ON PASCAL'S TRIANGLE MODULO 2 IN FIBONACCI REPRESENTATION

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(Submitted July 2001–Final Revision June 2002)

INTRODUCTION

In 1977, D. Hewgill published in this QUARTERLY a paper entitled "A Relationship Between Pascal's Triangle and Fermat's Numbers." [2] There he shows that if the entries of Pascal's triangle are reduced modulo 2 and each row is interpreted as a binary number, then each is a product of distinct Fermat numbers. That is, the members of the sequence

$$1, 3, 5, 15, \ldots, \sum_{i=0}^{n} \binom{n}{i}_{(\mathrm{mod}\ 2)} 2^i, \ldots \tag{1}$$

(see Figure 1) are related by a common characteristic involving Fermat numbers.

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 0 & & 1 & \\ 1 & & 1 & & 1 & & 1 \\ & & & \cdots & & & \end{array} \qquad \begin{array}{ccccccc} & & & 2^0 & & & \\ & & 2^1 & + & 2^0 & & \\ & 2^2 & & + & & 2^0 & \\ 2^3 & + & 2^2 & + & 2^1 & + & 2^0 \\ & & & \cdots & & & \end{array}$$

Figure 1. Rows of Pascal's triangle, reduced modulo 2 and interpreted as binary numbers.

In the present paper we consider an analogous problem, but instead of using the binary number system as Hewgill did, we interpret *every second* row of modulo 2 reduced Pascal's triangle in *the Fibonacci number system.* Moreover, we are interested about the whole family of sequences where the even indexed rows specify the most significant digits *("fibits")* of the Zeckendorf expansion of an integer, with an arbitrary, but fixed amount of zeros appended to the right. For example, if we let the rightmost 1 of each row stand for the least significant "fibit" ($= F_2$) in a such expansion (that is, no zeros appended), we obtain the sequence

$$1, 4, 9, 33, \ldots, \sum_{i=0}^{2n} \binom{2n}{i}_{(\mathrm{mod}\ 2)} F_{i+2}, \ldots \tag{2}$$

(see Figure 2.)

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & 0 & 1 & & \\ & 1 & 0 & 0 & 0 & 1 & \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ & & & \cdots & & & \end{array} \qquad \begin{array}{ccccccc} & & & F_2 & & & \\ & & F_4 & + & F_2 & & \\ & F_6 & & + & & F_2 & \\ F_8 & + & F_6 & + & F_4 & + & F_2 \\ & & & \cdots & & & \end{array}$$

Figure 2. Even indexed rows of Pascal's triangle, reduced modulo 2 and interpreted as numbers in the Fibonacci number system.

The following question is now asked: are the members of a sequence in this family related by some characteristic analogous to Hewgill's observation about the rows of Pascal's triangle reduced modulo 2? The purpose of this paper is to provide an affirmative answer to this question and to detail the relationship.

TRIANGLE TRIVIA

The elements of Pascal's triangle are binomial coefficients, thus its elements reduced modulo 2 are the coefficients of the successive powers $n = 0, 1, 2, 3, \ldots$ of the $\mathbb{Z}_2[X]$ polynomial $x + 1$ in the case of triangle 1, and of the polynomial $x^2 + 1$ in the case of triangle 2. As noted by Wolfram [5] and other authors before him, these triangles, if viewed from infinitely far, with the details getting infinitely small, form a self-similar fractal image called Sierpinski Sieve. What's more important for us, the triangle 2 can be viewed also as showing the successive generations of the one-dimensional linear cellular automation rule 90 (as specified in [5]) beginning from the seed of single active cell in the generation 0 on the top, with the rule saying that in subsequent generations (i.e. rows of the triangle) each cell will be 1 (*alive*) only if either (*but not both*) of its two immediate neighbours was alive in the previous generation. That is, each cell's state is modulo two sum of its two nearest neighbors on the previous step. Keeping this cellular automata connection in mind, we will now and then use such "organic" phrasings like that each successive row of the triangle (or associated Zeckendorf Expansion) "has grown from the previous one".

Furthermore, for each pattern of alive and dead cells there can exist no more than one predecessor which contains finite number of alive cells. This is easily seen when we realize that such finite predecessor of any pattern can be found by dividing the associated polynomial with x^2+1 (and it exists only if the remainder is zero), and when we remember that the polynomial ring $\mathbb{Z}_2[X]$ is a Unique Factorization Domain.

