could require that $h^{-1}(X) \subseteq L$ is an ω-language from $\mathcal{C}$. We will only encounter classes $\mathcal{C}$ of ω-languages such that the following closure property holds: if $H \in \mathcal{C}$ and R is an ω-automatic relation, then also $R(H) = \{y \mid \exists x \in H : (x, y) \in R\} \in \mathcal{C}$. Since $h^{-1}h(H) = R_\approx(H)$, all our results also hold for this alternative requirement $h^{-1}(X) \in \mathcal{C}$.

3.2.2. *Ramsey's theorem for ω-automatic partitions.* We want to prove

$$(2^{\aleph_0}, \omega\mathsf{A}) \to (2^{\aleph_0}, \text{co-}\omega\mathrm{erCF} \cap \mathbf{\Lambda})_\ell^2$$

for all $\ell \geq 2$. Hence we will be concerned with edge-labeled graphs. The proof will not be done via the Ramsey tree. Instead, we will identify a common substructure of all ω-automatic edge-labeled graphs (see Example 3.5) and show that this substructure gives rise to homogeneous sets as required.

EXAMPLE 3.5. Let $V = \{0,1\}^\omega$ consist of all ω-words. Two distinct such words x and y are connected by an edge (i.e., $\{x,y\}$ belongs to E_1) if and only if $\mathrm{supp}(x)\setminus\mathrm{supp}(y)$ and $\mathrm{supp}(y)\setminus\mathrm{supp}(x)$ are both infinite; $E_2 = [V]^2\setminus E_1$ denotes the set of non-edges. We construct an instructive E_1-homogeneous set of size $2^{\aleph_0}$. For this, let $u_0, u_1, u_2, \ldots$ be the list of words from $\{0,1\}^*$ in length-lexicographic order (i.e., u comes before v if $|u| < |v|$ or $|u| = |v|$, $u = x0y$ and $v = x1z$). Intuitively, we list the nodes of the complete binary tree $(\{0,1\}^*, \leq_{\mathrm{pref}})$ level-wise where each level is listed from left to right. Let the ω-language N consist of all ω-words $x \in \{0,1\}^\omega$ such that

$$\{u_i \mid i \in \mathrm{supp}(x)\}$$

is an infinite branch in the complete binary tree $(\{0,1\}^*, \leq_{\mathrm{pref}})$. Then, clearly, $N \subseteq V$ has size $2^{\aleph_0}$ since its elements correspond to infinite branches. An ω-word $x \in V$ does not belong to N if and only if

- $0 \notin \mathrm{supp}(x)$ or
- $\exists i \in \mathrm{supp}(x) : |\{2i+1, 2i+2\} \cap \mathrm{supp}(x)| \neq 1$ or
- $\exists i \in \mathrm{supp}(x) \setminus \{0\} : \lfloor \frac{i-1}{2} \rfloor \notin \mathrm{supp}(x)$.

Since this condition relates positions i and $2i$, this is a typical context-free condition, hence N is the complement of an eventually regular context-free language. Finally, any two branches have a finite intersection, so $\mathrm{supp}(x) \cap \mathrm{supp}(y)$ is finite for any $x, y \in N$ distinct. Since $\mathrm{supp}(x)$ and $\mathrm{supp}(y)$ are infinite, this implies $[N]^2 \subseteq E_1$. Hence we found an E_1-homogeneous set of size $2^{\aleph_0}$ whose complement is eventually regular context-free. Let $x, y \in N$ be distinct with $X = \{u_i \mid i \in \mathrm{supp}(x)\}$ and $Y = \{u_i \mid i \in \mathrm{supp}(y)\}$. Then $x <_{\mathrm{lex}} y$ if and only if the branch X is right of the branch Y. Since $(\mathbb{R}, \leq)$ can be embedded into the set of branches (ordered naturally from right to left), there is also an embedding into $(N, \leq_{\mathrm{lex}})$. Hence $N \in \mathbf{\Lambda}$.

Proposition 3.7 below will express that this example is typical for ω-automatic $(2, \ell)$-partitions. But for its proof, we first need the following.

