Extended Midy's Theorem

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Theorem (Original)

If the period of a repeating decimal for a/p, where p is prime and a/p is a reduced fraction, has an even number of digits, then dividing the repeating portion into halves and adding gives a string of 9s. For example, $1/7 = 0.\overline{142857}$, and $142 + 857 = 999$.

Theorem (Extended)

If the period of a repeating decimal for a/p, where p is prime and a/p is a reduced fraction, is $h = m \times n$, then dividing the repeating portion into n parts and adding gives $c_n(a,p) \times 9$ m's, where c is a constant depending on p, a and n.

Proof

We will show some interesting properties for various types of primes p.

Let a/p have period $h > 1$ with $a < p$, and assume $a/p = 0.\overline{a_1 \dots a_{m\times n}}$. Thus, $p \neq 2, 5$.

Then, $10^i \times a/p - \lfloor 10^i \times a/p \rfloor = 0.\overline{a_{i+1} \dots a_{m\times n} a_1 \dots a_i} < 1$, since $a < p$.

Thus, $p \times (10^i \times a/p - \lfloor 10^i \times a/p \rfloor) = p \times 0.\overline{a_{i+1} \dots a_{m\times n} a_1 \dots a_i} < p \Rightarrow 10^i \times a - \lfloor 10^i \times a/p \rfloor \times p = a \times 10^i \text{ (mod p)}, \forall a, i \ni 0 \leq i \leq m \times n - 1, 1 \leq a \leq p - 1$.

And, $\{a \times 10^i - \lfloor 10^i \times a/p \rfloor \times p, \forall i \ni 0 \leq i \leq m \times n - 1\} = \{a, a \times 10, a \times$

$10^2, \ldots, a \times 10^{m \times n-1}\}$ (mod p) $\subseteq \{1, \ldots, p-1\}$.

And hence, there exists $\frac{p-1}{h}$ distinct cyclic generators $(\{e = e_1, e_2, \ldots, e_{\frac{p-1}{h}}\})$ such that $\bigcup_{i=1}^{\frac{p-1}{h}} \{e_i, e_i^2, \ldots, e_i^{m \times n-1}\} = \{1, \ldots, p-1\}$.

Let

$$\begin{aligned} S_n &= \sum_{i=0}^{n-1} 0.\overline{a_{m\times i+1} \ldots a_{m\times n} a_1 \ldots a_{m\times i}} \\ &= \sum_{i=0}^{n-1} 0.\overline{a_{m\times i+1} \ldots a_{m\times (i+1)}} \\ &= \sum_{i=0}^{n-1} a_{m\times i+1} \ldots a_{m\times (i+1)} / (10^m - 1) \end{aligned}$$

Example:

$0.\overline{142857} + 0.\overline{571428} + 0.\overline{285714} = 0.\overline{141414} + 0.\overline{282828} + 0.\overline{575757} = 14/99 + 28/99 + 57/99.$

Thus,

$$\begin{aligned} p \times S_n &= p \times \sum_{i=0}^{n-1} 0.\overline{a_{m\times i+1} \ldots a_{m\times n} a_1 \ldots a_{m\times i}} \\ &= \sum_{i=0}^{n-1} ((a \times 10^{m\times i})(mod p)) \\ &= p \times \sum_{i=0}^{n-1} a_{m\times i+1} \ldots a_{m\times (i+1)} / (10^m - 1) \end{aligned}$$

Thus, $c_n(a, p) = \sum_{i=0}^{n-1} a_{m\times i+1} \ldots a_{m\times (i+1)} = \frac{10^m - 1}{p} \times \sum_{i=0}^{n-1} ((a \times 10^{m\times i})$ (mod p)).

