

# Five families of rapidly convergent evaluations of zeta values

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This work derives 5 methods to evaluate families of odd zeta values by combining a power of  $\pi$  with Lambert series whose ratios of successive terms tend to  $e^{-\pi\sqrt{a}}$  with integers  $a \geq 7$ , outperforming Ramanujan's results with merit  $a = 4$ . Families with  $a = 7$  and  $a = 8$  evaluate  $\zeta(2n+1)$ . Families with  $a = 9$  and  $a = 16$  evaluate  $\zeta(4n+1)$  with faster convergence. A fifth family with  $a = 12$  evaluates  $\zeta(6n+1)$  and gives the fastest convergence for  $\zeta(6n+7)$ . Members of three of the families were discovered empirically by Simon Plouffe. An intensive new search strongly suggests that there are no more than 5 families with integers  $a \geq 7$ . There are at least 18 families that involve involve Lambert series with rational  $a > 4$ . Quasi-modular transformations of Lambert series resolve rational sequences that were discovered empirically. Expansions of Lambert series in polylogarithms, familiar from quantum field theory, provide proofs of all known evaluations with rational merit  $a > 4$ .

## 1 Introduction

I derive 5 families of reductions of  $\zeta(k)$ , for odd  $k > 1$ , to algebraic multiples of  $\pi^k$  and rational multiples of very rapidly convergent Lambert series of the form

$$S_k(q) = \sum_{n=1}^{\infty} \frac{1}{n^k} \left( \frac{q^n}{1-q^n} \right) = \sum_{m=1}^{\infty} \text{Li}_k(q^m) \quad (1)$$

with expansions in polylogarithms  $\text{Li}_k(x) = \sum_{n>0} x^n n^{-k}$  [17]. For each of these 5 families, the Lambert series have spectacularly fast convergence, with arguments  $q = e^{-\pi\sqrt{a}} < \frac{1}{4071}$  for integers  $a \geq 7$ . All 5 families give evaluations that converge faster than the classical evaluations [7, 13, 19, 22]

$$\zeta(3) = \frac{7\pi^3}{180} - 2 \sum_{n=1}^{\infty} \frac{1}{n^3} \left( \frac{1}{e^{2\pi n} - 1} \right) \quad (2)$$

$$\zeta(5) = \frac{\pi^5}{294} - \frac{2}{35} \sum_{n=1}^{\infty} \frac{1}{n^5} \left( \frac{36}{e^{2\pi n} - 1} + \frac{1}{e^{2\pi n} + 1} \right) \quad (3)$$

with lesser merit  $a = 4$  and hence  $q = e^{-2\pi} \approx \frac{1}{535}$  in Ramanujan's formula (2).

Section 2 describes the families. For each, I determine the rational coefficients of Lambert series, by solving recurrence relations for integer sequences. Section 3 comments on computational efficiency. Section 4 gives proofs of the results stated in Section 2 and also derives the coefficients of powers of  $\pi$ . Section 5 offers comments and conclusions.

## 2 Families that outperform Ramanujan

I assign merit  $a$  to a series with an asymptotic ratio of terms  $e^{-\pi\sqrt{a}}$ . Evaluations (2,3) have merit  $a = 4$ . There are 5 families that have integer merit  $a \geq 7$ , with  $a = 7, 8, 9, 16, 12$ . The respective square-free divisors of  $a$  are  $s = 7, 2, 1, 1, 3$  and the Lambert series are of the form  $S_k(q_s^d)$  where  $q_s = e^{-\pi\sqrt{s}}$  and  $d|m$  is a divisor of a least common multiple  $m = 4, 6, 12, 20, 12$ , with  $d\sqrt{s} > 2$  to ensure convergence faster than Ramanujan's result (2). I conjecture that there are precisely 5 families with integers  $a \geq 7$  and exhibit 13 families with Lambert series of fractional merit  $a > 4$ . The order of merit of the 18 families with rational  $a > 4$  is given in (74).

### 2.1 Family A: evaluations of $\zeta(2n+1)$ with merit 7 using divisors of 4

For odd  $k > 1$ , there is an infinite family of evaluations of the form

$$\zeta(k) = A_{k,0}\sqrt{7}\pi^k + \sum_{d \in \{1,2,4\}} A_{k,d}S_k(q_7^d), \quad q_7 = \exp(-\pi\sqrt{7}). \quad (4)$$

Using the procedure `lindep` in `Pari/GP` [18], I obtained

$$\zeta(3) = \frac{29\sqrt{7}}{1980}\pi^3 + \frac{24}{11}S_3(q_7) - \frac{52}{11}S_3(q_7^2) + \frac{6}{11}S_3(q_7^4) \quad (5)$$

$$\zeta(5) = \frac{5\sqrt{7}}{3906}\pi^5 + \frac{64}{31}S_5(q_7) - \frac{130}{31}S_5(q_7^2) + \frac{4}{31}S_5(q_7^4) \quad (6)$$

$$\zeta(7) = \frac{851\sqrt{7}}{6747300}\pi^7 + \frac{240}{119}S_7(q_7) - \frac{1927}{476}S_7(q_7^2) + \frac{15}{476}S_7(q_7^4) \quad (7)$$

and extended these results to  $\zeta(127)$ .

The existence of this family was inferred by Simon Plouffe [20]. I was then able to determine that the coefficients of the Lambert series in (4) satisfy

$$A_{2n+1,1} = 2^{2n}A_{2n+1,4} = 2 + \frac{2}{a_n}, \quad \sum_{d \in \{1,2,4\}} A_{2n+1,d} = -2 \quad (8)$$

with  $a_n = 11, 31, 119, 543, 1991, 8239, 32855, 130623, 525287, 2095951, \dots$

This integer sequence obeys the recurrence relation

$$a_{n+3} = a_{n+2} + 8a_{n+1} + 16a_n \quad (9)$$

for  $n \geq 0$ , with  $a_0 = 0$ . Observing that the roots of  $x^3 = (x+4)^2$  are  $x = 4$  and  $x = \frac{1}{2}(-3 \pm \sqrt{-7})$ , I inferred that

$$a_n = 2^{2n+1} - \left(\frac{1 + \sqrt{-7}}{2}\right)^{2n} - \left(\frac{1 - \sqrt{-7}}{2}\right)^{2n}. \quad (10)$$

Using the coefficient of  $\sqrt{7}\pi^k$  in (4), I define the rational sequence

$$A_n = \frac{(2n+3)!}{2^{2n-1}} a_n A_{2n+1,0} \quad (11)$$

with  $A_n = 29/3, 25, 851/5, 6451/3, 1088813/35, 684521, 846243643/45, \dots$

**Conjecture 1:** The denominator of  $A_n$  is divisible by no prime greater than  $n+2$ .

**Comment 1:** For  $\zeta(291)$ , I found that  $3 \cdot 5 \cdot 13 \cdot 19 \cdot 59 A_{145}$  is a 459-digit prime. For  $\zeta(2821)$ , the numerator of  $A_{1410}$  is divisible by a 7128-digit probable prime.

## 2.2 Family B: evaluation of $\zeta(2n+1)$ with merit 8 using divisors of 6

Following work by Linas Vepštas [22], Plouffe found faster convergence in evaluations of the form [20]

$$\zeta(k) = B_{k,0} \sqrt{2} \pi^k + \sum_{d \in \{2,3,6\}} B_{k,d} S_k(q_2^d), \quad q_2 = \exp(-\pi \sqrt{2}). \quad (12)$$

Then `lindep` gave

$$\zeta(3) = \frac{17\sqrt{2}}{620} \pi^3 - \frac{60}{31} S_3(q_2^2) - \frac{4}{31} S_3(q_2^3) + \frac{2}{31} S_3(q_2^6) \quad (13)$$

$$\zeta(5) = \frac{191\sqrt{2}}{79695} \pi^5 - \frac{516}{253} S_5(q_2^2) + \frac{8}{253} S_5(q_2^3) + \frac{2}{253} S_5(q_2^6) \quad (14)$$

$$\zeta(7) = \frac{3197\sqrt{2}}{13538700} \pi^7 - \frac{612}{307} S_7(q_2^2) - \frac{16}{2149} S_7(q_2^3) + \frac{2}{2149} S_7(q_2^6) \quad (15)$$

and results up to  $\zeta(127)$ .

The coefficients of the Lambert series in (12) satisfy

$$B_{2n+1,6} = \frac{B_{2n+1,3}}{(-2)^n} = \frac{2}{b_n}, \quad \sum_{d \in \{2,3,6\}} B_{2n+1,d} = -2 \quad (16)$$

with  $b_n = 31, 253, 2149, 19633, 177661, 1593601, 14346013, 129151873, \dots$

This integer sequence obeys the recurrence relation

$$b_{n+4} = 5b_{n+3} + 23b_{n+2} + 99b_{n+1} + 162b_n \quad (17)$$

for  $n \geq 0$ , with  $b_0 = 0$ . Observing that the roots of  $x^4 = 5x^3 + 23x^2 + 99x + 162$  are  $x = 9$ ,  $x = -2$  and  $x = -1 \pm \sqrt{-8}$ , I inferred that

$$b_n = 3^{2n+1} - (-2)^n - (1 + \sqrt{-2})^{2n} - (1 - \sqrt{-2})^{2n}. \quad (18)$$

Using the coefficient of  $\sqrt{2}\pi^k$  in (12), I define the rational sequence

$$B_n = \frac{(2n+3)!}{2^{2n-1}} b_n B_{2n+1,0} \quad (19)$$

with  $B_n = 51, 382, 28773/5, 145566, 181903711/35, 247890198, \dots$

**Conjecture 2:** The denominator of  $B_n$  is divisible by no prime greater than  $n + 2$ .

**Comment 2:** For  $\zeta(189)$ , I found that  $77B_{94}/3298100598$  is a 287-digit prime. For  $\zeta(619)$ , the numerator of  $B_{309}$  is divisible by a 1268-digit prime.

