

ON BINOMIAL IDENTITIES IN ARBITRARY BASES

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ABSTRACT. We extend the digital binomial identity as given by Nguyen et al. to an identity in an arbitrary base b , by introducing the b -ary binomial coefficients. We then study the properties of these coefficients such as orthogonality, a link to Lucas' theorem and the corresponding b -ary Pascal triangles.

1. INTRODUCTION

In a series of recent articles, H.D. Nguyen [6, 7] himself and also together with T. Mansour [4, 5], have introduced different versions of the binomial identity, in which the usual integer powers are replaced by arithmetical functions that count the digits of these powers in some base b . See for example [2, Section 3].

Choose an integer n in base b , then denote $S_b^{(k)}(n)$ the number of k 's in the b -ary expansion

$$(1.1) \quad n = \sum_i n_i b^i$$

and $S_b(n)$ the sum of these digits,

$$(1.2) \quad S_b(n) = \sum_{j=0}^{b-1} j S_b^{(j)}(n).$$

The main extension of the binomial identity as given by Nguyen is for base $b = 2$ and reads as follows [6, 7]

$$(X + Y)^{S_2(n)} = \sum_{\substack{0 \leq k \leq n \\ (k, n-k) \text{ carry-free}}} X^{S_2(k)} Y^{S_2(n-k)},$$

where the sum is over all values of k such that the addition of k and $n - k$ in base b is carry-free; remarking that this condition is verified if and only if $S_2(k) + S_2(n - k) = S_2(n)$, Nguyen's formula is also stated equivalently in the beautifully symmetric form

$$(X + Y)^{S_2(n)} = \sum_{S_2(k) + S_2(n-k) = S_2(n)} X^{S_2(k)} Y^{S_2(n-k)}.$$

This result is proven in [7] using a polynomial generalization of the Sierpinski triangle, which is the Pascal triangle modulo 2.

An extension of this result to an arbitrary base b is given in [6] in the form

$$(1.3) \quad \prod_{i=0}^{N-1} \binom{x + y + n_i - 1}{n_i} = \sum_{0 \leq k \leq_b n} \prod_{i=0}^{N-1} \binom{x + k_i - 1}{k_i} \binom{x + n_i - k_i - 1}{n_i - k_i},$$

where the summation range $0 \leq k \leq_b n$ is over all integers k such that the digits of which satisfy $k_i \leq n_i$, for all $i \in \{0, \dots, N-1\}$.

The aim of this paper is to show that these results are a consequence of an elementary identity stated in the next section. This approach makes Nguyen's results more accessible and provides a number of generalizations. The paper is organized as follows.

In Section 2, we state a general formula on the sum over digits of an integer number. In Section 3, we provide a new version of the binomial expansion in base b that involves a corresponding b -ary binomials coefficients; some of the properties of these coefficients such as the link with Kummer's theorem and orthogonality are presented in Section 4. The last section exhibits the construction rules for a Pascal type triangle built from these coefficients.

2. A GENERAL FORMULA

Formula (1.3) is proved by Nguyen using a polynomial extension of Sierpinski's matrices. We show here another more elementary approach.

We start to remark that the formula (1.3) can be restated equivalently as

$$\prod_{i=0}^{N-1} \frac{(x+y)_{n_i}}{n_i!} = \prod_{i=0}^{N-1} \sum_{0 \leq k_i \leq n_i} \frac{(x)_{k_i}}{k_i!} \frac{(y)_{n_i-k_i}}{(n_i-k_i)!},$$

where $(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)}$ is the Pochhammer symbol. Next we realize that this identity holds in fact component-wise, i.e. for all $i \in \{0, \dots, N-1\}$ and arbitrary integer n_i , it holds that

$$(2.1) \quad \frac{(x+y)_{n_i}}{n_i!} = \sum_{k_i=0}^{n_i} \frac{(x)_{k_i}}{k_i!} \frac{(y)_{n_i-k_i}}{(n_i-k_i)!}$$

which is actually the Chu-Vandermonde identity. Since this identity is a consequence of the fact that the Pochhammer sequence $(x)_k$ is a binomial-type sequence¹, it suggests the following result.