LEMMA 3.6. *Let $L \subseteq \Sigma^\omega$ be regular and $\approx$ an automatic equivalence relation on L with $|L/{\approx}| \geq \aleph_1$. Then there exist finite words $u_i, v_i \in \Sigma^+$ such that $u_i v_i^\omega \in L$ and $[u_i v_i^\omega]_{\sim_e} \cap [u_j v_j^\omega]_\approx = \emptyset$ for all $0 \leq i < j$.*

PROOF. Let $n \geq 0$ and suppose we have constructed pairs (u_i, v_i) with the desired properties for $0 \leq i < n$. We want to find (u_n, v_n). Suppose, for all $x \in L$, there exists $0 \leq i < n$ with $[u_i v_i^\omega]_{\sim_e} \cap [x]_\approx \neq \emptyset$. In other words, the countable set $\bigcup_{0 \leq i < n}[u_i v_i^\omega]_{\sim_e}$ intersects every equivalence class of $\approx$. But this contradicts $|L/\{\approx\}| > \aleph_0$. Hence there exists an element $x \in L$ with $[u_i v_i^\omega]_{\sim_e} \cap [x]_\approx = \emptyset$ for all $i < n$, i.e., the formula

$$\bigwedge_{0 \leq i < n} \neg \exists y : u_i v_i^\omega \sim_e y \approx x$$

is satisfiable in the structure $(L, (u_i v_i^\omega), \sim_e, \approx)$. Since the relations $\sim_e$, $\approx$, and $\{u_i v_i^\omega\}$ for $0 \leq i < n$ are automatic, also the set of ω-words x satisfying this formula is automatic and therefore a nonempty regular ω-language [**2**]. Hence it contains some ultimately periodic ω-word, i.e., there are $u_n, v_n \in \Sigma^+$ as desired. $\square$

Let u, v, w be some finite words. For $x \in \{0,1\}^\omega$, let $f_{v,w}(x)$ denote the ω-word obtained from x by replacing every occurrence of 0 by v and every occurrence of 1 by w. Furthermore, $H_{u,v,w} = uf_{v,w}(N)$ where N is the language from Example 3.5.

PROPOSITION 3.7. *Let $P = (L, E_0, E_1, \ldots, E_\ell)$ be some $(2, 1 + \ell)$-partition with injective ω-automatic presentation (L, id) such that $\{(x, y) \mid \{x, y\} \in E_0\} \cup \{(x, x) \mid x \in L\}$ is an equivalence relation (denoted $\approx$) on L with at least $\aleph_1$ equivalence classes. Then there exist nonempty words u, v, and w with v and w distinct, but of the same length, such that $H_{u,v,w}$ is E_i-homogeneous for some $1 \leq i \leq \ell$.*

This proposition is stated in [11] where I explain how to obtain it from a proof in [1]. Here, we give a self-contained proof that follows and simplifies the ideas from [1, 11].

PROOF. There are finite ω-semigroups S and T and homomorphisms

$$\varphi : \Sigma^\infty \to S \text{ and } \psi : (\Sigma \times \Sigma)^\infty \to T$$

such that $\varphi^{-1}\varphi(L) = L$ and

$$\psi(x \otimes y) = \psi(x' \otimes y') \implies (\{x, y\} \in E_i \iff \{x', y'\} \in E_i)$$

for all $0 \leq i \leq \ell$. Since the semigroups S_+ and T_+ are finite, there exists $\alpha \geq 1$ such that x^α is idempotent for all $x \in S_+ \cup T_+$.

From Lemma 3.6, we find finite words $u_i, v_i \in \Sigma^+$ such that $u_i v_i^\omega \in L$ and

$$[u_i v_i^\omega]_{\sim_e} \cap [u_j v_j^\omega]_\approx = \emptyset$$

for all $0 \leq i < j \leq |S_+| \cdot |T_+|$. Let m be the maximal length of a word u_i for $0 \leq i \leq |S_+| \cdot |T_+|$. By extending the word u_i by the prefix of v_i^ω of length $m - |u_i|$ (and rotating the word v_i accordingly), we can assume $|u_i| = |u_j|$ for all $0 \leq i < j \leq |S_+| \cdot |T_+|$ without changing the words $u_i v_i^\omega$. Furthermore, we can replace v_i by its

$$\left(\alpha \cdot \prod_{\substack{0 \leq j \leq |S_+| \cdot |T_+| \\ i \neq j}} |v_j| \right)^{th}$$

power, i.e., we can assume that $|v_i| = |v_j|$ and that all the elements

$$\varphi(v_i) \in S_+ \text{ and } \psi(v_i \otimes v_j) \in T_+$$

are idempotent.

