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The Classical Decision Problem

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Preface

This book is addressed to all those – logicians, computer scientists, mathematicians, philosophers of science as well as the students in all these disciplines – who may be interested in the development and current status of one of the major themes of mathematical logic in the twentieth century, namely the classical decision problem known also as Hilbert's *Entscheidungsproblem*. The text provides a comprehensive modern treatment of the subject, including complexity theoretic analysis.

We have made an effort to combine the features of a research monograph and a textbook. Only the basic knowledge of the language of first-order logic is required for understanding of the main parts of the book, and we use standard terminology. The chapters are written in such a way that various combinations of them can be used for introductory or advanced courses on undecidability, decidability and complexity of logical decision problems. This explains a few intended redundancies and repetitions in some of the chapters. The annotated bibliography, the historical remarks at the end of the chapters and the index allow the reader to use the text also for quick reference purposes.

The book is the result of an effort which went over a decade. Many people helped us in various ways: with English, with pictures and latex, with comments and information. It is a great pleasure to thank David Basin, Bertil Brandin, Martin Davis, Anatoli Degtyarev, Igor Durdanovic, Dieter Ebbinghaus, Ron Fagin, Christian Fermüller, Phokion Kolaitis, Alex Leitsch, Janos Makowsky, Karl Meinke, Jim Huggins, Silvia Mazzanti, Vladimir Orevkov, Martin Otto, Eric Rosen, Rosario Salomone, Wolfgang Thomas, Jurek Tyszkiewicz, Moshe Vardi, Stan Wainer and Suzanne Zeitman. This list is incomplete and we apologize to those whose names have been inadvertently omitted. We are specially thankful to Saharon Shelah for his help with the Shelah case and to Cyril Allauzen and Bruno Durand for providing an appendix with a new, simplified proof for the unsolvability of the domino problem. Also, we use this opportunity to thank Springer Verlag, the Omega group and in particular Gert Müller for the patience and belief in our longstanding promise to write this book.

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1. Introduction: The Classical Decision Problem

1.1 The Original Problem

The original *classical decision problem* can be stated in several equivalent ways.

- The *satisfiability problem* (or the *consistency problem*) for first-order logic: given a first-order formula, decide if it is consistent.
- The *validity problem* for first-order logic: given a first-order formula, decide if it is valid.
- The *provability problem* for a sound and complete formal proof system for first-order logic: given a first-order formula, decide if it is provable in the system.

Recall that a formula is *satisfiable* (or *consistent*) if it has a model. It is *valid* (or *logically true*) if it holds in all models where it is defined. A proof system is *sound* if every provable formula is valid; it is *complete* if every valid formula is provable.

It was Hilbert who drew attention of mathematicians to the classical decision problem and made it into a central problem of mathematical logic. He called it *das Entscheidungsproblem*, literally "the decision problem". In the beginning of this century, he was developing the formalist programme for the foundations of mathematics (see [263, 264, 525]) and thus was interested in axiomatizing various branches of mathematics by means of finitely many first-order axioms. In principle, such an axiomatization reduces proving a mathematical statement to performing a mechanical derivation in a fixed formal logical system; see below. Obviously, the *Entscheidungsproblem* is very important in this context:

> *...stellt sich ...die Frage der Widerspruchsfreiheit als ein Problem der reinen Prädikaten-Logik dar ...Eine solche Frage ...fällt unter das "Entscheidungsproblem".*[1] [267, page 8]

Hilbert and Ackermann formulated a sound formal proof system for first-order logic and conjectured that the system is complete [266]. Later Gödel

[1] ...the question of consistency presents itself as a problem of the pure predicate logic ... Such a question ... falls under the "Entscheidungsproblem".

proved the completeness [184]. The proof is found in standard logic textbooks, e.g. [57, 142, 146, 307, 471]. For our purposes, the details of a formal system are of no importance. We will simply assume that some sound and complete formal proof system for first-order logic has been fixed. Notice that there is a mechanical procedure that derives all valid first-order formulae in some order.

