Crux Mathematicorum

VOLUME 40, NO. 6 JUNE / JUIN 2014

Editorial Board

Editor-in-Chief	**Kseniya Garaschuk**	University of British Columbia
Contest Corner Editor	**Robert Bilinski**	Collège Montmorency
Olympiad Editors	**Nicolae Strungaru**	Grant MacEwan University
	Carmen Bruni	Hamilton, ON
Book Reviews Editor	**Robert Bilinski**	Collège Montmorency
Articles Editor	**Robert Dawson**	Saint Mary's University
Problems Editors	**Edward Barbeau**	University of Toronto
	Chris Fisher	University of Regina
	Anna Kuczynska	University of the Fraser Valley
	Edward Wang	Wilfrid Laurier University
	Dennis D. E. Epple	Berlin, Germany
	Magda Georgescu	University of Toronto
Assistant Editors	**Chip Curtis**	Missouri Southern State University
	Lino Demasi	Ottawa, ON
	Allen O'Hara	University of Western Ontario
Guest Editors	**Alejandro Erickson**	Durham University
	Joseph Horan	University of Victoria
	Robin Koytcheff	University of Victoria
	Amanda Malloch	University of Victoria
Editor-at-Large	**Bill Sands**	University of Calgary
Managing Editor	**Johan Rudnick**	Canadian Mathematical Society

Copyright © Canadian Mathematical Society, 2015

IN THIS ISSUE / DANS CE NUMÉRO

Crux Mathematicorum

Founding Editors / Rédacteurs-fondateurs: Léopold Sauvé & Frederick G.B. Maskell
Former Editors / Anciens Rédacteurs: G.W. Sands, R.E. Woodrow, Bruce L.R. Shawyer, Shawn Godin

Crux Mathematicorum with Mathematical Mayhem

Former Editors / Anciens Rédacteurs: Bruce L.R. Shawyer, James E. Totten, Václav Linek, Shawn Godin

EDITORIAL

Dear ***Crux*** readers,

Summer is conference time for us academic types. So I have been meeting lots of new people and catching up with many old friends. And I've been talking to people about ***Crux*** , about what they think the journal is and what they think it should be: there indeed are as many opinions as there are people! But one thing is for certain – with the departure of Mayhem, the publication has lost some of its younger audiences, younger whether in age or problem solving savvy. So I would like to open the ***Crux*** door just a little wider to invite more people into the world of mathematical problem solving.

I have also gotten great response on the recent ***Crux*** materials and I am happy to continue to include articles on the introduction to various research areas through problem solving. As I mentioned before, I will also be featuring a "From the Archives" section that will include materials published in Russian in *Kvant*. If you know any other non-English journals that you feel the ***Crux*** audience would benefit from, please let me know. Finally, I am looking forward to publishing historical notes that highlight mathematical development through the years and expose us to new (or, rather, quite old!) ways of looking at familiar objects.

As always, I'm happy to hear from you, so drop me a line at
`crux-editors@cms.math.ca`.

Kseniya Garaschuk

Math Quotes

Perhaps the most surprising thing about mathematics is that it is so surprising. The rules which we make up at the beginning seem ordinary and inevitable, but it is impossible to foresee their consequences. These have only been found out by long study, extending over many centuries. Much of our knowledge is due to a comparatively few great mathematicians such as Newton, Euler, Gauss, or Riemann; few careers can have been more satisfying than theirs. They have contributed something to human thought even more lasting than great literature, since it is independent of language.

Titchmarsh, E. C. in N. Rose "Mathematical Maxims and Minims", Rome Press Inc., 1988.

Copyright © Canadian Mathematical Society, 2015

THE CONTEST CORNER

No. 26

Olga Zaitseva

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'un concours mathématique de niveau secondaire ou de premier cycle universitaire, ou en ont été inspirés. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1er octobre 2015** *; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

La rédaction souhaite remercier André Ladouceur, Ottawa, ON d'avoir traduit les problèmes.

CC126. Anna et Ben se rendent dans un archipel de 2009 îles. Certaines îles sont reliées par bateaux qui font la navette dans les deux sens. Pendant leur voyage, Anna et Ben s'amusent avec le jeu suivant : Anna choisit l'île d'arrivée qu'ils atteindront par avion. Ben choisit ensuite la deuxième île qu'ils visiteront. Chacun choisit ensuite l'île suivante à tour de rôle. Lorsqu'ils arrivent dans une île qui n'est reliée qu'à des îles déjà visitées, la personne qui doit choisir l'île suivante perd. Démontrer qu'Anna peut toujours gagner, peu importe comment Ben choisit ou comment les îles sont reliées.

CC127. Dans le château de Camelot, il y a une table ronde de n places. Merlin y convoque n chevaliers pour une conférence. Chaque jour, chaque chevalier se voit assigner une place en particulier. À partir de la 2^e journée, deux chevaliers qui deviennent voisins peuvent changer de place entre eux, à moins qu'ils n'aient été voisins la 1^re journée. Si les chevaliers réussissent à s'asseoir dans le même ordre cyclique que celui de la 1^re journée, la conférence se terminera la journée suivante. Quel est le nombre maximal de journées que Merlin peut assurer pour la conférence ?

CC128. Dans un royaume, on utilise des grains d'or et des grains de platine comme monnaie. Deux entiers strictement positifs, g et p, sont utilisés pour fixer le taux d'échange : x grammes d'or valent y grammes de platine si $x : y = p : g$ (x et y ne sont pas nécessairement des entiers). Un jour où $g = p = 1001$, on annonce que chaque jour suivant, g ou p sera diminué de 1 de sorte qu'après 2000 jours, on aura $g = p = 1$. Or, l'ordre dans lequel les nombres diminueront n'est pas annoncé d'avance. Le jour de l'annonce, un banquier possède 1 kg de grains d'or et 1 kg de grains de platine. Le banquier aimerait faire des échanges de sorte qu'après 2000 jours, il ait au moins 2 kg de grains d'or et 2 kg de grains de platine. Le banquier peut-il atteindre son but en réalisant des échanges astucieux ?

CC129.

On trace dans le plan une ligne brisée fermée qui se recoupe. Chaque segment de cette ligne est recoupé exactement une fois et trois segments ne peuvent se recouper en un même point. De plus, aucun point d'intersection de segments ne coïncide avec des sommets et il n'y a aucun chevauchement de segments. Est-il possible que tous les points d'intersection divisent les deux segments en leurs milieux ?

CC130. Soit $P(x)$ un polynôme tel que $P(0) = 1$ et $(P(x))^2 = 1+x+x^{100}Q(x)$, $Q(x)$ étant un polynôme. Démontrer que le coefficient du terme x^{99} du polynôme $(P(x)+1)^{100}$ est nul.

. . . .

CC126. Anna and Ben decided to visit Archipelago with 2009 islands. Some islands are connected by boats which run both ways. Anna and Ben are playing the following game during the trip: Anna chooses the first island on which they arrive by plane; then Ben chooses the next island to visit. Thereafter, they take turns choosing a new island. When they arrive at an island connected only to islands they have already visited, whoever's turn it is to choose the next island loses. Prove that Anna can always win, no matter how Ben plays and how islands are connected.

CC127. In the Great Hall of Camelot there is the Round Table with n seats. Merlin summons n knights of Camelot for a conference. Every day, he assigns seats to the knights. From the second day on, any two knights who become neighbours may switch their seats unless they were neighbours on the first day. If the knights manage to sit in the same cyclic order as on one of the previous days, the next day the conference ends. What is the maximal number of days of the conference Merlin can guarantee?

CC128. In one kingdom gold sand and platinum sand are used as a currency. An exchange rate is defined by two positive integers g and p; namely, x grams of gold sand are equivalent to y grams of platinum sand if $x : y = p : g$ (x and y are not necessarily integers). At a day when $g = p = 1001$, the Treasury announced that on each of the following days one of the numbers, either g or p would be decreased by 1 so that after 2000 days $g = p = 1$. However, the exact order in which the numbers decreased is kept unknown. At the day of the announcement a banker had 1 kg of gold sand and 1 kg of platinum sand. The banker's goal is to make exchanges so that at the end of this period he would have at least 2 kg of gold sand and 2 kg of platinum sand. Can the banker reach his goal for certain as a result of some clever exchanges?

Copyright © Canadian Mathematical Society, 2015

CC129. A closed broken self-intersecting line is drawn in the plane. Each of the links of this line is intersected exactly once and no three links intersect at the same point. Further, there are no self-intersections at the vertices and no two links have a common segment. Can it happen that every point of self-intersection divides both links in half?

CC130. Let $P(x)$ be a polynomial such that $P(0) = 1$ and $(P(x))^2 = 1 + x + x^{100}Q(x)$, where $Q(x)$ is also a polynomial. Prove that the coefficient of x^{99} of the polynomial $(P(x) + 1)^{100}$ is zero.

Croquet

This young lady is wondering how can she get the ball through every gate, starting from A and ending with Z with the smallest number of moves. Can you help her?

Puzzle by Sam Loyd as used in Kvant, 1976 (2).

CONTEST CORNER SOLUTIONS

CC76. The point $P(a, b)$ lies in the first quadrant. A line, drawn through P, cuts the axes at Q and R such that the area of triangle OQR is $2ab$, where O is the origin. Prove that there are three such lines that satisfy these criteria.

Originally 1996 Invitational Mathematics Challenge, Grade 11, problem 4.

We present the solution of Michel Bataille.

Without loss of generality, we suppose that Q is on the x-axis and R on the y-axis. Then, $Q(c, 0)$ and $R(0, d)$ where c, d are nonzero real numbers. The equation of the line QR is $\frac{x}{c} + \frac{y}{d} = 1$ and P is on this line if and only if $\frac{a}{c} + \frac{b}{d} = 1$, that is, $ad + bc = cd$.

The area of OQR being $\frac{|c||d|}{2}$, the criterion about the area is satisfied if and only if $|cd| = 4ab$, that is, $cd = 4ab$ or $cd = -4ab$.

Thus, setting $u = \frac{a}{c}$ and $v = \frac{b}{d}$, the criteria are satisfied if and only if $u + v = 1$ and ($\frac{1}{u} + \frac{1}{v} = 4$ or $\frac{1}{u} + \frac{1}{v} = -4$).

Now, the first system formed by the equations $u + v = 1$ and $\frac{1}{u} + \frac{1}{v} = 4$ can be rewritten as $u + v = 1$ and $uv = \frac{1}{4}$, whose unique solution is given by $u = v = \frac{1}{2}$ (whence $c = 2a, d = 2b$).

The second system formed by the equations $u + v = 1$ and $\frac{1}{u} + \frac{1}{v} = -4$ can be rewritten as $u + v = 1$ and $uv = -\frac{1}{4}$, which has two solutions for (u, v), namely $(\frac{1+\sqrt{2}}{2}, \frac{1-\sqrt{2}}{2})$ and $(\frac{1-\sqrt{2}}{2}, \frac{1+\sqrt{2}}{2})$, whence $(c, d) = (2(\sqrt{2} - 1)a, -2(\sqrt{2} + 1)b)$ or $(c, d) = (-2(\sqrt{2} + 1)a, 2(\sqrt{2} - 1)b)$.

Altogether, three lines satisfy the criteria.

Editor's comments. Note that it suffices to solve the problem for $P(1, 1)$ since solutions for this problem correspond bijectively to those for the problem $P(a, b)$ via the linear transformation $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$.

Some solvers considered three cases of solutions $Q(c, 0)$ and $R(0, d)$: (1) $c > 0, d > 0$, (2) $c < 0, d > 0$, and (3) $c > 0, d < 0$. If one reduces the problem to $P(1, 1)$, one can show that there is a unique solution with $c < 0, d > 0$ if and only if there is a unique solution with $c > 0, d < 0$, since such solutions are in bijective correspondence via the reflection over the line $y = x$.

CC77. The three following circles are tangent to each other: the first has radius a, the second has radius b, and the third has radius $a + b$ for some $a, b \in \mathbb{R}$ with $a, b > 0$. Find the radius of a fourth circle tangent to each of these three circles.

Copyright © Canadian Mathematical Society, 2015

Inspired by the comments to question 10 of the 2013 Manitoba Mathematical Contest.