Now there are $0 \leq i < j \leq |S_+| \cdot |T_+|$ with $\varphi(v_i) = \varphi(v_j)$ and $\psi(v_i \otimes v_i) = \psi(v_j \otimes v_j)$. We may assume $i = 0$ and $j = 1$ and write, for simplicity,

$$s = \varphi(v_0) = \varphi(v_1), \qquad\qquad t = \psi(v_0 \otimes v_0) = \psi(v_1 \otimes v_1),$$
$$t_{01} = \psi((v_0 v_0 v_0 \otimes v_0 v_1 v_0)^\alpha), \text{ and} \qquad t_{10} = \psi((v_0 v_1 v_0 \otimes v_0 v_0 v_0)^\alpha).$$

We will use the following equalities repeatedly:

$$t_{01} = t_{01} \cdot t = t \cdot t_{01} = t_{01}^2 \qquad t_{10} = t_{10} \cdot t = t \cdot t_{10} = t_{10}^2 \qquad t = t^2$$

Finally, for $X \subseteq \mathbb{N}$ we define a word W_X as follows:

$$W_X = u_1 v_1 \prod_{i \geq 0} \left\{ \begin{array}{ll} (v_0 v_0 v_0)^\alpha & \text{if } i \in X \\ (v_0 v_1 v_0)^\alpha & \text{if } i \notin X \end{array} \right\}$$

Note that, with $u = u_1 v_1$, $v = (v_0 v_1 v_0)^\alpha$, and $w = (v_0 v_0 v_0)^\alpha$, we get $W_X = u f_{v,w}(\chi_X)$ where $\chi_X \in \{0, 1\}^\omega$ is the characteristic word of X.

We will now prove three claims concerning the words W_X.

CLAIM 1. $W_X \in L$ for all $X \subseteq \mathbb{N}$.

PROOF OF CLAIM 1. We have $\varphi(W_X) = \varphi(u_1)\varphi(v_1)s^\omega = \varphi(u_1 v_1^\omega)$ since $W_X \in u_1 v_1 \{v_0, v_1\}^\omega$. Now $W_X \in L$ follows from $u_1 v_1^\omega \in L$.   q.e.d.

CLAIM 2. *Let $X, Y \subseteq \mathbb{N}$ such that $X \setminus Y$ and $Y \setminus X$ are both infinite. Then*

$$\psi(W_X \otimes W_Y) = \psi(u_1 \otimes u_1) \begin{cases} (t_{01} t_{10})^\omega & \text{if } \min(X \triangle Y) \in X \\ (t_{10} t_{01})^\omega & \text{if } \min(X \triangle Y) \in Y \end{cases}$$

PROOF OF CLAIM 2.

$$\psi(W_X \otimes W_Y) = \psi(u_1 v_1 \otimes u_1 v_1) \prod_{0 \leq i} \left\{ \begin{array}{ll} \psi(v_0 v_0 v_0 \otimes v_0 v_0 v_0)^\alpha & \text{if } i \in X \cap Y \\ \psi(v_0 v_0 v_0 \otimes v_0 v_1 v_0)^\alpha & \text{if } i \in X \setminus Y \\ \psi(v_0 v_1 v_0 \otimes v_0 v_0 v_0)^\alpha & \text{if } i \in Y \setminus X \\ \psi(v_0 v_1 v_0 \otimes v_0 v_1 v_0)^\alpha & \text{otherwise} \end{array} \right\}$$

$$= \psi(u_1 \otimes u_1)\, t \prod_{0 \leq i} \left\{ \begin{array}{ll} t_{01} & \text{if } i \in X \setminus Y \\ t_{10} & \text{if } i \in Y \setminus X \\ t^{3\alpha} = t & \text{otherwise} \end{array} \right\}$$

Now the claim follows since there are infinitely many factors t_{01} and t_{10} (since $X \setminus Y$ and $Y \setminus X$ are both infinite), since these two factors are idempotent, and since multiplication of t with t_{10} or t_{01} absorbs t.   q.e.d.

