CARLEMAN'S INEQUALITY OVER PRIME NUMBERS

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Abstract

Motivated by studying Carleman's inequality over the prime numbers and over the reciprocal of the prime numbers, we consider the sequences $\{C_n\}_{n=1}^{\infty}$ with general term $C_n = \sum_{k=1}^{n}(p_1 \cdots p_k)^{1/k} / \sum_{k=1}^{n} p_k$ and $\{C'_n\}_{n=1}^{\infty}$ defined similarly by replacing the numbers p_k by $1/p_k$ in C_n. Based on the recently obtained results concerning the arithmetic and geometric means of the prime numbers, we obtain asymptotic expansions and also explicit bounds for the sequences C_n and C'_n.

1. Introduction

For positive real numbers $a_1, \ldots, a_n$, Carleman's inequality [6, 11] asserts that

$$\sum_{k=1}^{n}(a_1 \cdots a_k)^{\frac{1}{k}} \leq e \sum_{k=1}^{n} a_k. \tag{1.1}$$

The constant e in the inequality is the best possible, that is, the inequality does not always hold if e is replaced by a smaller number. However, this constant can be improved for some particular sequences $\{a_n\}_{n=1}^{\infty}$. In this paper we study this possibility for sequences of prime numbers and their reciprocals. As usual, let p_k denotes the kth prime number. We define the sequences $\{C_n\}_{n=1}^{\infty}$ and $\{C'_n\}_{n=1}^{\infty}$ respectively by

$$C_n = \frac{\sum_{k=1}^{n}(p_1 \cdots p_k)^{\frac{1}{k}}}{\sum_{k=1}^{n} p_k}, \qquad \text{and} \qquad C'_n = \frac{\sum_{k=1}^{n}\left(\frac{1}{p_1} \cdots \frac{1}{p_k}\right)^{\frac{1}{k}}}{\sum_{k=1}^{n} \frac{1}{p_k}}.$$

We denote by A_n and G_n the arithmetic and geometric means of the prime numbers $p_1, \ldots, p_n$, respectively. It is known [13] that

$$A_n = \frac{p_n}{2} + O(n), \qquad \text{and} \qquad G_n = \frac{p_n}{e} + O(n).$$

Also, we write $B_n = \sum_{k=1}^{n} p_k^{-1}$, for which we have $B_n = \log\log n + O(1)$. These approximations and the prime number theorem in the form $p_n \sim n \log n$ as $n \to \infty$ imply that

$$C_n = \frac{\sum_{k=1}^{n} G_k}{nA_n} = \frac{\sum_{k=1}^{n} (p_k + O(k))}{enA_n} = \frac{nA_n + O(n^2)}{enA_n} = \frac{1}{e} + O\left(\frac{1}{\log n}\right). \quad (1.2)$$

Hence, the constant e of Carleman's inequality over prime numbers is not the best possible. In our first result, we obtain a more precise asymptotic formula of C_n compared with Equation (1.2).

Theorem 1. *As $n \to \infty$, we have*

$$C_n = \frac{1}{e} - \frac{1}{e \log n} + \frac{\log\log n - 3}{e \log^2 n} + O\left(\frac{(\log\log n)^2}{\log^3 n}\right).$$

Based on some explicit bounds concerning A_n and G_n in [13], and the nth prime number p_n in [14], recently the second author [12] showed that the inequalities

$$\frac{1}{e} - \frac{4}{\log n} < C_n < \frac{1}{e} + \frac{4}{\log n} \quad (1.3)$$

hold for every integer $n \geq 2$. Our first goal is to improve the inequalities given in Equation (1.3) in the direction of Theorem 1. In order to do this we use some results of [4] to get the following lower and upper bound for C_n.

Theorem 2. *We have*

$$\frac{1}{e} - \frac{1}{e \log n} + \frac{\log\log n - 5.485}{e \log^2 n} < C_n < \frac{1}{e} - \frac{1}{e \log n} + \frac{\log\log n - 0.345}{e \log^2 n},$$

where the left-hand side inequality holds for every integer $n \geq 2$ and the right-hand side inequality is valid for every integer $n \geq 55$.

