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The density of 0's in recurrence double sequences

Yossi Moshe

Department of Mathematics, Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel

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Abstract

Let $A = (A(i,j))_{i=0,\, j=-\infty}^{\infty,\infty}$ be a double sequence over a finite field $\mathbb{F} = GF(q)$ satisfying a linear recurrence with constant coefficients, with at most finitely many nonzero elements on each row. Given a nonzero element g of $\mathbb{F}$, we show how to obtain an explicit formula for the number of g's in the first q^n rows of A. We also characterize the cases when the density of 0's is 1.

MSC: primary 11B05; secondary 11B37

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1. Introduction

The distribution of the elements in Pascal's triangle modulo a prime p has been extensively studied (cf. [1,9,11]). In particular, many papers deal with the asymptotic distribution of the set of elements satisfying a certain congruence (cf. [3–5,7,14]) or system of congruences (cf. [12]). Using Lucas's Theorem [15] it easily follows that the number of nonzero elements in the first p^l rows of Pascal's triangle modulo p is $(1 + \cdots + p)^l = (\frac{p(p+1)}{2})^l$. In particular, since the number of elements in these rows is $\frac{p^l(p^l+1)}{2}$, the percentage of the nonzero elements in the first n rows approaches 0 as $n \to \infty$ [8]. In other words, the density of 0's in the triangle is 1.

Klostermeyer et al. [13] defined Pascal's rhombus as the double sequence $(a_{i,j})_{i=0,\, j=-\infty}^{\infty,\infty}$ over $\mathbb{Z}$, defined by the recurrence

$$a_{i,j} = a_{i-1,j} + a_{i-1,j-1} + a_{i-1,j-2} + a_{i-2,j-2}, \quad i \geqslant 2,\ j \in \mathbb{Z},$$

E-mail address: moshey@cs.bgu.ac.il.

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and the initial conditions

$a_{0,0} = a_{1,0} = a_{1,1} = a_{1,2} = 1,$

$a_{0,j} = 0, \ j \neq 0,$

$a_{1,j} = 0, \ j \neq 0, 1, 2.$

They conjectured that the density of 0's in the infinite triangle $\{a_{i,j} (\mathrm{mod}\, 2) : i \in \mathbb{N}, \ 0 \leqslant j \leqslant 2i\}$ is 1. This conjecture was settled affirmatively by Goldwasser et al. [10].

Pascal's triangle and Pascal's rhombus, when viewed modulo a prime, are particular instances of the following general family of (double) sequences. A double array $(A(i,j))_{i=0, j=-\infty}^{\infty, \infty}$ over a finite field $\mathbb{F}$ is a *double linear recurrence sequence of order d with bounded initial conditions* (henceforth *DLR*) if:

(1) $(A(i,j))_{i,j}$ satisfies a recurrence of the form

$$A(i,j) = \sum_{k=1}^{r} c_k A(i - i_k, j - j_k), \quad i \geqslant d, \ j \in \mathbb{Z}. \tag{1.1}$$

Here $c_k \in \mathbb{F} \backslash \{0\}$, $j_k \in \mathbb{Z}$ and $1 \leqslant i_k \in \mathbb{N}$ are constants, and $d = \max_{1 \leqslant k \leqslant r} i_k$.

(2) for every $i < d$ there are only finitely many elements $j \in \mathbb{Z}$ such that $A(i,j) \neq 0$.

An alternative way to view a *DLR* is as a sequence $(P_i(X))_{i=0}^{\infty}$ (where $P_i(X) = \sum_{j=-\infty}^{\infty} A(i,j) X^j$) over the ring $\mathbb{F}[X, X^{-1}]$ of Laurent polynomials, which satisfies a recurrence with constant coefficients

$$P_i(X) = \sum_{k=1}^{d} C_k(X) P_{i-k}(X), \quad i \geqslant d.$$

For example, in Pascal's triangle, $P_i(X) = (1+X)^i$, and the recurrence is $P_i(X) = (1+X)P_{i-1}(X)$. Similarly, the recurrence corresponding to Pascal's rhombus is $P_i(X) = (1 + X + X^2)P_{i-1}(X) + X^2 P_{i-2}(X)$.

We remark that first-order *DLR*'s appear in the literature as double sequences generated by linear cellular automata (cf. [2,6]).

In view of the above-mentioned results concerning the density of 0's in Pascal's triangle and rhombus, it is natural to pose the following

Question. *Is the density of* 0*'s in every DLR equal to* 1?