From Claim 2, we now derive that $H_{u,v,w}$ is homogeneous in the partition P: Let $A, B \in [H_{u,v,w}]^2$. Then there exist sets $X, Y, X', Y' \subseteq \mathbb{N}$ with $A = \{W_X, W_Y\}$ and $B = \{W_{X'}, W_{Y'}\}$. Furthermore, the differences $X \setminus Y$, $Y \setminus X$, $X' \setminus Y'$, and $Y' \setminus X'$ are all infinite. We assume $\min(X \triangle Y) \in X$ and $\min(X' \triangle Y') \in X'$. Then, by Claim 2, $\psi(W_X \otimes W_Y) = \psi(W_{X'} \otimes W_{Y'})$ proving $A \in E_i \iff B \in E_i$ for all $0 \leq i \leq \ell$. Hence $H_{u,v,w}$ is homogeneous. To show that it is not 0-homogeneous, we use the following claim.

CLAIM 3. $\psi(W_{2\mathbb{N}} \otimes W_{4\mathbb{N}+2}) = \psi(u_1 v_1 v_0^\omega \otimes u_1 (v_1 v_0)^\omega) = \psi(u_1 (v_1 v_0)^\omega \otimes u_1 v_1^\omega)$

PROOF OF CLAIM 3.

$$\psi(W_{2\mathbb{N}} \otimes W_{4\mathbb{N}+2}) = \psi(u_1 v_1 \otimes u_1 v_1)$$

$$\cdot \prod_{0 \leq i} \left\{ \begin{array}{ll} \psi(v_0 v_0 v_0 \otimes v_0 v_0 v_0)^\alpha & \text{if } i \in 4\mathbb{N}+2 \\ \psi(v_0 v_0 v_0 \otimes v_0 v_1 v_0)^\alpha & \text{if } i \in 4\mathbb{N} \\ \psi(v_0 v_1 v_0 \otimes v_0 v_1 v_0)^\alpha & \text{if } i \text{ odd} \end{array} \right\}$$

$$= \psi(u_1 \otimes u_1) t \prod_{0 \leq i} \left\{ \begin{array}{ll} t_{01} & \text{if } i \in 4\mathbb{N} \\ t^{3\alpha} & \text{otherwise} \end{array} \right\}$$

$$= \psi(u_1 \otimes u_1) t_{01}^\omega \text{ since } t t_{01} = t_{01}$$

$$\psi(u_1 v_1 v_0^\omega, u_1 (v_1 v_0)^\omega) = \psi(u_1 v_1 \otimes u_1 v_1) \psi(v_0 v_0 \otimes v_0 v_1)^\omega$$

$$= \psi(u_1 \otimes u_1) t (t \psi(v_0 \otimes v_1))^\omega$$

$$= \psi(u_1 \otimes u_1) (t \psi(v_0 \otimes v_1) t)^\omega \text{ since } t^2 = t$$

$$= \psi(u_1 \otimes u_1) t_{01}^\omega \text{ since } (t \psi(v_0 \otimes v_1) t)^\alpha = t_{01}$$

$$\psi(u_1 (v_1 v_0)^\omega \otimes u_1 v_1^\omega) = \psi(u_1 \otimes u_1)(t \psi(v_0 \otimes v_1))^\omega$$

$$= \psi(u_1 \otimes u_1)(t \psi(v_0 \otimes v_1) t)^\omega \text{ since } t^2 = t$$

$$= \psi(u_1 \otimes u_1) t_{01}^\omega \text{ since } (t \psi(v_0 \otimes v_1) t)^\alpha = t_{01}$$

<div align="right">q.e.d.</div>

Suppose $[H_{u,v,w}]^2 \subseteq E_0$. Let $A \in [H_{u,v,w}]^2$, i.e., $A = \{W_X, W_Y\}$ for some $X, Y \subseteq \mathbb{N}$ with $X \setminus Y$ and $Y \setminus X$ infinite. We assume $\min(X \triangle Y) \in X$. From Claim 2, we obtain

$$\psi(W_X \otimes W_Y) = \psi(W_{2\mathbb{N}} \otimes W_{2\mathbb{N}+1}) = \psi(W_{2\mathbb{N}+1} \otimes W_{4\mathbb{N}+2}).$$