Proposition 2.1. *Assume that $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are three sequences related as*

$$(2.2) \quad c_n = \sum_{k=0}^n \binom{n}{k} a_k b_{n-k},$$

then

$$(2.3) \quad \prod_{i=0}^{N-1} \frac{c_{n_i}}{n_i!} = \sum_{0 \leq k \leq n} \prod_{i=0}^{N-1} \frac{a_{k_i}}{k_i!} \frac{b_{n_i-k_i}}{(n_i-k_i)!}.$$

We now apply this general formula to obtain a base b generalization of the binomial identity.

3. A GENERALIZED BINOMIAL IDENTITY

Theorem 3.1. *With the notations (1.1) and (1.2), the identity*

$$(3.1) \quad (X+Y)^{S_b(n)} = \sum_{k=0}^n \binom{n}{k}_b X^{S_b(k)} Y^{S_b(n-k)}.$$

holds for all $X, Y \in \mathbb{C}$, where the b -ary binomial coefficients $\binom{n}{k}_b$ are defined as follows. Suppose n and k have expansions

$$\begin{cases} n = \sum_{l=0}^{N_n-1} n_l b^l, \\ k = \sum_{l=0}^{N_k-1} k_l b^l, \end{cases}$$

in base b , where N_n and N_k are the number of digits of n and k , respectively in base b . Then letting $N := \max\{N_n, N_k\}$, we have

$$(3.2) \quad \binom{n}{k}_b = \prod_{l=0}^{N-1} \binom{n_l}{k_l}.$$

Proof. It only suffices to assume that $\forall l = 0, \dots, N-1$, $0 \leq k_l \leq n_l$, since other cases gives $\binom{n}{k}_b = 0$. In this case, we could compute

$$n - k = \sum_{l=0}^{N-1} (n_l - k_l) b^l,$$

which is equivalent to both

$$S_b(k) + S_b(n-k) = S_b(n)$$

and also the assumption that the addition of k and $n-k$ is carry-free in base b , as mentioned before.

¹a sequence of polynomials $p_n(x)$ is of binomial type (see [8, p.26]) if it satisfies the convolution identity

$$p_n(x+y) = \sum_{k=0}^n \binom{n}{k} p_k(x) p_{n-k}(y).$$

Applying the convolution relation (2.3) to get

$$(3.3) \quad \prod_{l=0}^{N-1} c_{n_l} = \prod_{l=0}^{N-1} n_l! \sum_{k_l=0}^{n_l} \frac{a_{k_l}}{j_k!} \cdot \frac{b_{n_l-k_l}}{(n_l-k_l)!} = \sum_{S_b(k)+S_b(n-k)=S_b(n)} \left(\prod_{l=0}^{N-1} \binom{n_l}{k_l} a_{k_l} b_{n_l-k_l} \right).$$

Now the choice

$$\begin{cases} c_{n_l} := (X + Y)^{n_l} & , \\ a_{k_l} := X^{k_l} & , \\ b_{n_l-k_l} := Y^{n_l-k_l} & , \end{cases}$$

satisfies the property (2.2); with this choice, we obtain

$$\prod_{l=0}^{N-1} (X + Y)^{n_l} = \sum_{S_b(k)+S_b(n-k)=S_b(n)} \left(\prod_{l=0}^{N-1} \binom{n_l}{k_l} X^{k_l} Y^{n_l-k_l} \right).$$

Notice that the left-hand side reads

$$\prod_{l=0}^{N-1} (X + Y)^{n_l} = (X + Y)^{\sum_{l=0}^{N-1} n_l} = (X + Y)^{S_b(n)},$$

while the right-hand side is

$$\prod_{l=0}^{N-1} \sum_{k_l=0}^{n_l} \binom{n_l}{k_l} X^{k_l} Y^{n_l-k_l} = \sum_{k=0}^n \left(\prod_{l=0}^{N-1} \binom{n_l}{k_l} \right) X^{\sum_{l=0}^{N-1} k_l} Y^{\sum_{l=0}^{N-1} (n_l-k_l)} = \sum_{k=0}^n \left(\prod_{l=0}^{N-1} \binom{n_l}{k_l} \right) X^{S_b(k)} Y^{S_b(n-k)}.$$