Several correct solutions and one incorrect solution were received. We present the solution of Neculai Stanciu and Titu Zvonaru.

Building a circle tangent to three circles is the problem of Apollonius. In the particular case in which the three given circles are tangent in pairs, the problem admits two solutions, known as the inner and outer Soddy circles. A formula for the radii of the Soddy circles in terms of the radii of the given circles can be found, for example, on `http://mathworld.wolfram.com/SoddyCircles.html`. It is

$$r^2 = \frac{r_1r_2r_3}{r_1r_2 + r_1r_3 + r_2r_3 \pm \sqrt{r_1r_2r_3(r_1 + r_2 + r_3)}}.$$

In our case, $r_1 = a$, $r_2 = b$, and $r_3 = a + b$.

This solves the problem if the three given circles are tangent externally. If, instead, the two smaller circles are interior to the larger circle, then the centers of the three given circles are collinear. Let O, O_1, O_2 be the centers of the circles with radii a, b, and $a+b$, respectively. Let T be the center and r the radius of the circle tangent to these three. We apply Stewart's Theorem to triangle TO_1O_2 with cevian TO, obtaining

$$(OO_1)(TO_2)^2 + (OO_2)(TO_1)^2 = (O_1O_2)(TO)^2 + (OO_1)(OO_2)^2 + (OO_2)(OO_1)^2.$$

Using $OO_1 = b$, $OO_2 = a$, $O_1O_2 = a + b$, $TO_1 = r + a$, $TO_2 = r + b$, and $TO = a + b - r$, we have

$$b(r + b)^2 + a(r + a)^2 = (a + b)(a + b - r)^2 + ba^2 + ab^2.$$

Solving for r gives

$$r = \frac{ab(a + b)}{a^2 + b^2 + ab}.$$

CC78. Let $g(x) = x^3 + px^2 + qx + r$, where p, q and r are integers. Prove that if $g(0)$ and $g(1)$ are both odd, then the equation $g(x) = 0$ cannot have three integer roots.

Originally from 2001 Canadian Open Mathematics Challenge, problem B3b.

We present two solutions. Two solvers followed the strategy of the first solution and the rest provided the second solution.

Solution 1, by Šefket Arslanagić.

We prove a stronger generalization:

> Suppose that $n \geq 2$, $g(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + x_0$ where each a_i is an integer, and $g(0)$ and $g(1)$ are both odd. Then $g(x)$ does not have an integer root.

Suppose otherwise that $g(x)$ has an integer root n. Then $g(x) = (x - n)f(x)$ for some monic polynomial $f(x)$ with integer coefficients. But then $g(0) = -nf(0)$ and $g(1) = -(n-1)f(1)$ cannot both be odd since one of n and $n-1$ is even. The result follows.

Solution 2, by Matei Coiculescu.

Since $g(0) = r$ and $g(1) = 1 + p + q + r$ are both odd, then r and $p + q$ are both odd. Suppose that $g(x)$ has three integer roots a, b and c. Then $abc = -r$ is odd, so each of a, b, c is odd. Then so are $p = -(a+b+c)$ and $q = ab+bc+ca$, making $p + q$ even. But this is a contradiction and the result follows.

CC79. Show that if n is an integer greater than 1, then $n^4 + 4$ is not prime.

Originally question 2 of 1979 APICS Math Competition.

We present the solution by Edward Wang.

The solution to this problem comes directly from a particular case of the well-known Sophie Germain identity:

$$x^4 + 4y^4 = (x^2 + 2y^2)^2 - (2xy)^2 = (x^2 - 2xy + 2y^2)(x^2 + 2xy + 2y^2)$$

Setting $x = n$ and $y = 1$ yields $n^4 + 4 = (n^2 - 2n + 2)(n^2 + 2n + 2)$. Since $n > 1$, we have $1 < n^2 - 2n + 2 < n^2 + 2n + 2$, so $n^4 + 4$ is a composite.

Editor's comment. Edward Wang noticed that using similar arguments one can prove the following:

(i) For $a, b \in \mathbb{N}$, $a^4 + 4b^4$ is a prime iff $a = b = 1$.

(ii) For $n \in \mathbb{N}$, $n^4 + 4^n$ is a prime iff $n = 1$.

CC80. Alphonse and Beryl play a game involving n safes. Each safe can be opened by a unique key and each key opens a unique safe. Beryl randomly shuffles the n keys, and after placing one key inside each safe, she locks all of the safes with her master key. Alphonse then selects m of the safes (where $m < n$), and Beryl uses her master key to open just the safes that Alphonse selected. Alphonse collects all of the keys inside these m safes and tries to use these keys to open up the other $n - m$ safes. If he can open a safe with one of the m keys, he can then use the key in that safe to try to open any of the remaining safes, repeating the process until Alphonse successfully opens all of the safes, or cannot open any more. Let $P_m(n)$ be the probability that Alphonse can eventually open all n safes starting from his initial selection of m keys. Determine a formula for $P_2(n)$.

Originally 2014 Sun Life Financial Repèchage Competition, problem 2d).

No solutions to this problem were received.

THE OLYMPIAD CORNER

No. 324

Nicolae Strungaru and Carmen Bruni

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1er octobre 2015** *; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

La rédaction souhaite remercier André Ladouceur, Ottawa, ON d'avoir traduit les problèmes.

OC186. Démontrer que s'il existe des entiers non négatifs m, n, N et k qui vérifient l'équation

$$(n^2+1)^{2^k} \cdot (44n^3+11n^2+10n+2) = N^m,$$

il faut que $m = 1$.

OC187. Soit PA et PB des tangentes à un cercle de centre O et C un point sur le petit arc AB. La perpendiculaire à PC menée au point C coupe les bissectrices des angles AOC et BOC aux points respectifs D et E. Démontrer que $CD = CE$.

OC188. Déterminer la valeur maximale de M pour laquelle

$$a^3+b^3+c^3-3abc \geq M(ab^2+bc^2+ca^2-3abc)$$

pour tous les réels a, b et c strictement positifs.

OC189. Soit n et k des entiers strictement positifs, $n \geq k$. On considère n personnes. Chaque personne appartient à exactement un groupe, soit le groupe 1, le groupe $2, \cdots$, le groupe k et chaque groupe contient au moins une personne. Démontrer qu'il est possible de donner n^2 bonbons à n personnes de manière à satisfaire aux conditions suivantes :

1. Chaque personne reçoit au moins un bonbon.
2. Les entiers a_i représentent chacun le nombres de bonbons donnés à chaque personne du groupe i $(1 \leq i \leq k)$ de manière que si $1 \leq i < j \leq k$, alors $a_i > a_j$.

OC190. Soit a, m et n des entiers strictement positifs, $m \leq n$. Démontrer que si un des nombres a^m et a^n est divisible par m, l'autre nombre doit être divisible par n.

. . . .

OC186. Show that if for non-negative integers m, n, N, k, the equation

$$(n^2+1)^{2^k} \cdot (44n^3 + 11n^2 + 10n + 2) = N^m$$

holds, then necessarily $m = 1$.

OC187. Let PA and PB be tangents to a circle centered at O and C a point on the minor arc AB. The perpendicular from C to PC intersects internal angle bisectors of $\angle AOC$ and $\angle BOC$ at D and E. Show that $CD = CE$.

OC188. Find the maximum value of M for which for all positive real numbers a, b, c, we have

$$a^3 + b^3 + c^3 - 3abc \geq M(ab^2 + bc^2 + ca^2 - 3abc).$$

OC189. Let n, k be positive integers with $n \geq k$. There are n people, each person belongs to exactly one of group 1, group $2, \cdots$, group k and at least one person belongs to each group. Show that n^2 candies can be given to n people in such a way that all of the following conditions are satisfied:

1. at least one candy is given to each person;
2. a_i candies are given to each person belonging to group i $(1 \leq i \leq k)$ such that if $1 \leq i < j \leq k$, then $a_i > a_j$.

OC190. Let a, m and n be positive integers with $m \leq n$. Prove that if one of the numbers a^m and a^n is divisible by m, then the other number must be divisible by n.

Copyright © Canadian Mathematical Society, 2015

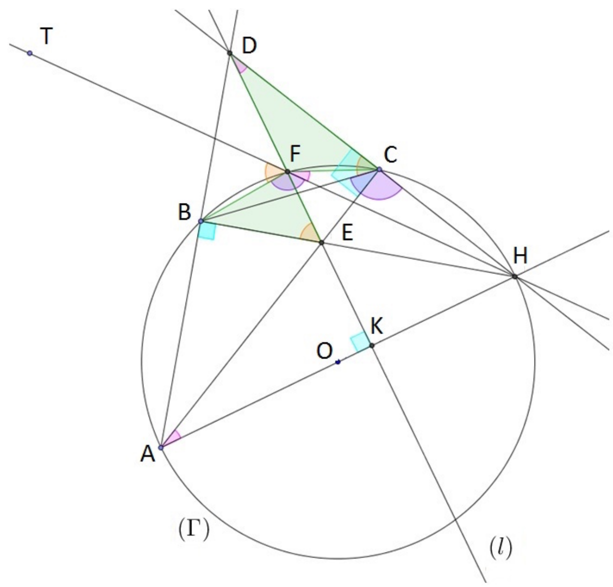
OLYMPIAD SOLUTIONS

OC126. Let A, B and C be points lying on a circle Γ with center O. Assume that $\angle ABC > 90°$. Let D be the point of intersection of the line AB with the perpendicular at C on AC. Let l be the perpendicular from D onto AO. Let E be the point of intersection of l with the line AC, and let F be the point of intersection of Γ with l that lies between D and E. Prove that the circumcircles of triangles BFE and CFD are tangent at F.

Originally problem 1 from 2012 Balkan Mathematical Olympiad.

We give the (translated) solution of Zouhair Ziani.

The solution follows from a few lemmas.



Lemma I. Lines BE, AO, and DC are concurrent at a point H belonging to the circle Γ and E is the orthocenter of ΔAHD.

Proof. Let H be the second intersection point of DC and Γ. We have $DC \perp AC$, and since H lies on DC we have $AC \perp CH$ thus $\angle ACH = 90°$. Since A, C, H all lie on Γ, we see that segment HA is a diameter of the circle Γ. Since O is the center of Γ, we conclude that O lies on segment AH and hence H is on line OA.

On the other hand, since segment AH is a diameter of Γ and B lies on Γ, we see that $BH \perp AB$. Since D lies on AB, we have $BH \perp AD$, and BH is a height of ΔAHD. Moreover, since $DE = \ell$, $\ell \perp OA$ and $OA = AH$ we see that DE is the height of ΔAHD. Knowing that ℓ and AC intersect at the point E, we conclude that E is the orthocenter of ΔAHD. Thus E lies on BH and consequently, H lies on BE. This completes the proof. ■

Lemma II. Point E lies inside ΔAHD (which is the orthocenter).

Proof. Since $\angle ACD = 90°$, we have that $\angle ADC + \angle DAC = 90°$. Thus $\angle ADC < 90° < \angle ABC$. Since A, B, and D are collinear, we have that B lies on segment AD.

From Lemma I, is known that C, D, and H are collinear. Since D and H are not on the same side of BC we have that C lies on segment DH ($90° = \angle ABH < \angle ABC < \angle ABD = 180°$). It is known that HB and AC are altitudes of ΔADH, so E, the orthocenter of ΔADH, is located inside the triangle. This completes the proof of the lemma. ■

Lemma III. Let K be the intersection of DE and AH. Then the quadrilateral $AKCD$ is convex.

Proof. We know that $DE \perp AH$, so $\angle AKD = 90°$, and $AC \perp CD$, so $\angle CDA = 90°$. Since C and K are the same side of AD by Lemma II, we conclude this lemma. ■

Lemma IV. Quadrilateral $DBEC$ is a convex cyclic quadrilateral.

Proof. Since $AC \perp CD$ and E lies on AC, we have $\angle EDC = 90°$. As HA is a diameter Γ and B lies on Γ we have $AB \perp BH$. Since E lies on BH and D lies on AB, then $BE \perp DB$. Therefore $\angle EBD = 90°$.