### 2.3 Family C: evaluation of $\zeta(4n + 1)$ with merit 9 using divisors of 12

For  $k = 4n + 1$ , I found faster convergence for evaluations of the form

$$\zeta(k) = C_{k,0}\pi^k + \sum_{d \in \{3,4,6,12\}} C_{k,d}S_k(q_1^d), \quad q_1 = \exp(-\pi). \quad (20)$$

Then `lindep` gave

$$\zeta(5) = \frac{682}{201285}\pi^5 + \frac{296}{355}S_5(q_1^3) - \frac{488}{355}S_5(q_1^4) - \frac{1073}{710}S_5(q_1^6) + \frac{37}{710}S_5(q_1^{12}) \quad (21)$$

$$\zeta(9) = \frac{5048}{150155775}\pi^9 - \frac{2272}{1605}S_9(q_1^3) - \frac{5624}{1605}S_9(q_1^4) + \frac{37559}{12840}S_9(q_1^6) - \frac{71}{12840}S_9(q_1^{12}) \quad (22)$$

and results up to  $\zeta(125)$ .

The coefficients of the Lambert series in (20) satisfy

$$C_{4n+1,12} = \frac{C_{4n+1,3}}{2^{4n}} = \left(1 - \frac{2^{4n+1} + 1}{(-4)^n}\right) \frac{2}{c_n}, \quad (23)$$

$$\frac{C_{4n+1,6}}{C_{4n+1,12}} = -(2^{4n+1} + (-4)^n + 1), \quad \sum_{d \in \{3,4,6,12\}} C_{4n+1,d} = -2 \quad (24)$$

with  $c_n = 355, 11235, 2114515, 95520195, 12606788275, 709832878755, \dots$

This integer sequence obeys the recurrence relation

$$c_{n+3} = 33c_{n+2} + 4192c_{n+1} - 82944c_n \quad (25)$$

for  $n \geq 0$ , with  $5|c_n$  and  $c_0 = 0$ . Observing that the roots of  $x^3 = 33x^2 + 4912x - 82944$  are  $x = 81$ ,  $x = -64$  and  $x = 16$ , I inferred that

$$c_n = 3^{4n+1} - 2(-4)^{3n} - 2^{4n}. \quad (26)$$

Using the coefficient of  $\pi^k$  in (20), I define the rational sequence

$$C_n = \frac{(4n)!(2n+1)(4n+3)}{2^{4n}} c_n C_{4n+1,0} \quad (27)$$

with  $C_n = 341/9, 88340/27, 26827985/3, 18438055674, \dots$

**Conjecture 3:** The denominator of  $C_n$  is divisible by no prime greater than  $2n + 1$  and by no prime congruent to 1 modulo 4.

**Comment 3:** For  $\zeta(285)$ , I found that  $19 \cdot 23 \cdot 27C_{71}/28362151103$  is a 483-digit prime. For  $\zeta(1449)$ , the numerator of  $C_{362}$  is divisible by a 3495-digit probable prime.

## 2.4 Family D: evaluation of $\zeta(4n+1)$ with merit 16 using divisors of 20

For  $k = 4n+1$ , Plouffe found even faster convergence for evaluations of the form [20]

$$\zeta(k) = D_{k,0}\pi^k + \sum_{d \in \{4,5,10,20\}} D_{k,d}S_k(q_1^d), \quad q_1 = \exp(-\pi). \quad (28)$$

Then `lindep` gave

$$\zeta(5) = \frac{694}{204813}\pi^5 - \frac{6280}{3251}S_5(q_1^4) + \frac{296}{3251}S_5(q_1^5) - \frac{1073}{6502}S_5(q_1^{10}) + \frac{37}{6502}S_5(q_1^{20}) \quad (29)$$

and results up to  $\zeta(125)$ .

The coefficients of the Lambert series in (28) satisfy

$$D_{4n+1,20} = \frac{D_{4n+1,5}}{2^{4n}} = \left(1 - \frac{2^{4n+1} + 1}{(-4)^n}\right) \frac{2}{d_n}, \quad (30)$$

$$\frac{D_{4n+1,10}}{D_{4n+1,20}} = -(2^{4n+1} + (-4)^n + 1), \quad \sum_{d \in \{4,5,10,20\}} D_{4n+1,d} = -2 \quad (31)$$

with  $d_n = 3251, 1945731, 1221199811, 762905503491, 476839323944771, \dots$

This integer sequence obeys a fifth order recurrence relation whose characteristic polynomial is  $(x - 625)(x + 64)(x - 16)(x^2 + 14x + 625)$ , with  $d_0 = 0$ , from which I inferred that

$$d_n = 5^{4n+1} - 2(-4)^{3n} - 2^{4n} - (2 + \sqrt{-1})^{4n} - (2 - \sqrt{-1})^{4n}. \quad (32)$$

Using the coefficient of  $\pi^k$  in (28), I define the rational sequence

$$D_n = \frac{(4n)!(2n+1)(4n+3)}{2^{4n}} d_n D_{4n+1,0} \quad (33)$$

with  $D_n = 347, 15297320/27, 5164889285, 1030773088821534/7, \dots$

**Conjecture 4:** The denominator of  $D_n$  is divisible by no prime greater than  $2n-1$  and by no prime congruent to 1 modulo 4.

**Comment 4:** For  $\zeta(273)$ , I found that  $19 \cdot 23 \cdot 31 \cdot 21^2 D_{68}/6706150$  is a 524-digit prime. For  $\zeta(897)$ , the numerator of  $D_{224}$  is divisible by a 2173-digit probable prime.

## 2.5 Family E: evaluation of $\zeta(6n+1)$ with merit 12 using divisors of 12

For  $k = 6n+1$ , I found evaluations of the form

$$\zeta(k) = E_{k,0}\sqrt{3}\pi^k + \sum_{d \in \{2,3,6,12\}} E_{k,d}S_k(q_3^d), \quad q_3 = \exp(-\pi\sqrt{3}), \quad (34)$$

with results up to  $\zeta(127)$ . The first member of this family is the evaluation

$$\zeta(7) = \frac{1}{1043} \left( \frac{3257\sqrt{3}\pi^7}{16200} - 2215S_7(q_3^2) - 129S_7(q_3^3) + \frac{129^2}{64}S_7(q_3^6) - \frac{129}{64}S_7(q_3^{12}) \right) \quad (35)$$

with merit  $a = 12$  and hence faster convergence than (15), with lesser merit  $a = 8$ .

The coefficients of the Lambert series in (34) satisfy

$$E_{6n+1,2} + 2 = E_{6n+1,3} = 2^{6n}E_{6n+1,12} = -\frac{f_n}{e_n}, \quad (36)$$

$$E_{6n+1,6} = -f_nE_{6n+1,12}, \quad f_n = 2^{6n+1} + 1, \quad (37)$$

with  $e_n = 1043, 792701, 580878431, 423627261785, 308835631574603, \dots$

This integer sequence obeys a recurrence relation whose characteristic polynomial is  $(x - 3^6)(x + 3^3)(x - 2^6)$ , with  $7|e_n$  and  $e_0 = 0$ , from which I inferred that

$$e_n = \frac{3^{6n+1} - (-3)^{3n}}{2} - 2^{6n}. \quad (38)$$

Using the coefficient of  $\sqrt{3}\pi^k$  in (34), I define the rational sequence

$$E_n = \frac{(6n)!(6n+3)(6n+4)}{2^{6n-4}} e_n E_{6n+1,0} \quad (39)$$

with  $E_n = 3257, 212373152/3, 325348272142978/15, \dots$

**Conjecture 5:** The denominator of  $E_n$  is divisible by no prime greater than  $2n - 1$  and by no prime congruent to 1 modulo 6.

**Comment 5:** For  $\zeta(235)$ , I found that  $47 \cdot 59 \cdot 75E_{39}/153734$  is a 385-digit prime. For  $\zeta(2881)$ , the numerator of  $E_{480}$  is divisible by a 7794-digit probable prime.

## 2.6 Families with Lambert series of fractional merit

I found 13 families that involve Lambert series  $S_k(q_s^d)$  with  $q_s = e^{-\pi\sqrt{s}}$ , fractional  $s \in \{\frac{3}{5}, \frac{5}{3}, \frac{1}{3}, \frac{1}{2}, \frac{1}{5}, \frac{1}{11}, \frac{7}{5}, \frac{5}{7}, \frac{1}{7}, \frac{1}{23}\}$  and  $d\sqrt{s} > 2$ . They result from complex multiplication with discriminants  $D \in \{-3, -7, -8, -11, -15, -20, -23, -35\}$ .