Then $A \in E_0$ implies

$$\{W_{2\mathbb{N}}, W_{2\mathbb{N}+1}\}, \{W_{2\mathbb{N}+1}, W_{4\mathbb{N}+2}\} \in E_0,$$

in other words,

$$W_{2\mathbb{N}} \approx W_{2\mathbb{N}+1} \approx W_{4\mathbb{N}+2}.$$

Since $\approx$ is an equivalence relation, this implies $W_{2\mathbb{N}} \approx W_{4\mathbb{N}+2}$. Now from Claim 3, we obtain $u_1 v_1 v_0^\omega \approx u_1 (v_1 v_0)^\omega \approx u_1 v_1^\omega$. Since $u_1 v_1 v_0^\omega \sim_e u_0 v_0^\omega$, this contradicts our choice of the words u_i and v_i. $\qquad\square$

THEOREM 3.8 (Kuske [**11**]). *For all $\ell \geq 2$, we have*

$$(2^{\aleph_0}, \omega\mathsf{A}) \to (2^{\aleph_0}, \text{co-}\omega\text{erCF} \cap \mathbf{\Lambda})_\ell^2.$$

PROOF. Let $P' = (V, E_1', \ldots, E_\ell')$ be some $(2, \ell)$ partition with ω-automatic presentation (L, h). To apply Prop. 3.7, consider the following $(2, 1+\ell)$-partition $P = (L, E_0, \ldots, E_\ell)$:

- The underlying set is the ω-language L,
- E_0 consists of all sets $\{x, y\}$ with $h(x) = h(y)$ and $x \neq y$, and
- E_i (for $1 \leq i \leq \ell$) consists of all sets $\{x, y\}$ with $\{h(x), h(y)\} \in E_i'$.

Then (L, id) is an injective ω-automatic presentation of the $(2, 1+\ell)$-partition P. By Prop. 3.7, there exists $1 \leq i \leq \ell$ and words u, v and w such that $v \neq w$, $|v| = |w|$, and such that $H_{u,v,w}$ is E_i-homogeneous for some $1 \leq i \leq \ell$. Since $(E_0, \ldots, E_\ell)$ is a partition of $[L]^2$, we have $\{x, y\} \notin E_0$ (and therefore $h(x) \neq h(y)$) for all $x, y \in H_{u,v,w}$ distinct. Hence h is injective on $H_{u,v,w}$. Furthermore $[H_{u,v,w}]^2 \subseteq E_i$ implies $[h(H_{u,v,w})]^2 \subseteq E_i'$. Hence $h(H_{u,v,w})$ is an E_i'-homogeneous set of size $2^{\aleph_0}$. From $N \in \text{co-}\omega\text{erCF} \cap \mathbf{\Lambda}$, we get immediately $H_{u,v,w} \in \text{co-}\omega\text{erCF} \cap \mathbf{\Lambda}$. $\qquad\square$

From Theorem 3.8, we now derive a necessary condition for a partial order of size $2^{\aleph_0}$ to be ω-automatic. A partial order $(V, \sqsubseteq)$ is ω-automatic iff there exists a regular ω-language L and a surjection $h : L \to V$ such that the relations $R_= = \{(x, y) \in L^2 \mid h(x) = h(y)\}$ and $R_\sqsubseteq = \{(x, y) \in L^2 \mid h(x) \sqsubseteq h(y)\}$ are ω-automatic.

COROLLARY 3.9. *If $(V, \sqsubseteq)$ is an ω-automatic partial order with $|V| \geq \aleph_1$, then $(\mathbb{R}, \leq)$ or an antichain of size $2^{\aleph_0}$ embeds into $(V, \sqsubseteq)$.*

PROOF. Let $(V, \sqsubseteq)$ be a partial order, $L \subseteq \Sigma^\omega$ a regular ω-language and $h : L \to V$ a surjection such that $R_=$ and $R_\sqsubseteq$ are ω-automatic. Define an injective ω-automatic $(2, 4)$-partition $G = (L, E_0, E_1, E_2, E_3)$:

- E_0 comprises all pairs $\{x, y\} \in [L]^2$ with $h(x) = h(y)$,
- E_1 comprises all pairs $\{x, y\} \in [L]^2$ with $h(x) \sqsubset h(y)$ and $x <_{\text{lex}} y$,
- E_2 comprises all pairs $\{x, y\} \in [L]^2$ with $h(x) \sqsupset h(y)$ and $x <_{\text{lex}} y$, and
- $E_3 = [L]^2 \setminus (E_0 \cup E_1 \cup E_2)$ comprises all pairs $\{x, y\} \in [L]^2$ such that $h(x)$ and $h(y)$ are incomparable.

