

# Pandigital and penholodigital numbers

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## Abstract

Pandigital and penholodigital numbers are numbers that contain every digit or nonzero digit respectively. We study properties of pandigital or penholodigital numbers that are also squares or prime.

## 1 Introduction

Pandigital [1] and penholodigital numbers are defined as numbers that contain every digit or every nonzero digit, respectively. More precisely,

**Definition 1.** *A number  $n$  is a pandigital number in base  $b$  if  $n$  expressed as a base  $b$  number contains each of the  $b$  different digits at least once. A number  $n$  is a strict pandigital number in base  $b$  if  $n$  expressed as a base  $b$  number contains each of the  $b$  different digits exactly once.*

A strict pandigital number is pandigital and clearly there are a finite number of strict pandigital numbers for each base  $b$ .

**Definition 2.** *A number  $n$  is a penholodigital number in base  $b$  if  $n$  expressed as a base  $b$  number does not contain the zero digit<sup>1</sup> and contains each of the  $b - 1$  different nonzero digits at least once. A number  $n$  is a strict penholodigital number in base  $b$  if  $n$  expressed as a base  $b$  number does not contain the zero digit and contains each of the  $b - 1$  different nonzero digits exactly once.*

For example, in base 10, 1023456798 is a strict pandigital number and 10023546789 is a pandigital number. Similarly, 123456798 is a strict penholodigital number and 1323546789 is a penholodigital number in base 10.

Many of these numbers are listed as sequences in the On-line Encyclopedia of Integer Sequences (OEIS) [2]. Examples of pandigital and strict pandigital numbers in base 10 are listed in OEIS sequences A171102 and A050278 respectively. Examples of penholodigital numbers in base 10 are listed in OEIS sequence A050289. The smallest and largest strict pandigital numbers in base  $b$  are listed in OEIS sequences A049363 and A062813 respectively. The smallest and largest strict penholodigital numbers in base  $b$  are listed in OEIS sequences A023811 and A051846 respectively.

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<sup>1</sup>Such numbers are also called zeroless numbers.

## 2 Pandigital and penholodigital squares

Let  $s_b(n)$  be the sum of the digits of  $n$  in base  $b$ . Since  $b \equiv 1 \pmod{b-1}$  this means that  $b^k \equiv 1 \pmod{b-1}$ , which in turn implies that  $s_b(n) \equiv n \pmod{b-1}$ .

Note that for a strict pandigital or a strict penholodigital number  $n$ ,  $s_b(n) = b(b-1)/2$ . This implies directly the following:

**Theorem 1.** *Let  $A_b$  be the set of modular square roots of  $b(b-1)/2$  modulo  $b-1$ , i.e. it is the set of integers  $0 \leq m < b-1$  such that  $m^2 \equiv b(b-1)/2 \pmod{b-1}$ . If  $n^2$  is a strict pandigital or a strict penholodigital square, then  $n \equiv m \pmod{b-1}$  for some  $m \in A_b$ .*

In general, finding square roots modulo  $m$  is a difficult problem, as difficult as factoring  $m$  [3], but for some values of  $b$ , we can explicitly find  $A_b$ .

**Theorem 2.**  *$b$  is odd and  $b-1$  has an even 2-adic valuation<sup>2</sup> if and only if  $A_b = \emptyset$ . If  $b$  is an even squarefree number, then  $A_b = \{0\}$ . If  $b$  is an odd squarefree number, then  $A_b = \{(b-1)/2\}$ .*

*Proof.* If  $b$  is even,  $b(b-1)/2 \equiv 0 \pmod{b-1}$  and  $0 \in A_b$ . Let  $b = 2^{2k+1}q + 1$  for some odd  $q$ . Then  $b-1 = 2^{2k+1}q$ ,  $b(b-1)/2 \equiv 2^{2k}q \pmod{b-1}$  and  $(2^kq)^2 + \frac{1-q}{2}2^{2k+1}q = 2^{2k}q$ , i.e.  $(2^kq)^2 \equiv 2^{2k}q \pmod{2^{2k+1}q}$  and thus  $2^kq \in A_b$ . So in both these cases  $A_b \neq \emptyset$ .