To explain how proving a mathematical statement reduces to performing a mechanical derivation, assume that T is a finitely axiomatizable mathematical theory. Without loss of generality, the axioms have no free individual variables (that is, are sentences); indeed, if an axiom has free individual variables, replace it with its universal closure. Let α be the conjunction of the axioms, β another first-order sentence (a mathematical claim in the terminology of Hilbert), and γ the implication $\alpha \to \beta$. Then β is a theorem of T if and only if γ is valid if and only if γ is provable in the fixed formal proof system. Thus the mathematical question whether β is a theorem of T reduces to the logical question whether γ is valid which, in its turn, reduces to the question whether the mechanical procedure mentioned above derives γ.

Many important mathematical problems reduce to logic this way [266, 267]. Let us add another example.

Example. Reduction of the Riemann Hypothesis to the validity problem for some first-order sentence γ. Recall that a Diophantine equation is an equation $P(x_1, \ldots, x_k) = 0$ where P is a polynomial with integers coefficients and the variables x_i range over integers. In [98], the authors exhibit a Diophantine equation E that is solvable if and only if the Riemann Hypothesis fails. It suffices to find a finitely axiomatizable theory T and a sentence β such that β is provable in T if and only if E is solvable; the desired γ is then the implication $\alpha \to \beta$ where α is the conjunction of (the universal closures of) the axioms of T.

Recall that the standard arithmetic $\mathcal{A}$ is the set of natural numbers with distinguished element 0, the successor function, addition, multiplication and the order relation $\leq$. Let L be the first-order language of $\mathcal{A}$. Robinson's system Q is a finitely axiomatizable theory in L such that an arbitrary existential L-sentence φ is provable in Q if and only if it holds in $\mathcal{A}$ [307]. (A similar theory is called N in [471].)

Choose T to be Q. It suffices to construct an existential L-sentence β in such a way that E is solvable if and only if β holds in $\mathcal{A}$.

In fact, an arbitrary Diophantine equation D can be expressed by an existential formula β_D in such a way. Since a disjunction of existential sentences is equivalent to an existential sentence, it suffices to check that an existential L-sentence can express the given equation $P(x_1, \ldots, x_k) = 0$ together with an atomic constraint $x_i \geq 0$ or $x_i \leq 0$ for every variable x_i. But this is obvious. For example, an equation $x^3 - y^5 + 1 = 0$ with constraints $x \leq 0, y \leq 0$ is equivalent to an equation $(-x)^3 - (-y)^5 + 1 = 0$ with constraints $x \geq 0, y \geq 0$

which is equivalent to an equation $y^5 + 1 = x^3$ with constraints $x \geq 0, y \geq 0$ which is obviously expressible by an existential L-sentence.

The classical decision problem is called the main problem of mathematical logic by Hilbert and Ackermann:

> *Das Entscheidungsproblem ist gelöst, wenn man ein Verfahren kennt, das bei einem vorgelegten logischen Ausdruck durch endlich viele Operationen die Entscheidung über die Allgemeingültigkeit bzw. Erfüllbarkeit erlaubt. (...) Das Entscheidungsproblem muss als das Hauptproblem der mathematischen Logik bezeichnet werden.*[2] [266, pp 73ff]

Hilbert and Ackermann were not alone in their evaluation of the importance of the classical decision problem. Their attitude has been shared by other leading logicians of the time. Bernays and Schönfinkel wrote:

> *Das zentrale Problem der mathematischen Logik, welches auch mit den Fragen der Axiomatik im engsten Zusammenhang steht, ist das* Entscheidungsproblem.[3] [35].

Herbrand's paper [253] starts with:

> *We could consider the fundamental problem of mathematics to be the following. Problem A: What is the necessary and sufficient condition for a theorem to be true in a given theory having only a finite number of hypotheses?*

The paper ends with:

> *The solution of this problem would yield a general method in mathematics and would enable mathematical logic to play with respect to classical mathematics the role that analytic geometry plays with respect to ordinary geometry.*

In [254], Herbrand adds:

> *In a sense it* [the classical decision problem – BGG] *is the most general problem of mathematics.*

Ramsey wrote that his paper was

> *concerned with a special case of one of the leading problems in mathematical logic, the problem of finding a regular procedure to determine the truth or falsity of any given logical formula.* [435, p. 264]

[2] The Entscheidungsproblem is solved when we know a procedure that allows for any given logical expression to decide by finitely many operations its validity or satisfiability. (...) The Entscheidungsproblem must be considered the main problem of mathematical logic.