This completes the proof. \square

Remark 3.2. When $b > n$, then in the b -ary expansion, n has only one digit, i.e., $n = n_0$. And so do k and $n - k$, since $k \leq n$. Then, we have

$$\begin{cases} S_b(n) = n_0 = n & , \\ S_b(k) = k, \quad S_b(n-k) = n - b & , \\ \binom{n}{k}_b = \binom{n}{k} & , \end{cases}$$

so that 3.1 reduces to

$$(X + Y)^n = \sum_{k=0}^n \binom{n}{k} X^k Y^{n-k},$$

namely the usual binomial identity.

Corollary 3.3. *When $b = 2$, i.e. in the binary case, we recover the identity*

$$(X + Y)^{S_2(n)} = \sum_{S_2(k)+S_2(n-k)=S_2(n)} X^{S_2(k)} Y^{S_2(n-k)}$$

since the coefficients $\binom{n}{k}_2$ take the value $\binom{0}{0} = \binom{1}{0} = \binom{1}{1} = 1$ and 0 otherwise.

In the case $b = 3$, all $\binom{n}{k}_3 = 0$ if $S_3(k) + S_3(n-k) \neq S_3(n)$; otherwise, $\binom{n}{k}_3 = 1$ except for $\binom{2}{1} = 2$. Therefore, we have the equivalent expression

$$(X + Y)^{S_3(n)} = \sum_{S_3(k)+S_3(n-k)=S_3(n)} 2^{S_3^{(2)}(n)-S_3^{(2)}(k)-S_3^{(2)}(n-k)} X^{S_3(k)} Y^{S_3(n-k)}.$$

Remark 3.4. Different choices of $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ lead to different identities. For instance, another possible choice is

$$\begin{cases} c_n = (X + Y)_n & , \\ a_n = (X)_n & , \\ b_n = (Y)_n & . \end{cases}$$

The generating function

$$\sum_{n \geq 0} \frac{(X)_n}{n!} z^n = \frac{1}{(1-z)^X}$$

shows that the convolution property (2.2) holds. Then from (3.3), we deduce

$$\prod_{l=0}^{N-1} \frac{(X+Y)_{n_l}}{n_l!} = \sum_{S_b(k)+S_b(n-k)=S_b(n)} \left(\prod_{l=0}^{N-1} \frac{X_{j_l}}{j_l!} \cdot \frac{Y_{n_l-j_l}}{(n_l-j_l)!} \right),$$

or equivalently

$$\prod_{l=0}^{N-1} \binom{X+Y+n_l-1}{n_l} = \sum_{0 \leq k \leq_b n} \left[\prod_{l=0}^{N-1} \binom{X+j_l-1}{j_l} \binom{Y+n_l-j_l-1}{n_l-j_l} \right],$$

which appears as [6, Thm.2]. Then, for the binary case $b = 2$, $n_l \in \{0, 1\}$ yields $n_l! = 1$ for all l , and from

$$(X+Y)_{n_l} = \begin{cases} X+Y, & \text{if } n_l = 1 \\ 0, & \text{if } n_l = 0 \end{cases}$$

we deduce

$$(X+Y)^{S_2(n)} = \sum_{S_2(k)+S_2(n-k)=S_2(n)} X^{S_2(k)} Y^{S_2(n-k)}.$$

4. PROPERTIES OF THE BINOMIAL COEFFICIENTS

We state in this sections some properties of the generalized binomial coefficients defined by (3.2). The first one is a generating function for these coefficients.