For  $D = -15$ , there are two independent families. The first begins with

$$\zeta(3) = \frac{10\sqrt{15}}{999}\pi^3 - \frac{3}{37}S_3(q_{5/3}^2) + \frac{80}{37}S_3(q_{15}) - \frac{171}{37}S_3(q_{15}^2) + \frac{20}{37}S_3(q_{15}^4) \quad (40)$$

$$\zeta(5) = \frac{1327\sqrt{15}}{1516725}\pi^5 - \frac{63}{535}S_5(q_{5/3}^2) + \frac{208}{107}S_5(q_{15}) - \frac{2112}{535}S_5(q_{15}^2) + \frac{13}{107}S_5(q_{15}^4) \quad (41)$$

with terms of merit  $a \geq \frac{20}{3}$ . In the denominators there is an integer sequence  $r_n = 37, 535, 8047, 130495, 2103727, 33561055, 536581327, \dots$  with solution

$$r_n = 2^{4n+1} - \Re(\rho^{2n} + (-3\rho)^n), \quad \rho = \frac{1}{2}(1 + \sqrt{-15}). \quad (42)$$

A better family for  $D = -15$  has terms of merit  $a \geq \frac{48}{5}$  and begins with

$$\zeta(3) = \frac{16351}{3302\sqrt{15}}\pi^3 + \frac{2}{121}S_3(q_{3/5}^4) - \frac{6}{605}S_3(q_{5/3}^4) + \frac{342}{605}(4S_3(q_{15}) + S_3(q_{15}^4)) - \frac{2924}{605}S_3(q_{15}^2). \quad (43)$$

Studying denominators of the rational coefficients, I encountered an integer sequence  $\tilde{r}_n = 605, 32957, 2107325, 134451197, 8588427965, \dots$  with an intricate solution

$$\tilde{r}_n = 2^{2n+1}g_n - b_n(5^n + (-3)^n + b_nf_n) \quad (44)$$

$$b_n = 2\Re(\rho^n), \quad f_n = 2^{2n+1} + 1, \quad g_n = (f_n - 2^n)(f_n + 2^n). \quad (45)$$

For  $k = 2n + 1$ , the general form is

$$\frac{1}{2}\tilde{r}_n\zeta(k) = a_n\sqrt{15}\pi^k + b_n\left(5^n S_k(q_{3/5}^4) + (-3)^n S_k(q_{5/3}^4)\right) + \sum_{d \in \{1,2,4\}} c_{n,d} S_k(q_{15}^d) \quad (46)$$

$$c_{n,1} = 2^{2n}c_{n,4} = f_n(g_n - b_n^2), \quad 2^{2n}c_{n,2} = (b_nf_n - g_n)(b_nf_n + g_n) \quad (47)$$

with rational  $a_n$ . I found that  $a_{1035}$  is divisible by a 6181-digit probable prime.

For  $D = -3$ , there is family whose terms have merit  $a \geq \frac{16}{3}$ , beginning with

$$\zeta(3) = \frac{133\sqrt{3}}{5940}\pi^3 - \frac{6}{11}S_3(q_{1/3}^4) - \frac{18}{11}S_3(q_3^2) + \frac{2}{11}S_3(q_3^4) \quad (48)$$

$$\zeta(5) = \frac{17\sqrt{3}}{8694}\pi^5 + \frac{18}{23}S_5(q_{1/3}^4) - \frac{66}{23}S_5(q_3^2) + \frac{2}{23}S_5(q_3^4) \quad (49)$$

$$\zeta(7) = \frac{10309\sqrt{3}}{53468100}\pi^7 - \frac{2322}{6601}S_7(q_{1/3}^4) - \frac{10966}{6601}S_7(q_3^2) + \frac{86}{6601}S_7(q_3^4). \quad (50)$$

For  $k = 2n + 1$ , with  $3 \nmid n$ , the denominator of the final coefficient is  $2^{2n+1} - (-3)^n$ .

For  $k = 6n + 1$ , the denominator sequence  $s_n = 6601, 20377357, 95065741729, \dots$  obeys a fourth order recurrence relation, with  $7|s_n$  and  $s_0 = 0$ , solved by

$$s_n = \frac{1}{3}((2^{6n+2} - 1)2^{6n} - (2^{6n+1} + 1)(-3)^{3n}). \quad (51)$$

For  $D = -8$ , there is family whose terms have merit  $a \geq \frac{9}{2}$ , beginning with

$$\zeta(3) = \frac{181\sqrt{2}}{660}\pi^3 - \frac{16}{55}S_3(q_{1/2}^3) - \frac{4}{5}S_3(q_2^2) - \frac{52}{55}S_3(q_2^4) + \frac{2}{55}S_3(q_2^{12}) \quad (52)$$

$$\zeta(5) = \frac{5171\sqrt{2}}{2158380}\pi^5 + \frac{128}{571}S_5(q_{1/2}^3) - \frac{812}{571}S_5(q_2^2) - \frac{460}{571}S_5(q_2^4) + \frac{2}{571}S_5(q_2^{12}). \quad (53)$$

Studying denominators of the rational coefficients, I encountered an integer sequence  $t_n = 55, 571, -3557, -1277, 1212475, -4016045, -82965653, \dots$  with solution

$$t_n = 3^{2n+1} - (-2)^{3n} - (2^{2n+2} + (-2)^{n+1})\Re((1 + \sqrt{-2})^{2n}). \quad (54)$$

For  $D = -20$ , there is family whose terms have merit  $a \geq 5$ , beginning with

$$\zeta(3) = \frac{13\sqrt{5}}{750}\pi^3 + \frac{2}{5}S_3(q_5) - \frac{1}{4}S_3(q_{1/5}^6) - \frac{23}{10}S_3(q_5^2) + \frac{1}{10}S_3(q_5^4) + \frac{1}{20}S_3(q_5^6) \quad (55)$$

$$\zeta(5) = \frac{809\sqrt{5}}{533925}\pi^5 + \frac{8}{113}S_5(q_5) + \frac{25}{113}S_5(q_{1/5}^6) - \frac{521}{226}S_5(q_5^2) + \frac{1}{226}S_5(q_5^4) + \frac{1}{113}S_5(q_5^6). \quad (56)$$

Studying denominators of the rational coefficients, I encountered an integer sequence  $u_n = 20, 113, 980, 10793, 86360, 774053, 7471820, 62858993, \dots$  with solution

$$u_n = \frac{1}{2}(3^{2n+1} - (-5)^n) - \Re((2\sigma)^n), \quad \sigma = (-2 + \sqrt{-5}). \quad (57)$$

A better family for  $D = -20$  has terms of merit  $a \geq \frac{36}{5}$  and begins with

$$\zeta(3) = \frac{18491\sqrt{5}}{1066500}\pi^3 - \frac{10}{79}S_3(q_{1/5}^6) - \frac{2}{79}S_3(q_{1/5}^{12}) + \frac{18}{395}S_3(q_5^{4/3}) + \frac{2}{395} \sum_{d \in \{2,3,6,12\}} \tilde{g}_d S_3(q_5^d) \quad (58)$$

with  $\tilde{g}_d = -384, 4, 5, 1$  for  $d = 2, 3, 6, 12$ . The denominators yield an integer sequence  $\tilde{u}_n = 790, 107111, 20593510, 3025683569, 511718343910, \dots$  with a 12th order recurrence relation, solved by

$$\begin{aligned} \tilde{u}_n = & 2^{n-1} (3^{4n+2} - 3^{2n+1}(-5)^n + 3^{2n} - (-5)^n) \\ & - \Re(\sigma^n(2^{2n+1}(-5)^n - 3^{2n} + 2^{2n} + (2\sigma)^n)). \end{aligned} \quad (59)$$

For  $D = -11$ , there is a family whose terms have merit  $a \geq \frac{44}{9}$ , beginning with

$$\zeta(3) = \frac{623}{4860\sqrt{11}}\pi^3 + \frac{22}{45}S_3(q_{1/11}^{12}) + \frac{2}{45} \sum_{d|36, d>1} f_d S_3(q_{11}^{d/3}) \quad (60)$$

with integers  $f_d = 81, 132, -9, -297, -4, 33, 9, -1$ , for  $d = 2, 3, 4, 6, 9, 12, 18, 36$ . Studying denominators of the rational coefficients, I encountered an integer sequence  $v_n = 1, 29, 993, 38293, 1329305, 48535917, 1740077137, \dots$  with solution

$$v_n = \frac{1}{45} (36^n + (-11)^n - 2\Re((1 + \sqrt{-11})^{2n})). \quad (61)$$

For  $D = -35$ , there is a family whose terms have merit  $a \geq \frac{144}{35}$ , beginning with

$$\begin{aligned} \zeta(3) = & \frac{2969}{12960\sqrt{35}}\pi^3 + \frac{35}{24}S_3(q_{1/35}^{12}) - \frac{7}{24}S_3(q_{5/7}^4) \\ & - \frac{15}{8}S_3(q_{7/5}^2) + \frac{5}{24}S_3(q_{7/5}^4) + \frac{1}{24} \sum_{d|36, d>1} g_d S_3(q_{35}^{d/3}) \end{aligned} \quad (62)$$

with integers  $g_d = 81, 112, -9, -252, -4, 28, 9, -1$ , for  $d = 2, 3, 4, 6, 9, 12, 18, 36$ . Studying denominators of the rational coefficients, I encountered an integer sequence  $w_n = 16, 5096, -140379, -12384944, 774803876, 34673015481, \dots$  with solution

$$w_n = \frac{1}{3} (36^n + (-35)^n - 2\Re(2^{2n+1}(-5\omega)^n - (7\omega)^n)), \quad \omega = \frac{1}{2}(1 + \sqrt{-35}). \quad (63)$$