[3] The cental problem of mathematical logic, which is also most closely related to the questions of axiomatics, is the *Entscheidungsproblem*.

The roots of the classical decision problem can be traced while back. Philosophers were interested in a general problem-solving method. The medieval thinker Raimundus Lullus called such a method *ars magna*. Leibniz was the first to realize that a comprehensive and precise symbolic language (*characteristica universalis*) is a prerequisite for any general problem solving method. He thought about a calculus (*calculus ratiocinator*) to resolve mechanically questions formulated in the universal language. A universal symbolic language, restricted to mathematics, had to wait until 1879 when Frege published [171]; the language allowed Russel and Whitehead [446] to embed virtually the whole body of then known mathematics into a formal framework.[4] Leibniz distinguished between two different versions of *ars magna*. The first version, *ars inveniendi* , finds all true scientific statements. The other, *ars iudicandi* , allows one to decide whether any given scientific statement is true or not [255].

In the framework of first-order logic, an *ars inveniendi* exists: the collection of valid first-order formulae is recursively enumerable, hence there is an algorithm that lists all valid formulae. The classical decision problem can be viewed as the *ars iudicandi* problem in the first-order framework. It can be sharpened to a yes/no question: Does there exist an algorithm that decides the validity of any given first-order formula? Some logicians felt sceptical about ever finding such an algorithm. It wasn't clear, however, whether the scepticism could be justified by a theorem. John von Neumann wrote:

> *Es scheint also, daß es keinen Weg gibt, um das allgemeine Entscheidungskriterium dafür, ob eine gegebene Normalformel a beweisbar ist, aufzufinden. (Nachweisen können wir freilich gegenwärtig nichts. Es ist auch gar kein Anhaltspunkt dafür vorhanden, wie ein solcher Unentscheidbarkeitsbeweis zu führen wäre.) (...) Und die Unentscheidbarkeit ist sogar die Conditio sine qua non dafür, daß es überhaupt einen Sinn habe, mit den heutigen heuristischen Methoden Mathematik zu treiben. An dem Tage, an dem die Unentscheidbarkeit aufhörte, würde auch die Mathematik im heutigen Sinne aufhören zu existieren; an ihre Stelle würde eine absolut mechanische Vorschrift treten, mit deren Hilfe jedermann von jeder gegebenen Aussage entscheiden könnte, ob diese bewiesen werden kann oder nicht.*
> *Wir müssen uns also auf den Standpunkt stellen: Es ist allgemein unentscheidbar, ob eine gegebene Normalformel beweisbar ist oder nicht. Das einzige, was wir tun können, ist, (...), beliebig viele beweisbare Normalformeln aufzustellen. (...) Auf diese Art können wir von vielen Normalformeln feststellen, daß sie beweisbar sind. Aber auf diesem Weg kann uns niemals die Feststellung gelingen, daß eine Normalformel nicht beweisbar ist.* [5] [525, pp 11–12]

[4] See the forthcoming book by M. Davis [96] in this connection.

[5] It appears thus that there is no way of finding the general criterion for deciding whether or not a well-formed formula a is provable. (We cannot, however, at

Gödel's Incompleteness Theorem [185] was a breakthrough in logic. Can one use a similar method to prove the nonexistence of a decision algorithm for the classical decision problem? In an appendix to his paper "The fundamental problem of mathematical logic" Herbrand wrote:

> *Note finally that, although at present it seems unlikely that the decision problem can be solved, it has not yet been proved that it is impossible to do so.* [254]

Herbrand, Gödel and Kleene developed a very general notion of recursive functions [307]. In 1936, Church put forward a bold thesis: Every computable function from natural numbers to natural numbers is recursive in the sense of Herbrand-Gödel-Kleene. He showed that no recursive function could decide the validity of first-order sentences and concluded that that there was no decision algorithm for the classical decision problem [80].