Now suppose  $b = 2^{2k}q + 1$  for some  $k > 0$  and odd  $q$ . First note that  $b-1 = 2^{2k}q$  and  $2^{2k}$  and  $q$  are coprime. Since  $b-1$  is even,  $b(b-1)/2 \equiv (b-1)/2 \pmod{b-1}$ . Let  $r = (b-1)/2 = 2^{2k-1}q$ . If  $k = 1$ ,  $b-1 = 4q$ ,  $r = 2q$  and thus  $r \equiv 2 \pmod{4}$  is a quadratic nonresidue modulo 4. By the Chinese Remainder Theorem,  $r$  is a quadratic nonresidue modulo  $b-1$ . If  $k > 1$ , Gauss showed [4, 3] that a nonzero number  $r$  is a residue modulo  $2^{2k}$  if and only if  $r$  is of the form  $2^{2m}(8j+1)$ . Since  $r = 2^{2k-1}q$ ,  $r$  is a quadratic nonresidue modulo  $2^{2k}$ . Thus if  $b$  is odd and  $b-1$  has an even 2-adic valuation, then  $A_b = \emptyset$ .

Next, suppose  $b-1$  is odd and squarefree. Then  $b$  is even,  $b-1$  divides  $b(b-1)/2$  and  $m^2 \equiv 0 \pmod{b-1}$ . Since  $b-1 = \prod_i p_i$  for distinct odd primes  $p_i$ , and  $m^2 \equiv 0 \pmod{p_i}$  if and only if  $m \equiv 0 \pmod{p_i}$ , this implies that  $m \equiv 0 \pmod{b-1}$  by the Chinese Remainder Theorem, i.e.  $A_b = \{0\}$ . Similarly, if  $b-1$  is even and squarefree, then  $m^2 \equiv b(b-1)/2 \equiv (b-1)/2 \pmod{b-1}$  and  $(b-1)/2 = \prod_i p_i$  for distinct odd primes  $p_i$ , i.e.,  $m^2 \equiv 0 \pmod{p_i}$  and  $m^2 \equiv 1 \pmod{2}$ . This implies that  $m \equiv 0 \pmod{p_i}$  and  $m \equiv 1 \pmod{2}$ . Again by the Chinese Remainder Theorem,  $m \equiv (b-1)/2 \pmod{b-1}$  and  $A_b = \{(b-1)/2\}$ .  $\square$

Theorems 1 and 2 result in the following immediate consequences.

**Corollary 1.** *If  $b$  is odd and  $b-1$  has an even 2-adic valuation, then there are no strict pandigital nor strict penholodigital squares in base  $b$ .*

Corollary 1 for the case of pandigital squares was shown in [5] directly using a different technique. We conjecture that a strict pandigital square and a strict penholodigital square in base  $b$  exists if and only if  $b$  is odd and  $b-1$  has an even 2-adic valuation.

**Corollary 2.** *Let  $m^2$  be a strict pandigital or a strict penholodigital square in base  $b$ . If  $b$  is an even squarefree number, then  $m \equiv 0 \pmod{b-1}$ . If  $b$  is an odd squarefree number, then  $m \equiv (b-1)/2 \pmod{b-1}$ .*

The number of strict pandigital and strict penholodigital squares for each base  $b$  are listed in OEIS sequences A258103 and A370950 respectively.

## 3 Pandigital and penholodigital primes

Since  $s_b(n) = b(b-1)/2$ , this means that  $n \equiv 0 \pmod{b-1}$  if  $b$  is even and  $n \equiv 0 \pmod{(b-1)/2}$  if  $b$  is odd. This implies that there are no strict pandigital nor strict penholodigital prime numbers in base  $b > 3$ , i.e. a pandigital prime must be larger or equal to  $\frac{b^b - b^2 + b - 1}{(b-1)^2} + b^b$  (i.e. the base  $b$  representation is 10123... $(b-1)$ )

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<sup>2</sup>The 2-adic valuation of  $n$  is the largest power of 2 that divides  $n$ .

and a penholodigital prime must be larger or equal to  $\frac{b^b - b^2 + b - 1}{(b-1)^2} + b^{b-1}$  (i.e. the base  $b$  representation is  $1123\dots(b-1)$ ). In other words, we have the following lower bounds:

**Theorem 3.** *Let  $b > 3$ . If  $n$  is a pandigital prime in base  $b$ , then  $n \geq \frac{b^b - b^2 + b - 1}{(b-1)^2} + b^b$ . If  $n$  is a penholodigital prime in base  $b$ , then  $n \geq \frac{b^b - b^2 + b - 1}{(b-1)^2} + b^{b-1}$ .*

These lower bounds can be improved for bases of the form  $b = 4k + 3$ .