From Lemma II, it is observed that C and B are not on the same side of ED, and from the above two points the lemma is concluded. ■

To solve the problem, we use the above lemmas to show that circles BEF and CDF both have a common tangent line at F. According to Lemma II, we have C lies on DH, so $\angle BCD + \angle BCH = 180°$. According to Lemma II, we have $\angle BED = \angle BCD$. Since F lies on segment DE, we have $\angle BEF = \angle BCD$. Therefore $\angle BEF + \angle BCH = 180°$. From Lemma I, it is noted that B, C, F, H all lie on Γ. Since F lies on segment DE and C lies on segment DH by Lemma II and both E and H lie on BH, we have that F and C are on the same side as BH, and then $\angle BFH = \angle BCH$, therefore $\angle BEF + \angle BFH = 180°$. Now T is a point of the half-line opposite ray FH, different from F. Thus $\angle BFT + \angle BFH = 180°$, so $\angle BEF = \angle BFT$.

According to Lemma II, it is known that E lies on BH and both E and H are thus on the same side of BF, and hence T belongs to the opposite side.

Finally, according to the above two points, it is concluded that $FH = FT$ is a straight line tangent to the circle C given by points BEF at F.

From Lemma I we have that A, F, C, H all lie on Γ. Since F and A are on the same side of HC (E lies on segment AC from Lemma II and F lies on segment DE) we have $\angle CAH = \angle CFH$, and since K lies on AH we have $\angle CAK = \angle CFH$. According to Lemma III we have $\angle CAK = \angle CDK$ and therefore $\angle CDK = \angle CFH$. Since E lies on DK by Lemma II and F lies on segment DE, we have $\angle CDF = \angle CFH$.

Copyright © Canadian Mathematical Society, 2015

According to Lemma II we have C lies on DH, so D and H are not on the same side of CF. This and the above fact combine to conclude that the straight line FH is tangent to the circle CDF at F.

Hence circles BEF and CDF both have a common tangent line at F.

OC127. Let p and k be positive integers such that p is prime and $k > 1$. Prove that there is at most one pair (x, y) of positive integers such that $x^k + px = y^k$.

Originally problem 4 from 2012 South Africa National Olympiad, Senior Round 3.

No correct solutions were submitted.

OC128. Let n be a positive integer. Prove that the equation

$$\sqrt{x} + \sqrt{y} = \sqrt{n}$$

has solution (x, y) with x, y positive integers, if and only if n is divisible by some m^2 where $m > 1$ is an integer.

Originally question 3 from Day 2 of the 2012 Indonesia National Science Olympiad.

We give the solution of Michel Bataille.

If $n = am^2$ for some integers a, m with $a \geq 1$ and $m > 1$, then the pair of positive integers $(x, y) = (a(m-1)^2, a)$ is a solution to $\sqrt{x} + \sqrt{y} = \sqrt{n}$.

Conversely, we suppose that $\sqrt{x} + \sqrt{y} = \sqrt{n}$ where x and y are positive integers. Clearly, we must have $n \geq 4$. Let $d = \gcd(x, y, n)$ and x', y', n' be defined by $x = dx', y = dy', n = dn'$. Then $\gcd(x', y', n') = 1$ and $\sqrt{x'} + \sqrt{y'} = \sqrt{n'}$. Thus, we may (and will) suppose that $\gcd(x, y, n) = 1$ from the outset.

From the equation, we deduce $n = x + y + 2\sqrt{xy}$ and then $4xy = (n - x - y)^2$ which rewrites as

$$(x - y)^2 = n(2x + 2y - n). \tag{1}$$

If n is a power of 2, say $n = 2^\alpha$ where $\alpha \geq 2$ (since $n \geq 4$), then n is divisible by 2^2. Otherwise, n is divisible by an odd prime p. From (1), p also divides $x - y$. Should p divide $2x + 2y - n$, p would divide $2(x + y)$, hence $x + y$ and also x, y (as an odd divisor of both $x - y$ and $x + y$), in contradiction to $\gcd(x, y, n) = 1$. It follows that p^2, which divides $(x - y)^2$, must divide n. Thus n is also divisible by a square greater than 1 in this case.

OC129. Find all functions $f : (0, +\infty) \to (0, +\infty)$ which are onto and for which

$$2x \cdot f(f(x)) = f(x) \cdot (x + f(f(x))) \, \forall x \in (0, \infty) \, .$$

Originally question 3 from 2012 Brasil Mathematical Olympiad, Day 2.

We give the solution of Oliver Geupel.

The function $f(x) = x$ is a solution, and we show that it is unique.

Assume that f is any solution and x_1 is any positive real number. Then there is a positive number x_0 such that $x_1 = f(x_0)$ because f is onto. Also, the given functional equation with $x = x_0$ implies

$$2x_0 f(x_1) = x_1(x_0 + f(x_1));$$

whence

$$x_0(2f(x_1) - x_1) = x_1 f(x_1).$$

Since $x_1 f(x_1) \neq 0$, we also have $2f(x_1) - x_1 \neq 0$ and

$$x_0 = \frac{x_1 f(x_1)}{2f(x_1) - x_1}.$$

Hence, f is injective.

For $k \in \mathbb{Z}$, consider the numbers

$$a_k = \frac{1}{f^k(x_0)},$$

where $f^0(x) = x$, $f^{k+1}(x) = f(f^k(x))$, and $f^{k-1}(x) = f^{-1}(f^k(x))$. The given functional equation with $x = f^k(x_0)$ rewrites as

$$\frac{2}{a_k a_{k+2}} = \frac{1}{a_{k+1}}\left(\frac{1}{a_k} + \frac{1}{a_{k+2}}\right);$$

whence

$$a_{k+1} - a_k = a_{k+2} - a_{k+1}$$

for every integer k.

If $|a_{k+1} - a_k| > 0$ then not all of a_k can be positive, a contradiction. Thus, $a_2 - a_1 = 0$, so that

$$f(x_1) = x_1.$$

Since x_1 was arbitrarily chosen, the desired result follows. □

OC130. Initially there are $n+1$ monomials on the blackboard: $1, x, x^2, \ldots, x^n$. Every minute each of k boys simultaneously writes on the blackboard the sum of some two polynomials that were written before. After m minutes among others there are the polynomials $S_1 = 1+x, S_2 = 1+x+x^2, S_3 = 1+x+x^2+x^3, \ldots, S_n = 1 + x + x^2 + \ldots + x^n$ on the blackboard. Prove that $m \geq \frac{2n}{k+1}$.

Originally problem 4 from Day 1, Grade X level of the 2012 All Russia Olympiad.

No solutions were submitted.

Copyright © Canadian Mathematical Society, 2015

BOOK REVIEWS

Robert Bilinski

The proof and the pudding: What Mathematicians, Cooks, and You Have in Common by Jim Henle
ISBN: 9781400865680
Published by Princeton University Press, 2015, 164 pages

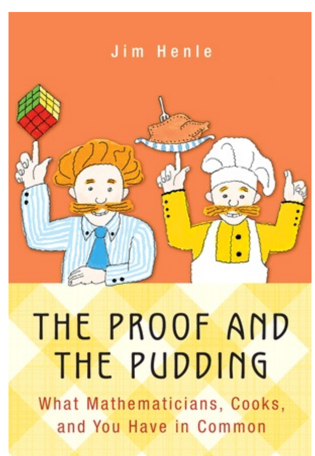
Reviewed by ***Robert Bilinski****, Collège Montmorency.*

Jim Henle is a mathematics teacher at Smith College in Massachusetts who specializes in set theory and logic. When I first saw his book, I thought it would be about mathematical aspects of dietetics or nutrition. This impression was quickly dashed as I read the first lines of the preface:

> The premise of this book is that if you look at Mathematics and gastronomy the right way, they are amazingly alike. The goal of this book is to explore the two and to reveal their essential similarity.

You could then wonder if the author will use stereotypes like "math is just following recipes" and you would still be wrong. Dr. Henle cooks in a messy, non-recipe-following way. It is more like problem solving : having a goal, being a bit stubborn and having a go at it. This strategy allows him to slowly build up parallels between being a cook and being a mathematician.

The book is structured a bit like a train running on two tracks. On the one side, we have anecdotes of his personal life centered on cooking, eating, partying and creating either new flavours or vegetarian "imitations" of classic meals. On the other, we follow a professor building knowledge on a mathematical problem that has been a pole of his recent research and his side project. You wouldn't know any of this from the titles of the chapters which are very metaphorical. Here are a few examples taken among the 25 chapters of the book : 1 - The Mad Scientist, 2 - The Arrogant Chef, 7 - Gluttony, 18 - Just to be Weird. Each chapter is like a train station where we explore the mindset of a mathematician and that of a cook. The aim of the book, the end of the train trip so to speak, is to hopefully convince the mathematically minded reader to start cooking. Hmm, not really, but it is less trodden than convincing "lay people" that mathematics is fun and doable by anyone, particularly someone who already cooks since then they can transversely transpose their cooking skills to do math. I really have a problem imagining a math phobic person reading this book and suddenly deciding to do mathematics, but I digress.



The main recurring mathematical problem that is explored throughout most of the chapters of the book is the "pool table" problem. In other words, we have a closed form (not necessarily convex or 2D), most often a rectangle, in which we study a path that bounces off the sides, usually with the normal reflection laws. Naturally, to fill up as many as 8 chapters with a single context, one basically has to ask many questions in that setting to nourish the text: "Is the path cyclical?", "What is the longest path?", "Does the path end in a corner?", etc. The tone of the presentation is very laid back and entertaining, even personal. The solutions are built step by step from particular cases, probably to make them more concrete and comprehensible by a wider audience. Interesting results are obtained and interesting parallels are made with other fields of mathematics, such as geometry, number theory and topology. The author definitely found a very rich problem for a popular math book. The book also offers a website and some bibliography to explore the presented problems in more detail thereby encouraging the reader to use the book as a stepping stone to further learning, an approach I endorse fully. On top of being a good read, the book offers a good overall reader experience.

The book is structured with one math problem or factoid per chapter with a corresponding anecdote around a recipe. Other problems which are explored are based on games like flip-out, sudoku, a new kind of game invented by the author and his students called "clueless sudoku", 1-2-3 takeaway, problems related to unconventional number representations, Cheney's card trick [1] (not to be confused with Dick Cheney's Iraq Illusion trick), the lamp problem [2] and other related problems. The factoids generally are not trivial, but make for cute math trivia.

The proposed mathematical trip is quite amusing to follow. As an added bonus, the book is full of recipes, most of them experiments by the author. I haven't tried the recipes, but having read them and knowing a thing or two about cooking, I find that they are full of originality and will become side projects in the future (ex-benedict, blue pizza, loxitaw, etc.). This book will surely please problem solvers, specifically teenage ones, who have never explored the aforementioned problems. The laid back approach may not be enough for the hardcore problem solvers though, but it might make a good gift for a mathematically gifted student who may not want to continue into math. All in all, this book is written from the heart and will please anyone that needs convincing that math is also an emotional pursuit and not a sterile field.

Happy reading!

References

[1] Colm Mulcahy, Fitch Cheney's Five Card Trick and Generalizations, Math Horizons, February 2003, pp. 10-13.

[2] Laurent Bartholdi, lamps, factorizations, and Finite Fields , American Mathematical Monthly 187, no. 5 (2000), pp. 429-436.

Copyright © Canadian Mathematical Society, 2015

Sums of equal powers of natural numbers

V. S. Abramovich

If someone asked you if you can calculate the sum of equal powers of consecutive natural numbers, you'd probably shrug and say sure, everyone can add. But there are many different ways to add. You probably know the story about young Gauss: while his classmates were labouring over calculating the sum $1 + 2 + \ldots + 100$ term by term in order, he noticed that $1 + 100 = 2 + 99 = \ldots = 50+51 = 101$ and quickly got the answer of $100 \cdot 101/2 = 5050$. [Ed.: Many authors describe the Gauss classroom story, sometimes with different details. Check out this article in American Scientist `http://www.americanscientist.org/issues/pub/gausss-day-of-reckoning` for more historical details.]