In this paper we show that the answer to this question is positive for first-order *DLR*s but negative in general. In fact, we show (Theorem 6) how to obtain an explicit formula for the number of entries in the first q^n rows equal to any field element, where $q = |\mathbb{F}|$. This formula enables us to decide in each case whether the density of occurrences of any element is 0 or not.

In Section 2 we present the main results. Section 3 is devoted to the proof that in every nontrivial first-order *DLR* the density of 0's (in the "appropriate" triangle) is 1. In Section 4 we describe an algorithm which determines, for a given *DLR*, whether the density of 0's is 1.

2. Notations and main results

Let $A(i,j)$ be a *DLR* as in Section 1. Due to the nature of our questions we may assume, by shifting the rows appropriately, that $j_k \geqslant 0$ for each k and that $A(i,j) = 0$ for $j < 0$. Thus, we may consider A as a double array of the form $(A(i,j))_{i=0,\, j=0}^{\infty,\infty}$. Denote $M = \max_{k \leqslant t} j_k$. In the case $M = 0$, the recurrence relation is *trivial*.

It is convenient to view the initial conditions as determined by a function $B : \{0, 1, \ldots, d-1\} \times \mathbb{N} \to \mathbb{F}$, with $B(i,j) \neq 0$ for at most finitely many pairs (i,j), by the requirement $A(i,j) = B(i,j)$, $i < d$, $j \geqslant 0$. (Another possibility is to view B as an infinite matrix with d rows.) The *length* of the initial conditions is defined by $l(B) = \max\{ j\colon\ \exists i < d;\ B(i,j) \neq 0\}$, where we agree to put $l(B) = -1$ if $B = 0$.

The double array defined by the same recurrence relation as A and any initial conditions B' will be denoted by $A_{B'}$. (Thus A itself will be denoted by A_B when we need to emphasize the initial conditions.) The sub-array $(A_B(i,j))_{i \in \mathbb{N}, 0 \leqslant j \leqslant l(B)+iM}$ is the *triangle corresponding to* A_B (although it is usually a trapezium). This triangle contains every nonzero element of A_B.

As mentioned, our main interest in this paper is to characterize the situation where every nonzero element of $\mathbb{F}$ occurs with density 0 in this triangle (so that $0 \in \mathbb{F}$ occurs with density 1). Here $g \in \mathbb{F}$ occurs with density α if the frequency of occurrences of g in the first m rows of the triangle approaches α as the number of rows increases:

$$\frac{|\{(i,j)\colon\ 0 \leqslant i \leqslant m-1,\ 0 \leqslant j \leqslant l(B)+iM,\ A_B(i,j) = g\}|}{|\{(i,j)\colon\ 0 \leqslant i \leqslant m-1, 0 \leqslant j \leqslant l(B)+iM\}|} \xrightarrow[m \to \infty]{} \alpha. \tag{2.1}$$

Remark. (1) In many cases, the density α does not exist (cf. Example 8).

(2) To verify that $\alpha = 0$, it suffices to take the frequencies on the left hand side of (2.1) as m increases along powers of q.

The following theorem generalizes Fine's result [8] about the density of 0's in Pascal's triangle modulo a prime.

Theorem 1. *Let A_B be a nontrivial first-order DLR over a finite field. Then the density of* 0*'s in the corresponding triangle is* 1.

Considering a *DLR* as a sequence over $\mathbb{F}[X, X^{-1}]$, we obtain

Corollary 2. *Let $G = G(X)$ be a polynomial of degree $k > 0$ over a finite field. Let $(a_{i,j})_{i \in \mathbb{N},\ 0 \leqslant j \leqslant ik}$ be the double array whose ith row consists of the coefficients of G^i (i.e., $G^i(X) = \sum_{j=0}^{ik} a_{i,j} X^j$). Then the density of* 0*'s in the double array $(a_{i,j})$ is* 1.

Remark. In contrast to Corollary 2, it is not true that the density of 0's in the polynomial $G(X)^n$ approaches 1 as $n \to \infty$. For example, if $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ and $G(X) = 1 + X$, then for every $k \in \mathbb{N}$ and $n = 2^k - 1$, all the coefficients of $G(X)^n$ are 1.

Let g be an arbitrary fixed nonzero element in $\mathbb{F}$. Our goal is to compute the number $\#(A_B, n)$ $(= \#_g(A_B, n))$ of occurrences of g in the first q^n rows of A_B.