We list few formal identities concerning these two variants of Pascal's Triangle computed modulo 2, of which the last (9) provides the crux move of our proof.

$$\binom{2^j}{i} \equiv 0 \pmod 2, 0 < i < 2^j \tag{3}$$

$$\binom{2^k + r}{i} \equiv \binom{r}{i} \pmod 2, 0 \leq i \leq r \tag{4}$$

$$\binom{2^k + r}{2^k + r - i} \equiv \binom{r}{i} \pmod 2, 0 \leq i \leq r \tag{5}$$

$$\binom{2^k + r}{i} \equiv \binom{2^k + r}{2^k + i} \pmod 2, 0 \leq i \leq r \tag{6}$$

$$\binom{2n}{2k + 1} \equiv 0 \pmod 2 \tag{7}$$

$$\binom{2^k n}{2^k i} \equiv \binom{n}{i} \pmod 2, k \geq 0 \tag{8}$$

$$\binom{4n}{4k+i} \equiv 0 \pmod 2, 0 < i < 4 \tag{9}$$

The identity (3) follows from a more general fact that $\binom{p^j}{i} \pmod p = 0$, when $0 < i < p^j$, and p is a prime. The identities (4), (5) and (6) are its corollaries, which is best seen in CA-terms: the identity (3) says that the population of automaton collapses to just two active cells in every generation 2^j, and being separated from each other by $2^{j+1}-1$ vacant cells, they both have just enough time to spawn two exact replicas of the tip of the triangle before the population collapses again, at the step 2^{j+1}.

The identity (7), which guarantees that there are no consecutive 1's on any even row of Pascal's Triangle mod 2, follows from:

$$\binom{2n}{2k} = \binom{2n-2}{2k-2} + \binom{2n-2}{2k-1} + \binom{2n-2}{2k-1} + \binom{2n-2}{2k} \tag{10}$$

and as the two middlemost terms cancel out when computed modulo 2 we get the equation:

$$\binom{2n}{2k} \equiv \binom{2n-2}{2k-2} + \binom{2n-2}{2k} \pmod 2 \tag{11}$$

and similarly:

$$\binom{2n}{2k+1} \equiv \binom{2n-2}{2k-1} + \binom{2n-2}{2k+1} \pmod 2 \tag{12}$$

which means that in any such triangle obeying the rule 90, the even and odd-positioned cells have separate evolutions. And as the single initial seedling $\binom{0}{0}$ is located at even position, the odd-positioned cells will stay quiescent for ever and always.

The identity (8) follows by applying the well-known [1], [4] (c.f. entry *factorial*) function ε_p for computing the highest power of prime p dividing $h!$ which in case $p = 2$ reduces to

$$\varepsilon_2(h) = h - \sum_{j=0}^{\infty} bit_j(h) \tag{13}$$

where

$$bit_j(n) \stackrel{\text{def}}{=} \lfloor \frac{n}{2^j} \rfloor \pmod 2. \tag{14}$$

When we apply it to a binomial coefficient $\binom{n}{i} = \frac{n!}{(n-i)!i!}$ we observe that

$$\begin{aligned}(n - \sum_{j=0}^{\infty} bit_j(n)) - (n - i - \sum_{j=0}^{\infty} bit_j(n-i)) - (i - \sum_{j=0}^{\infty} bit_j(i)) \\ = \sum_{j=0}^{\infty} bit_j(n-i) + bit_j(i) - bit_j(n)\end{aligned} \tag{15}$$

from which we see that the result does not depend on the magnitude of the integers n and i, but only the digital sums (in base 2) of them and their difference. Thus, as $\binom{2^k n}{2^k i} = \frac{(2^k n)!}{(2^k(n-i))!(2^k i)!}$ and multiplying by 2's power does not change the bit sum, the stated identity follows.

The identity (9) is a corollary of the identities (7) and (8) and it says that on every fourth row of Pascal's triangle modulo 2 ones can occur only in positions divisible by four. Applying the same analysis repeatedly yields a similar result for all higher powers of 2 as well.