Independently, Alan Turing introduced computing devices which are called now Turing machines. He put forward a similar thesis: a function from strings to strings is computable if and only if it is computable by a Turing machine [513]. He showed that no Turing machine could decide the validity of first-order sentences and also concluded that there is no decision algorithm for the classical decision problem. The equivalence of Church's and Turing's theses was quickly established. The Church-Turing thesis was largely accepted and thus it was accepted that the yes/no version of the classical decision problem was solved negatively by Church and Turing.

1.2 The Transformation of the Classical Decision Problem

By the time of Church's and Turing's theses, the area of the classical decision problem had already a rich and fruitful history. Numerous fragments of first-order logic were proved decidable for validity and numerous fragments were shown to be as hard as the whole problem. What does it mean that a fragment F is as hard for validity as the whole problem? This means that there exists

the moment demonstrate this. Indeed, we have no clue as to how such a proof of undecidability would go.) (...) The undecidability is even the *conditio sine qua non* for the contemporary practice of mathematics, using as it does heuristic methods, to make any sense. The very day on which the undecidability would cease to exist, so would mathematics as we now understand it; it would be replaced by an absolutely mechanical prescription, by means of which anyone could decide the provability or unprovability of any given sentence.

Thus we have to take the position; it is generally undecidable, whether a given well-formed formula is provable or not. The only thing we can do is (...) to construct an arbitrary number of provable formulae. In this way, we can establish for many well-formed formulae that they are provable. But in this way we never succeed to establish that a well-formed formula is not provable.

an algorithm A that transforms an arbitrary formula φ into a formula in F in such a way that $A(\varphi)$ is valid if and only if φ is so; such a fragment is called a *reduction class* for validity. Actually, it had been more common to speak about satisfiability and finite satisfiability, that is satisfiability in a finite structure. Reduction classes for satisfiability (respectively finite satisfiability) are defined similarly.

To convey a feeling of the field, let us quote some early results on fragments of pure first-order predicate logic (first-order logic without function symbols or equality). But first let us recall that a *prenex* formula is a formula with all its quantifiers up front. View a string in the four-letter alphabet $\{\forall, \exists, \forall^*, \exists^*\}$ as a regular expression denoting a collection of strings in the two-letter alphabet $\{\forall, \exists\}$. For example, $\forall^3\exists^*$ denotes the collection of strings of the form $\forall^3\exists^j$ where j is an arbitrary natural number, and $\exists^*\forall^2\exists^*$ denotes the collection of strings of the form $\exists^i\forall^2\exists^j$ where i and j are arbitrary natural numbers.

In 1915, Löwenheim [365] gave a decision procedure for the satisfiability of predicate formulae with only unary predicates. He proved also that formulae with binary predicates form a reduction class for satisfiability. In 1931, Herbrand [254] sharpened the latter result showing that just three binary predicates suffice. In 1936, Kalmár [295] showed that one binary predicate suffices.

In 1920, Skolem [477] showed that $\forall^*\exists^*$ sentences form a reduction class for satisfiability. In 1928, Bernays and Schönfinkel [35] gave a decision procedure for the satisfiability of $\exists^*\forall^*$ sentences. In 1928, Ackermann [16] gave a decision procedure for the satisfiability of $\exists^*\forall\exists^*$ sentences. Gödel [186], Kalmár [293] and Schütte [457], separately in 1932, 1933 and 1934 respectively, discovered decision procedures for the satisfiability of pure $\exists^*\forall^2\exists^*$ sentences. In another paper, Gödel proved that every satisfiable $\exists^*\forall^2\exists^*$ sentence has a finite model and that $\forall^3\exists^*$ sentences form a reduction class for satisfiability [187]. (See [234] for a popular introduction to the classical decision problem.)