To study Carleman's inequality over the reciprocal of the prime numbers, first we observe that Equation (1.1) asserts that

$$C'_n \leq e$$

for every positive integer n. Considering the approximations $G_n = p_n/e + O(n)$ and $p_n \sim n \log n$, we get $G_n^{-1} = e/p_n + O(1/n\log^2 n)$. Hence,

$$C'_n = \frac{\sum_{k=1}^{n} G_k^{-1}}{B_n} = \frac{eB_n + O(1)}{B_n} = e + O\left(\frac{1}{\log\log n}\right).$$

Thus, the constant e in Equation (1.1) is the best possible for the sequence $\{C'_n\}_{n=1}^{\infty}$. We give the following explicit estimates for this sequence.

Theorem 3. *We have*

$$e - \frac{1.14951}{\log\log n} < C'_n < e - \frac{0.71884}{\log\log n}, \tag{1.4}$$

where the left-hand side inequality holds for every integer $n \geq 4$ and the right-hand side inequality is valid for every integer $n \geq 6$.

Finally, we mention that computations running over the values of C_n and C'_n lead us to formulate the following conjecture.

Conjecture 1. The sequence $\{C_n\}_{n=1}^{\infty}$ is strictly increasing for $n \geq 298$. Also, the sequence $\{C'_n\}_{n=1}^{\infty}$ is strictly increasing for $n \geq 1$.

2. Proof of Theorem 1

In the following proof of Theorem 1, we use some recent asymptotic results, due to the first author, concerning the sum of the first n prime numbers and the sums

$$\sum_{k=1}^{n} \frac{p_k}{\log^j p_k},$$

where $j \in \{1, 2, 3\}$.

Proof of Theorem 1. By [5, Theorem 1.4], we have

$$\sum_{k=1}^{n} p_k = \frac{n^2}{2}\left(\log n + \log_2 n - \frac{3}{2} + \frac{\log_2 n - \frac{5}{2}}{\log n} - \frac{(\log_2 n)^2 - 7\log_2 n + \frac{29}{2}}{2\log^2 n} + s(n)\right),$$

where $\log_2 x = \log\log x$ and

$$s(n) = O\left(\frac{(\log_2 n)^3}{\log^3 n}\right).$$

Further, we use the power series for $\exp(x)$ to obtain

$$\exp\left(\frac{1}{\log p_k} + \frac{3}{\log^2 p_k} + \frac{13}{\log^3 p_k}\right) = 1 + \frac{1}{\log p_k} + \frac{7}{2\log^2 p_k} + \frac{97}{6\log^3 p_k} + O\left(\frac{1}{\log^4 p_k}\right).$$

If we combine this with [4, Proposition 2.5], we get

$$eG_k = p_k\left(1 - \frac{1}{\log p_k} - \frac{5}{2\log^2 p_k} - \frac{61}{6\log^3 p_k}\right) + O\left(\frac{p_k}{\log^4 p_k}\right).$$

Hence

$$(eC_n - 1)\sum_{k=1}^{n} p_k$$
$$= -\sum_{k=1}^{n} \frac{p_k}{\log p_k} - \frac{5}{2}\sum_{k=1}^{n} \frac{p_k}{\log^2 p_k} - \frac{61}{6}\sum_{k=1}^{n} \frac{p_k}{\log^3 p_k} + O\left(\sum_{k=1}^{n} \frac{p_k}{\log^4 p_k}\right).$$

Now we use the asymptotic expansions obtained in [3, p. 9] to see that

$$(eC_n - 1)\sum_{k=1}^{n} p_k = \frac{n^2}{2}\left(-1 - \frac{3}{2\log n} + \frac{18\log_2 n - 89}{12\log^2 n} + r(n)\right),$$

where

$$r(n) = O\left(\frac{(\log_2 n)^2}{\log^3 n}\right).$$

It follows that

$$eC_n - 1 = \frac{-1 - \frac{3}{2\log n} + \frac{18\log_2 n - 89}{12\log^2 n} + r(n)}{\log n + \log_2 n - \frac{3}{2} + \frac{\log_2 n - \frac{5}{2}}{\log n} - \frac{(\log_2 n)^2 - 7\log_2 n + \frac{29}{2}}{2\log^2 n} + s(n)}.$$