We will decompose A_B into q^2 double arrays $(A_B^{s,t})_{s=0,t=0}^{q-1,q-1}$ according to the values of the two indices modulo q: for each (s,t) with $0 \leqslant s,t < q$, let $A_B^{s,t}: \mathbb{N} \times \mathbb{N} \to \mathbb{F}$ be defined by

$$A_B^{s,t}(i,j) = A_B(iq+s, jq+t), \quad i \geqslant 0, \ j \geqslant 0.$$

Denote by $B^{s,t}$ the restriction of $A_B^{s,t}$ to $\{0,\dots,d-1\} \times \mathbb{N}$ (that is, $B^{s,t}$ consists of the first d rows of $A_B^{s,t}$).

Proposition 3. *If A_B is a DLR of order d over $\mathbb{F}$, then so is $A_B^{s,t}$. Moreover, $A_B^{s,t}$ satisfies the same recurrence as A_B. In other words, $A_B^{s,t} = A_{B^{s,t}}$.*

We define X as the closure of $\{B\}$ under the operations $C \mapsto C^{s,t}$. Namely, for $i \geqslant 0$ let X_i be the set of initial conditions which is given by

$$X_0 = \{B\}, \ X_{i+1} = \{C^{s,t}: \ C \in X_i, \ 0 \leqslant s,t < q\}, \quad i \geqslant 0.$$

Take $X = \bigcup_{i \in \mathbb{N}} X_i$. The following proposition shows that X is finite.

Proposition 4. *Let $L = \max(\frac{M(qd-1)}{q-1}, l(B))$. Then $l(C) \leqslant L$ for every $C \in X$. In particular, $|X| \leqslant q^{d(L+1)}$.*

A function $C: \{0,1,\dots,d-1\} \times \mathbb{N} \to \mathbb{F}$ defining initial conditions is *g-trivial* if the double array A_C contains no g's. Let $X' = \{C \in X: \ C \text{ is not } g\text{-trivial}\}$.

Remark. (1) The removal of the g-trivial elements from X is essential for the proof of Lemma 14 (and thus also for Theorem 10). However, it has no effect on our main result—Theorem 6.

(2) In order to determine whether an element C of X is g-trivial, it is enough to check whether $\#(A_C, |X|) = 0$. (Another approach is given in the proof of Theorem 11.) Hence the set X' can be effectively computed.

(3) We will assume that $X' \neq \emptyset$ (otherwise $\#(A_B, n) = 0$ for every $n \geqslant 0$).

Example 5. Consider Pascal's triangle modulo 3, which forms a first-order *DLR* over $\mathbb{Z}/3\mathbb{Z}$. Take $g = 2$.

Here, the initial conditions are given by $B(0,0) = 1$, and $B(0,j) = 0$ for every $j \neq 0$. It turns out that each of the double arrays $A_B^{s,t}$ is a scalar multiplication of A_B. (This is also the case in Pascal's triangle modulo other primes.) Thus, $X = \{aB: \ a \in \mathbb{Z}/3\mathbb{Z}\}$ and $X' = \{B_1, B_2\}$ where $B_1 = B$ and $B_2 = 2B$ (Fig. 1).

Let us enumerate the elements of X', say $X' = \{B_1,\dots,B_m\}$, where $B_1 = B$. For (i,j) such that $1 \leqslant i,j \leqslant m$, let $D_{i,j}$ be the number of pairs (s,t) (with $0 \leqslant s,t < q$) such that $B_j^{s,t} = B_i$. The $m \times m$ matrix $D = (D_{i,j})_{i,j=1}^m$ will play an important role in the

$$\begin{array}{llllllllllllll}
1 & & & & & & & 2 & & & & & & \\
1 & 1 & & & & & & 2 & 2 & & & & & \\
1 & 2 & 1 & & & & & 2 & 1 & 2 & & & & \\
1 & 0 & 0 & 1 & & & & 2 & 0 & 0 & 2 & & & \\
1 & 1 & 0 & 1 & 1 & & & 2 & 2 & 0 & 2 & 2 & & \\
1 & 2 & 1 & 1 & 2 & 1 & & 2 & 1 & 2 & 2 & 1 & 2 & \\
\vdots & & & & & & \ddots & \vdots & & & & & & \ddots
\end{array}$$

Fig. 1. The first rows of A_{B_1} and A_{B_2}.

sequel. Let $\{\vec{e}_i\colon 1 \leqslant i \leqslant m\}$ be the standard basis of $\mathbb{Z}^m$, the vectors being considered as column vectors. Let $\vec{v} = (v_i)_{i=1}^m$ be the row vector defined by $v_i = \#(A_{B_i}, 0)$.