**Theorem 4.** *If  $b = 4k + 3$  for  $k > 0$ , then a pandigital prime in base  $b$  is larger than or equal to  $n \geq \frac{b^b - b^2 + b - 1}{(b-1)^2} + b^b + b^{b-2}$  and a penholodigital prime in base  $b$  is larger than or equal to  $n \geq \frac{b^b - b^2 + b - 1}{(b-1)^2} + b^{b-1} + b^{b-2}$ .*

*Proof.* For a pandigital prime, if  $b = 4k + 3$ , then  $b(b-1)/2 + 1 = 2(4k^2 + 5k + 2)$  and  $b-1$  are both even. Thus if  $s_b(n) = b(b-1)/2 + 1$ , then  $n > 2$  is even and thus not prime. Thus  $s_b(n) \geq b(b-1)/2 + 2$  and thus  $n$  is larger than or equal to  $10223\dots(n-1)$  in base  $b$ .

Similarly, for a penholodigital prime,  $n$  is larger than or equal to  $1223\dots(n-1)$  in base  $b$ . □

The smallest pandigital and penholodigital primes are listed in OEIS A185122 and A371194 respectively. Numerical experiments suggest that for  $b > 3$ , the smallest pandigital prime or penholodigital prime  $n$  satisfy  $s_b(n) = b(b-1)/2 + 2$  if  $b$  is of the form  $4k + 3$  and  $s_b(n) = b(b-1)/2 + 1$  otherwise.

## 4 Subpandigital and subpenholodigital numbers

We can also consider numbers whose digits in base  $b$  include all (nonzero) digits up to  $b-2$ .

**Definition 3.** *A number  $n$  is a subpandigital number in base  $b$  if  $n$  expressed as a base  $b$  number does not contain the digit  $b-1$  and contains each of the  $b-1$  digits  $0, 1, \dots, b-2$  at least once. A number  $n$  is a strict subpandigital number in base  $b$  if  $n$  expressed as a base  $b$  number does not contain the digit  $b-1$  and contains each of the  $b-1$  digits  $0, 1, \dots, b-2$  exactly once.*

**Definition 4.** *A number  $n$  is a subpenholodigital number in base  $b$  if  $n$  expressed as a base  $b$  number does not contain the zero digit nor the digit  $b-1$  and contains each of the  $b-2$  digits  $1, \dots, b-2$  at least once. A number  $n$  is a strict subpenholodigital number in base  $b$  if  $n$  expressed as a base  $b$  number does not contain the zero digit nor the digit  $b-1$  and contains each of the  $b-2$  digits  $1, \dots, b-2$  exactly once.*

For example, in base 10, 120345687 is a strict subpandigital number as it contains all digits except 9 exactly once and 87654123 is a strict subpenholodigital number as it contains all nonzero digits except 9 exactly once. Since there are no subpenholodigital number in base 2 and the only subpandigital number in base 2 is 0, we only consider bases  $b > 2$  in this section. As  $s_b(n) = (b-1)(b-2)/2 = b(b-1)/2 - (b-1)$  for a strict subpandigital or a strict subpenholodigital number  $n$  and thus  $(b-1)(b-2)/2 \equiv b(b-1)/2 \pmod{b-1}$ , we have the following analog result to Theorem 1:

**Theorem 5.** *Let  $A_b$  be as defined in Theorem 1. If  $n^2$  is a strict subpandigital or a strict subpenholodigital square, then  $n \equiv m \pmod{b-1}$  for some  $m \in A_b$ .*

**Corollary 3.** *If  $b$  is odd and  $b-1$  has an even 2-adic valuation, then there are no strict subpandigital nor strict subpenholodigital squares in base  $b$ .*