4.1. Generating function.

Theorem 4.1. *A generating function for the b -ary binomial coefficients $\binom{n}{k}_b$ is*

$$(4.1) \quad \sum_{k=0}^n \binom{n}{k}_b x^k = \prod_{l=0}^{N-1} (1 + x^{b^l})^{n_l}.$$

Proof. Define the right hand side as $P(x)$, a polynomial with degree

$$\deg P = \sum_{l=0}^{N-1} n_l b^l = n.$$

Denote $a_k^{(n)}$ the coefficient of x^k in $P(x)$ and remark that $a_k^{(n)} = 0$ if and only if there is a carry in the addition of k and $n - k$ in base b . An elementary enumeration shows that

$$a_k^{(n)} = \binom{n_{N-1}}{k_{N-1}} \binom{n_{N-2}}{k_{N-2}} \cdots \binom{n_0}{k_0}$$

which gives the desired result. □

4.2. Multinomial version. The next property is the extension of the previous result to the multinomial case.

Theorem 4.2. *[The Multinomial Version] Define the multinomial coefficient*

$$\binom{n}{k_1, \dots, k_m}_b = \prod_{l=0}^{N-1} \binom{n_l}{(k_1)_l, \dots, (k_m)_l}_b$$

where $(k_i)_l$ denotes the rank- l digit in the expression of k_i in base b .

Then

$$(X_1 + \cdots + X_m)^{S_b(n)} = \sum_{k_1, \dots, k_m} \binom{n}{k_1, \dots, k_m}_b X_1^{S_b(k_1)} \cdots X_m^{S_b(k_m)}.$$

Proof. The proof is straightforward from the binomial expansion (3.1) and (3.2). □

4.3. Symmetries.

Theorem 4.3. *The b -ary binomial coefficients satisfy*

(1) the symmetry property

$$\binom{n}{k}_b = \binom{n}{n-k}_b$$

(2) the recurrence

$$\binom{n}{k}_b = \binom{n-b}{k-b}_b + \binom{n-b}{k}_b.$$

Proof. The symmetry property is easily deduced from the definition (3.2) and the invariance $k_l \mapsto n_l - k_l$ of each term.

For the recurrence property, assume that each k and n have a non-zero rank first digit, i.e. $k_0 > 0$, $n_0 > 0$. Then, $(n-b)_0 = n_0 - 1$ and $(k-b)_0 = k_0 - 1$ and

$$\binom{n-b}{k-b}_b = \binom{n_{N-1}}{k_{N-1}} \cdots \binom{n_0-1}{k_0-1}, \quad \binom{n-b}{k}_b = \binom{n_{N-1}}{k_{N-1}} \cdots \binom{n_0-1}{k_0}$$

and we deduce

$$\binom{n-b}{k-b}_b + \binom{n-b}{k}_b = \binom{n_{N-1}}{k_{N-1}} \cdots \left[\binom{n_0-1}{k_0-1} + \binom{n_0-1}{k_0} \right] = \binom{n_{N-1}}{k_{N-1}} \cdots \binom{n_0}{k_0} = \binom{n}{k}_b.$$

This extends easily to the case where one or two of the numbers (k_0, n_0) is equal to 0. \square

4.4. Link with Lucas' theorem. The definition (3.2) will look familiar to those readers who have already met Lucas' famous theorem which we restate here (see also [3])

Theorem 4.4. *[Lucas] For p a prime number, the binomial coefficients satisfy*

$$\binom{n}{k} \equiv \binom{n_N}{k_N} \cdots \binom{n_0}{k_0} \pmod{p}.$$

Under the same condition, this theorem is concisely rephrased in our notations as

$$\binom{n}{k} \equiv \binom{n}{k}_p \pmod{p}.$$

Remark 4.5. The Sierpinski matrix that is used by Nguyen et al. to prove their results contains the coefficients $\binom{n}{k} \pmod{p}$, or equivalently $\binom{n}{k}_p \pmod{p}$, which do not coincide with the coefficients $\binom{n}{k}_p$ studied here, but are congruent to them.

4.5. Orthogonality Relations. There are many elementary identities involving the usual binomial coefficients that can be transferred to the case of the b -ary binomial coefficients. Here, we only show one as an example.