Note. The identities (7) and (8) give us also an alternative sum formula for the sequence given in (2), i.e.

$$\sum_{i=0}^{2n} \binom{2n}{i}_{(\text{mod } 2)} F_{i+2} = \sum_{i=0}^{n} \binom{n}{i}_{(\text{mod } 2)} F_{2i+2} \tag{16}$$

THE THEOREM

Theorem:

The identity

$$\sum_{i=0}^{2n} \binom{2n}{i}_{(\text{mod } 2)} F_{i+d} = E_{n+d} \prod_{i=0}^{\infty} L_{2^i}^{bit_i(n)} \tag{17}$$

where E_{n+d} stands for F_{n+d} (the $n+d$:th Fibonacci number) if n is even, and L_{n+d} (the $n+d$:th Lucas number) if n is odd, holds for all integers $n \geq 0, d \geq 0$.

Note. The integer d gives the "displacement". For example, with $d = 2$ we get the sequence (2) shown in the introduction, and with $d = 3$ there's one extra zero appended to the right of each such Zeckendorf expansion. With $d = 1$ the standard Zeckendorf expansion does not anymore contain the same pattern of ones and zeros as the corresponding row in a modulo 2 reduced Pascal's triangle, because of the corrupting "carry effect" $F_1 + F_3 = F_4$, but the soon defined *odd indiced Fibonacci representation* shows the pattern faithfully here. With $d = 0$ the standard Zeckendorf expansion works again, except that the rightmost 1 of each row is annihilated as $F_0 = 0$.

We prove our identity in a sequence of four lemmas. Most of the time our workhorse is the identity (c.f. for example the entry *Fibonacci Number* in [4])

$$L_m F_p = F_{m+p} + (-1)^{p+1} F_{m-p} \tag{18}$$

which essentially says that under certain conditions, the pattern of 0's and 1's in Zeckendorf or a similar expansion of a positive integer can be duplicated in a mirror-image fashion by multiplying it with an appropriate Lucas number. We immediately see that an easy way to obtain such conditions is to make sure that all p's (i.e. the positions of 1's in each expansion) are odd and less than m. This gives us a reason to introduce the following concept.

Definition: We say that a natural number n has a unique *odd indiced Fibonacci representation* if it can be represented as a sum of distinct odd-indiced Fibonacci numbers.

Remark: First such numbers are:

$$\begin{aligned} F_1 &= 1 \\ F_3 &= 2 \\ F_1 + F_3 &= 3 \\ F_5 &= 5 \\ F_1 + F_5 &= 6 \\ &\text{etc.} \end{aligned}$$

Note that for every other such integer (i.e. ones with no F_1 in the sum, 2, 5, 7, 13, 15, 18, 20, 34, 36, ...) the indices of Fibonacci numbers present give also the positions of ones in the Zeckendorf Expansion of that integer, and these are precisely all the integers whose Zeckendorf Expansion contain ones only at odd positions.

Definition: Let c be the sequence

$$1, 3, 7, 21, 47, 141, 329, 987, 2207, 6621, 15449, 46347, 103729, \ldots$$

obtained as

$$c[n] \stackrel{\text{def}}{=} \sum_{i=0}^{2r(n)} \binom{2r(n)}{i}_{(\text{mod } 2)} F_{n-i} \tag{19}$$

where n ranges through all the positive odd integers $1, 3, 5, 7, 9, 11, \ldots$ and the auxiliary function r (introduced for the notational convenience) gives the remainder after the most significant 1-bit of its positive integer argument has been "toggled off":

$$r(n) \stackrel{\text{def}}{=} n - 2^{\lfloor \log_2(n) \rfloor}. \tag{20}$$

The first lemma and its corollary shows that the terms of the sequence c occur at the product side of our theorem (17).

Lemma 1: *If k is a positive integer ≥ 1, and n is an odd positive integer less than 2^k, then*

$$c[2^k + n] = L_{2^k} c[n] \tag{21}$$

Proof: We need three conditions to get the induction working:

(1) $c[n]$ has an *odd indiced Fibonacci representation.*
(2) The index i_1 of the least Fibonacci number in such representation of $c[n]$ is
$$2^{\lfloor \log_2(n)+1 \rfloor} - n.$$
(3) The index i_u of the greatest Fibonacci number in such representation of $c[n]$ is n.