The reaction of the logicians to the discoveries of Church and Turing was that the classical decision problem was wider than the yes/no version of it. Here is one of the earliest reactions:

> *Solche Reduktionen des Entscheidungsproblems werden hoffentlich vorteilhaft sein für systematische Untersuchungen über die Zählausdrücke, z.B. wenn man versuchen will eine Übersicht zu bekommen, für welche Klassen von solchen man das Entscheidungsproblem wirklich lösen kann. Bekanntlich hat A. Church bewiesen, dass eine allgemeine Lösung dieses Problems nicht möglich ist.*[6] [482]

[6] Such reductions [a reference to the reductions proposed by Skolem in the paper cited — BGG] will hopefully be advantageous for systematic investigations of first-order formulae, for example if one would like to try to arrive at a complete picture, for which classes of such formulae one can really solve the Entschei-

The logicians started to think about the classical decision problem as a classification problem.

- Which fragments are decidable for satisfiability and which are undecidable?
- Which fragments are decidable for finite satisfiability and which are undecidable?
- Which fragments have the finite model property and which contain axioms of infinity (that is satisfiable formulae without finite models)?

For a long time the classical decision problem remained a central problem of mathematical logic. With the development of computational complexity theory, the problem has been refined. If a fragment of first-order logic is decidable for satisfiability, then indeed there is an absolutely mechanical procedure, that is an algorithm, for deciding the satisfiability or unsatisfiability of any given sentence. But what is the computational complexity of determining satisfiability? Similarly, if a given fragment is decidable for finite satisfiability, what is the computational complexity of determining finite satisfiability?

Of course, the unrestricted classifiability problem is hopeless. There are just too many fragments. Some of them are of no interest to anybody. Some of them involve particular branches of mathematics. Consider for example the satisfiability problem for sentences $\alpha \wedge \beta$ where α is (the universal closure of) the conjunction of the axioms of fields and β is an arbitrary formula in the vocabulary of fields; this problem rightfully belongs to field theory rather than logic.

Eventually, the classical decision problem became to mean the restriction of the classification problem described above to traditional fragments. This description is admittedly not precise but it gives a good guidance which we will follow. One can argue that the complexity issue does not really belong to the traditional classical decision problem. This is true too, but it is impossible to ignore the complexity issue these days, in particular because of the relevance of the logical decision procedures to theorem proving and model checking methods. We will try to cover the known complexity results.

As we have mentioned above, for a long time the classical decision problem remained a central problem of mathematical logic. The literature on the subject is huge and contains a great wealth of material. The classical decision problem served as a laboratory for various logic methods[7] and especially reduction methods. The classification results have been used not only in logic but also in theoretical computer science. In particular, they have been used as a guide to the study of zero-one laws for fragments of second-order logic. Classical techniques inspired some proofs on the zero-one laws and some of classical techniques have been further extended. See [235, 313, 314, 315, 414, 415] in this connection.

dungsproblem. As it is known, A. Church has proved that a general solution of this problem is not possible.

[7] By the way, Ramsey proved his famous combinatorial lemma in a paper on the classical decision problem [435].

There is a number of books devoted to the classical decision problem. In the 1950s, Ackermann gave a comprehensive treatment of the solvable cases known at the time [18], and Surányi gave a complementary comprehensive treatment of reduction classes known at the time [498]. The book [133] of Dreben and Goldfarb illustrates the potential of the so-called Herbrand expansion technique in establishing solvability. The complementary book [351] of Lewis covers many reduction results on classical fragments of pure predicate logic. Together the two books give a systematic treatment of decision problems for predicate logic without functions or equality.

Nevertheless, much of the wealth has never appeared in a book form. Moreover, by now, the work on the classical decision problem is by and large completed (though some open problems remain of course) and most of the major classifications have not been ever covered in book form. That is exactly what we intend to do in this book.

1.3 What Is and What Isn't in this Book

We give most attention to the most traditional fragments of first-order logic, namely, to collections of prenex formulae given by restrictions on the quantifier prefix and/or vocabulary. (Recall that there is a simple algorithm for transforming an arbitrary first-order formula to an equivalent one in the prenex form.)

Strings in the two-letter alphabet $\{\forall, \exists\}$ will be called *prefixes*. A *prefix set* is a set of prefixes. An *arity sequence* is a function p from the set of positive integers to the set of non-negative integers augmented with the first infinite ordinal ω.