Example 4.6. If two sequences $\{a_n\}$ and $\{c_n\}$ are related as

$$a_n = \sum_{k=0}^n (-1)^k \binom{n}{k} c_k$$

then

$$c_n = \sum_{k=0}^n (-1)^k \binom{n}{k} a_k;$$

these identities are known as inverse relations. Note that they are equivalent to the orthogonality conditions

$$\sum_{k=j}^n \binom{n}{k} \binom{k}{j} (-1)^{k+j} = \delta_{n,j} = \begin{cases} 1 & j = n \\ 0 & \text{otherwise} \end{cases}.$$

The generalization to the b -ary case is as follows.

Theorem 4.7. *The b -ary binomial coefficients satisfy the orthogonality relations*

$$(4.2) \quad \sum_{k=j}^n \binom{n}{k}_b \binom{k}{j}_b (-1)^{S_b(k)+S_b(j)} = \delta_{n,j},$$

namely

$$a_{S_b(n)} = \sum_{k=0}^n (-1)^{S_b(k)} \binom{n}{k}_b c_{S_b(n)} \Rightarrow c_{S_b(n)} = \sum_{k=0}^n (-1)^{S_b(k)} \binom{n}{k}_b a_{S_b(n)}.$$

Proof. Since

$$\binom{n}{k}_b = \prod_{l=1}^{N-1} \binom{n_j}{k_l}_b, \quad \binom{k}{j}_b = \prod_{l=1}^{N-1} \binom{k_l}{j_l}_b, \quad S_b(k) = \sum_{l=0}^{N-1} k_l \text{ and } S_b(j) = \sum_{l=0}^{N-1} j_l,$$

we get

$$\sum_{k=j}^n \binom{n}{k}_b \binom{k}{j}_b (-1)^{S_b(k)+S_b(j)} = \prod_{l=1}^{N-1} \sum_{k_l=j_l}^{n_l} \binom{n_j}{k_l}_b \binom{k_l}{j_l}_b (-1)^{k_l+j_l} = \prod_{l=1}^{N-1} \delta_{n_l,j_l} = \delta_{n,j}.$$

□

5. PASCAL TRIANGLES

In this last section, we look at the equivalent of Pascal triangles that can be built from the b -ary binomial coefficients $\binom{n}{k}_b$. Let us start with two examples, where we systematically replace each null binomial entry with a dot (.) symbol to make the structure of the triangle more apparent; remember that these null entries correspond to the couples (n, k) for which the addition of k and $n - k$ is not carry-free in base b .

Example 5.1. In base $b = 3$, the first 9 rows of the Pascal triangle are

$$T_2^{(3)} = \begin{array}{cccccccccc} & & & & & & & 1 & & & & \\ & & & & & & & & 1 & & 1 & \\ & & & & & & 1 & & 2 & & 1 & \\ & & & & 1 & & . & & . & & 1 & \\ & & & 1 & & 1 & & . & 1 & & 1 & \\ & & 1 & & 2 & & 1 & & 1 & & 2 & & 1 & \\ & 1 & & 1 & & . & & . & 2 & & . & & . & & 1 & \\ 1 & & 1 & & 1 & & . & & 2 & & 2 & & . & & 1 & & 1 & \\ & & 1 & & 2 & & 1 & & 2 & & 4 & & 2 & & 1 & & 2 & & 1 \end{array}$$

Denote as

$$T_1^{(3)} = \begin{array}{ccc} & & 1 \\ & 1 & & 1 \\ & & 1 & & 1 \end{array}$$

the first 3 top rows of this triangle and

$$* = \begin{array}{cc} . & . \\ . & . \end{array} = \begin{array}{cc} 0 & 0 \\ & 0 \end{array}$$

the elementary reverse triangle built with zero entries, and notice that

$$T_2^{(3)} = \begin{array}{ccccc} & & T_1^{(3)} & & \\ & T_1^{(3)} & * & T_1^{(3)} & \\ T_1^{(3)} & * & 2T_1^{(3)} & * & T_1^{(3)} \end{array}.$$

It appears that, with obvious notations,

$$T_2^{(3)} = T_1^{(3)} \otimes T_1^{(3)}$$

from which we deduce

$$T_m^{(3)} = [T_1^{(3)}]^{\otimes m}.$$

The operator \otimes will be defined and discussed in the Proposition 5.3 below.