Of course computing sums of squares, cubes, fourth powers, and so on of natural numbers is quite a bit harder. In this article, we will consider three ways of constructing summation formulas that you can use to calculate any such sum. The sums we will be dealing with are of the form

$$S_q(n) = 1^q + 2^q + \cdots + n^q, \tag{1}$$

where q is a nonnegative integer; so, for example

$$S_2(4) = 1^2 + 2^2 + 3^2 + 4^2 = 30 \quad \text{and} \quad S_{10}(2) = 1^{10} + 2^{10} = 1025.$$

Clearly, $S_0(n) = n$ for any $n \in \mathbb{N}$.

1 What about the binomial theorem?

One technique we can use when calculating sums as in (1) is the binomial theorem. Consider it in the following form:

$$(k-1)^{q+1} = k^{q+1} - \binom{q+1}{1} k^q + \binom{q+1}{2} k^{q-1} - \cdots + (-1)^q \binom{q+1}{q} k - (-1)^q. \tag{2}$$

Let us now manipulate the sum (2) in the following way: move the first term of the right side to the left, let $k = 1, 2, \ldots n$, add all the resulting expressions and slightly massage them to get (using notations from (1))

$$n^{q+1} = \binom{q+1}{1} S_q(n) - \binom{q+1}{2} S_{q-1}(n) + \cdots - (-1)^q \binom{q+1}{q} S_1(n) - (-1)^q S_0(n).$$

Solving this equation for $S_q(n)$, we have:

$$S_q(n) = \frac{1}{q+1}\left(n^{q+1} + \binom{q+1}{2} S_{q-1}(n) - \cdots - (-1)^q S_0(n)\right). \tag{3}$$

Setting $q = 1$ gives $S_1(n) = \frac{1}{2}\left(n^2 + \binom{2}{2}S_0(n)\right) = \frac{1}{2}(n^2 + n)$. Now, substitute this expression for $S_1(n)$ as well as $q = 2$ into (3):

$$S_2(n) = \frac{1}{3}\left(n^3 + \binom{3}{2}S_1(n) - S_0(n)\right) = \frac{1}{3}\left(n^3 + \frac{3}{2}n^2 + \frac{1}{2}n\right).$$

Similarly, using (3), we can derive expressions for $S_3(n), S_4(n)$ and so on.

To experience the efficacy of these formulas, simply perform some calculations. For example, using the derived expression for $S_2(n)$, a complicated and clumsy sum $1^2 + 2^2 + \ldots + 100^2$ can be quickly computed as $S_2(100) = 338350$. Can you imagine how long it would take you to actually compute this sum term by term?

2 One term is better than q terms

From Section 1 we see that you can use (3) to obtain the formulas for computing $S_q(n)$, $q = 1, 2, \ldots$. Such a relation where each term of the sequence is defined using its preceding terms is called a *recurrence relation*. Clearly, the best recurrence relations are the ones that contain as few preceding terms as possible, preferably just the one immediately preceding the term you are computing. Luckily for us, we can find such a minimal recurrence relation for $S_q(n)$.

Let $P_m(n)$ be the following polynomial in terms of n of degree m:

$$P_m(n) = a_0n^m + a_1n^{m-1} + \cdots + a_{m-1}n + a_m.$$

Let $P_m^*(n)$ be the degree $m+1$ polynomial that you get by replacing n^k in $P_m(n)$ with the expressions

$$\frac{n^{k+1} - n}{k+1}, \quad \text{for} \quad k = 0, 1, \ldots m.$$

The following relation is fairly straighforward: given two polynomials $P(n)$ and $Q(n)$,

$$(P(n) + Q(n))^* = P(n)^* + Q(n)^*. \tag{4}$$

Theorem 1 *The following two statements are true.*

1. The sum $S_q(n)$ is a polynomial in terms of n of degree $q+1$ not containing the constant term.
2. The sum $S_q(n)$ can be defined by the recurrence

$$S_q(n) = n + S_{q-1}^*(n), \quad \textit{with } S_{-1} = 0. \tag{5}$$

Proof. We prove both assertions using mathematical induction.

1. Since $S_0(n) = n$, the recursion (5) is true for $q = 0$. Suppose it is also true for $S_0(n), S_1(n), \ldots, S_{q-1}(n)$. Then, plugging into the right-hand side of (3), we get a polynomial of degree $q+1$ with no constant term. Therefore, the expression holds for $S_q(n)$ and by induction it is true for all $q \geq 0$.

Copyright © Canadian Mathematical Society, 2015

2. Clearly, (5) holds for $q = 1$:

$$S_1(n) = n + n^* = n + \frac{n^2 - n}{2} = \frac{n(n+1)}{2}.$$

Suppose (5) holds for $q = 1, 2, \ldots, m-1$. Substituting $q = m$ into (3), we get:

$$\begin{aligned} S_m(n) &= \frac{1}{m+1}\left[n^{m+1} + \binom{m+1}{2}[n + (m-1)S^*_{m-2}(n)] - \cdots (-1)^m \binom{m+1}{m+1} n\right] \\ &= \frac{1}{m+1}\left[n^{m+1} + n\left(\binom{m+1}{2} - \binom{m+1}{3} + \cdots + (-1)^{m+1}\binom{m+1}{m+1}\right)\right. \\ &\qquad \left. +(m-1)\binom{m+1}{2} S^*_{m-2}(n) - \cdots + (-1)^m \binom{m+1}{m} S^*_0(n)\right]. \end{aligned}$$

Now note that since

$$1 - \binom{m+1}{1} + \binom{m+1}{2} - \cdots + (-1)^{m+1}\binom{m+1}{m+1} = 0,$$

we get

$$\binom{m+1}{2} - \binom{m+1}{3} + \cdots + (-1)^{m+1}\binom{m+1}{m+1} = \binom{m+1}{1} - 1 = m,$$

Also noting that

$$\binom{m+1}{k}\frac{m+k-1}{m+1} = \binom{m}{k}$$

and using (4), we finally get the following expression for $S_m(n)$:

$$S_m(n) = \frac{n^{m+1} + nm}{m+1} + \left[\binom{m}{2} S_{m-2}(n) - \cdots + (-1)^m \binom{m}{m} S_0(n)\right]^*.$$

Setting $q = m - 1$ in (3), we see that the expression in square brackets above equals $mS_{m-1}(n) - n^m$ and thus

$$\begin{aligned} S_m(n) &= \frac{n^{m+1} + nm}{m+1} + [mS_{m-1}(n) - n^m]^* \\ &= \frac{n^{m+1} + nm}{m+1} + mS^*_{m-1}(n) - \frac{n^{m+1} - n}{m+1} \\ &= n + mS^*_{m-1}(n). \end{aligned}$$

Therefore the relation holds for $q = m$ and the proof is complete. ■

Example 1 *Given $S_3 = \frac{1}{4}(n^4 + 2n^3 + n^2)$ (check this), we can use Theorem 1 to find $S_4(n)$:*

$$\begin{aligned} S_4(n) &= n + 4S^*_3(n) = n + (n^4 + 2n^3 + n^2)^* \\ &= n + \frac{n^5 - n}{5} + 2\cdot\frac{n^4 - n}{4} + \frac{n^3 - n}{3} = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}. \end{aligned}$$

3 The general formula

The binomial theorem can give us much more than (3). For example, consider the following transformations, where $m, n \in \mathbb{N}$:

$$m^n - 1 = ((m-1)+1)^n - 1 = (m-1)\binom{n}{1} + (m-1)^2\binom{n}{2} + \dots + (m-1)^n\binom{n}{n}.$$

Suppose $m \neq 1$ and divide both sides by $m-1$:

$$1 + m + m^2 + \ldots + m^{n-1} = \binom{n}{1} + (m-1)\binom{n}{2} + \dots + (m-1)^{n-1}\binom{n}{n}. \quad (6)$$

Problem 1 *Prove that*

$$a_0 + a_1 + \ldots + a_{n-1} = a_0\binom{n}{1} + (a_1 - a_0)\binom{n}{2} + (a_2 - 2a_1 + a_0)\binom{n}{3} + \cdots$$
$$+ \left[a_{n-1} - a_{n-2}\binom{n-1}{1} + a_{n-3}\binom{n-1}{2} - \dots + a_0(-1)^{n-1}\right]\binom{n}{n}. \quad (7)$$

Theorem 2 *In the general case, we have the following formula for $S_q(n)$:*

$$S_q(n) = \binom{n}{1} + (2^q - 1)\binom{n}{2} + (3^q - 2\cdot 2^q + 1)\binom{n}{3} + \cdots$$
$$+ \left((q+1)^q - \binom{q}{1}q^q + \binom{q}{2}(q-1)^q - \dots + (-1)^q\right)\binom{n}{q+1}. \quad (8)$$

Sketch of the proof. In (7), let $a_k = (k+1)^q, k = 0, 1, \ldots n-1$ and first consider $n \le q+1$. Since in that case $\binom{n}{p} = 0$ for $p > q+1$, (8) follows from (7). However, both sides of the expression are polynomials in n of degree $q+1$ with no constant terms, meaning they have to coincide for all n. ■

Formula (8) gives $S_q(n)$ as a linear combination of $\binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{q+1}$:

$$S_0(n) = \binom{n}{1}, \qquad S_1(n) = \binom{n}{1} + \binom{n}{2},$$
$$S_2(n) = \binom{n}{1} + 3\binom{n}{2} + 2\binom{n}{3}, \quad S_3(n) = \binom{n}{1} + 7\binom{n}{2} + 12\binom{n}{3} + 6\binom{n}{4},$$

and so on.

We now offer you, the reader, some exercises. Note that, for convenience, we use notation S_q to denote $S_q(n)$.

Problem 2 *Show that*

$$2S_1^2 = 2S_3,$$
$$4S_1^3 = 3S_5 + S_3,$$
$$8S_1^4 = 4S_7 + 4S_5,$$

Copyright © Canadian Mathematical Society, 2015

and, in general, for $k = 1, 2, \ldots$

$$2^{k-1}S_1^k = \binom{k}{1}S_{2k-1} + \binom{k}{3}S_{2k-3} + \binom{k}{5}S_{2k-5} + \cdots .$$

Moreover, the last term on the right-hand side of the above expression equals S_k *or* kS_{k-1} *for odd and even* k*, respectively.*

Problem 3 *Show that for* $k = 1, 2, \ldots$

$$3 \cdot 2^{k-1}S_2S_1^{k-1} = \left(\binom{k}{0} + 2\binom{k}{1}\right)S_{2k} + \left(\binom{k}{2} + 2\binom{k}{3}\right)S_{2k-2} + \cdots .$$

Moreover, the last term on the right-hand side of the above expression is equal to $(k+2)S_{k+1}$ *or* S_k *for odd and even* k*, respectively.*

Problem 4 *Show that*

$$S_3 = S_1^2,$$
$$S_5 = S_1^2 \cdot \frac{4S_1 - 1}{3},$$
$$S_7 = S_1^2 \cdot \frac{6S_1^2 - 4S_1 + 1}{3},$$

and, in general, S_{2k-1} *(for* $2k - 1 \geq 3$*) is a polynomial in terms of* $S_1 = \frac{n(n+1)}{2}$ *of degree* k *divisible by* S_1^2*.*

Problem 5 *Show that*

$$S_4 = S_2 \cdot \frac{6S_1 - 1}{5},$$
$$S_6 = S_2 \cdot \frac{12S_1^2 - 6S_1 + 1}{7},$$
$$S_8 = S_2 \cdot \frac{40S_1^3 - 40S_1^2 + 18S_1 - 3}{15},$$

and, in general, S_{2k}/S_2 *is a polynomial in terms of* S_1 *of degree* k*.*

Problem 6 *Prove that for* $k \geq 1$*, we have* $S_k(-x-1) = (-1)^{k-1}S_k(x)$*.*

Problem 7 *Find the sum* $1 + 27 + 125 + \cdots + (2n-1)^3$*.*

Problem 8 *Find the sum* $2^2 + 5^2 + 8^2 + \cdots + (3n-1)^2$*.*

This article originally appeared in Kvant, 1973 (5). It has been translated and adapted with permission.