Theorem 6. *For every $n \geqslant 0$ and $i \leqslant m$ we have $\#(A_{B_i}, n) = \vec{v}D^n\vec{e}_i$.*

Example 5 (Continued). *For Pascal's triangle modulo* 3, *the matrix D and the vectors $\vec{v}$ and $\vec{e}_1$ are*

$$D = \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}, \quad \vec{v} = (0, 1), \quad \vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

For example, note that

$$A_{B_1}^{0,0} = A_{B_1}^{1,0} = A_{B_1}^{1,1} = A_{B_1}^{2,0} = A_{B_1}^{2,2} = A_{B_1}, \quad A_{B_1}^{2,1} = A_{B_2}, \quad A_{B_1}^{s,t} = 0, \ s < t,$$

and thus the first row of D is $(5, 1)^T$. Using Theorem 6 we show routinely that the number of 2's in the first 3^n rows of the triangle is (as is well known) $\vec{v}D^n\vec{e}_1 = \frac{1}{2}(6^n - 4^n)$. A similar calculation shows that the number of 1's in the same rows is $\frac{1}{2}(6^n + 4^n)$.

Example 7. Consider the double array A_{B_1} over $\mathbb{Z}/2\mathbb{Z}$, generated by the recurrence

$$A(i,j) = A(i-1,j) + A(i-1,j-1) + A(i-2,j-1), \quad i \geqslant 2, \quad j \geqslant 0,$$

and the initial conditions

$$B_1 = \begin{matrix} 0, 0, 0, 0, \ldots \\ 1, 0, 0, 0, \ldots \end{matrix}.$$

It can be observed directly that the nth row of A_{B_1} consists of n consecutive 1's and thus $\#(A_{B_1}, n) = 0 + 1 + \cdots + 2^n - 1 = 2^{n-1}(2^n - 1)$. In this example $X' = \{B_1, B_2\}$, where

$$B_2 = \begin{matrix} 1, 0, 0, 0, \ldots \\ 1, 1, 0, 0, \ldots \end{matrix}$$

and

$$D = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}, \ \vec{v} = (0,1), \quad \vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

An easy calculation shows that $\vec{v}D^n\vec{e}_1 = 2^{n-1}(2^n - 1)$. Thus, Theorem 6 gives again $\#(A_B, n) = 2^{n-1}(2^n - 1)$. Obviously, the density of 0's in the corresponding triangle is 0.

Example 8. Consider the *DLR* of order 2 generated by

$$A(i,j) = A(i-1,j) + A(i-1,j-1) + A(i-2,j) + A(i-2,j-1) + A(i-2,j-2), \quad i \geqslant 2, \ j \geqslant 0,$$

and the initial conditions

$$B = \begin{matrix} 0,0,0,0,\ldots \\ 1,0,0,0,\ldots \end{matrix}.$$

In this case $X' = X = \{B_1, B_2, B_3, B_4\}$, where B_1, B_2, B_3 and B_4 are given by Fig. 2.

The matrix D and the vectors $\vec{v}$ and $\vec{e}_1$ are

$$D = \begin{pmatrix} 3 & 2 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix}, \quad \vec{v} = (0,1,0,1), \quad \vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

By Theorem 6, the number of 1's in the first 2^n rows of A_{B_1} is $\vec{v}D^n\vec{e}_1$. A routine calculation shows that $\#(A_B, n) = \frac{1}{30}(9 \cdot 4^n - 5 \cdot 2^n - 4 \cdot (-1)^n)$. Thus, the proportion of 1's in the first 2^n rows of the triangle $A_B(i,j)_{0 \leqslant j \leqslant i}$ goes to

$$\lim_{n \to \infty} \frac{\frac{1}{30}(9 \cdot 4^n - 5 \cdot 2^n - 4 \cdot (-1)^n)}{\frac{2^n(2^n+1)}{2}} = \frac{3}{5}.$$

It is interesting to note that if we take the proportions along other geometric sequences we may obtain other limits. Consider for example the number of 1's in the first $3 \cdot 2^n$ rows. Using a theorem from [16] (which gives a general formula for the

A_{B_1}	A_{B_2}	A_{B_3}	A_{B_4}
0 0 0 0 0...	1 0 0 0 0...	0 0 0 0 0...	1 0 0 0 0...
1 0 0 0 0...	0 0 0 0 0...	0 1 0 0 0...	1 1 0 0 0...
1 1 0 0 0...	1 1 1 0 0...	0 1 1 0 0...	0 1 0 0 0...
0 1 0 0 0...	1 0 0 1 0...	0 0 1 0 0...	1 1 1 1 0...