Example 5.2. In base 4, the first 16 rows of the binomial triangle $T_4^{(2)}$ are as follows.

Starting from the elementary triangles,

$$T_1^{(4)} = \begin{pmatrix} & & 1 & & \\ & 1 & & 1 & \\ & & 2 & & \\ 1 & & & 3 & \\ & 1 & & & 1 \end{pmatrix}, * = \begin{pmatrix} & & & & 0 & 0 & 0 \\ & & & & & 0 & 0 \\ & & & & & & 0 \\ & & & & & & \\ & & & & & & \end{pmatrix}$$

$$T_2^{(4)} = \begin{array}{ccccccc} & & & T_1^{(4)} & & & \\ & & & * & & T_1^{(4)} & \\ & T_1^{(4)} & & 2T_1^{(4)} & & * & T_1^{(4)} \\ & & * & * & & * & \\ T_1^{(4)} & & 3T_1^{(4)} & * & & 3T_1^{(4)} & * & T_1^{(4)} \\ & & & & & & & \end{array}$$
$$T_n^{(4)} = \left[T_1^{(4)} \right]^{\otimes n}.$$

Proposition 5.3. *The structure of the triangle built from the coefficients $\binom{n}{k}_b$ satisfies*

- (1) $T_1^{(b)}$ is made of the first b rows of the usual Pascal triangle;
- (2) for $m > 1$, $T_m^{(b)}$ is obtained from $T_{m-1}^{(b)}$ by the associative operation

$$T_m^{(b)} = T_{m-1}^{(b)} \otimes T_1^{(b)},$$

$$\binom{0}{0}T_{m-1}^{(b)}$$

$$T_m^{(b)} = \begin{matrix} & & \binom{1}{0} T_{m-1}^{(b)} & & \binom{1}{1} T_{m-1}^{(b)} \\ & \ddots & & \ddots & \\ \binom{b-1}{0} T_{m-1}^{(b)} & & \cdots & \binom{b-1}{j} T_{m-1}^{(b)} & \cdots & \binom{b-1}{b-1} T_{m-1}^{(b)} \end{matrix}$$

$$T_m^{(b)} = \left[T_1^{(b)} \right]^{\otimes m};$$

- (3) $T_m^{(b)}$ has b^m rows since

$$\#rows\left(T_m^{(b)}\right)=b \times \#rows\left(T_{m-1}^{(b)}\right) ;$$

- (4) the bottom row of T_m^b has b^m entries.

Proof. These properties are a direct consequence of (3.2). Since properties (3) and (4) are elementary, only properties (1) and (2) need to be verified and here we use induction.

- (i) For $m = 1$, $T_1^{(b)}$ contains the first b rows of the triangle, made of coefficients $\binom{n}{k}_b$ where

$$0 \leq n \leq b-1,$$

so that the b -ary expression of n consists of a single digit: $n = n_0$. In this case, $\binom{n}{k}_b$ coincides with $\binom{n}{k}$ making the first b rows coincide with those of the Pascal triangle (see Remark 3.2)

- (ii) Consider $T_m^{(b)}$ by assuming that properties (1) and (2) hold for $T_1^{(b)}, \dots, T_{m-1}^{(b)}$. Then, the b^m elements are exactly the case

$$0 \leq n \leq b + \dots + b^m - 1,$$

which implies that n could have at most m digits, namely

$$n = n_{m-1}b^{m-1} + \cdots + n_0.$$

Thus,

$$\binom{n}{k}_b = \binom{n_{m-1}}{k_{m-1}} \underbrace{\prod_{l=0}^{m-2} \binom{n_l}{k_l}}_{\text{Copy of } T_{m-1}^{(b)}}.$$

Since $0 \leq n_{m-1} \leq b-1$, $\binom{n_{m-1}}{k_{m-1}}$ gives a copy of $T_1^{(b)}$ while the rest of the product gives a copy of $T_{m-1}^{(b)}$. Thus, by induction

$$T_m^{(b)} = \left[T_1^{(b)} \right]^{\otimes m}.$$

□

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