UNSOLVED CRUX PROBLEMS

As remarked in the problem section, no problem is ever closed. We always accept new solutions and generalizations to past problems. Chris Fisher published a list of unsolved problems from ***Crux*** *[2010 : 545, 547]. Below is a sample of two of these unsolved problems.*

1615. [1991: 44; 1992: 82–83]

Proposed by Clark Kimberling, University of Evansville, Evansville, IN, USA.

Consider the following array:

$$\begin{array}{cccccccccccc}
\mathbf{1} & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \ldots \\
2 & \mathbf{3} & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & \ldots \\
4 & 2 & \mathbf{5} & 6 & 7 & 8 & 9 & 10 & 11 & 12 & \ldots \\
6 & 2 & 7 & \mathbf{4} & 8 & 9 & 10 & 11 & 12 & 13 & \ldots \\
8 & 7 & 9 & 2 & \mathbf{10} & 6 & 11 & 12 & 13 & 14 & \ldots \\
6 & 2 & 11 & 9 & 12 & \mathbf{7} & 13 & 8 & 14 & 15 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

For example, to produce row 5 from row 4, write down, in order: the 1st number to the right of 4, the 1st number to the left of 4, the 2nd to the right of 4, the 2nd to the left of 4, the 3rd to the right of 4, the 3rd to the left of 4, and then the 4th, 5th,... numbers to the right of 4. Notice that a number will be expelled from a row if and only if it is the diagonal element in the previous row (the bold numbers in the array), and once missing it of course never appears.

a) Is 2 eventually expelled? [Ed.: see Volume 18, 1992, p. 82-83.]

b)$\star$ Is every positive integer eventually expelled?

2399$\star$. [1998: 506; 1999: 531]

Proposed by David Singmaster, South Bank University, London, England.

In James Dodson's *The Mathematical Repository*, 2nd ed., J. Nourse, London, 1775, pp 19 and 31, are two variations on the classic "Ass and Mule" problem:

> What fraction is that, to the numerator of which 1 be added, the value will be 1/3; but if 1 be added to the denominator, its value is 1/4?

This is easily done and it is easy to generalize to finding x/y such that $(x+1)/y = a/b$ and $x/(y+1) = c/d$, giving $x = c(a+b)/(ad-bc)$ and $y = b(c+d)/(ad-bc)$. We would normally take $a/b > c/d$, so that $ad - bc > 0$, and we can also assume a/b and c/d are in lowest terms.

Copyright © Canadian Mathematical Society, 2015

> A butcher being asked, what number of calves and sheep be had bought, replied, 'If I had bought four more of each, I should have four sheep for every three calves; and if I had bought four less of each, I should have three sheep for every two calves'. How many of each did he buy?

That is, find x/y such that $(x+4)/(y+4) = 4/3$ and $(x-4)/(y-4) = 3/2$. Again, this is easily done and it is easy to solve the generalization, $(x+A)/(y+A) = a/b$ and $(x-A)/(y-A) = c/d$, getting $x = A(2ac-bc-ad)/(bc-ad)$ and $y = A(ad+bc-2bd)/(bc-ad)$. We would normally take $a/b < c/d$ so that $bc-ad > 0$, and we can also assume a/b and c/d are in lowest terms.

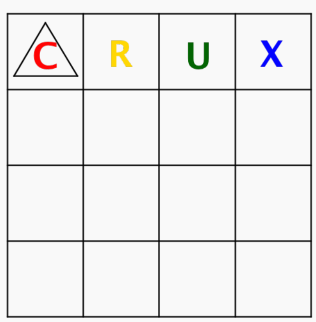
In either problem, given that a, b, c and d are integers, is there a condition (simpler than computing x and y) to ensure that x and y are integers?

Alternatively, is there a way to generate all the integer quadruples a, b, c, d, which produce integer x and y?

A magic Crux square

Given a 4 by 4 square, can you fill out its entries with letters C, R, U, X enclosed in 4 different shapes (square, circle, triangle and rhombus) and coloured with 4 different colours (red, yellow, green and blue) so that the following conditions hold:

- every row contains every letter, every shape and every colour;
- every letter appears exactly once in an entry of any given colour;
- every shape contains every letter and every colour.



Inspired by cover of Van Lint and Wilson's "A Course in Combinatorics", a Kvant problem and designs in general.

Exercises in Inductive Processes

Olga Zaitseva

A process is a sequence of operations or steps applied to an object. So-called *inductive processes* are often used in problems when one is asked "to prove that (something) is possible", to create a strategy or to build an algorithm, to estimate a variable, to gradually simplify a problem and to construct an object with the given properties. As the first example, let us consider a problem which many readers probably know from childhood.

Problem 1 *A platoon of soldiers needs to cross a river. The soldiers flag down two boys sailing on a small boat, which can only hold either two boys or one soldier at a time. How can the platoon cross the river?*

Solution. Assume that there are n soldiers and they are all initially on a shore A. Two boys sail to destination shore B, one boy stays there while the second boy sails to shore A. He then remains on shore A while one of the soldiers sails to shore B. The boy on shore B sails to shore A to pick up the other boy. Now, we have the original problem, the only difference is that the number of soldiers on the shore A is decreased by 1. It is clear that if this process will be repeated $n-1$ times, all soldiers will be on destination shore B. □

Problem 2 *Ali Baba and the 40 thieves want to cross Bosporus strait. They make a line so that any two people standing next to each other are friends. Ali Baba is the first man in line; he is friends with the thief next to him and his neighbour. There is a single boat that can carry 2 or 3 people at a time, but these people must all be friends. Can Ali Baba and the 40 thieves always cross the strait if a single person cannot sail? (35th Tournament of Towns, Spring of 2014, O-Level, Juniors.)*

Solution. The answer is "yes" and, as in the first problem, let us organize a process showing how to do it.

Assume that the number of thieves is n and the line of thieves is $A, T_1, T_2, \ldots, T_n$. Since any two people standing next to each other must be friends, we know that the thief T_i is a friend with T_{i-1} and T_{i+1}, but they both could be his only friends in the whole gang; also note that T_{i-1} and T_{i+1} could be foes. We refer to the initial shore as Asia and destination shore as Europe (Bosporus strait separates Asia and Europe).

Consider the case of n even. First, A, T_1, T_2 sail to Europe (it is the only case where we know three thieves can sail together), and A, T_1 sail back leaving T_2 in Europe. Then T_3, T_4 sail to Europe and T_2, T_3 sail back, leaving T_4 in Europe (so T_4 replaces T_2). Continuing this process (T_{2k-1}, T_{2k} sail to Europe and T_{2k-2}, T_{2k-1} come back), we end up with T_n in Europe while $A, T_1, \ldots, T_{n-1}$ and the boat are in Asia.

In case when n is odd, we slightly change the pattern of sailing: A, T_1, T_2 sail to Europe, then A, T_2 sail back leaving T_1 in Europe. Then T_2, T_3 sail to Europe, T_1, T_2

Copyright © Canadian Mathematical Society, 2015

sail back and now T_3 is in Europe. Continuing this process (T_{2k}, T_{2k+1} sail to Europe and T_{2k-1}, T_{2k} sail back), we end up with T_n in Europe, and $A, T_1, \ldots, T_{n-1}$ with the boat are in Asia.

Therefore, starting with n thieves on initial shore, we get the same problem as the original one for $n-1$, $n-2$, and so on. When we have only three thieves (A, T_1, T_2) remain in Asia, we know that they all can sail to Europe. □

When organizing a process, one must be sure that it eventually stops. Although there are cases when it is obvious (for instance, in previous examples), there are also cases when one needs to justify it. Usually it is proven by constructing a so-called semi-invariant, that is a variable that changes only in one direction. Let us consider examples of these problems.

Problem 3 *Suppose you have a natural number each digit of which is either* 0 *or* 1. *You can replace any occurrence of "10" within it by "00" or "01". Prove that this process cannot continue forever.*

Solution. Observe that no matter the choice of replacement, each operation decreases the number. Since a natural number cannot be decreased infinitely, the process eventually stops. □

The solution of the next problem (Tournament of Towns 2000, Spring, A-Level, Seniors) is based on the solution of the previous problem.

Problem 4 *In a deck of 52 cards, some are face up and some are face down. Pete takes out one or several cards such that both the top and the bottom cards are face up, turns over his selection and puts it back in at the same place. (In case of one card, the card should be face up). Prove that eventually all cards will be face down.*

HINT: given the deck of cards, assign 0 to every card facing down and 1 for every card facing up.

Problem 5 *Each lamp in a rectangular array is either in "on" or "off" position. You can simultaneously change the positions of all lamps in any row or any column of the array. Prove that applying this operation several times, you can always get an array such that in every column and every row no less than half of the lamps are in "on" position.*

Solution. Let us organize the process: if a row/column has less than half the lamps in "on" position, then we apply the operation to this row/column. We now need to show that after a finite number of operations, we reach the required condition. At any moment, let us consider S, the number of lamps in the array in "on" position. Clearly, S is a positive integer and S is increasing with every operation. It is also clear that S is bounded from above by the total number of lamps. Therefore, the process eventually stops. □

Now, let us consider an application of an inductive process to the class of problems when one needs to estimate a variable or to gradually simplify a problem: we call this approach a Bulldog-Rhino principle. Assume we know how to fit a rhino into a fridge. Can we fit a bulldog into a fridge of the same size if we do not know which of the two animals is larger? If we can prove that the bulldog is no bigger than a jackal, which is no bigger than a lion, which is no bigger than a rhino, then the answer to the original problem is yes! Let us consider some examples.

Problem 6 *Compare* $\sqrt{(x^2+y^2)/2}$ *and* $2/(1/x+1/y)$*, where* x *and* y *are positive real numbers.*

Solution sketch. Let us first prove that $\sqrt{(x^2+y^2)/2} > (x+y)/2$ (*Root Mean Square – Arithmetic Mean Inequality*). We compare $\sqrt{(x^2+y^2)/2}$ and $(x+y)/2$, which is equivalent to comparing the following expressions:

$$\begin{array}{rcl} (x^2+y^2)/2 & \text{versus} & (x+y)^2/4 \\ (x^2+y^2)/2 & \text{versus} & (x^2+2xy+y^2)/4 \\ 2(x^2+y^2) & \text{versus} & x^2+2xy+y^2 \\ (x^2+y^2) & \text{versus} & 2xy. \end{array}$$

The latter is the well-known *Cauchy inequality* $(x^2+y^2) \geq 2xy$, which follows directly from the fact that $(x-y)^2 \geq 0$.

We leave it to the reader to prove that $(x+y)/2 \geq \sqrt{xy}$ (*Arithmetic Mean–Geometric Mean inequality*) and that $\sqrt{xy} \geq 2/(1/x+1/y)$ (*Geometric Mean–Harmonic Mean inequality*) and to complete the proof. □

Problem 7 *Pick four points in a square with side lengths 1. Prove that there are two points at distance of no more than 1 away from each other.*

Solution idea. Gradually transform the problem by increasing pairwise distances between points so that eventually all four points lie on the boundaries of square. The latter problem is simpler and can be solved algebraically. Then, if there are two points on the boundary at distance of no more than 1 away from each other, it is obviously true for the original set of four points.

We now offer the reader several problems on which to exercise the above techniques on.

1. There are several gas stations along the Big Circle Road. The total gas supply at these stations is enough to circle the road. Prove that a vehicle with an empty tank can start at any one station and circle the entire road while filing up with gas along the way. [Ed.: For an entertaining take on this problem and its solution, see an article by Jason Siefken in CMS Student Committee's *Notes from the Margin*, Volume VIII, at `http://issuu.com/cms-studc/docs/margin-fall2014`.]

2. A hare hides at one of the vertices of a cube such that the three hunters can not see it. Each time hunters choose three vertices and shoot simultaneously. If the hare happens to be at one of these vertices, it is hit; otherwise, the hare escapes to an adjacent vertex. Is there a strategy the hunters can follow to guarantee hare for dinner?

Copyright © Canadian Mathematical Society, 2015

3. There are n bushes in a row and a hare hides under one of them. A hunter cannot see it and shoots at any bush of his choice. If the hare happens to be under that bush, it is hit; otherwise, the hare escapes to a neighbouring bush. Is there a strategy that guarantees that hunter can hit the hare?

4. In a group of n people, everyone challenges exactly one person to a duel. Prove that the group can be split into three smaller groups such that in each group no one has challenged anyone else.