By Prop. 3.7, there exists $H \subseteq L$ 1-, 2- or 3-homogeneous with $(\mathbb{R}, \le) \hookrightarrow (H, \le_{\mathrm{lex}})$. Since $[H]^2 \subseteq E_1 \cup E_2 \cup E_3$ and since G is a partition of L, the mapping h acts injectively on H. If $[H]^2 \subseteq E_1$ (the case $[H]^2 \subseteq E_2$ is symmetrical) then $(\mathbb{R}, \le) \hookrightarrow (H, \le_{\mathrm{lex}}) \cong (h(H), \sqsubseteq)$. If $[H]^2 \subseteq E_3$, then $h(H)$ is an antichain of size $2^{\aleph_0}$. □

A linear order $(L, \sqsubseteq)$ is *scattered* if $(\mathbb{Q}, \le)$ cannot be embedded into $(L, \sqsubseteq)$. Automatic partial orders are defined similarly to ω-automatic partial orders with the help of finite automata instead of Büchi-automata.

COROLLARY 3.10. *Any scattered ω-automatic linear order $(V, \sqsubseteq)$ is countable. Hence,*

- *a scattered linear order is ω-automatic if and only if it is automatic, and*
- *an ordinal α is ω-automatic if and only if $\alpha < \omega^\omega$.*

PROOF. If $(V, \sqsubseteq)$ is not countable, then it embeds $(\mathbb{R}, \le)$ by the previous corollary and therefore in particular $(\mathbb{Q}, \le)$. The remaining two claims follow immediately from [**1**] ("countable ω-automatic structures are automatic") and [**4**] ("an ordinal is automatic iff it is properly smaller than ω^{ω^ω}"), resp. □

3.2.3. *Sierpiński's theorem for ω-automatic partitions.* Compare Theorem 3.8 to Sierpiński's theorem 3.1 and the theorem of Erdős and Rado: if an edge-labeled graph is ω-automatic, already $2^{\aleph_0}$ nodes guarantee the existence of a homogeneous set of size $2^{\aleph_0}$. Compared to Rubin's theorem 2.12, there are two shortcomings: We would like to extend the result to (k, ℓ)-partitions for $k > 2$, and we would like to find ω-regular homogeneous sets (and not only complements of context-free ones). The following two examples prove that this is not possible, i.e., that Theorem 3.8 is best possible.

EXAMPLE 3.11. Let $\Sigma = \{0, 1\}$, $L = \{0, 1\}^\omega$. For $H \subseteq L$, we write $\bigwedge H \in \Sigma^\infty$ for the longest common prefix of all ω-words in H, $\bigwedge\{x, y\}$ is also written $x \wedge y$. By $\le_{\mathrm{lex}}$, we denote the lexicographic order on the set Σ^ω (with some, implicitly assumed linear order on the letters from Σ). Then let E_1 consist of all 3-sets $\{x, y, z\} \in [L]^3$ with $x <_{\mathrm{lex}} y <_{\mathrm{lex}} z$ and $x \wedge y <_{\mathrm{pref}} y \wedge z$; E_2 is the complement of E_1. This finishes the construction of the $(3, 2)$-partition (L, E_1, E_2) of size $2^{\aleph_0}$ with injective ω-automatic presentation (L, id).