If above conditions hold (as is true with the base case $c[1] = 1$), then by applying the identity (18) we get:

$$L_{2k} c[n] = \sum_{i=0}^{2r(n)} \binom{2r(n)}{i}_{(\text{mod } 2)} F_{2^k+n-i} + F_{2^k-n+i} \tag{22}$$

We see that the most significant (leftmost) one present in the odd indiced Fibonacci representation of $L_{2^k}\ c[n]$ will be located at the position $2^k + n$ and furthermore, the pattern formed by the ones at the lefthand side will be the exact replica of the original pattern in

$c[n]$. Likewise, the least significant (rightmost) one present in the expansion of $L_{2^k}\ c[n]$ will be located at $2^k - n$, and the pattern of ones at the righthand side will also duplicate *(in reverse order)* the pattern found in $c[n]$. Furthermore, from the identities (4) and (5) we see that this is exactly the pattern of ones found in the odd indiced Fibonacci representation of

$$c[2^k + n] = \sum_{i=0}^{2n} \binom{2n}{i}_{(\text{mod } 2)} F_{2^k+n-i} \tag{23}$$

Thus, if $c[n]$ satisifes the three conditions given above, then any value $c[m]$ will also satisfy them, with respect to a new index $m = 2^k + n$, where $n < 2^k$. Here the reader may wish consult the appendix which gives the odd indiced Fibonacci representation of the sequence $c[n]$ upto the term $c[31]$.

Now it easily follows by induction:

Corollary: If n is an odd positive integer then

$$c[n] = \prod_{i=0}^{\infty} L_{2^i}^{bit_i(n)} \tag{24}$$

Lemma 2: *If n if an odd positive integer then*

$$L_{n+d}c[n] = \sum_{i=0}^{2n} \binom{2n}{i}_{(\text{mod } 2)} F_{i+d} \tag{25}$$

Proof: Proceeding as with the previous lemma, we get

$$L_{n+d}c[n] = \sum_{i=0}^{2r(n)} \binom{2r(n)}{i}_{(\text{mod } 2)} F_{2n-i+d} + F_{i+d} \tag{26}$$

We see that the union of all the selected indices that appear in the above sum

$$\bigcup_{i=0}^{2r(n)} \{2n - i + d \cup i + d | \binom{2r(n)}{i} \textit{ is odd}\} \tag{27}$$

is disjoint, all the indices are of the same parity (odd if d is odd, even if d is even), and the difference of the least and the largest is $2n$, and this set of indices consists of two disjoint copies of the pattern present in the odd indiced Fibonacci representation of $c[n]$.

Noticing that $2n = 2^{\lfloor \log_2(2n) \rfloor} + 2r(n)$ and then applying the identities (4) and (5) we get the identities $\binom{2n}{i} \equiv \binom{2r(n)}{i} \pmod 2$ and $\binom{2n}{2n-i} \equiv \binom{2r(n)}{i} \pmod 2$, from which we see this is just the pattern present on the row $2n$ of Pascal's triangle computed modulo 2.

However, to proceed to the case where n is even, we have to completely change our view of who is the "object" and who is the "subject" when we apply the identity (18). In contrast to above lemma (case n odd), where we were duplicating the pattern of digits (of the odd indiced Fibonacci representation) in $c[n]$ (the "object") by multiplying it with L_{n+d} (the "subject", "duplicator"), we now must let the factor L_{n+d} or F_{n+d} be the "object" which is duplicated, and dig the "duplicators" from the term $c[n]$ itself. The next lemma guarantees that this is possible.

Lemma 3: *All terms $c[n]$, where n is an odd number, can be also represented as sums of distinct Lucas numbers, i.e. more formally, $c[1] = 1 = L_1$ and for all odd values of $n \geq 3$:*

$$\sum_{i=0}^{2r(n)} \binom{2r(n)}{i}_{(\text{mod } 2)} F_{n-i} = \sum_{i=0}^{2r(n)-2} \binom{2r(n)-2}{i}_{(\text{mod } 2)} L_{n-i-1} \tag{28}$$

Proof: We know that $2r(n) \equiv 2 \pmod 4$ (for all odd values of $n \geq 3$) thus $2r(n) - 2$ is divisible by 4 and from (9) we see that such rows of the Pascal's Triangle mod 2 may not contain ones nearer than four positions from each other, thus each subpattern $\ldots 00100 \ldots$ on the row $4k$ grows (*by the rule 90*) unhindered to subpattern $\ldots 01010 \ldots$ on the row $4k + 2$, and as $L_n = F_{n-1} + F_{n+1}$ the lemma follows. We are now ready for the finale.