Fig. 2. The first rows of A_{B_1}, A_{B_2}, A_{B_3}, A_{B_4}.

number $F_g(A_B, k)$ of g's in the first k rows of A_B), it follows that this number is $(3,4,3,4)D^n\vec{e}_1$. Hence, a simple computation shows that the number of 1's in the first $3 \cdot 2^n$ rows is $\frac{1}{30}(99 \cdot 4^n - 5 \cdot 2^n - 4 \cdot (-1)^n)$, and we conclude that the proportion of 1's in those rows approaches $\frac{99 \cdot 4^n}{30} \frac{2}{(3 \cdot 2^n)^2} = \frac{11}{15}$. Thus, the density of 0's in this DLR does not exist.

Proposition 9. *Let $\rho = \rho(D)$ be the spectral radius of D. Then there exist positive numbers C, e such that $C\rho^n \leqslant \#(A_B, n) \leqslant n^e \rho^n$ for all sufficiently large n.*

Theorem 10. *The following properties are equivalent*:

(1) *The density of g's in the triangle corresponding to A_B is* 0.
(2) $\rho(D) < q^2$.
(3) *The matrix D has no square submatrix D′ (i.e., $D' = (D_{i,j})_{i,j \in I}$ for some $I \subseteq \{1, \dots, m\}$), such that the sum of the entries in any column in D′ is q^2.*

We remark that since the sum of the entries in any column of D is at most q^2, we must have $\rho(D) \leqslant q^2$. Moreover, $\rho(D) < q^2$ unless q^2 itself is an eigenvalue of D (in which case $\rho(D) = q^2$).

The equivalence (2) $\Leftrightarrow$ (3) in Theorem 10 is certainly well known, but for the sake of completeness the proof is given among the proofs of the other results.

Theorem 11. (1) *There exists an algorithm which, given a recurrence relation of the form* (1.1), *bounded initial conditions and $g \neq 0$ in $\mathbb{F}$, decides whether the density of g's in the corresponding triangle is* 0. *In particular, there exists an algorithm which decides whether the density of* 0*'s is* 1.

(2) *There exists an algorithm which, given a recurrence relation of the form* (1.1), *checks if there exist bounded initial conditions for which the density of* 0*'s is not* 1.

Examples 5,7,8 (Continued). Let us demonstrate Theorem 10 on the previous examples. In Examples 7,8, the sum of the entries in each column of D is $4 = 2^2$, so that D has a submatrix satisfying the third property in Theorem 10 (take $I = \{1, 2, \dots, m\}$ and $D' = D$). In particular, the density of 1's is not 0, and thus the density of 0's is not 1.

However, in Example 5, the sum of the entries in any column of D is less than 9. Thus D has no submatrix as above, and the density of 2's is 0. Similarly, the density of 1's is 0, which implies that the density of 0's is 1.

3. First-order *DLR*s

We begin the section with several claims on general *DLR*s. Then we restrict ourselves to first-order *DLR*s, and prove the results stated in Section 2 on such *DLR*s.

Lemma 12. *Let A_B be a DLR of order d. Then A_B satisfies also the following recurrence of order qd*:

$$A_B(i,j) = \sum_{k=1}^{r} c_k A_B(i - qi_k, j - qj_k), \quad i \geqslant qd, \ j \geqslant 0.$$

Proof. It is well known that a double sequence $(a_{i,j})_{i=0,\, j=-\infty}^{\infty,\infty}$ satisfies a recurrence relation of the form (1.1) if and only if its generating function $G(X,Y) = \sum_{i=0,\, j=-\infty}^{\infty,\infty} a_{i,j} X^i Y^j$ is a rational function of the form $\frac{P}{1-Q}$, where $Q = Q(X,Y) = \sum_{k=1}^{r} c_k X^{i_k} Y^{j_k}$, and $P = P(X,Y)$ is a polynomial of degree less than d in X.