5. Each of n identical jars is filled with paint to $(n-1)/n$ of its volume. No two jars contain the same kind of paint. You can pour any amount of paint from one jar to another jar. Prove that you can get the same mixture of paint in every jar. (Paint is not disposable and there are no other empty vessels.)

6. During the school year, Andrew recorded his marks in maths. He called his upcoming mark (2, 3, 4, or 5) *unexpected* if, until the moment he received it, it appeared less often than any other possible mark. For instance, if he had marks $3, 4, 2, 5, 5, 5, 2, 3, 4, 3$ on his list, then unexpected marks would be the first 5 and then 4. It happened that at the end of the year Andrew's record contained forty marks and each possible mark was repeated exactly 10 times (the order of marks is unknown). Is it possible to determine the number of unexpected marks Andrew received? (Tournament of Towns 2014, Fall, O-Level, Juniors.)

7. There are 100 pots with honey, each containing no more than 1/10-th of the total amount. Every day Winnie the Pooh chooses 10 pots and eats the same amount of honey from each pot. Prove that Winnie can eat all the honey in a finite number of days. (Tournament of Towns 2005, Fall, A-Level, Juniors.)

8. Anna, Ben and Chris sit at a round table passing nuts. Initially, Anna has all the nuts. She divides them equally between Ben and Chris and eats a leftover if there is any. Then Ben does the same with his pile, then Chris with his. The game continues infinitely: each child divides his/her pile of nuts equally between two other children and eats a leftover if there is any. Given that the initial number of nuts is greater than 3, prove that at least one nut is eaten. (Tournament of Towns 2000, Spring, A-Level, Seniors.)

PROBLEMS

Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème présenté dans cette section. De plus, nous les encourageons à soumettre des propositions de problèmes. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1er octobre 2015***; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

Un astérisque ($\star$) *signale un problème proposé sans solution.*

La rédaction souhaite remercier Rolland Gaudet, Université de Saint-Boniface, d'avoir traduit les problèmes.

3951. *Proposé par Michel Bataille.*

Soit n un entier positif. Évaluer l'expression suivante, en forme close

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n+k}{2k}k\cdot 2^{2k}.$$

3952. *Proposé par Bill Sands.*

Trois points se trouvent sur un terrain plat, avec coordonnées $O(0,0), A(-a,0)$ et $B(0,-b)$, où $a>b>0$. Une montgolfière se trouve à une certaine distance en haut du point X sur le terrain, où $X=(x,-x)$ pour un certain x réel. À un observateur dans la mongolfière, le triangle OAB “paraît” équilatéral. Démontrer que

$$x=\frac{ab(a+b)^2}{2(a^3-b^3)}$$

et déterminer l'altitude de la montgolfière.

3953. *Proposé par An Zhen-ping.*

Soit ABC un triangle avec côtés a,b et c et surface Δ. Démontrer que

$$a^2\cos\frac{\angle B-\angle C}{2}+b^2\cos\frac{\angle C-\angle A}{2}+c^2\cos\frac{\angle A-\angle B}{2}\geq 4\sqrt{3}\Delta.$$

3954. *Proposé par Dao Hoang Viet.*

Résoudre l'inégalité

$$\sqrt{4x^2-8x+5}+\sqrt{3x^2+12x+16}\geq 6\sqrt{x}-x-6.$$

Copyright © Canadian Mathematical Society, 2015

3955. *Proposé par Roy Barbara.*

Soit p un nombre premier. Supposer que $\sqrt{p} = a + b$, où a et b sont des nombres algébriques réels. Prouver ou prouver le contraire de l'affirmation suivante : $\sqrt{p}$ doit appartenir à au moins un des corps $\mathbb{Q}(a)$ et $\mathbb{Q}(b)$.

3956. *Proposé par Dragoljub Milošević.*

Soient m_a, m_b et m_c les longueurs des médianes et w_a, w_b et w_c les longueurs des bissectrices d'angles, puis r et R les rayons des cercles inscrit et circonscrit. Démontrer que

a) $\dfrac{m_a}{w_a} + \dfrac{m_b}{w_b} + \dfrac{m_c}{w_c} \leq 1 + \dfrac{R}{r}$ est vrai pour tout triangle aigu,

b) $\dfrac{13}{4} - \dfrac{r}{2R} \leq \dfrac{m_a}{w_a} + \dfrac{m_b}{w_b} + \dfrac{m_c}{w_c}$ est vrai pour tout triangle.

3957. *Proposé par Zhihan Gao, Edward Wang et Dexter Wei.*

Un total de n^2 lampes forment un grillage carré $n \times n$. Chaque lampe possède un commutateur spécial qui change le satut de la lampe, et en même temps le statut de toutes les lampes dans la même rangée et de toutes les lampes dans la même colonne, d'allumé à éteint ou vice-versa. Ceci sera dénoté "opération". Initialement, toutes les lampes sont éteintes.

Démontrer qu'il existe une suite finie d'opérations permettant d'allumer toutes les n^2 lampes ; déterminer la valeur de $f(n)$ qui représente le plus petit nombre d'opérations requises pour que les n^2 lampes soient allumées.

3958. *Proposé par Marcel Chiriţă.*

Soient $A, B \in M_2(\mathbb{R})$, c'est-à-dire que A et B sont des matrices 2×2 de nombres réels. Démontrer que si $BA = 0$ et $\det(A^2 + AB + B^2) = 0$, alors $\det(A + xB) = 0$ pour tout nombre réel positif x.

3959. *Proposé par Dragoljub Milošević.*

Supposer que dans $\triangle ABC$, on a $\angle A = 40°$ et $\angle B = 60°$. Si $\dfrac{BC}{AC} = k$, démontrer que $3k^3 - 3k + 1 = 0$.

3960. *Proposé par George Apostolopoulos.*

Soient a, b, c des nombres réels non négatifs tels que $a + b + c = 4$. Démontrer que

$$\frac{a^2 b}{3a^2 + b^2 + 4ac} + \frac{b^2 c}{3b^2 + c^2 + 4ab} + \frac{c^2 a}{3c^2 + a^2 + 4bc} \geq \frac{1}{2}.$$

. . . .

3951. *Proposed by Michel Bataille.*

Let n be a positive integer. Evaluate in closed form

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n+k}{2k}k\cdot 2^{2k}.$$

3952. *Proposed by Bill Sands.*

Three points O, A, B are on a flat field at coordinates $O(0,0), A(-a,0), B(0,-b)$, where $a > b > 0$. A balloon is hovering some distance above a point X in the field, where $X = (x, -x)$ for some real number x. To an observer in the balloon, the triangle OAB "looks" equilateral. Show that

$$x = \frac{ab(a+b)^2}{2(a^3-b^3)}\ ,$$

and find the height of the balloon.

3953. *Proposed by An Zhen-ping.*

Let ABC be a triangle with sides a, b, c and area Δ. Prove that

$$a^2\cos\frac{\angle B-\angle C}{2}+b^2\cos\frac{\angle C-\angle A}{2}+c^2\cos\frac{\angle A-\angle B}{2}\geq 4\sqrt{3}\Delta.$$

3954. *Proposed by Dao Hoang Viet.*

Solve the inequality

$$\sqrt{4x^2-8x+5}+\sqrt{3x^2+12x+16}\geq 6\sqrt{x}-x-6.$$

3955. *Proposed by Roy Barbara.*

Let p be a prime number. Suppose that $\sqrt{p} = a + b$, where a and b are algebraic real numbers. Prove or disprove that $\sqrt{p}$ must lie in (at least) one of the fields $\mathbb{Q}(a)$, $\mathbb{Q}(b)$.

3956. *Proposed by Dragoljub Milošević.*

Let m_a, m_b and m_c be the lengths of medians, w_a, w_b and w_c be the lengths of the angle bisectors, r and R be the inradius and the circumradius. Prove that

a) $\dfrac{m_a}{w_a}+\dfrac{m_b}{w_b}+\dfrac{m_c}{w_c}\leq 1+\dfrac{R}{r}$ is true for all acute triangles,

b) $\dfrac{13}{4}-\dfrac{r}{2R}\leq\dfrac{m_a}{w_a}+\dfrac{m_b}{w_b}+\dfrac{m_c}{w_c}$ is true for triangles.

Copyright © Canadian Mathematical Society, 2015

3957. *Proposed by Zhihan Gao, Edward Wang and Dexter Wei.*

There are n^2 lamps arranged in an $n \times n$ array. Each lamp is equipped with a switch, which, when pressed, will change the status of the lamp as well as all the lamps in the same row and the same column from on to off or vice versa. This is called an "operation". Initially, all n^2 lamps are off.

Prove that there is a finite sequence of operations which will turn all the n^2 lamps on and determine the value $f(n)$ of the smallest number of operations necessary for all the n^2 lamps to be on.

3958. *Proposed by Marcel Chiriţă.*

Let $A, B \in M_2(\mathbb{R})$, that is A and B are 2×2 matrices over reals. Prove that if $BA = 0$ and $\det(A^2 + AB + B^2) = 0$, then $\det(A + xB) = 0$ for all positive real numbers x.

3959. *Proposed by Dragoljub Milošević.*

Suppose that in a $\triangle ABC$, we have $\angle A = 40^\circ$ and $\angle B = 60^\circ$. If $\dfrac{BC}{AC} = k$, prove that $3k^3 - 3k + 1 = 0$.

3960. *Proposed by George Apostolopoulos.*

Let a, b, c be nonnegative real numbers such that $a + b + c = 4$. Prove that

$$\frac{a^2b}{3a^2+b^2+4ac} + \frac{b^2c}{3b^2+c^2+4ab} + \frac{c^2a}{3c^2+a^2+4bc} \geq \frac{1}{2}.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3851. *Proposed by Billy Jin and Edward Wang.*

Let $U = \{1, 2, 3, \ldots, n\}$ where $n \in \mathbb{N}$, and let $S \subseteq U$ with $|S| = k$ where $0 < k \leq n$. Determine the number of unordered pairs (X, Y) such that $S = X\Delta Y$ where X and Y are subsets of U, and $X\Delta Y = (X - Y) \cup (Y - X) = (X \cup Y) - (X \cap Y)$ is the symmetric difference of X and Y.

We present the solution by the Missouri State University Problem Solving Group.

We will first determine the number of ordered pairs. First note that if X and S are given, then we must have $Y = X\Delta S$, i.e. Y is determined by X and S. This means we have as many ordered pairs as subsets of U, namely 2^n. As $S \neq \emptyset$, $X \neq Y$, we just need to divide by 2 to obtain the number of unordered pairs: 2^{n-1}.

Editor's note. If we allow the possibility that $S = \emptyset$, then in that case, every possible pair (X, Y) satisfies $X = Y$, so the number of unordered pairs is the same as the number of ordered pairs: 2^n.

3852. *Proposed by Václav Konečný.*

Given the graphs of two positive, continuous, increasing functions f, g, satisfying $0 < f(x) < g(x)$ for all $x \geq 0$. Consider the following system of equations

$$f(x_1) + f(x_2) = K$$
$$g(x_1) + g(x_2) = L.$$

If $x_1 > 0$ and $K > 0$ are given, find $x_2 > 0$ and $L > 0$ by the Classical Greek construction (compass and straightedge) such that the system of equations is satisfied.

All the submitted solutions were essentially the same. We present the solution by Oliver Geupel.

Let us additionally assume that the coordinate axes are given with origin O. As a monotonic function, f is injective so that there is at most one real number $x_2 > 0$ such that the first equation is satisfied. Consequently, there is at most one solution of the system of equations, which can be constructed as follows.

First, construct the perpendicular to the x-axis at the point $P_1(x_1, 0)$. It intersects the graphs of f and g in points $F_1(x_1, f(x_1))$ and $G_1(x_1, g(x_1))$, respectively. Next, we draw the line $y = K - f(x_1)$; that is, we construct the length $K - |P_1F_1|$ and the point $(x_1, K - |P_1F_1|)$, then draw the line parallel to the x-axis through that point.