Example:

$1/17 = 0.\overline{0588235294117647}$, and

	m	n	c_n
$0+5+8+8+2+3+5+2+9+4+1+1+7+6+4+7 = 72 \equiv 0 \pmod 9$	1	16	8
$05+88+23+52+94+11+76+47 = 396 \equiv 0 \pmod{99}$	2	8	4
$0588+2352+9411+7647 = 19998 \equiv 0 \pmod{9999}$	4	4	2
$05882352+94117647 = 99999999 \equiv 0 \pmod{99999999}$	8	2	1

$1/19 = 0.\overline{052631578947368421}$, and

	m	n	c_n
$0+5+2+6+3+1+5+7+8+9+4+7+3+6+8+4+2+1 = 81 \equiv 0 \pmod 9$	1	18	9
$05+26+31+57+89+47+36+84+21 = 396 \equiv 0 \pmod{99}$	2	9	4
$052+631+578+947+368+421 = 2997 \equiv 0 \pmod{999}$	3	6	3
$052631+578947+368421 = 999999 \equiv 0 \pmod{999999}$	6	3	1
$052631578 + 947368421 = 999999999 \equiv 0 \pmod{999999999}$	9	2	1

$1/757 = 0.\overline{001321003963011889035667107}$, and

	m	n	c_n
$0+0+1+3+2+1+0+0+3+9+6+3+0+1+1+8+8+9+0+3+5+6+6+7+1+0+7 = 90 \equiv 0 \pmod 9$	1	27	10
$001+321+003+963+011+889+035+667+107 = 2997 \equiv 0 \pmod{99}$	3	9	3
$001321003+963011889+035667107 = 999999999 \equiv 0 \pmod{999}$	9	3	1

Trivia:

The decimal expansion of $1/19$ is equal to the sum of the powers of 2 in reverse.

$1/19 = 0.\overline{052631578947368421}$

Compare it to $1/49$.

Summary

Assuming $n \mid h$, where $10^h \equiv 1$ (mod p) and $1 \leq a \leq p-1$.

If $p = 3$, then $c_n(a, 3) = a \times n/3$.

If $n = 1$, then $c_1(a, p) = 1/p$.

If $n = 2$, then $c_n(a, p) \times p < (p-1) + (p-2) = 2 \times p - 3 < 2 \times p$. Thus, $c_2(a, p) = 1$ (Midy's Theorem).

If $n = 3$ and a $= 1$, then $c_n(1, p) \times p < 1 + (p-2) + (p-3) = 2 \times p - 2 < 2 \times p$. Thus, $c_3(1, p) = c_3(2, p) = c_3(3, p) = c_3(4, p) = 1$. However, these may not be unique.

If $n = p - 1$, then $c_n(a, p) \times p = \sum_{i=1}^{p-1} i$. Thus, $c_{p-1}(a, p) = \frac{p-1}{2}$ (Full Reptend Prime).

If we replaced p with any number $b \neq \{2, 5\}$, then adding the n equal partitions of the repeating decimal of a/b is also equal to $c_n(a, b) \times 9$ m's, where $c_n(a, b)$ is a rational number.

Open Questions

1. $c_n(1, p) = \left\lfloor \frac{c_h(1,p)}{h/n} \right\rfloor$, $\forall\ n \mid h$.
   When p is a full reptend prime, $c_n(a, p) = c_n(1, p)$, $\forall\ a \neq 1$.

2. If $2 \mid h$, then $h/2 \leq c_h(a, p) \leq C$, where $C = \{\sup x \ni \left\lfloor \frac{x}{h/2} \right\rfloor = 1\}$.

If $3 \mid h$, then $h/3 \leq c_h(a,p) \leq C$, where $C = \{\sup x \ni \left\lfloor \frac{x}{h/3} \right\rfloor = 1\}$.
If $6 \mid h$, then $h/2 \leq c_h(a,p) \leq C$, where $C = \{\sup x \ni \left\lfloor \frac{x}{h/3} \right\rfloor = 1\}$.
Examples: $9 \leq c_{27}(a,757) \leq 17$, $9 \leq c_{18}(a,19) \leq 11$ and $3 \leq c_6(a,7) \leq 3$.

3. $c_n(1,p) \leq c_n(a,p)$, $\forall\ 1 < a < p$.
   This is equivalent to $\sum_{i=0}^{n-1}(10^{m\times i} \pmod{\text{p}}) \leq \sum_{i=0}^{n-1}((a \times 10^{m\times i}) \pmod{\text{p}})$.

References

1. Eric W. Weisstein. "Midy's Theorem." From *MathWorld*–A Wolfram Web Resource. http://mathworld.wolfram.com/MidysTheorem.html

2. Eric W. Weisstein. "Full Reptend Prime." From *MathWorld*–A Wolfram Web Resource. http://mathworld.wolfram.com/FullReptendPrime.html