A better family for  $D = -35$  has terms of merit  $a \geq \frac{28}{5}$  and begins with

$$\begin{aligned} \zeta(3) = & \frac{4573}{19920\sqrt{35}}\pi^3 - \frac{315}{166}S_3(q_{7/5}^2) + \frac{1}{664}(4S_3(q_{35}) - 9S_3(q_{35}^2) + S_3(q_{35}^4)) \\ & + \frac{1}{664}(-4F_3 + 28F_4 + 9F_6 - F_{12}), \quad F_d = 5S_3(q_{7/5}^d) - 7S_3(q_{5/7}^d). \end{aligned} \quad (64)$$



Studying denominators of the rational coefficients, I encountered an integer sequence  $\tilde{w}_n = 332, 45583, 8942693, 1562702887, 284184726617, \dots$  with solution

$$\tilde{w}_n = \frac{1}{4} (6(180)^n + 20^n - (3^{2n+1} + 2^{2n} + 1)(-7)^n) - \frac{1}{2} \Re((-4\omega)^n). \quad (65)$$

For  $D = -7$  there is a family with terms of merit  $a \geq \frac{36}{7}$ , inferior to Family A with  $a \geq 7$ . At  $k = 3$ , the result with lower merit is expressed most simply as

$$73 \zeta(3) = \frac{15}{2\sqrt{7}} \pi^3 - 175 S_3(q_{1/7}^6) + 21 S_3(q_{1/7}^{12}) - 27 S_3(q_7^{4/3}) + \sum_{d|12} \tilde{a}_d S_3(q_7^d) \quad (66)$$

with integers  $\tilde{a}_d = -12, 40, -12, -3, 25, -3$  for  $d = 1, 2, 3, 4, 6, 12$ . Identity (66) may be combined with any rational multiple of (5), which contains  $S_3(q_7^d)$  with  $d|4$ .

For  $D = -23$ , there is a family whose terms have merit  $a \geq \frac{144}{23}$ , beginning with

$$\zeta(3) = \frac{93}{500\sqrt{23}} \pi^3 - \frac{23}{25} S_3(q_{1/23}^{12}) + \frac{1}{25} \sum_{d|36, d>1} h_d S_3(q_{23}^{d/3}) \quad (67)$$

with integers  $h_d = -72, -60, 9, 114, 4, -15, -8, 1$ , for  $d = 2, 3, 4, 6, 9, 12, 18, 36$ . Studying denominators of the rational coefficients, I encountered an integer sequence  $x_n = 25, 85132, -50570780, 66436773424, -39027031565300, \dots$  with solution

$$x_n = \Im \left( \frac{\alpha(\alpha + 2\beta - 2^{2n+1}\gamma_+ + \gamma_-) - \beta\gamma_+}{2^{2n+2}\sqrt{23}} \right), \quad (68)$$

$$\alpha = (1 + \sqrt{-23})^{2n}, \quad \beta = (5 - \sqrt{-23})^{2n}, \quad \gamma_{\pm} = 36^n \pm (-23)^n. \quad (69)$$

For  $k = 6n + 1$  and  $D = -3$ , a family with terms of merit  $a \geq \frac{25}{3}$  begins with

$$970 \zeta(7) = \frac{21203\sqrt{3}}{113400} \pi^7 - \frac{29 \cdot 449}{7} S_7(q_3^2) + \frac{43}{2^{77}} \sum_{d|12} G_d S_7(q_{1/3}^{5d}) \quad (70)$$

with familiar integers  $G_d = 12^3, -3^3 129, -64, 27, 129, -1$  for  $d = 1, 2, 3, 4, 6, 12$ . Studying denominators of the rational coefficients, I encountered an integer sequence  $y_n = 970, 14460979, 227188159336, 3547683233832985, \dots$  with solution

$$y_n = \frac{1}{84} (5^{6n+1} - 2^{6n+1} - (2^{6n+1} + 1)(-3)^{3n}). \quad (71)$$

A better family has terms of merit  $a \geq \frac{49}{3}$  and begins with

$$44071 \zeta(7) = \frac{5734\sqrt{3}}{675} \pi^7 + \frac{1}{2^7} \sum_{d_1 \in \{1, 2, 4\}} \sum_{d_2 \in \{7, 9, 21\}} H_{d_1 d_2} S_7(q_{1/3}^{d_1 d_2}) \quad (72)$$

with integers  $H_d$  for  $d \in \{7, 9, 14, 18, 21, 28, 36, 42, 84\}$  and  $H_7/12^3 = -H_{84} = 443$ . Studying denominators of the rational coefficients, I encountered an integer sequence  $z_n = 44071, 4786291978, 571088556271897, \dots$  with solution

$$z_n = \frac{1}{20} (7^{6n+1} - 3^{6n} (2 - (-3)^{3n+1}) - 2 \Re((2 + \sqrt{-3})^{6n})). \quad (73)$$

## 2.7 Order of merit

In summary, there are 18 known families with  $a > 4$ . Their order of merit is

$$\frac{49}{3}, 16, 12, \frac{48}{5}, 9, \frac{25}{3}, 8, \frac{36}{5}, 7, \frac{20}{3}, \frac{144}{23}, \frac{28}{5}, \frac{16}{3}, \frac{36}{7}, 5, \frac{44}{9}, \frac{9}{2}, \frac{144}{35} \quad (74)$$

with  $D = -3$  appearing 4 times. Each  $D \in \{-4, -7, -8, -15, -20, -35\}$  appears twice. Each  $D \in \{-11, -23\}$  appears just once. It seems probable that there are finitely many independent families with rational merit  $a > 4$ . Whether there be more than these 18 remains an open question.

## 2.8 Logarithms at $k = 1$

In the singular case  $k = 1$ , Families A to E give

$$S_1(q_7) - 2S_1(q_7^2) + S_1(q_7^4) = -\frac{\sqrt{7}}{24}\pi + \frac{1}{2}\log(2) \quad (75)$$

$$2S_1(q_2^2) - S_1(q_2^3) - S_1(q_2^6) = \frac{5\sqrt{2}}{24}\pi + \frac{1}{4}\log(2) - \log(3) \quad (76)$$

$$S_1(q_1^3) + 2S_1(q_1^4) - 4S_1(q_1^6) + S_1(q_1^{12}) = \frac{1}{24}\pi + \log(2) - \frac{3}{4}\log(3) \quad (77)$$

$$2S_1(q_1^4) + S_1(q_1^5) - 4S_1(q_1^{10}) + S_1(q_1^{20}) = \frac{7}{24}\pi + \log(2) - \log(5) \quad (78)$$

$$S_1(q_3^2) + S_1(q_3^3) - 3S_1(q_3^6) + S_1(q_3^{12}) = \frac{\sqrt{3}}{24}\pi + \frac{1}{3}\log(2) - \frac{5}{12}\log(3). \quad (79)$$

Similarly, for the fractional families, the logarithms of 7, 11 and 23 also appear. The fastest convergence occurs in the identity

$$\frac{5\sqrt{3}}{36}\pi + \frac{13}{12}\log(3) - \log(7) = L_7 - 3L_9 + L_{21}, \quad L_d = S_1(q_{1/3}^d) - 2S_1(q_{1/3}^{2d}) + S_1(q_{1/3}^{4d}) \quad (80)$$

with 9 Lambert series, each no bigger than  $S_1(q_{1/3}^7) < 3.062 \times 10^{-6}$ .

## 3 Computational efficiency

The evaluation [19, 22]

$$\zeta(7) = \frac{19\pi^7}{56700} - 2 \sum_{n=1}^{\infty} \frac{1}{n^7(e^{2\pi n} - 1)} \quad (81)$$

has merit  $a = 4$ , inferior to (35) in Family E, with merit  $a = 12$ . Yet 4 Lambert series are needed for latter. I estimate the cost of Family E as

$$T_E = \frac{2}{\sqrt{3}} \left( \frac{1}{2} + \frac{1}{3} + \frac{1}{6} + \frac{1}{12} \right) \approx 1.251 \quad (82)$$

relative to the cost of (81). The relative costs of Families A and B are

$$T_A = \frac{2}{\sqrt{7}} \left(1 + \frac{1}{2} + \frac{1}{4}\right) \approx 1.323, \quad T_B = \frac{2}{\sqrt{2}} \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{6}\right) \approx 1.414. \quad (83)$$

Hence I conclude that the family beginning with Ramanujan's discovery (2) is preferable in the case of  $\zeta(4n-1)$ .