Copyright © Canadian Mathematical Society, 2015

It has at most one intersection with the graph of f. If there is no intersection, then no solution exists. Otherwise, the line intersects the graph of f at the point $F_2(x_2, f(x_2))$. The parallel to the y-axis through F_2 intersects the x-axis and the graph of g in points $P_2(x_2, 0)$ and $G_2(x_2, g(x_2))$, respectively. As the result, we have $x_2 = |OP_2|$ and $L = |P_1G_1| + |P_2G_2|$.

3853. *Proposed by Dragoljub Milošević.*

Let a, b, c be positive real numbers such that $a + b + c = 3$. Prove that

$$\frac{a}{b(2c+a)} + \frac{b}{c(2a+b)} + \frac{c}{a(2b+c)} \geq 1.$$

We present the proof provided by Oliver Geupel and Mark Girard (independently).

Applying the AM-GM inequality twice, we have

$$\begin{aligned} \frac{a}{b(2c+a)} + \frac{b}{c(2a+b)} + \frac{c}{a(2b+c)} &\geq 3 \cdot \frac{1}{\sqrt[3]{(2c+a)(2a+b)(2b+c)}} \\ &\geq 3 \cdot \frac{3}{(2c+a)+(2a+b)+(2b+c)} \\ &= \frac{9}{3(a+b+c)} = 1. \end{aligned}$$

Clearly, equality holds if and only if $a = b = c = 1$.

Editor's comment. Wagon gave a proof using Mathematica's FindInstance and pointed out that the condition can be relaxed to $a + b + c \leq 3$.

3854. *Proposed by Paul Yiu.*

Show that the parabola tangent to the internal and external bisectors of angles B and C of triangle ABC has focus at vertex A and directrix the line BC.

We present the nearly identical solutions by Michel Bataille and Omran Kouba (done independently).

First note that there exists a unique conic tangent to the five following lines: the two bisectors of angle B, the two bisectors of angle C, and the line at infinity; this conic is a parabola $\mathcal{P}$. The subject of the problem is the determination of the focus and directrix of $\mathcal{P}$. A well-known result about parabolas is the following:

The locus of the point of intersection of perpendicular tangents to the parabola is its directrix.

Since the internal and external bisectors of any angle are perpendicular, we see that B and C lie on the directrix of $\mathcal{P}$. Thus BC is the directrix of $\mathcal{P}$. Another classical result about parabolas is

The reflection of the directrix in any tangent line passes through the focus.

Here the reflection of the directrix BC in a bisector of angle B is the line BA and similarly, the reflection of BC in a bisector of angle C is the line CA. It follows that the focus of $\mathcal{P}$ is A.

Editor's comment. The proposer based his solution on the fact that the circumcircle of the triangle bounded by three tangents to a parabola passes through the focus of the parabola. The AN-anduud PSG produced a nice solution using coordinates.

3855. *Proposed by Leonard Giugiuc.*

Let a and b be real numbers with $0 < a < b$ and $\dfrac{1+ab}{b-a} \leq \sqrt{3}$. Prove that

$$(1+a^2)(1+b^2) \geq 4a(a+b).$$

When does equality hold?

We present the solution by D. M. Bătineţu-Giurgiu, Neculai Stanciu and Titu Zvonaru.

Let $x, y \in (0, \frac{\pi}{2})$ with $x < y$ such that $\tan x = a$ and $\tan y = b$. Then

$$\frac{1+ab}{b-a} \leq \sqrt{3}$$

is equivalent to

$$\frac{b-a}{1+ab} \geq \frac{1}{\sqrt{3}} \iff \tan(y-x) \geq \frac{1}{\sqrt{3}} \iff y - x \geq \frac{\pi}{6}$$

and the inequality to be proven is equivalent, in succession, to

$$(1+\tan^2 x)(1+\tan^2 y) \geq 4\tan x(\tan x + \tan y),$$

$$\frac{1}{\cos^2 x} \cdot \frac{1}{\cos^2 y} \geq 4 \cdot \frac{\sin x}{\cos x} \cdot \frac{\sin(x+y)}{\cos x \cos y},$$

$$4 \sin x \cos y \sin(x+y) \leq 1,$$

$$\sin(x+y)(\sin(x+y) - \sin(y-x)) \leq \frac{1}{2}. \qquad (1)$$

Since $0 < \sin(x+y) \leq 1$ and $\frac{\pi}{6} \leq y - x < \frac{\pi}{2}$ implies $\sin(y-x) \geq \sin\frac{\pi}{6} = \frac{1}{2}$, we have

$$\sin(x+y)(\sin(x+y) - \sin(y-x)) \leq 1 - \sin(y-x) \leq \frac{1}{2},$$

so (1) holds.

Equality holds if and only if $x + y = \frac{\pi}{2}$ and $y - x = \frac{\pi}{6}$; that is, if and only if $x = \frac{\pi}{6}$ and $y = \frac{\pi}{3}$ or $a = \frac{1}{\sqrt{3}}$ and $b = \sqrt{3}$.

Copyright © Canadian Mathematical Society, 2015

Editor's comment. Wagon's solution uses Mathematica to confirm that the minimum value of $\frac{(1+a^2)(1+b^2)}{a(a+b)}$ is 4.

3856. *Proposed by Nguyen Ngoc Giag.*

Given a triangle ABC with internal angle bisectors AA', BB', CC'. Bisector CC' meets $A'B'$ at F, and bisector BB' meets $C'A'$ at E. Prove that if $BE = CF$ then triangle ABC is isosceles.

We present the solution by Miguel Amengual Covas.

Let $a = BC$, $b = CA$, $c = AB$, and $2s = a + b + c$. Because AA' bisects $\angle A$, on the side BC we have $BA' = \frac{ac}{b+c}$; similarly, $BC' = \frac{ca}{a+b}$. For the bisector BE of triangle $C'BA'$, we thus have

$$\begin{aligned} BE &= \frac{2\cdot BC'\cdot BA'}{BC'+BA'}\cos\frac{B}{2} \\ &= \frac{2\cdot\frac{ca}{a+b}\cdot\frac{ac}{b+c}}{\frac{ca}{a+b}+\frac{ac}{b+c}}\cos\frac{B}{2} \\ &= \frac{2ca}{a+2b+c}\cos\frac{B}{2} \\ &= \frac{2ca}{2s+b}\cos\frac{B}{2} \end{aligned}$$

Similarly, for the bisector CF of $\triangle A'CB'$, we have $CF = \frac{2ab}{2s+c}\cos\frac{C}{2}$.

Hence

$$\begin{aligned} BE = CF \quad &\Leftrightarrow \quad \frac{c}{2s+b}\cos\frac{B}{2} = \frac{b}{2s+c}\cos\frac{C}{2} \\ &\Leftrightarrow \quad \frac{\sin C}{2s+b}\cos\frac{B}{2} = \frac{\sin B}{2s+c}\cos\frac{C}{2} \qquad \text{(by the Law of Sines)} \\ &\Leftrightarrow \quad \frac{\sin\frac{C}{2}}{2s+b} = \frac{\sin\frac{B}{2}}{2s+c} \qquad \text{(dividing through by } 2\cos\tfrac{B}{2}\cos\tfrac{C}{2}) \end{aligned}$$

Now suppose, to the contrary, that the sides b and c are not equal; for definiteness suppose $b > c$. In this case, the angle B, opposite the longer side b, is greater than angle C. Thus $\frac{\pi}{2} > \frac{B}{2} > \frac{C}{2}$, and we have $\sin\frac{B}{2} > \sin\frac{C}{2}$ giving

$$\frac{\sin\frac{B}{2}}{2s+c} > \frac{\sin\frac{C}{2}}{2s+b},$$

which contradicts our supposition that $BE = CF$. Hence, b and c cannot be unequal, and triangle ABC must be isosceles.

3857. *Proposed by George Apostolopoulos.*

Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{a^{n+2}}{a^n+(n-1)b^n} + \frac{b^{n+2}}{b^n+(n-1)c^n} + \frac{c^{n+2}}{c^n+(n-1)a^n} \geq \frac{3}{n}$$

for each positive integer n.

We present the solution by AN-anduud Problem Solving Group.

Applying the AM-GM inequality twice, we have

$$\begin{aligned}\sum \frac{a^{n+2}}{a^n+(n-1)b^n} &= \sum a^2 - (n-1)\sum \frac{a^2b^n}{a^n+(n-1)b^n}\\ &\geq \sum a^2 - (n-1)\sum \frac{a^2b^n}{n\cdot\sqrt[n]{a^n\cdot(b^n)^{n-1}}}\\ &= \sum a^2 - \frac{n-1}{n}\sum ab\\ &= \frac{1}{n}\left(a^2+b^2+c^2+\frac{1}{2}(n-1)\left((a-b)^2+(b-c)^2+(c-a)^2\right)\right)\\ &\geq \frac{1}{n}(a^2+b^2+c^2) \geq \frac{1}{n}\cdot 3\cdot\sqrt[3]{a^2b^2c^2} = \frac{3}{n}.\end{aligned}$$

Clearly, equality holds if and only if $a = b = c = 1$.

3858. *Proposed by Michel Bataille.*

Let a, b be positive real numbers with $a \neq b$. Solve the system

$$a^2x^2 - 2abxy + b^2y^2 - 2a^2bx - 2ab^2y + a^2b^2 = 0$$
$$abx^2 + (a^2-b^2)xy - aby^2 + ab^2x - a^2by = 0$$

for $(x, y) \in \mathbb{R}^2$.

We present the solution by Titu Zvonaru and Neculai Stanciu. One submitted solution had a valid approach but led to an answer that could not be verified.

It is straightforward to solve the equations when $ab = 0$, so we will assume henceforth that $ab \neq 0$. The two equations can be rewritten as

$$(ax-by)^2 - 2ab(ax+by) + a^2b^2 = 0 \qquad (1)$$
$$(bx+ay)(ax-by) + ab(bx-ay) = 0. \qquad (2)$$

Let $2abu = ax + by$ and $2abv = ax - by$. Then

$$\begin{aligned}x &= b(u+v),\\ y &= a(u-v),\\ bx+ay &= (b^2+a^2)u + (b^2-a^2)v,\\ bx-ay &= (b^2-a^2)u + (b^2+a^2)v.\end{aligned}$$

Then (1) becomes $4u = 4v^2 + 1$ and (2) leads to

$$\begin{aligned}0 &= 4[(b^2+a^2)u + (b^2-a^2)v]2abv + ab[(b^2-a^2)u + (b^2+a^2)v]\\ &= 2abv[(b^2+a^2)(4v^2+1) + 4(b^2-a^2)v] + ab[(b^2-a^2)(4v^2+1) + 4(b^2+a^2)v]\\ &= ab[b^2(8v^3+12v^2+6v+1) + a^2(8v^3-12v^2+6v-1)]\\ &= ab[b^2(2v+1)^3 + a^2(2v-1)^3].\end{aligned}$$

Copyright © Canadian Mathematical Society, 2015

Suppose that $w = [-b^2/a^2]^{1/3}$. Then there is one real solution for the equations given by $(2v-1)/(2v+1) = w$ or

$$v = \frac{1+w}{2(1-w)} = \frac{1-w^2}{2(1-w)^2}$$
$$u = \frac{1+w^2}{2(1-w)^2}.$$

It follows that

$$(x,y) = \left(\frac{b}{(1-w)^2}, \frac{aw^2}{(1-w)^2}\right) = \left(\frac{a^{4/3}b}{(a^{2/3}+b^{2/3})^2}, \frac{b^{4/3}a}{(a^{2/3}+b^{2/3})^2}\right).$$

Editor's comment. What happens when $a = b \neq 0$? The system then becomes $0 = (x-y)^2 - 2a(x+y) + a^2 = a^2(x-y)[(x+y)+a]$. We cannot have $x+y = -a$, for this would make $(x-y)^2$ negative, so the only possibility is $x = y = a/4$. The proposer imposed the condition $a \neq b$ to avoid an exceptional case in his solution.

3859. *Proposed by Jung In Lee.*

The sequence $\{F_n\}$ is defined by $F_1 = F_2 = 1$, $F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$. For any natural number m, define $v_2(m)$ as $v_2(m) = n$ if $2^n \mid m$ and $2^{n+1} \nmid m$. Find all positive integer n that satisfy the equation

$$v_2(n!) = v_2(F_1F_2\cdots F_n).$$

We present the solution by Brian Beasley.