Note that 1^*0^ω is a countable E_1-homogeneous set and that 0^*1^ω is a countable E_2-homogeneous set. But there is no uncountable homogeneous set: First suppose $H \subseteq L$ is infinite and $x \wedge y <_{\mathrm{pref}} y \wedge z$ for all $x <_{\mathrm{lex}} y <_{\mathrm{lex}} z$ from H. Let $u \in \Sigma^*$ such that $H \cap u0\Sigma^\omega$ and $H \cap u1\Sigma^\omega$ are both nonempty and let $x, y \in H \cap u0\Sigma^\omega$ with $x \le_{\mathrm{lex}} y$ and $z \in H \cap u1\Sigma^\omega$. Then $x \wedge y >_{\mathrm{pref}} u = y \wedge z$ and therefore $x = y$ (for otherwise, we would have $x <_{\mathrm{lex}} y <_{\mathrm{lex}} z$ in H with $x \wedge y >_{\mathrm{pref}} y \wedge z$). Hence we showed $|H \cap u0\Sigma^\omega| = 1$. Let $u_0 = \bigwedge H$ and $H_1 = H \cap u_0 1\Sigma^\omega$. Since $H \cap u_0 0\Sigma^\omega$ is finite, the set H_1 is infinite. We proceed by induction: $u_n = \bigwedge H_n$ and $H_{n+1} = H_n \cap u_n 1\Sigma^\omega$ satisfying $|H_n \cap u_n 0\Sigma^\omega| = 1$. Then $u_0 <_{\mathrm{pref}} u_0 1 \le_{\mathrm{pref}} u_1 <_{\mathrm{pref}} u_1 1 \le_{\mathrm{pref}} u_2 \cdots$ with

$$H = \bigcup_{n \ge 0} (H \cap u_n 0\Sigma^\omega) \cup \bigcap_{n \ge 0} (H \cap u_n 1\Sigma^\omega).$$

Then any of the sets $H \cap u_n 0\Sigma^\omega = H_n \cap u_n 0\Sigma^\omega$ and $\bigcap(H \cap u_n 1\Sigma^\omega)$ is a singleton, proving that H is countable. Thus, there cannot be an uncountable E_1-homogeneous set.

So let $H \subseteq L$ be infinite with $x \wedge y \geq_{\mathrm{pref}} y \wedge z$ for all $x <_{\mathrm{lex}} y <_{\mathrm{lex}} z$. Since we have only two letters, we get $x \wedge y >_{\mathrm{pref}} y \wedge z$ for all $x <_{\mathrm{lex}} y <_{\mathrm{lex}} z$ which allows to argue symmetrically to the above. Thus, indeed, there is no uncountable homogeneous set in L. This proves $(2^{\aleph_0}, \omega\mathsf{iA}) \not\to (\aleph_1, \omega\mathrm{LANG})_2^3$; using the trick from the end of the proof of Sierpiński's theorem 3.1 (where $x_1 < x_2$ is replace by $x_1 <_{\mathrm{lex}} x_2$), we get $(2^{\aleph_0}, \omega\mathsf{iA}) \not\to (\aleph_1, \omega\mathrm{LANG})_\ell^k$ for all $k \geq 3$ and $\ell \geq 2$.

EXAMPLE 3.12. Let L denote the regular ω-language $(1^+0^+)^\omega$. Recall that the ω-words x and y are ultimately equal, briefly $x \sim_e y$, if there exist finite words u and v of the same length and an ω-word z with $x = uz$ and $y = vz$ (i.e., from some point on, the two ω-words coincide). Let $E_1 \subseteq [L]^2$ consists of all 2-sets $\{x, y\} \subseteq L$ such that $\mathrm{supp}(x) \cap \mathrm{supp}(y)$ is finite or $x \sim_e y$. The set E_2 is the complement of E_1 in $[L]^2$. This completes the construction of the $(2, 2)$-partition $G = (L, E_1, E_2)$. Note that (L, id_L) is an injective ω-automatic presentation of G.

Now let $n \in \mathbb{N}$ and $U_i, V_i \subseteq \{0, 1\}^+$ be languages, set

$$(3.1) \qquad\qquad H = \bigcup_{1 \leq i \leq n} U_i V_i^\omega \,,$$

and assume that H has size $2^{\aleph_0}$. We show that H is not homogeneous: Since H is infinite, there are $1 \leq i \leq n$ and $x, y \in U_i V_i^\omega$ distinct with $x \sim_e y$ and therefore $\{x, y\} \in E_1$.