Definition 1.3.1 (Prefix-Vocabulary Classes). For any prefix set Π and any arity sequences p and f, $[\Pi, p, f]$ (respectively, $[\Pi, p, f]_=$) is the collection of all prenex formulae φ of first-order logic without equality (respectively with equality) such that

- the prefix of φ belongs to Π,
- the number of n-ary predicate symbols in φ is $\leq p(n)$, and
- the number of n-ary function symbols in φ is $\leq f(n)$.
- φ has no nullary predicate symbols with the exception of the logic constants *true* and *false*, no nullary function symbols and no free variables.

Let us explain the last clause. We will speak about logic without equality but the same applies to logic with equality. It is easy to see that the status (decidable or undecidable) of the (finite) satisfiability problem for a prefix-vocabulary class does not change if nullary predicate symbols are allowed. Now let us consider the rôle of nullary function symbols, that is individual constants. Let $C = [\Pi, p, f]$ and C' be the version of C when one is allowed to use say 7 individual constants. It is easy to see that the status of the (finite) satisfiability problem for C' is that of the (finite) satisfiability problem for $[\Pi', p, f]$ where $\Pi' = \{\exists^7\pi : \pi \in \Pi\}$. Instead of individual constants, we could speak about free individual variables. Thus allowing individual constants or free individual variables does not give us more classes either.

The definition of prefix-vocabulary classes above seems to be excessively general. Call a prefix set *closed* if it contains all substrings (even not contiguous substrings) of its prefixes. Clearly, one can restrict attention to closed prefix sets. Further, call a prefix set Π *standard* if either it is the set of all prefixes or else it can be given by a string w in the four-letter alphabet $\{\forall, \exists, \forall^*, \exists^*\}$. In the first case Π is denoted *all*. Thus, every standard prefix set has a succinct notation. Furthermore, we can require without loss of generality that w is *reduced* in the following sense: $\forall^*$ cannot have $\forall$ as a neighbor, and similarly $\exists^*$ cannot have $\exists$ as a neighbor. For example, a string $\forall^*\forall\exists\exists^*$ reduces to $\forall^*\exists^*$; clearly the two strings define the same prefix set.

Call an arity sequence p *standard* if it satisfies the following condition: $p(n) = \omega$ whenever the sum $p(n) + p(n+1) + \cdots$ is infinite. Every standard sequence can be given a succinct notation. The standard arity sequence that assigns ω to each n will be denoted *all*. Any other standard sequence p has a tail of zeroes, $0 = p(m) = p(m+1) = \cdots$, and will be denoted by the sequence $(p(1), p(2), \ldots, p(m-1))$. In case $m = 1$, for readability, we denote p with (0) rather than $()$. Similar notation can be used for non-standard sequences with a tail of zeroes. Notice that every arity sequence reduces (in a sense made more precise in Sect. 2.3) to a standard arity sequence. For example, $[all, (0, \omega), (0)] \subseteq [all, (\omega, \omega), (0)]$ and every sentence $\varphi \in [all, (\omega, \omega), (0)]$ can be easily rewritten as an equivalent sentence in $[all, (0, \omega), (0)]$: just replace formulae $R(x)$ with formulae $R'(x, x)$ where R' is a binary predicate symbol that does not occur in φ.

Definition 1.3.2. A prefix-vocabulary class $[\Pi, p, f]$ or $[\Pi, p, f]_=$ is *standard* if Π, p and f are standard.

The classification problem for the prefix-vocabulary fragments admits a complete solution in a form of a finite table. In particular, there are only finitely many minimal undecidable fragments with closed prefix sets, and all these minimal fragments are standard. This follows from the Classifiability Theorem of Gurevich proved in Sect. 2.3. Accordingly, in the main body of the book, the prefix-vocabulary classes of interest will be almost exclusively standard classes. The Classifiability Theorem has provided guidance for research and it provides guidance for this book.