In the case of  $\zeta(4n+1)$ , the cost of the family beginning with (3) is  $1 + \frac{1}{2} = 1.5$ , since it may be written in terms of  $S_5(q_1^2)$  and  $S_5(q_1^4)$ , with  $q_1 = e^{-\pi}$ . In comparison, the costs of Families C and D are

$$T_C = 2 \left(\frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{12}\right) \approx 1.667, \quad T_D = 2 \left(\frac{1}{4} + \frac{1}{5} + \frac{1}{10} + \frac{1}{20}\right) = 1.2. \quad (84)$$

Hence I conclude that Family D is preferable in the case of  $\zeta(4n+1)$ .

### 3.1 Comparison with other methods

For  $\zeta(3)$ , the hypergeometric method of Amdeberhan and Zeilberger [1] is the fastest so far exploited. Its convenience comes from the fact that the summands are rational. For  $\zeta(5)$ , there is a fast method with rational summands given by Broadhurst in Eq. 72 of [10], which consists of 22 terms of the type found by Bailey, Borwein and Plouffe (BBP), in their 4-term formula [6]

$$\pi = \sum_{k=0}^{\infty} \frac{1}{2^{4k}} \left( \frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right). \quad (85)$$

In the case of  $\zeta(5)$ , there are 8 terms with  $2^{-4k}$  in their summands, 8 terms with  $2^{-12k}$  and 6 terms with  $2^{-20k}$ . These inverse powers of 2 are well suited to numerical computation, with 20 billion decimal digits of  $\zeta(5)$  obtainable in 25 hours on a machine with 14 cores [23]. This record was eclipsed in 2023 using a hypergeometric formula, discovered by Y. Zhao and proved by Kam Cheong Au [2], which yielded 200 billion decimal digits of  $\zeta(5)$  in 30 hours on a machine with 128 cores [23].

Using PSLQ [4], Bailey has compiled a compendium [3] of BBP-type identities, including some in base 3 found in [11]. None of these goes beyond polylogarithmic weight 5. In [10], it was shown how to extend base-2 BBP identities up to weight 11, with  $\zeta(7)$ ,  $\zeta(9)$  and  $\zeta(11)$  determined by increasingly refined combinations of binary BBP terms with products of  $\pi^2$  and  $\log(2)$ . This base-2 polylogarithmic ladder terminates with  $\zeta(11)$  in Eq. 83 of [10]. Since  $\pi$  and  $\log(2)$  have very fast numerical evaluations, it is probable that by diligent binary programming of identities in [10] one might evaluate odd zeta values up to  $\zeta(11)$  by BBP methods that are asymptotically faster than those for the families in this article.

With a base in an algebraic number field, it is possible to reach weight 17, using polylogarithms  $\text{Li}_{17}(\alpha^{-k})$ , with  $k|630$ , and products of  $\pi^2$  and  $\log(\alpha)$ , where  $\alpha \approx 1.17628$  is smallest known Salem number [5]. However, ladders with irrational bases such as  $\alpha$  or the golden ratio  $\frac{1}{2}(1 + \sqrt{5})$  are uncompetitive with the methods of the families identified in the current work, where convergence is governed by a much larger transcendental constant  $e^{\pi\sqrt{a}} > 4071$ , with integer  $a \geq 7$ .

I suggest that for  $k = 4n + 1 > 9$  the most efficient known method for evaluating  $\zeta(k)$  is provided by Family D in (28). For  $k = 4n - 1 > 11$ , I suggest that the best method is provided by the Ramanujan family [13, 22]

$$\zeta(4n - 1) = -2S_{4n-1}(q_1^2) - \frac{(2\pi)^{4n-1}}{2} \sum_{m=0}^{2n} (-1)^m \frac{\mathcal{B}_{2m}}{(2m)!} \frac{\mathcal{B}_{4n-2m}}{(4n-2m)!} \quad (86)$$

where  $\mathcal{B}_k$  is the  $k$ th Bernoulli number and  $q_1 = e^{-\pi}$ . It converges faster than the method of [14] for the alternating sum  $(2^{1-k} - 1)\zeta(k) = \sum_{n>0} (-1)^n n^{-k}$ .

## 4 Resolving the families

Consider the periodic function  $P_k(z) = S_k(e^{2\pi iz}) = P_k(z + 1)$  with complex  $z$  in the upper half-plane  $\Im z > 0$ . For odd  $k > 1$  the transformation  $z \rightarrow -1/z$  gives the quasi-modular relation

$$z^{k-1} P_k(-1/z) = P_k(z) - R_k(z) \quad (87)$$

with a correction term [16, 22]

$$R_k(z) = \frac{z^{k-1} - 1}{2} \zeta(k) + \frac{(2\pi i)^k}{2z} \sum_{m=0}^{(k+1)/2} z^{2m} \frac{\mathcal{B}_{2m}}{(2m)!} \frac{\mathcal{B}_{k+1-2m}}{(k+1-2m)!}. \quad (88)$$

Moreover, (87) holds at  $k = 1$ , with

$$R_1(z) = \frac{1}{2} \log(z) + \frac{\pi i}{12} \left( z - 3 + \frac{1}{z} \right). \quad (89)$$

Since  $z^{k-1} R_k(-1/z) = -R_k(z)$ , the aperiodic function  $M_k(z) = P_k(z) - \frac{1}{2} R_k(z)$  satisfies  $z^{k-1} M_k(-1/z) = M_k(z)$ . Identity (86) is then obtained from  $M_{4n-1}(i) = 0$  which shows that  $P_k(i) = \frac{1}{2} R_k(i)$  for  $k = 4n - 1$ .

Further progress may be made by using the doubling relation

$$P_k(z - \tfrac{1}{2}) + P_k(z) = (2 + 2^{1-k}) P_k(2z) - 2^{1-k} P_k(4z) \quad (90)$$

which holds for all real  $k$ . To prove this, expand the left hand side in polylogarithms to obtain the even function  $\sum_{n=1}^{\infty} \text{Li}_k((-q)^n) + \sum_{n=1}^{\infty} \text{Li}_k(q^n)$ , with  $q = e^{2\pi iz}$ . Then the right hand side results from separating cases with even and odd  $n$  and using the doubling relation  $\text{Li}_k(-x) + \text{Li}_k(x) = 2^{1-k} \text{Li}_k(x^2)$  for polylogarithms [17].

For  $k = 4n + 1$ , one obtains a family with merit  $a = 4$  beginning with  $\zeta(5)$  in (3) by setting  $z = \frac{1}{2}i$  in (90). Then  $P_k(i)$  and  $P_k(2i)$  appear on the right hand side. On the left, (87) relates  $P_k(\frac{1}{2}(i - 1))$  to  $P_k(i + 1) = P_k(i)$  and  $P_k(\frac{1}{2}i)$  to  $P_k(2i)$ . Hence  $P_k(i)$  is related to  $P_k(2i)$  for  $k = 4n + 1$ .

## 4.1 Resolution of Family A

For Family A in Section 2.1, I use the identity

$$P_k(z - \frac{1}{4}) + P_k(z + \frac{1}{4}) = ((2 + 2^{1-k})^2 - 2^{1-k}) P_k(4z) - (2 + 2^{1-k}) (P_k(2z) + 2^{1-k} P_k(8z)) \quad (91)$$

obtained by two applications of (90). Set  $z = \frac{1}{4}\sqrt{-7}$  in (91). Then  $S_k(q_7^d)$  with  $d|4$  appear on the right hand side. On the left, the quasi-modular relation (87) transforms  $P_k(\frac{1}{4}(\sqrt{-7} \pm 1))$  to  $P_k(\frac{1}{2}(\sqrt{-7} \mp 1))$  and (90) gives the same targets.

Cleaning up, I proved the empirical determinations of Section 2.1 and obtained

$$A_n = \frac{(2n+3)2^{2n+3}}{\sqrt{7}} \Im \mathcal{H}_n \left( \frac{1 + \sqrt{-7}}{4} \right) \quad (92)$$

$$\mathcal{H}_n(z) = (-1)^{n+1} \sum_{m=0}^{n+1} z^{2m-1} \binom{2n+2}{2m} \mathcal{B}_{2m} \mathcal{B}_{2n+2-2m} \quad (93)$$

for the rational sequence in Conjecture 1, now checked up to  $n = 1501$  for  $\zeta(3003)$ .

## 4.2 Resolution of Family B

To resolve Family B in Section 2.2, I use the trebling relation

$$P_k(z - \frac{1}{3}) + P_k(z) + P_k(z + \frac{1}{3}) = (3 + 3^{1-k}) P_k(3z) - 3^{1-k} P_k(9z). \quad (94)$$

Set  $z = \frac{1}{3}\sqrt{-2}$  in (94). Then  $S_k(q_2^2)$  and  $S_k(q_2^6)$  appear on the right hand side. On the left, transform  $P_k(z)$  to  $S_k(q_2^3)$ . Transform  $P_k(\frac{1}{3}(\sqrt{-2} \pm 1))$  to  $P_k(\sqrt{-2} \mp 1) = S_k(q_2^2)$ .

Cleaning up, I proved the empirical determinations of Section 2.2 and obtained

$$B_n = \frac{4(2n+3)3^{2n}}{\sqrt{2}} \Im \left( 2\mathcal{H}_n \left( \frac{1 + \sqrt{-2}}{3} \right) + \mathcal{H}_n \left( \frac{\sqrt{-2}}{3} \right) \right) \quad (95)$$

for the rational sequence in Conjecture 2, now checked up to  $n = 1501$ .