Lemma 4: *If n is an even non-negative integer, then*

$$F_{n+d}c[n+1] = \sum_{i=0}^{2n} \binom{2n}{i}_{(\text{mod } 2)} F_{i+d}. \tag{29}$$

Proof: Again we consider the case $n = odd$ first. If we multiply both sides of the equation (28) by L_{n+d} we get the following equation that holds for all odd values of $n \geq 3$:

$$L_{n+d}c[n] = \sum_{i=0}^{2r(n)-2} \binom{2r(n)-2}{i}_{(\text{mod } 2)} L_{n-i-1}(F_{n-1+d} + F_{n+1+d}) \tag{30}$$

(by the definition of $c[n]$ and on the right hand side by decomposing L_{n+d} into $F_{n-1+d}+F_{n+1+d}$ and transferring it inside the sum.)

One more time we apply the identity (18), and resolve the negative indices produced with the help of identities $F_{-2i} = -F_{2i}$ and $F_{-(2i+1)} = F_{2i+1}$ to get

$$L_{n+d}c[n] = \sum_{i=0}^{2r(n)-2} \binom{2r(n)-2}{i}_{(\text{mod } 2)} (F_{2n-i+d} + F_{2n-2-i+d} + F_{i+2+d} + F_{i+d}) \tag{31}$$

Obviously, the right hand sides of the equations (26) and (31) are equal in the usual sense, but we must also check they are equal "indexwise", i.e. that their equality is not result of any intricate carry-effect. For that, it suffices to show that the union of the indices

$$\bigcup_{i=0}^{2r(n)-2} \{2n - i + d \cup 2n - 2 - i + d \cup i + 2 + d \cup i + d | \binom{2r(n)-2}{i} \textit{is odd}\} \tag{32}$$

is disjoint and doesn't contain two consecutive indices. This is easily seen to be true when we remember what was said in Lemma 3 and notice that $2r(n) - 2 \leq n - 3$. In other words this is precisely the same set of indices (27) as what we obtained with the straightforward method used in Lemma 2.

Now, if we multiply both sides of the equation (28) (n being odd) with F_{n-1+d} instead of L_{n+d}, we get

$$F_{n-1+d}c[n] = \sum_{i=0}^{2r(n)-2} \binom{2r(n)-2}{i}_{(\mathrm{mod}\ 2)} L_{n-i-1}F_{n-1+d} \tag{33}$$

i.e. "the duplicator" L_{n-i-1} at the right hand side is the same as in (30), but now only the other one (F_{n-1+d}) remains to make copies of.

We see that the pattern in the set of the indices produced now

$$\bigcup_{i=0}^{2r(n)-2} \{2n-2-i+d \cup i+d | \binom{2r(n)-2}{i} \textit{is odd}\} \tag{34}$$

is two positions shorter, and the indices are located in such way that after one generation of the rule 90 the pattern is exactly the same as in the set (32). Remembering the uniqueness of the finite predecessors we know it must be idenitical with the pattern of ones found at the row $2(n-1)$ of Pascal's Triangle computed modulo 2.

Finally, seeing that $L_{2^0} = 1$, we can write for all even $n \geq 0$

$$F_{n+d}c[n+1] = F_{n+d} \prod_{i=0}^{\infty} L_{2^i}^{bit_i(n)} \tag{35}$$

and we are done. □

APPENDIX

Below we show the odd indiced Fibonacci representation for the first few terms of the sequence $c[n]$. The rightmost place stands for F_1. E.g. the term $c[29] = F_3 + F_5 + F_{11} + F_{13} + F_{19} + F_{21} + F_{27} + F_{29} = 726103$.

$$\begin{aligned}
1 &= c[1]\\
101 &= c[3]\\
10100 &= c[5]\\
1010101 &= c[7]\\
101000000 &= c[9]\\
10101010000 &= c[11]\\
1010000010100 &= c[13]\\
101010101010101 &= c[15]\\
10100000000000000 &= c[17]\\
1010101000000000000 &= c[19]\\
101000001010000000000 &= c[21]\\
10101010101010100000000 &= c[23]\\
1010000000000000101000000 &= c[25]\\
101010100000000010101010000 &= c[27]\\
10100000101000001010000010100 &= c[29]\\
1010101010101010101010101010101 &= c[31]
\end{aligned}$$

ACKNOWLEDGEMENT

The author thanks the anonymous referee who suggested a better introduction for this paper.

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AMS Classification Numbers: 11B39, 11B65, 11B85

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