Let us review briefly the contents of the book. The main part of Chapter 2 is devoted to the reduction theory which we explain from scratch and develop to a certain depth. The reduction theory helps us to give simpler proofs and proper lower complexity bounds. The rest of Chapter 2 is devoted to the Classifiability Theorem.

In Chapters 3 and 4, we give a complete treatment of the undecidable prefix-vocabulary fragments of first-order logic (with or without function symbols, with or without equality). In Chapter 5, we present various other undecidable fragments mainly defined in terms of additional restrictions on the propositional structure of the formulae; we study in particular Krom and Horn formulae which have played an important rôle in the theory of logic programming.

In Chapters 6 and 7, we treat the decidable prefix-vocabulary fragments of first-order logic (with or without function symbols, with or without equality). Together with the results of Chapters 3 and 4 this gives a complete classification of the decidable and undecidable prefix-vocabulary classes. Tables 1.1 and 1.2 summarize the decidability/undecidability results on prefix-vocabulary fragments.

Undecidable Cases

A: Pure predicate logic (without functions, without =)

(1)	$[\forall\exists\forall, (\omega, 1), (0)]$	(Kahr 1962)
(2)	$[\forall^3\exists, (\omega, 1), (0)]$	(Surányi 1959)
(3)	$[\forall^*\exists, (0, 1), (0)]$	(Kalmár-Surányi 1950)
(4)	$[\forall\exists\forall^*, (0, 1), (0)]$	(Denton 1963)
(5)	$[\forall\exists\forall\exists^*, (0, 1), (0)]$	(Gurevich 1966)
(6)	$[\forall^3\exists^*, (0, 1), (0)]$	(Kalmár-Surányi 1947)
(7)	$[\forall\exists^*\forall, (0, 1), (0)]$	(Kostyrko-Genenz 1964)
(8)	$[\exists^*\forall\exists\forall, (0, 1), (0)]$	(Surányi 1959)
(9)	$[\exists^*\forall^3\exists, (0, 1), (0)]$	(Surányi 1959)

B: Classes with functions or equality

(10)	$[\forall, (0), (2)]_=$	(Gurevich 1976)
(11)	$[\forall, (0), (0, 1)]_=$	(Gurevich 1976)
(12)	$[\forall^2, (0, 1), (1)]$	(Gurevich 1969)
(13)	$[\forall^2, (1), (0, 1)]$	(Gurevich 1969)
(14)	$[\forall^2\exists, (\omega, 1), (0)]_=$	(Goldfarb 1984)
(15)	$[\exists^*\forall^2\exists, (0, 1), (0)]_=$	(Goldfarb 1984)
(16)	$[\forall^2\exists^*, (0, 1), (0)]_=$	(Goldfarb 1984)

Table 1.1. Minimal Undecidable Standard Classes

Decidable Cases

A: Classes with the finite model property

(1)	$[\exists^*\forall^*, all, (0)]_=$	(Ramsey 1930)
(2)	$[\exists^*\forall^2\exists^*, all, (0)]$	(Gödel 1932, Kalmár 1933, Schütte 1934)
(3)	$[all, (\omega), (\omega)]$	(Löb 1967, Gurevich 1969)
(4)	$[\exists^*\forall\exists^*, all, all]$	(Maslov-Orevkov 1972, Gurevich 1973)
(5)	$[\exists^*, all, all]_=$	(Gurevich 1976)

B: Classes with infinity axioms

(6)	$[all, (\omega), (1)]_=$	(Rabin 1969)
(7)	$[\exists^*\forall\exists^*, all, (1)]_=$	(Shelah 1977)

Table 1.2. Maximal Decidable Standard Classes

We give also a fairly complete complexity analysis of the decidable cases. One open problem is to find the exact complexities of the satisfiability and finite satisfiability problems for the Shelah class. For most of the maximal decidable standard fragments, the satisfiability problem has a very high computational complexity, typically deterministic or nondeterministic exponential time, the complexity is even non-elementary in the case of the Rabin class. At the end of Chapter 6 we also present a classification of the standard classes that have the finite model property and of those having infinity axioms. The decidability results in Chapter 7 rely (in our exposition) on a reduction to S2S, the monadic second-order theory of the infinite binary tree. The decidability of S2S, proved by Rabin [430], is one of the most important and difficult decidability theorems for mathematical theories. We give a complete proof of this result in Sect. 7.1. In Chapter 8 we present some other decidable cases of the decision problem. In addition, the book contains a quite extensive annotated bibliography and an appendix, written by Cyril Allauzen and Bruno Durand, containing a new simplified proof for the unsolvability of the unconstrained domino problem which is used at many places in this book.