### 4.3 Resolution of Family C

To resolve Family C in Section 2.3, with  $k = 4n + 1$ , set  $z = \frac{2}{3}i$  in the trebling relation (94). Then  $S_k(q_1^4)$  and  $S_k(q_1^{12})$  appear on the right hand side. On the left, transform  $P_k(\frac{2}{3}i)$  to  $P_k(\frac{3}{2}i) = S_k(q_1^3)$ . It remains to consider the real combination  $P_k(\frac{1}{3}(2i - 1)) + P_k(\frac{1}{3}(2i + 1)) = P_k(\frac{2}{3}(i + 1)) + P_k(\frac{2}{3}(i - 1))$ , which (87) converts to the real combination  $P_k(\frac{3}{4}(i - 1)) + P_k(\frac{3}{4}(i + 1))$ , since  $(i - 1)^{k-1} = (i + 1)^{k-1}$ . Translate this to  $P_k(\frac{1}{4}(3i + 1)) + P_k(\frac{1}{4}(3i - 1))$  and use (91) to obtain  $S_k(q_1^d)$  with  $d \in \{3, 6, 12\}$ .

Cleaning up, I proved the empirical determinations of Section 2.3 and obtained

$$C_n = \frac{(4n + 3)3^{4n}}{4n + 1} \Im \left( 2\mathcal{H}_{2n} \left( \frac{2 + 2i}{3} \right) + \mathcal{H}_{2n} \left( \frac{2i}{3} \right) \right) \quad (96)$$

for the rational sequence in Conjecture 3, now checked up to  $n = 750$  for  $\zeta(3001)$ .

### 4.4 Resolution of Family D

To resolve Family D in Section 2.4, with  $k = 4n + 1$ , set  $z = \frac{2}{5}i$  in the identity

$$\sum_{r=-2}^2 P_k \left( z + \frac{r}{5} \right) = (5 + 5^{1-k}) P_k(5z) - 5^{1-k} P_k(25z). \quad (97)$$

Then  $S_k(q_1^4)$  and  $S_k(q_1^{20})$  appear on the right hand side. On the left, transform  $P_k(\frac{2}{5}i)$  to  $S_k(q_1^5)$ . Observe that  $5 = (1 + 2i)(1 - 2i)$  and transform  $P_k(\frac{1}{5}(2i \pm 1))$  to  $S_k(q_1^4)$ . Transform  $P_k(\frac{1}{5}(2i \pm 2))$  to  $P_k(\frac{5}{4}(i \mp 1)) = P_k(\frac{1}{4}(5i \mp 1))$  and use (91) to obtain  $S_k(q_1^d)$  with  $d \in \{5, 10, 20\}$ .

Cleaning up, I proved the empirical determinations of Section 2.4 and obtained

$$D_n = \frac{(4n + 3)5^{4n}}{4n + 1} \Im \left( 2\mathcal{H}_{2n} \left( \frac{2 + 2i}{5} \right) + 2\mathcal{H}_{2n} \left( \frac{1 + 2i}{5} \right) + \mathcal{H}_{2n} \left( \frac{2i}{5} \right) \right) \quad (98)$$

for the rational sequence in Conjecture 4, now checked up to  $n = 750$ .

### 4.5 Resolution of Family E

To resolve Family E in Section 2.5, with  $k = 6n + 1$ , set  $z = \frac{1}{3}v$  in (94), with  $v = \sqrt{-3}$ . Then  $S_k(q_3^2)$  and  $S_k(q_3^6)$  appear on the right hand side. On the left, transform  $P_k(\frac{1}{3}v)$  to  $S_k(q_3^2)$ . Noting that  $(v + 1)^{k-1} = (v - 1)^{k-1}$ , transform  $P_k(\frac{1}{3}(v \pm 1))$  to  $P_k(\frac{3}{4}(v \mp 1)) = P_k(\frac{1}{4}(3v \pm 1))$  and use (91) to obtain  $S_k(q_3^d)$  with  $d \in \{3, 6, 12\}$ .

Cleaning up, I proved the empirical determinations of Section 2.5 and obtained

$$E_n = \frac{48(2n + 1)(3n + 2)3^{6n}}{(3n + 1)(6n + 1)\sqrt{3}} \Im \left( 2\mathcal{H}_{3n} \left( \frac{1 + \sqrt{-3}}{3} \right) + \mathcal{H}_{3n} \left( \frac{\sqrt{-3}}{3} \right) \right) \quad (99)$$

for the rational sequence in Conjecture 5, now checked up to  $n = 500$  for  $\zeta(3001)$ .

## 4.6 Resolution of fractional families

Here more ingenuity was needed. From 3 applications of (90), I proved that

$$P_k(z - \frac{3}{8}) + P_k(z - \frac{1}{8}) + P_k(z + \frac{1}{8}) + P_k(z + \frac{3}{8}) = 4P_k(8z) - 4(1 + 2^{-k} + 4^{-k})(P_k(4z) - (1 + 2^{1-k})P_k(8z) + 2^{1-k}P_k(16z)) \quad (100)$$

and by diligent application of (90,94) obtained the identity

$$\begin{aligned} P_k(z - \frac{1}{6}) + P_k(z + \frac{1}{6}) &= P_k(z) + 2^{1-k}P_k(4z) + 3^{1-k}P_k(9z) \\ - (2^k + 1) (2^{1-k}P_k(2z) + 6^{1-k}P_k(18z)) &- (3^k + 1) (3^{1-k}P_k(3z) + 6^{1-k}P_k(12z)) \\ + 6^{1-k} ((2^k + 1)(3^k + 1)P_k(6z) &+ P_k(36z)). \end{aligned} \quad (101)$$

For  $D = -15$ , set  $z = \frac{1}{8}t$  in (100) with  $t = \sqrt{-15}$ . Then  $S_k(q_{15}^d)$  with  $d|4$  appear on the right hand side. On the left, transform  $P_k(\frac{1}{8}(t \pm 1))$  to  $P_k(\frac{1}{2}(t \mp 1))$ , which doubling relates to  $S_k(q_{15}^d)$  with  $d|4$ . Transform  $P_k(\frac{1}{8}(t \pm 3))$  to  $P_k(\frac{1}{3}(t \mp 3)) = S_k(q_{5/3}^2)$  with merit  $a = \frac{20}{3}$ . The result is the family beginning with (40). Now set  $z = \frac{1}{4}t$  in (91). Then  $S_k(q_{15}^d)$  with  $d|4$  appear on the right hand side. On the left, transform  $P_k(\frac{1}{4}(t \pm 1)) = P_k(\frac{1}{4}(t \mp 3))$  to  $P_k(\frac{1}{6}(t \pm 3))$ , which doubling converts to  $S_k(q_{5/3}^d)$  with  $d|4$ . Transform  $S_k(q_{5/3})$  to  $S_k(q_{3/5}^4)$  with merit  $a = \frac{48}{5}$ . Eliminate  $S_k(q_{5/3}^2)$  using the first family. The result is a better family beginning with (43).

For  $D = -3$ , one should distinguish the easier case  $k = 2n+1$ , with  $3 \nmid n$ , from the harder case  $k = 6n+1$ . For the former, set  $z = \frac{1}{2}\sqrt{-3}$  in the doubling relation (90). Then there is relation between  $S_k(q_3^d)$  with  $d|4$  and  $P_k(\lambda)$  with  $\lambda = \frac{1}{2}(1 + \sqrt{-3})$  and  $\lambda^6 = 1$ . Hence (87) gives  $(1 - \lambda^{k-1})P_k(\lambda) = R_k(\lambda) \neq 0$  and the easier case is now resolved by transforming  $S_k(q_3)$  to  $S_k(q_{1/3}^4)$ . For the harder case, set  $z = \frac{1}{8}\sqrt{-3}$  in (100). Then  $S_k(q_3^d)$  with  $d|4$  appear on the right hand side. On the left, transform  $P_k(\frac{1}{8}(\sqrt{-3} \pm 1))$  to  $S_k(q_3^4)$ . Transform  $P_k(\frac{1}{8}(\sqrt{-3} \pm 3))$  to  $S_k(q_{1/3}^4)$ .

For  $D = -8$ , set  $z = \frac{2}{3}\sqrt{-2}$  in (94). Then  $S_k(q_2^4)$  and  $S_k(q_2^{12})$  appear on the right hand side. On the left, transform  $P_k(z)$  to  $S_k(q_{1/2}^3)$ . Transform  $P_k(z \pm \frac{1}{3}) = P_k(z \mp \frac{2}{3})$  to  $P_k(\frac{1}{2}(\sqrt{-2} \pm 1))$ , which doubling relates to  $S_k(q_2^d)$  with  $d|4$ . Transform  $S_k(q_2)$  to  $S_k(q_2^2)$ .