A straightforward but exhausting calculation shows that

$$eC_n - 1 = -\frac{1}{\log n} + \frac{\log\log n - 3}{\log^2 n} + O\left(\frac{(\log\log n)^2}{\log^3 n}\right).$$

This completes the proof. □

3. Proof of Theorem 2

In order to prove Theorem 2, we first note the following estimates for the sum of the first n prime numbers. The first one is due to Dusart [8, Lemme 1.7] and the second one due to the first author [5, Corollary 9.1].

Lemma 1 ([8, 5])**.** *For every integer $n \geq 305\,494$, we have*

$$\sum_{k=1}^{n} p_k > \frac{n^2}{2}\left(\log n + \log\log n - \frac{3}{2}\right), \tag{3.1}$$

and for every integer $n \geq 115\,149$, we have

$$\sum_{k=1}^{n} p_k < \frac{n^2}{2}\left(\log n + \log\log n - \frac{3}{2} + \frac{\log\log n - 5/2}{\log n}\right).$$

In order to find the explicit estimates for C_n stated in Theorem 2, we also note the following identities concerning the sums

$$\sum_{p\le x}\frac{1}{p\log p}, \qquad \text{and} \qquad \sum_{p\le x}\frac{1}{p\log^2 p}.$$

Lemma 2. *For every $x \geq 2$, we have*

$$\sum_{p\le x}\frac{1}{p\log p}=\frac{\pi(x)}{x\log x}+\int_2^x \pi(t)\left(\frac{1}{t^2\log t}+\frac{1}{t^2\log^2 t}\right)\,dt, \tag{3.2}$$

and

$$\sum_{p\le x}\frac{1}{p\log^2 p}=\frac{\pi(x)}{x\log^2 x}+\int_2^x \pi(t)\left(\frac{1}{t^2\log^2 t}+\frac{2}{t^2\log^3 t}\right)\,dt. \tag{3.3}$$

Proof. We set $y = 3/2$, $g(t) = 1/(t\log t)$, and $a(n) = \mathbf{1}_{\mathbb{P}}(n)$ in [1, Theorem 4.2] to obtain the first identity. In order to prove Equation (3.3), we set $y = 3/2$, $g(t) = 1/(t\log^2 t)$, and $a(n) = \mathbf{1}_{\mathbb{P}}(n)$ in [1, Theorem 4.2]. $\square$

Remark 1. The prime number theorem implies that the series

$$\sum_{p\in\mathbb{P}}\frac{1}{p\log p}, \qquad \text{and} \qquad \sum_{p\in\mathbb{P}}\frac{1}{p\log^2 p},$$

both converge. More precisely, Cohen [7, p. 6] showed that

$$\sum_{p\in\mathbb{P}}\frac{1}{p\log p}=1.63661632335126086856965800392186367118\ldots. \tag{3.4}$$

Further, Cohen used the method investigated in [7] to compute

$$\sum_{p\in\mathbb{P}}\frac{1}{p\log^2 p}=1.52097043993950086346142862861557952208\ldots. \tag{3.5}$$

In the following proof of Theorem 2, we use Lemma 1 and an upper bound for the logarithmic integral $\mathrm{li}(x)$ which is defined for $x > 1$ as

$$\mathrm{li}(x)=\int_0^x\frac{\mathrm{d}t}{\log t}=\lim_{\varepsilon\to 0+}\left\{\int_0^{1-\varepsilon}\frac{\mathrm{d}t}{\log t}+\int_{1+\varepsilon}^x\frac{\mathrm{d}t}{\log t}\right\}.$$