Since $|\mathbb{F}| = q$ it follows that $Q(X,Y)^q = Q(X^q, Y^q)$, and thus

$$G = \frac{P(1 + Q + \cdots + Q^{q-1})}{(1-Q)(1 + Q + \cdots + Q^{q-1})} = \frac{P(1 + Q + \cdots + Q^{q-1})}{1 - Q(X^q, Y^q)},$$

which implies that A_B satisfies the required recurrence. $\square$

Proof of Proposition 3. The proposition follows immediately from Lemma 12. $\square$

Lemma 13. *For all $n \geqslant 0$*:

$$\#(A_B, n+1) = \sum_{s,t=0}^{q-1} \#(A_{B^{s,t}}, n).$$

Proof. The sets $A_B^{s,t}$, $0 \leqslant s, t < q$, form a splitting of A_B into q^2 parts, in which the elements of the first q^{n+1} rows of A_B are divided between the first q^n rows of the sets $A_B^{s,t}$. $\square$

Proof of Proposition 4. Given initial conditions C with $l(C) \leqslant L$ and $s, t < q$, we will prove that $l(C^{s,t}) \leqslant L$. Then, by induction on i, it will follow that for every $C' \in X_i$ we have $l(C') \leqslant L$.

First note that $A_C(i,j) = 0$ for every pair (i,j) with $j > L + Mi$ (since these pairs do not belong to the corresponding triangle). In particular, we get $A_C(i,j) = 0$ for every $i \leqslant qd - 1$ and $j > L + M(qd-1)$. Thus, $A_C^{s,t}(i,j) = 0$ for all $i < d$ and $j > \frac{L + M(qd-1)}{q}$, and so $l(C^{s,t}) \leqslant \frac{L + M(qd-1)}{q}$. Now, since $L \geqslant \frac{M(qd-1)}{q-1}$, we get easily that $\frac{L + M(qd-1)}{q} \leqslant L$, and thus $l(C^{s,t}) \leqslant L$. $\square$

Proof of Theorem 1. Denote $M_n = \max_{C \in X} \#(A_C, n)$. Let C be an element of X. We will prove first that there exists a k with $\#(A_C, n) \leqslant (q^{2k} - 1) M_{n-k}$ for every $n \geqslant k$.

The main idea is to use repeatedly Lemma 13 in order to split $\#(A_C, n)$ into a sum of q^{2k} elements, each of which is $\#(A_E, n-k)$ for some $E \in X$. The key is to prove that at least one of these E's is 0, and thus we will get $\#(A_E, n-k) = 0$. For this purpose we focus on the decomposition of the first row of A_C (which is essentially C).

Define the *weight* $w(C)$ of $C \in X$ as the number of nonzero elements of C, namely, since C consists of a single row, $w(C) = |\{j : C(0,j) \neq 0\}|$. The elements of C are divided between the q systems of initial conditions $(C^{0,t})_{t<q}$, and thus, assuming $C \neq 0$, there must exist some t for which $w(C^{0,t}) < w(C)$. Using Lemma 13, we conclude that $\#(A_C, n)$ is a sum of q^2 (not necessarily distinct) elements from $\{\#(A_E, n-1)\colon E \in X\}$, and that $w(E) \leqslant w(C) - 1$ for at least one of these.

By induction on k, we get that $\#(A_C, n)$ is a sum of $(q^2)^k$ elements from $\{\#(A_E, n-k)\colon E \in X\}$, and that for at least one of them we have $w(E) \leqslant \min(w(C) - k, 0)$. Taking $k \geqslant w(C)$, we get that $w(E) = 0$ for one of these E's, and hence $E = 0$, which implies that $\#(A_E, n-k) = 0$. This enables us to write $\#(A_C, n)$ as a sum of $(q^2)^k - 1$ elements from the set $\{\#(A_E, n-k)\colon E \in X\}$, and thus $\#(A_C, n) \leqslant ((q^2)^k - 1)M_{n-k}$. Now, if $k \geqslant \max_{C \in X} w(C)$, then $\#(A_C, n) \leqslant ((q^2)^k - 1)M_{n-k}$ for all $C \in X$, and we have $M_n \leqslant ((q^2)^k - 1)M_{n-k}$.