For  $D = -20$ , set  $z = \frac{1}{3}\sqrt{-5}$  in (94). Then  $S_k(q_5^2)$  and  $S_k(q_5^6)$  appear on the right hand side. On the left, transform  $P_k(z)$  to  $S_k(q_{1/5}^6)$ . Transform  $P_k(\frac{1}{3}(\sqrt{-5} \pm 1))$  to  $P_k(\frac{1}{2}(\sqrt{-5} \mp 1))$ , which doubling relates to  $S_k(q_5^d)$  with  $d|4$ . The result is the family beginning with (55), with terms of merit  $a \geq 5$ . Now set  $z = \frac{1}{6}\sqrt{5}$  in (101). Then  $S_k(q_5^{d/3})$  with  $d|36$  appear on the right hand side. Transform  $S_k(q_5^{1/3})$  to  $S_k(q_{1/5}^{12})$ . Transform  $S_k(q_5^{2/3})$  to  $S_k(q_{1/5}^6)$ . Eliminate  $S_k(q_5)$  using the first family. On the

left, transform  $P_k(\frac{1}{6}(\sqrt{5} \pm 1))$  to  $S_k(q_5^2)$ . The result is the better family beginning with (58), with terms of merit  $a \geq \frac{36}{5}$ .

For  $D = -11$ , set  $z = \frac{1}{6}\sqrt{-11}$  in (101). Then  $S_k(q_{11}^{d/3})$  with  $d|36$  appear on the right hand side. Transform  $S_k(q_{11}^{1/3})$  to  $S_k(q_{1/11}^{12})$ . On the left, transform  $P_k(z \pm \frac{1}{6})$  to  $P_k(\frac{1}{2}(\sqrt{-11} \mp 1))$ , which doubling relates to  $S_k(q_{11}^d)$  with  $d|4$ .

For  $D = -35$ , set  $z = \frac{1}{6}v$  in (101) with  $v = \sqrt{-35}$ . Then  $S_k(q_{35}^{d/3})$  with  $d|36$  appear on the right hand side. Transform  $S_k(q_{35}^{1/3})$  to  $S_k(q_{1/35}^{12})$ . On the left, transform  $P_k(\frac{1}{6}(v \pm 1)) = P_k(\frac{1}{6}(v \mp 5))$  to  $P_k(\frac{1}{10}(v \pm 5))$ , which doubling relates to  $S_k(q_{7/5}^d)$  with  $d|4$ . Transform  $S_k(q_{7/5})$  to  $S_k(q_{5/7}^4)$  to obtain the family with terms of merit  $a \geq \frac{144}{35}$  beginning with (62). Now set  $z = \frac{1}{6}w$  in (101) with  $w = \sqrt{-7/5}$ . On the right hand side, transform  $S_k(q_{7/5}^{d/3})$  with  $d \in \{1, 2, 3, 4\}$  to obtain  $S_k(q_{5/7}^d)$  with  $d \in \{3, 4, 6, 12\}$ . On the left, transform  $P_k(\frac{1}{6}(w \pm 1))$  to  $P_k(\frac{5}{2}(w \mp 1))$ , which doubling relates to  $S_k(q_{35}^d)$  with  $d|4$ . This results in a family with terms of merit  $a \geq \frac{28}{5}$ , beginning with (64).

For  $D = -7$ , set  $z = \frac{1}{12}\sqrt{-7}$  in the identity

$$\begin{aligned} & P_k(z - \frac{5}{12}) + P_k(z - \frac{1}{12}) + P_k(z + \frac{1}{12}) + P_k(z + \frac{5}{12}) = \\ & (2^k + 1) (2^{1-k} P_k(2z) + 4^{1-k} P_k(8z) + 6^{1-k} P_k(18z) + 12^{1-k} P_k(72z)) \\ & - ((2^k + 1)^2 - 2^{k-1}) (4^{1-k} P_k(4z) - 12^{1-k} (3^k + 1) P_k(12z) + 12^{1-k} P_k(36z)) \\ & - 6^{1-k} (2^k + 1) (3^k + 1) (P_k(6z) + 2^{1-k} P_k(24z)) \end{aligned} \quad (102)$$

obtained by diligent application of (90,94). Then  $S_k(q_7^{d/3})$  with  $d|36$  appear on the right hand side. Transform  $S_k(q_7^{1/3})$  to  $S_k(q_{1/7}^{12})$ . Transform  $S_k(q_7^{2/3})$  to  $S_k(q_{1/7}^6)$ , with merit  $a = \frac{36}{7}$ . On the left, transform  $P_k(\frac{1}{12}(\sqrt{-7} \pm 1))$  to  $P_k(\frac{3}{2}(\sqrt{-7} \mp 1))$ , which doubling relates to  $S_k(q_7^{3d})$  with  $d|4$ . Transform  $P_k(\frac{1}{12}(\sqrt{-7} \pm 5)) = P_k(\frac{1}{12}(\sqrt{-7} \mp 7))$  to  $P_k(\frac{3}{2}(\sqrt{-1/7} \pm 1))$ , which doubling relates to  $S_k(q_{1/7}^{3d})$  with  $d|4$ . Transform  $S_k(q_{1/7}^{4/3})$  to obtain a relation with terms of merit  $a \geq \frac{36}{7}$ . At  $k = 3$ , identity (66) is obtained by using (5) in Family A, which contains  $S_k(q_7^d)$  with  $d|4$ .

For  $D = -23$ , set  $z = \frac{1}{6}\sqrt{-23}$  in (101). Then  $S_k(q_{23}^{d/3})$  with  $d|36$  appear on the right hand side. On the left, transform  $P_k(\frac{1}{6}(\sqrt{-23} \pm 1))$  to  $P_k(\frac{1}{4}(\sqrt{-23} \mp 1))$ . This pair occurs with complex coefficients. Their real sum is also related to targets  $S_k(q_{23}^{d/3})$  with  $d|36$  by setting  $z = \frac{1}{4}\sqrt{-23}$  in (91). Thus both of  $P_k(\frac{1}{4}(\sqrt{-23} \pm 1))$  are related to these 9 targets. Now set  $z = \frac{1}{12}\sqrt{-23}$  in (102). Then  $S_k(q_{23}^{d/3})$  with  $d|36$  appear on the right hand side. On the left, transform  $P_k(z \pm \frac{1}{12})$  to  $P_k(\frac{1}{2}(\sqrt{-23} \mp 1))$ , which doubling relates to  $S_k(q_{23}^d)$  with  $d|4$ . Transform  $P_k(z \pm \frac{5}{12})$  to  $P_k(\frac{1}{4}(\sqrt{-23} \mp 5)) = P_k(\frac{1}{4}(\sqrt{-23} \mp 1))$ . It follows that there is a relation between  $S_k(q_{23}^{d/3})$  with  $d|36$ . At  $d = 1$ , transform  $S_k(q_{23}^{1/3})$  to  $S_k(q_{1/23}^{12})$ , with merit  $a = \frac{144}{23}$ , to



obtain the family beginning with (67). Sequence (69) comes from solving a pair of simultaneous equations with complex coefficients.

For  $D = -3$  and  $k = 6n + 1$ , set  $z = \frac{1}{10}\sqrt{-3}$  in the identity

$$\begin{aligned} \sum_{r \in \{1,3\}} (P_k(z - \frac{r}{10}) + P_k(z + \frac{r}{10})) &= P_k(z) + 2^{1-k}P_k(4z) + 5^{1-k}P_k(25z) \\ - (2^k + 1) (2^{1-k}P_k(2z) + 10^{1-k}P_k(50z)) &- (5^k + 1) (5^{1-k}P_k(5z) + 10^{1-k}P_k(20z)) \\ + 10^{1-k} ((2^k + 1)(5^k + 1)P_k(10z) &+ P_k(100z)). \end{aligned} \quad (103)$$

Then  $S_k(q_3^{d/5})$  with  $d|100$  appear on the right hand side. Transform terms with  $d|4$  to obtain  $S_k(q_{1/3}^{5d})$  with  $d|4$ . At  $d = 5$ , transform  $S_k(q_3)$  to  $S_k(q_{1/3}^4)$  with merit  $a = \frac{16}{3}$ . Eliminate this using the family that begins with (50). All the terms on the right now have merit  $a \geq \frac{25}{3}$ . On the left, transform  $P_k(\frac{1}{10}(\sqrt{-3} \pm 1))$  to  $P_k(\frac{5}{2}(\sqrt{-3} \mp 1))$ , which doubling relates to  $S_k(q_3^{5d})$  with  $d|4$ . Transform  $P_k(\frac{1}{10}(\sqrt{-3} \pm 3))$  to  $P_k(\frac{5}{6}(\sqrt{-3} \mp 3))$ , which doubling relates to  $S_k(q_{1/3}^{5d})$  with  $d|4$ . The result is the family beginning with (70). To resolve a better family, with terms of merit  $a \geq \frac{49}{3}$ , set  $z = \frac{1}{7}\sqrt{-3}$  in

$$\sum_{r=-3}^3 P_k(z + \frac{r}{7}) = (7 + 7^{1-k}) P_k(7z) - 7^{1-k} P_k(49z) \quad (104)$$

with  $S_k(q_3^2)$  and  $S_k(q_3^{14})$  on the right hand side. On the left, transform  $P_k(z)$  to  $S_k(q_{1/3}^{14})$ . Transform  $P_k(z \pm \frac{1}{7})$  to  $P_k(\frac{7}{4}\sqrt{-3} \pm \frac{1}{4})$  and use (91) to produce  $S_k(q_3^{7d})$  with  $d|4$ . Transform  $P_k(z \pm \frac{2}{7})$  to  $P_k(\sqrt{-3} \mp 2) = S_k(q_3^2)$ . Transform  $P_k(z \pm \frac{3}{7})$  to  $P_k(\frac{7}{12}(\sqrt{-3} \mp 3)) = P_k(\frac{7}{12}\sqrt{-3} \pm \frac{1}{4})$  and use (91) to produce  $S_k(q_{1/3}^{7d})$  with  $d|4$ . Thus  $S_k(q_3^2)$  is related to terms of the form  $S_k(q_3^{7d})$  and  $S_k(q_{1/3}^{7d})$  with  $d|4$ . It is also related, by Family E in (34), to  $S_k(q_3^{3d})$  with  $d|4$ . By this method, I obtained relation (72) between  $\zeta(7)$ ,  $\sqrt{3}\pi^7$  and the 9 terms  $S_7(q_{1/3}^{d_1 d_2})$  with  $d_1 \in \{1, 2, 4\}$  and  $d_2 \in \{7, 9, 21\}$ . These 9 terms are less efficient than the 4 terms in Family E. The computational cost of the former, relative to the latter, is 19/13.