Since $|H| > \aleph_0$, there is $1 \leq i \leq n$ with $|U_i V_i^\omega| > \aleph_0$; we set $U = U_i$ and $V = V_i$. Let $u \in U$ be arbitrary (such a word exists since $UV^\omega \neq \emptyset$). From $|U| \leq \aleph_0$, we obtain $|V^\omega| > \aleph_0$. Hence there are $v_1, v_2 \in V^+$ distinct with $|v_1| = |v_2|$. Since $uv_1^\omega \in H$ and each element of H contains infinitely many occurrences of 1, the word v_1 belongs to $\{0, 1\}^*10^*$. Consider the ω-words $x' = u(v_1v_2)^\omega$ and $y' = u(v_1v_1)^\omega$ from $UV^\omega \subseteq H$. Then $x' \not\sim_e y'$ since $v_1 \neq v_2$ and $|v_1| = |v_2|$. At the same time, $\mathrm{supp}(x') \cap \mathrm{supp}(y')$ is infinite since v_1 contains an occurrence of 1. Hence $\{x', y'\} \in E_2$.

Thus, we found ω-words $x, y, x', y' \in H$ with $\{x, y\} \in E_1$ and $\{x', y'\} \notin E_1$ proving that H is not homogeneous.

Since all context-free and regular ω-language are of the form (3.1) (see [18]), this proves $(2^{\aleph_0}, \omega\mathsf{iA}) \not\to (2^{\aleph_0}, \omega\mathrm{CF})_\ell^k$ and $(2^{\aleph_0}, \omega\mathsf{iA}) \not\to (2^{\aleph_0}, \omega\mathrm{REG})_\ell^k$ for all $k, \ell \geq 2$.

References

[1] V. Bárány, Ł. Kaiser, and S. Rubin, *Cardinality and counting quantifiers on omega-automatic structures*, STACS'08, IFIB Schloss Dagstuhl, 2008, pp. 385–396.

[2] A. Blumensath, *Automatic structures*, Tech. report, RWTH Aachen, 1999.

[3] A. Blumensath and E. Grädel, *Finite presentations of infinite structures: Automata and interpretations*, Theory of Computing Systems **37** (2004), no. 6, 641–674.

[4] Ch. Delhommé, *Automaticité des ordinaux et des graphes homogènes*, C. R. Acad. Sci. Paris, Ser. I **339** (2004), 5–10.

[5] P. Erdős and R. Rado, *A problem on ordered sets*, Journal of the LMS **28** (1953), 426–438.

[6] ———, *A partition calculus in set theory*, Bull. AMS **62** (1956), 427–489.

[7] C.G. Jockusch, *Ramsey's theorem and recursion theory*, Journal of Symbolic Logic **37** (1972), 268–280.

[8] C.G. Jockusch and R.I. Soare, Π_1^0-*classes an degrees of theories*, Trans. Am. Math. Soc. **173** (1972), 33–56.

[9] B. Khoussainov and A. Nerode, *Automatic presentations of structures*, Logic and Computational Complexity, Lecture Notes in Comp. Science vol. 960, Springer, 1995, pp. 367–392.

[10] B. Khoussainov, S. Rubin, and F. Stephan, *On automatic partial orders*, LICS'03, IEEE Computer Society Press, 2003, pp. 168–177.

[11] D. Kuske, *Is Ramsey's theorem ω-automatic?*, STACS 2010, Leibniz International Proceedings in Informatics (LIPIcs) vol. 5, Schloss Dagstuhl–Leibniz-Zentrum für Informatik, 2010, pp. 537–548.

[12] D. Kuske and M. Lohrey, *Some natural problems in automatic graphs*, Journal of Symbolic Logic **75** (2010), no. 2, 678–710.

[13] D. Perrin and J.-E. Pin, *Infinite words*, Pure and Applied Mathematics vol. 141, Elsevier, 2004.

[14] F.P. Ramsey, *On a problem of formal logic*, Proc. London Math. Soc. **30** (1930), 264–286.

[15] S. Rubin, *Automata presenting structures: A survey of the finite string case*, Bulletin of Symbolic Logic **14** (2008), 169–209.

[16] J.R. Shoenfield, *On degrees of unsolvability*, Annals of Mathematics **69** (1959), no. 3, 644–653.

[17] W. Sierpiński, *Sur un problème de la thèorie des relations*, Ann. Scuola Norm. Sup. Pisa **2** (1933), no. 2, 285–287.

[18] L. Staiger, *ω-languages*, Handbook of Formal Languages Vol. 3 (G. Rozenberg and A. Salomaa, eds.), Springer, 1997, pp. 339–387.

Institut für Theoretische Informatik, TU Ilmenau, PF 100565, D-98684 Ilmenau, Germany

E-mail address: `dietrich.kuske@tu-ilmenau.de`