By induction, $M_{bk} \leqslant (q^{2k} - 1)^b M_0$ for every b. Hence the number $\#(A_B, bk)$ of g's in the first q^{bk} rows is at most $(q^{2k} - 1)^b M_0$. Since the total number of entries in these rows, belonging to the triangle corresponding to A_B, is approximately $\frac{M}{2} q^{2bk}$, the density of g's in the triangle is 0. □

4. Counting the number of occurrences of an element

In this section we prove the results relating to *DLR*s of any order.

Proof of Theorem 6. For every $n \geqslant 0$ define an row m-vector $\vec{v}^n = (v_i^n)_{i=1}^m$ by $v_i^n = \#(A_{B_i}, n)$ (so that $\vec{v} = \vec{v}^0$). Let us prove that $\vec{v}^{n+1} = \vec{v}^n D$ for every $n \geqslant 0$.

By Lemma 13, the ith entry of $\vec{v}^{n+1}$ is

$$v_i^{n+1} = \#(A_{B_i}, n+1) = \sum_{s,t=0}^{q-1} \#(A_{B_i^{s,t}}, n).$$

Thus, by the definition of D we have

$$v_i^{n+1} = \sum_{r=1}^{m} D_{r,i} \#(A_{B_r}, n),$$

which is the ith entry of $\vec{v}^n D$. Using induction we conclude that $\vec{v}^n = \vec{v}^0 D^n$, and in particular

$$\vec{v} D^n \vec{e}_i = \vec{v}^0 D^n \vec{e}_i = \vec{v}^n \vec{e}_i = v_i^n = \#(A_{B_i}, n). \qquad \square$$

Lemma 14. (1) *For every $1 \leqslant i \leqslant m$ there exists an $n_i \geqslant 0$ such that the ith entry of the row vector $\vec{v} D^{n_i}$ is positive.*

(2) *For every* $1 \leqslant j \leqslant m$ *there exists an* $l_j \geqslant 0$ *such that the* j*th entry of the column vector* $D^{l_j}\vec{e}_1$ *is positive.*

Proof. (1): By the definition of X', the initial conditions B_i cannot be g-trivial. Thus there exists $n_i \geqslant 0$ for which $\#(A_{B_i}, n_i) > 0$, and so $\vec{v}D^{n_i}\vec{e}_i > 0$.

(2): It can be easily proven by induction on n that the jth entry of $D^n\vec{e}_1$ is positive if and only if $B_j \in X_n$. Thus the claim follows from the observation that $B_j \in \bigcup_{k=0}^{\infty} X_k$. $\square$

Proof of Proposition 9. We use the following fact which is obtained easily from Jordan's form of a matrix D:

Let D *be a real square matrix and* ρ *the spectral radius of* D. *Then there exist positive numbers* C, e *such that* $C\rho^n \leqslant ||D^n||_\infty \leqslant n^e\rho^n$ *for all large enough* n (*where* $||E||_\infty = \max_{i,j}|e_{i,j}|$ *for a matrix* $E = (e_{i,j})$).

Since $\#(A_B, n) = \vec{v}D^n\vec{e}_1$, we conclude that

$$\#(A_B, n) \leqslant m \max(\vec{v}) \cdot ||D^n||_\infty \leqslant m \max(\vec{v}) n^e \rho^n.$$

Thus, for $e' > e$ we have $\#(A_B, n) \leqslant n^{e'}\rho^n$ for large enough n.

It remains to prove that there exists a positive constant C such that $\#(A_B, n) \geqslant C\rho^n$ for sufficiently large n. We prove first that there exists $N \geqslant 0$ such that for every $n \geqslant 0$ we have $\#(A_B, n+N) \geqslant ||D^n||_\infty$. Let n_i and l_j be as in the previous lemma and take $N = \max_{1 \leqslant i,j \leqslant m}(n_i + l_j)$. Let $n \geqslant 0$, and let $D^n_{i,j}$ be an entry of D^n such that $||D^n||_\infty = D^n_{i,j}$. By the previous lemma, the ith and jth entries of $\vec{v}D^{n_i}$ and $D^{l_j}\vec{e}_1$, respectively, are positive, and thus (noting that the entries of D are integers) they are at least 1. We conclude that

$$\#(A_B, n+N) \geqslant \#(A_B, n+n_i+l_j) = \vec{v}D^{n_i}D^nD^{l_j}\vec{e}_1 \geqslant D^n_{i,j} = ||D^n||_\infty.$$