We show that the only such positive integers are 1, 3, 6, and 7.

Given a positive integer n, let $f(n) = v_2(n!)$ and $g(n) = v_2(F_1F_2\cdots F_n)$. It is straightforward to verify that $f(1) = g(1) = 0$, $f(3) = g(3) = 1$, $f(6) = g(6) = 4$, and $f(7) = g(7) = 4$. Also, for any other $n \leq 12$, we have $f(n) > g(n)$. For $n > 12$, we show that $f(n) > g(n)$ by applying a result of Lengyel [1]:

$$v_2(F_n) = \begin{cases} 0 & \text{if } n \equiv 1,2 \pmod 3 \\ 1 & \text{if } n \equiv 3 \pmod 6 \\ 3 & \text{if } n \equiv 6 \pmod{12} \\ v_2(n)+2 & \text{if } n \equiv 0 \pmod{12} \end{cases}$$

Our approach is to establish that given any positive integer k and any r in $\{1,2,\ldots,12\}$, $f(12k+r) > g(12k+r)$. Since $f(12) > g(12)$, this will complete the proof that $f(n) > g(n)$ for every $n > 12$.

Given any positive integer k, we assume $f(12k) > g(12k)$ for induction, and we analyze f and g for consecutive values of m with $12k < m \leq 12(k+1)$ (using Lengyel's theorem: T. Lengyel, The Order of the Fibonacci and Lucas Numbers, *The Fibonacci Quarterly* 33(3), June-July 1995, 234-239.):

$$\begin{array}{lcll}
f(12k+1) = f(12k) & \& & g(12k+1) = g(12k) & \Rightarrow f(12k+1) > g(12k+1) \\
f(12k+2) = f(12k+1)+1 & \& & g(12k+2) = g(12k+1) & \Rightarrow f(12k+2) < g(12k+2)+1 \\
f(12k+3) = f(12k+2) & \& & g(12k+3) = g(12k+2)+1 & \Rightarrow f(12k+3) > g(12k+3) \\
f(12k+4) \geq f(12k+3)+2 & \& & g(12k+4) = g(12k+3) & \Rightarrow f(12k+4) > g(12k+4)+2 \\
f(12k+5) = f(12k+4) & \& & g(12k+5) = g(12k+4) & \Rightarrow f(12k+5) > g(12k+5)+2 \\
f(12k+6) = f(12k+5)+1 & \& & g(12k+6) = g(12k+5)+3 & \Rightarrow f(12k+6) > g(12k+6) \\
f(12k+7) = f(12k+6) & \& & g(12k+7) = g(12k+6) & \Rightarrow f(12k+7) > g(12k+7) \\
f(12k+8) \geq f(12k+7)+2 & \& & g(12k+8) = g(12k+7) & \Rightarrow f(12k+8) > g(12k+8)+2 \\
f(12k+9) = f(12k+8) & \& & g(12k+9) = g(12k+8)+1 & \Rightarrow f(12k+9) > g(12k+9)+1 \\
f(12k+10) = f(12k+9)+1 & \& & g(12k+10) = g(12k+9) & \Rightarrow f(12k+10) > g(12k+10)+2 \\
f(12k+11) = f(12k+10) & \& & g(12k+11) = g(12k+10) & \Rightarrow f(12k+11) > g(12k+11)+2
\end{array}$$

Finally, since $f(12k+12) = f(12k+11) + v_2(12k+12)$ and $g(12k+12) = g(12k+11) + v_2(12k+12) + 2$, we conclude $f(12k+12) > g(12k+12)$. In fact, we actually have $f(12k+12) > g(12k+12)+1$, since we did not use the fact that exactly one of $12k+4$ or $12k+8$ is divisible by 8. Thus as k approaches infinity, $f(12k) - g(12k)$ increases without bound.

3860. *Proposed by Ovidiu Furdui.*

Let $n \geq 3$ be an odd integer and let $A \in M_n(\mathbb{Z})$. Prove that the determinant of $3A + 4A^T$ is divisible by 7. Does the result hold when n is an even integer?

We present a combination of solutions by the Missouri State University Problem Solving Group and Michel Bataille.

Let $n \geq 3$ be an odd integer, and let $A \in M_n(\mathbb{Z})$. We prove a more general result: for any odd integer $m > 1$ and any k in the range $0 \leq k \leq m$, the determinant of $kA + (m-k)A^T$ is divisible by m. Set $B = kA + (m-k)A^T$; since $B + B^T = m(A + A^T)$, we have

$$B = -B^T \pmod{m}.$$

Using properties of the determinant,

$$\det(B) = \det(-I)\det(B^T) = (-1)^n \det(B) = -\det(B) \pmod{m},$$

since n is odd, and therefore $2\det(B) = 0 \pmod{m}$. Thus $m | 2\det(B)$, and consequently m divides $\det(B)$, as m is odd, as desired. The solution to the proposed problem is obtained by taking $m = 7$ and $k = 3$.

If n is even, then the property does not hold. Let

$$U_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Then $3U_2 + 4U_2^T = -U_2$, so that $\det(3U_2 + 4U_2^T) = \det(-U_2) = 1$, which is not a multiple of 7. Thus, U_2 provides a counter-example when $n = 2$.

More generally, if $n = 2m$ with $m > 1$, we consider the block-diagonal matrix $U_{2m} = \mathrm{diag}(U_2, \ldots, U_2)$, which is m copies of U_2 along the main diagonal. Again,

Copyright © Canadian Mathematical Society, 2015

$3U_{2m} + 4U_{2m}^T = -U_{2m}$, and $\det(3U_{2m} + 4U_{2m}^T) = \det(-U_{2m})^m = 1$, which is not divisible by 7.

Editor's comments: This problem may also be solved by manipulating the terms in a summation formula for the determinant, as some of the solvers did. This requires more dexterity than the simple (but clever) algebraic trick shown above, but is messier.

Math Quotes

Most of the arts, as painting, sculpture, and music, have emotional appeal to the general public. This is because these arts can be experienced by some one or more of our senses. Such is not true of the art of mathematics; this art can be appreciated only by mathematicians, and to become a mathematician requires a long period of intensive training.

The community of mathematicians is similar to an imaginary community of musical composers whose only satisfaction is obtained by the interchange among themselves of the musical scores they compose.

Cornelius Lanczos iH. Eves "Mathematical Circles Squared", Boston: Prindle, Weber and Schmidt, 1972.

Solvers and proposers appearing in this issue

(Bold font indicates featured solution.)

Proposers

George Apostolopoulos, Messolonghi, Greece: 3960
Roy Barbara, Lebanese University, Fanar, Lebanon: 3955
Michel Bataille, Rouen, France: 3951
Marcel Chiriţă, Bucharest, Romainia : 3958
Zhihan Gao, Edward Wang and Dexter Wei, Wilfrid Laurier University. Waterloo, Ontario : 3957
Dragoljub Milošević, Gornji Milanovac, Serbia: 3956, 3959
Bill Sands, University of Calgary, Calgary, AB: 3952
Dao Hoang Viet, Pleiku city, Viet Nam : 3954
An Zhen-ping, Xianyang Normal University, China : 3953

Solvers - individuals

Arkady Alt, San Jose, CA, USA: 3853, 3855, 3857
Miguel Amengual Covas, Cala Figuera, Mallorca, Spain : **3856**
George Apostolopoulos, Messolonghi, Greece: CC76, CC78, CC79, 3856, 3857
Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina: CC76, CC77, **CC78**, CC79, 3853, 3855, 3856 (2 solutions)
Roy Barbara, Lebanese University, Fanar, Lebanon: 3851, 3856
Michel Bataille, Rouen, France: **CC76**, CC78, CC79, **OC128**, OC130, 3851, 3853, **3854**, 3855, 3856, 3857, 3858, **3860**
Ricardo Barroso Campos, University of Seville, Seville, Spain: 3854
Brian D. Beasley, Presbyterian College, Clinton, USA: **3859**
Kennard Callender, Southeast State Missouri University, Cape Girardeau, MO : 3853
Matei Coiculescu, East Lyme High School, East Lyme, CT, USA: CC76, **CC78**, CC79
Prithwijit De, Homi Bhabha Centre for Science Education, Mumbai, India: 3853, 3855, 3856
Marian Dincă, Bucharest, Romania: 3853
Ovidiu Furdui, Campia Turzii, Cluj, Romania: 3860
Oliver Geupel, Brühl, NRW, Germany: **OC130**, 3851, **3852**, **3853**, 3856
Mark Girard : **3853**
Leonard Giugiuc, Romania: 3855
Richard I. Hess, Rancho Palos Verdes, CA, USA: CC76, CC78, CC79
Václav Konečný, Big Rapids, MI, USA: 3852
Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria: 3853, **3854**, 3855, 3856, 3858, 3860
Dimitrios Koukakis, Kilkis, Greece: OC128
Jung In Lee, Seoul Science High School, Seoul, Republic of Korea : 3859
Ang Li, Southeast State Missouri University, Cape Girardeau, MO : 3855
Salem Malikić, student, Simon Fraser University, Burnaby, BC: 3853, 3856, 3857, 3859
David E. Manes, SUNY at Oneonta, Oneonta, NY, USA: CC76, CC78, CC79, OC128
Phil McCartney, Northern Kentucky University, Highland Heights, KY, USA: 3857
Norvald Midttun, Royal Norwegian Naval Academy, Sjøkrigsskolen, Bergen, Norway:

Copyright © Canadian Mathematical Society, 2015

CC78, CC79
Dragoljub Milošević, Gornji Milanovac, Serbia: 3853
Madhav R. Modak, formerly of Sir Parashurambhau College, Pune, India: 3853, 3855, 3856
Nguyen Ngoc Giang, The Vietnam Institute of Educational Sciences : 3856
Jennifer O'Day, Southeast State Missouri University, Cape Girardeau, MO : 3860
Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy: 3853, 3855, 3857
Ricard Peiró i Estruch, IES "Abastos", València. Spain : 3856
C.R. Pranesachar, Indian Institute of Science, Bangalore, India: 3853, 3855, 3856
Henry Ricardo, Tappan, NY, USA: CC79, 3853
Cristóbal Sánchez–Rubio, I.B. Penyagolosa, Castellón, Spain: 3856
Maung Soe Htet : Southeast State Missouri University, Cape Girardeau, MO : 3857
Digby Smith, Mount Royal University, Calgary, AB: 3853, 3855, 3857, 3858, 3860
Jason Smith, Southeast State Missouri University, Cape Girardeau, MO : 3859
Albert Stadler, Herrliberg, Switzerland : 3853
Daniel Văcaru, Piteşti, Romania: CC76, CC78, CC79, OC128, 3853, 3857
Stan Wagon, Macalester College, St. Paul, MN, USA: 3853, 3855, 3858
Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON : CC76, CC78, **CC79**
Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA: 3854
Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA: CC76, CC78, CC79, OC128
Zouhair Ziani, Institut Privé Assahoua Benslimane, Marocco : **OC126**

Solvers - collaborations

AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia: 3853, 3854, 3855, **3857**, 3860
Missouri State University Problem Solving Group, Springfield, MO : **3851**, 3855, **3860**
Dionne Bailey, Elsie Campbell, and Charles R. Diminnie, Angelo State University, San Angelo, USA: 3853
Billy Jin, Waterloo Collegiate Institute, Waterloo, ON and Edward T. H. Wang, Wilfrid Laurier University, Waterloo, ON: 3851
D. M. Bătineţu, Bucharest, Neculai Stanciu, Buzău, and Titu Zvonaru, Comăneşti, Romania: 3853, **3855**, 3856
John Hawkins and David R. Stone, Georgia Southern University, Statesboro, USA: 3851, 3852
Daniel Văcaru, Piteşti, Romaniaand Răzvan Veliche, Baltimore, USA : 3853
Haohao Wang and Jerzy Wojdylo, Southeast Missouri State University, Cape Girardeau, USA: 3853
Neculai Stanciu, Buzău, and Titu Zvonaru, Comăneşti, Romania: CC76, **CC77**, CC78, CC79, OC126, OC128, 3852, 3853, **3858**