Some classifications appear for the first time in a book: For example, the classifications of prefix-vocabulary fragments in the cases of logic with equality, functions or both. All complexity results appear for the first time in a book. There are many new proofs, e.g. those (assisted by Shelah) related to the Shelah class. There are also many new results.

On the other hand, there are many closely related topics that we do not cover in this book. We are concerned here with fragments of first-order logic and do not deal with decision problems for second-order logic, higher-order logic, intuitionistic logic (see [385, 412]), linear logic (see the forthcoming book [362]) or any other logic. We do not deal with decision problems for mathematical theories formalized in first-order or any other logic; in this connection see [89, 97, 148, 166, 231, 432, 506].

Furthermore, even though the classical decision problem is more or less finished in its most classical form, there are various other natural versions and extensions of it that we do not deal with here systematically. For example, we do not deal with classifications based on the resolution calculus; in this connection see [163, 340]. But we do discuss various extensions of the classical decision problem and various open problems on our way. Let us mention some extensions and open problems here.

Extend the classifiability theorem in various directions. This is very important; without a proper direction, it is hard even to remember a myriad of specific results.

Extend the prefix-vocabulary classification to important undecidable mathematical theories; see [229] in this connection. Find the computational complexity of decidable prefix-vocabulary classes of important mathematical theories (see [201, 206]); in many cases even the computational complexity of the theory itself is unknown. It would also be interesting to extend the classification to different logics.

We were interested whether a given fragment contains a formula without finite models. Does a given fragment contain a formula without recursive models? This direction is still covered by the Classifiability Theorem; in particular there are finitely many minimal prefix-vocabulary classes with formulae without recursive models and each of them is standard. Instead of recursivity, one can speak about other kinds of descriptive or computational complexity. Similarly, does a given fragment contain an axiom of an essentially undecidable theory? Since the fragment may be not closed under conjunction, it is meaningful to ask if the fragment includes a finite set of sentences that form an axiomatization of an essentially undecidable theory. Also, one may restrict attention to infinite models of certain complexity: primitive recursive models, recursive models, models of such and such Turing degrees, Borel models, etc.

In cases of fragments of reasonably low complexity bound, develop practical solutions of the decision problem. This problem is well recognized as a major bottleneck for *e.g.* model checking [70], an important current method for computer verification of hardware and software correctness claims.

One extension of the classical decision problem is related to the strictness of reductions. If one cares only about satisfiability, it suffices to require that a reduction transforms a given formula α into a formula α' which is satisfiable if and only if α is so. We usually care about satisfiability and finite satisfiability and thus consider so called *conservative reductions* when it is required that (i) α' is satisfiable if and only if α is satisfiable, and (ii) α' is finitely satisfiable if and only if α is finitely satisfiable. One may be interested in even stricter reductions. For example, one may require that α and α' have the same spectra or – more generally – that there is a simple connection between the spectra. (On several occasions, Surányi insisted that there should exist a general method that transforms a given model of α' to a model of α.) On the other hand, one may consider not only recursive but also arithmetical, Borel, etc. transformations.

There are many more specific problems. One is to examine Boolean combinations of prefix-vocabulary classes; see Section 5.4 in this connection.

The book is addressed to a wide audience and not only to professional logicians. There are scattered remarks and exercises addressing more special audiences (logicians, people familiar with logic programming, etc.) but the main body requires only the familiarity with basic notions of mathematical logic. (This does not mean of course that all parts are easy to read; some proofs are quite involved even after much simplification). Finally, let us note that sometimes we will omit the adjective "first-order"; formulae, languages and theories are by default first-order in this book.