Let $n \geqslant N$, and denote $n_0 = n - N$. We have $\#(A_B, n_0 + N) \geqslant ||D^{n_0}||_\infty$, and thus

$$\#(A_B, n) = \#(A_B, n_0 + N) \geqslant C\rho^{n_0} = \left(\frac{C}{\rho^N}\right)\rho^n$$

for all large enough n. $\square$

Proof of Theorem 10. (1) $\Leftrightarrow$ (2): Let T_n be the number of elements in the first q^n rows of the triangle corresponding to A_B. Since

$$T_n = \frac{q^n(2(l(B)+1) + (q^n - 1)M)}{2},$$

there exist constants $C_1, C_2 > 0$ such that $C_1(q^2)^n \leqslant T_n \leqslant C_2(q^2)^n$. Since the number $\#(A_B, n)$ of g's in those rows satisfies $C\rho^n \leqslant \#(A_B, n) < n^e\rho^n$ for sufficiently large n, the density of g's is 0 if and only if $\rho < q^2$.

(2) $\Leftrightarrow$ (3): Let $E = D^T$ be the transpose of D. We prove that $\rho(E) \geqslant q^2$ if and only if E has a submatrix $E' = (E_{i,j})_{i,j \in I}$ such that the sum of the entries in any row in E' is q^2.

Assume that E has such a submatrix E', and define a column m-vector $\vec{w} = (w_i)_{i=1}^m$ by $w_i = 1$ if $i \in I$ and $w_i = 0$ otherwise. It turns out that $E\vec{w} = q^2\vec{w}$, and thus $\rho(E) \geqslant q^2$.

Conversely, assume that E has an eigenvalue λ with $|\lambda| \geqslant q^2$, and let $\vec{w} = (w_i)_{i=1}^m$ be a corresponding eigenvector. We may assume (by multiplying $\vec{w}$ by a scalar) that $\max_i |w_i| = 1$. Let $I = \{i : |w_i| = 1\}$. Since $E\vec{w} = \lambda\vec{w}$, we have $\sum_{j=1}^m E_{i,j} w_j = \lambda w_i$, which implies that $\sum_{j=1}^m |E_{i,j}||w_j| \geqslant q^2$ for all $i \in I$. Since the entries in the ith row of E are nonnegative, and their sum is at most q^2, we must have $\sum_{j \in I} E_{i,j} = q^2$ for all $i \in I$. $\square$

Proof of Theorem 11. (1) Note that the density of 0's is 1 if and only if the density of every nonzero element is 0. Thus, it suffices to give an algorithm which checks whether the density of any given nonzero $g \in \mathbb{F}$ is 0. This can be accomplished by the following steps.

(a) Construct the set X:

$X_0 := B,\ i := 0$
do $X_{i+1} := C^{s,t} : C \in X_i,\ 0 \leqslant s, t < q$
    $i := i + 1$
until $X_i \subseteq \bigcup_{k=0}^{i-1} X_k$
$X := \bigcup_{k=0}^{i-1} X_k$

(b) Remove the g-trivial elements from X: in order to identify the g-trivial elements of X, define a directed graph $G = (X, E)$ by letting $(C_1, C_2) \in E$ if and only if $C_1^{s,t} = C_2$ for some (s, t) with $0 \leqslant s, t < q$. Let U be the set of elements for which $\#(A_C, 0) \geqslant 1$. Using Lemma 13, it follows that the g-trivial elements of X are exactly the vertices of G from which there is no directed path to U. Thus, the following steps construct X'.

$X' := U$
while (there exist $v \in X \backslash X'$ and $u \in X'$ with $(v, u) \in E$)
        $X' := X' \cup v$

(c) Construct the matrix D and check if it has a submatrix as in Theorem 10:
This can be done as follows:

$D' := D$
while (there exists an column in D' whose sum of entries is less than q^2)
    remove this column and the corresponding row from D'
If (D' is the empty matrix)

then return "there is no such submatrix"

else

return "D' is a submatrix as in Theorem 10"

(2) Let B denote the initial conditions:

$$B(d-1,0)=1 \quad \text{and } B(i,j)=0 \text{ for all other } i<d \text{ and } j\geqslant 0.$$

Observe that any bounded initial conditions are obtained as a linear combination of translates of B. Thus, if there exists initial conditions C such that the density of 0's in A_C is less than 1, then the density of 0's in A_B is also less than 1. Consequently, we only have to apply the algorithm of (1) to A_B. □

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