Proof of Theorem 2. First we show that the inequality

$$C_n>\frac{1}{e}-\frac{1}{e\log n}+\frac{\log\log n-5.485}{e\log^2 n}$$

holds for every integer $n \geq 2$. Let $n_0 = 400\,765$ and consider first the case where $n \geq n_0$. Using [4, Corollary 5.7], we have

$$\sum_{k=1}^{n} G_k > G_1 + G_2 + \frac{1}{e}\sum_{k=1}^{n} p_k - \frac{2}{e} - \frac{1}{e}\sum_{k=1}^{n} k - \frac{3.74}{e}\sum_{k=3}^{n}\frac{k}{\log p_k}.$$

Since $G_1 + G_2 > 2/e$ and $\log p_k > \log k$, we get

$$\sum_{k=1}^{n} G_k > \frac{1}{e}\sum_{k=1}^{n} p_k - \frac{n^2}{2e} - \frac{n}{2e} - \frac{3.74}{e}\sum_{k=3}^{n}\frac{k}{\log k}. \tag{3.6}$$

Note that the function $x/\log x$ is increasing for every $x \geq e$. We combine this with [8, Lemme 1.6] to get

$$\sum_{k=3}^{n}\frac{k}{\log k} \leq \int_{3}^{n+1}\frac{x}{\log x}\,\mathrm{d}x = \mathrm{li}((n+1)^2) - \mathrm{li}(9).$$

By the mean value theorem, there exists a real number $\xi \in (n^2, (n+1)^2)$ such that $\mathrm{li}((n+1)^2) - \mathrm{li}(n^2) = (2n+1)/\log\xi$. Thus, we obtain $\mathrm{li}((n+1)^2) - \mathrm{li}(n^2) < (n+1/2)/\log n$ for any $n > 1$, and so

$$\sum_{k=3}^{n}\frac{k}{\log k} \leq \int_{3}^{n+1}\frac{x}{\log x}\,\mathrm{d}x < \mathrm{li}(n^2) - \mathrm{li}(9) + \frac{n}{\log n} + \frac{1}{2\log n}.$$

Since $\mathrm{li}(9) > 1/(2\log n)$, we obtain

$$\sum_{k=3}^{n}\frac{k}{\log k} < \mathrm{li}(n^2) + \frac{n}{\log n}. \tag{3.7}$$

Applying this inequality to Equation (3.6), we see that

$$\sum_{k=1}^{n} G_k > \frac{1}{e}\sum_{k=1}^{n} p_k - \frac{n^2}{2e} - \frac{n}{2e} - \frac{3.74}{e}\left(\mathrm{li}(n^2) + \frac{n}{\log n}\right).$$

Now we use the fact that $\mathrm{li}(x) < x/\log x + 1.09x/\log^2 x$ for every $x \geq n_0^2$ to obtain the inequality

$$\sum_{k=1}^{n} G_k > \frac{1}{e}\sum_{k=1}^{n} p_k - \frac{n^2}{2e} - \frac{n}{2e} - \frac{3.74n^2}{2e\log n} - \frac{1.01915n^2}{e\log^2 n} - \frac{3.74n}{e\log n}.$$

Since $C_n = \sum_{k=1}^{n} G_k / \sum_{k=1}^{n} p_k$, we can use Equation (3.1) to get

$$C_n > \frac{1}{e} - \frac{1}{e(\log n + \log\log n - 3/2)} - \frac{1}{en\log n} - \frac{3.74}{e\log^2 n} - \frac{2.0383}{e\log^3 n} - \frac{7.48}{en\log^2 n}. \tag{3.8}$$

Note that

$$-\frac{1}{\log n + \log\log n - \frac{3}{2}} \geq -\frac{1}{\log n} + \frac{\log\log n - \frac{3}{2}}{\log^2 n} - \frac{(\log\log n - \frac{3}{2})^2}{\log^3 n}.$$

Applying this inequality to Equation (3.8), we see that

$$\begin{aligned} C_n > \frac{1}{e} - \frac{1}{e\log n} + \frac{\log\log n - 5.24}{e\log^2 n} & \\ - \frac{(\log\log n - 3/2)^2 + 2.0383}{e\log^3 n} - \frac{1}{en\log n} - \frac{7.48}{en\log^2 n}. & \qquad (3.9) \end{aligned}$$

Finally, we apply the inequality $0.245 \geq ((\log\log x - 3/2)^2 + 2.0383)/\log x + (\log x + 7.48)/x$, which holds for every $x \geq n_0$, to Equation (3.9) and get the required inequality for every integer $n \geq n_0$. A computer check for smaller values of n completes the proof.