**Corollary 4.** *Let  $m^2$  be a strict subpandigital or a strict subpenholodigital square in base  $b$ . If  $b$  is an even squarefree number, then  $m \equiv 0 \pmod{b-1}$ . If  $b$  is an odd square free number, then  $m \equiv (b-1)/2 \pmod{b-1}$ .*

Similarly, there is an analog to Theorems 3-4:

**Theorem 6.** *Let  $b > 3$ . A subpandigital prime in base  $b$  must be larger than or equal to  $\frac{b^{b-1}-b}{(b-1)^2} + b^{b-1}$  and any subpenholodigital prime must be larger than or equal to  $\frac{b^{b-1}-b}{(b-1)^2} + b^{b-2}$ .*

*Proof.* Follows from the fact that  $\frac{b^{b-1}-b}{(b-1)^2} + b^{b-1}$  can be written as 10123...(b-2) in base  $b$  and  $\frac{b^{b-1}-b}{(b-1)^2} + b^{b-2}$  can be written as 1123...(b-2) in base  $b$ .  $\square$

Similarly, these lower bounds can be improved for bases of the form  $b = 4k + 3$ .

**Theorem 7.** *If  $b = 4k + 3$  for  $k > 0$ , then the smallest subpandigital prime in base  $b$  is larger than or equal to  $\frac{b^{b-1}-b}{(b-1)^2} + b^{b-1} + b^{b-3}$  and the subsmallest penholodigital prime in base  $b$  is larger than or equal to  $\frac{b^{b-1}-b}{(b-1)^2} + b^{b-2} + b^{b-3}$ .*

*Proof.* For a subpandigital prime, if  $b = 4k + 3$ , then  $(b-2)(b-1)/2 + 1 = 2(4k^2 + 3k + 1)$  and  $b-1$  are both even. Thus if  $s_b(n) = (b-2)(b-1)/2 + 1$ , then  $n > 2$  is even and thus not prime. Thus  $s_b(n) \geq (b-2)(b-1)/2 + 2$  and thus  $n$  is larger than or equal to 10223...(n-2) in base  $b$  which is equal to  $\frac{b^{b-1}-b}{(b-1)^2} + b^{b-1} + b^{b-3}$

Similarly, for a subpenholodigital prime,  $n$  is larger than or equal to 1223...(n-2) in base  $b$  which is equal to  $\frac{b^{b-1}-b}{(b-1)^2} + b^{b-2} + b^{b-3}$ .  $\square$

Table 1 shows the smallest subpandigital and subpenholodigital primes for various bases. This table suggests that, similar to Section 3, for  $b > 4$  the smallest subpandigital prime or smallest subpenholodigital prime  $n$  satisfy  $s_b(n) = (b-2)(b-1)/2 + 2$  for  $b$  of the form  $4k + 3$  and  $s_b(n) = (b-2)(b-1)/2 + 1$  otherwise.

base $b$	smallest subpandigital prime	written in base $b$	smallest subpenholodigital prime	written in base $b$
3	3	10	13	111
4	73	1021	37	211
5	683	10213	163	1123
6	8521	103241	1861	12341
7	123323	1022354	22481	122354
8	2140069	10123645	304949	1123465
9	43720693	101236457	5455573	11234567
10	1012356487	1012356487	112345687	112345687
11	26411157737	10223456798	2831681057	1223456987
12	749149003087	10123459a867	68057976031	1123458a967
13	23459877380431	1012345678a9b	1953952652167	112345678ba9
14	798411310382011	1012345678c9ab	61390449569437	11234567a8bc9
15	29471615863458281	1022345678a9cdb	2224884906436873	122345678acb9d
16	1158045600182881261	10123456789acbed	77181689614101181	1123456789ceabd
17	48851274656431280857	10123456789acdefb	3052505832274232281	1123456789acebfd
18	2193475267557861578041	10123456789abcefgd	129003238915759600789	1123456789abfcedg
19	104737172422274885174411	10223456789abcdfehg	6090208982148446231753	1223456789abchfedg
20	5257403213296398892278377	10123456789abcdgefih	276667213296398892309917	1123456789abcdgiefh

Table 1: Smallest subpandigital and subpenholodigital primes.

## References

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