## 4.7 Derivative families

Each family may be extended by taking derivatives. For odd  $k > 1$ , let

$$T_k(z) = P_k(z) - \frac{2zP'_k(z)}{k-1}, \quad U_k(z) = R_k(z) - \frac{2zR'_k(z)}{k-1}. \quad (105)$$

Then  $z^{k-1}U(-1/z) = U_k(z)$  and there is a quasi-modular relation

$$z^{k-1}T_k(-1/z) = U_k(z) - T_k(z). \quad (106)$$

Hence  $T_k(i) = \frac{1}{2}U_k(i)$  for  $k = 4n + 1$ , as recorded by Emil Grosswald, in Theorem A of [15], and by Henri Cohen, in Theorem 1.1 of [13]. At  $k = 1$ , the identity  $P'_1(-1/z) + P'_1(z) = R'_1(z)$ , with positive real  $x = z/i$ , gives

$$\frac{1}{\pi} = \frac{1+x^2}{6x} - x\mathcal{S}(x) - \frac{1}{x}\mathcal{S}\left(\frac{1}{x}\right), \quad \mathcal{S}(x) = \sum_{n=1}^{\infty} \frac{1}{\sinh^2(n\pi x)} \quad (107)$$

and hence the evaluation  $1/\pi = \frac{1}{3} - 2\mathcal{S}(1)$ , with merit  $a = 4$ .

In Families A and B, derivative results yield alternative methods of evaluating  $\zeta(2n+1)$ . They extend Families C and D to  $\zeta(4n-1)$  and Family E to  $\zeta(6n-3)$ .

At  $k = 3$ , I obtained the derivative identities

$$\zeta(3) = \frac{7\sqrt{7}}{480}\pi^3 + \frac{5}{2}T_3(\tfrac{1}{2}\sqrt{-7}) - \frac{41}{8}T_3(\sqrt{-7}) + \frac{5}{8}T_3(2\sqrt{-7}) \quad (108)$$

$$= \frac{47\sqrt{2}}{1710}\pi^3 - \frac{44}{19}T_3(\sqrt{-2}) + \frac{4}{19}T_3(\tfrac{3}{2}\sqrt{-2}) + \frac{2}{19}T_3(3\sqrt{-2}) \quad (109)$$

$$= \frac{7}{180}\pi^3 - 4T_3(\tfrac{3}{2}i) - 8T_3(2i) + 11T_3(3i) - T_3(6i) \quad (110)$$

$$= \frac{221}{5700}\pi^3 - \frac{232}{95}T_3(2i) - \frac{28}{95}T_3(\tfrac{5}{2}i) + \frac{77}{95}T_3(5i) - \frac{7}{95}T_3(10i) \quad (111)$$

$$= \frac{43\sqrt{3}}{1920}\pi^3 - \frac{25}{8}T_3(\sqrt{-3}) - \frac{9}{8}T_3(\tfrac{3}{2}\sqrt{-3}) + \frac{81}{32}T_3(3\sqrt{-3}) - \frac{9}{32}T_3(6\sqrt{-3}) \quad (112)$$

with similar derivative relations for the fractional families.

In the singular case  $k = 1$ , I obtained

$$\frac{1}{\pi} = \frac{1}{3} - \left(10 + 2\sqrt{7}\right)\mathcal{S}(\tfrac{1}{2}\sqrt{7}) + \left(32 + 2\sqrt{7}\right)\mathcal{S}(\sqrt{7}) - 24\mathcal{S}(2\sqrt{7}) \quad (113)$$

$$= \frac{9}{28} - 4\sqrt{2}\left(\mathcal{S}(\sqrt{2}) - \mathcal{S}(2\sqrt{2})\right) + \frac{27}{14}\left(\mathcal{S}(\tfrac{3}{2}\sqrt{2}) - 2\mathcal{S}(3\sqrt{2})\right) \quad (114)$$

$$= \frac{23}{72} + \tfrac{3}{4}\mathcal{S}(\tfrac{3}{4}\sqrt{2}) - \left(\tfrac{10}{3} + 4\sqrt{2}\right)\mathcal{S}(\sqrt{2}) + \left(\tfrac{20}{3} + 4\sqrt{2}\right)\mathcal{S}(2\sqrt{2}) - 6\mathcal{S}(6\sqrt{2}) \quad (115)$$

$$= \frac{3}{10} + \left(\tfrac{42}{5} + 2\sqrt{15}\right)\mathcal{S}(\tfrac{1}{3}\sqrt{15}) - \left(29 + 6\sqrt{15}\right)\mathcal{S}(\tfrac{1}{2}\sqrt{15}) \\ + \left(\tfrac{234}{5} + 4\sqrt{15}\right)\mathcal{S}(\sqrt{15}) - 28\mathcal{S}(2\sqrt{15}) \quad (116)$$

with amusing initial rational terms corrected by sums with merit  $a > 4$ . Impressively fast convergence, with  $a = 12$ , is achieved in this final *pièce de résistance*

$$\frac{11}{\sqrt{3}}\left(\frac{1}{\pi} - \frac{7}{22}\right) = 4(1 + \sqrt{3})\mathcal{S}(\sqrt{3}) - 3(6 + \sqrt{3})\left(\mathcal{S}(\tfrac{3}{2}\sqrt{3}) - 4\mathcal{S}(6\sqrt{3})\right) \\ - 8(5 - 2\sqrt{3})\mathcal{S}(2\sqrt{3}) - 18(1 + 2\sqrt{3})\mathcal{S}(3\sqrt{3}). \quad (117)$$

## 5 Comments and conclusions

On 12 December 2023, Simon Plouffe kindly informed me by email of his notable discoveries (6,13,29), which are members of Families A, B and D. Setting up a

systematic search with `linddep` in `Pari/GP`, I discovered Families C and E. Further search strongly suggested that there are only 5 independent families of relations with integer merit  $a \geq 7$ .

Intrigued by this apparent oligopoly, reminiscent of the mafiosi of New York in 1931, I decided to investigate the 5 known families, up to  $\zeta(127)$ , seeking explicit formulæ for their rational coefficients. This resulted in recurrence relations for intriguing integer sequences, solved in (10,18,26,32,38). Then (88) gave formulæ in (92,95,96,98,99) for coefficients of  $\pi^k$ . In the first instance, this was done by guesswork, based on correspondences between the  $\zeta(k)$  and  $\pi^k$  terms in corrections to modularity for Lambert series. These successful guesses for the arguments of the function  $\mathcal{H}_n(z)$  in (93) were the key to subsequent proofs, based on  $\sum_{r \in \mathbf{Z}_n^\times} P_k(z + \frac{r}{n})$  for groups of units modulo  $n \in \{2, 3, 4, 5, 6, 7, 8, 10, 12\}$ .

Here I was guided by work by Spencer Bloch, Matt Kerr and Pierre Vanhove [8, 9] on Feynman integrals in quantum field theory, where one encounters modular forms that can be written as infinite sums of  $\text{Li}_k(q^n)$  with  $k = 2$  in [8] and  $k = 3$  in [9]. In the latter case, Detchat Samart was able to prove a conjecture [12] of mine by using the quasi-modular transformation (87) and doubling relation (90) at  $k = 3$ , in Proposition 1 of [21]. Thus prepared, I was able to construct the proofs in Section 4, with discriminants  $D \in \{-3, -4, -7, -8, -11, -15, -20, -23, -35\}$

At heart, these proofs use elementary decompositions of divisors of 24, namely  $2 = 1 + 1$ ,  $3 = 2 + 1$ ,  $2^2 = 3 + 1 = 5 - 1$ ,  $6 = 5 + 1$ ,  $2^3 = 7 + 1 = 5 + 3$ ,  $12 = 11 + 1 = 7 + 5$  and  $24 = 23 + 1$ . As Kronecker remarked, *die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk*.

In conclusion, I remark that everything in this article is now proved, with the exception of Conjectures 1 to 5, on primes in the denominators of rational numbers produced by sums over products of Bernoulli numbers, and my conjecture that precisely 5 independent families achieve integer merit  $a \geq 7$ . I leave open the question as to whether more than 18 independent families have rational merit  $a > 4$ .

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