Next, we prove that the inequality

$$C_n < \frac{1}{e} - \frac{1}{e\log n} + \frac{\log\log n - 0.345}{e\log^2 n}$$

is valid for every integer $n \geq 55$. First, we set $n_1 = 387\,572$ and consider the case where $n \geq n_1$. Using [4, Corollary 5.4] and a computer, we have

$$G_k < \frac{p_k}{e} - \frac{k}{e} + \frac{1.1k}{e\log p_k}$$

for every integer $k \geq 47$. A direct computation shows that

$$\sum_{k=1}^{46} G_k - \frac{1}{e}\sum_{k=1}^{46} p_k + \frac{1}{e}\sum_{k=1}^{46} k - \frac{1.1}{e}\sum_{k=1}^{46}\frac{k}{\log p_k} \leq 28.602$$

and it follows that

$$\sum_{k=1}^{n} G_k < 28.602 + \frac{1}{e}\sum_{k=1}^{n} p_k - \frac{1}{e}\sum_{k=1}^{n} k + \frac{1.1}{e}\left(\frac{1}{\log 2} + \frac{2}{\log 3} + \sum_{k=3}^{n}\frac{k}{\log k}\right).$$

Now we can use Equation (3.7) to get

$$\sum_{k=1}^{n} G_k < 28.602 - \frac{n}{2e} + \frac{1.1n}{e\log n} + \frac{1.1}{e\log 2} + \frac{2.2}{e\log 3} + \frac{1}{e}\sum_{k=1}^{n} p_k - \frac{n^2}{2e} + \frac{1.1}{e}\,\mathrm{li}(n^2).$$

We have $x/(2e) \geq 28.602 + 1.1x/(e\log x) + 1.1/(e\log 2) + 2.2/(e\log 3)$ for any $x \geq 269$. Hence

$$\sum_{k=1}^{n} G_k < \frac{1}{e}\sum_{k=1}^{n} p_k - \frac{n^2}{2e} + \frac{1.1}{e}\,\mathrm{li}(n^2).$$

We apply the inequality $\mathrm{li}(x) < 1.05x/\log x$, which holds for every $x \geq n_1^2$, to obtain

$$\sum_{k=1}^{n} G_k < \frac{1}{e}\sum_{k=1}^{n} p_k - \frac{n^2}{2e} + \frac{1.155n^2}{2e\log n}.$$

Now we use the definition of C_n and the inequalities stated in Lemma 1 to get

$$C_n < \frac{1}{e} - \frac{1}{e(\log n + \log\log n - \frac{3}{2} + \frac{\log\log n - \frac{5}{2}}{\log n})} + \frac{1.155}{e\log n(\log n + \log\log n - \frac{3}{2})}.$$

Since $n \geq n_1 > \exp(\exp(2.5))$, we have

$$-\frac{1}{\log n + \log\log n - \frac{3}{2} + \frac{\log\log n - \frac{5}{2}}{\log n}} < -\frac{1}{\log n} + \frac{\log\log n - \frac{3}{2}}{\log^2 n} + \frac{\log\log n - \frac{5}{2}}{\log^2 n(\log n + \log\log n - \frac{3}{2})}.$$

Furthermore, the identity

$$\frac{1.155}{\log n(\log n + \log\log n - \frac{3}{2})} = \frac{1.155}{\log^2 n} - \frac{1.155(\log\log n - \frac{3}{2})}{\log^2 n(\log n + \log\log n - \frac{3}{2})}$$

holds. So we get

$$C_n < \frac{1}{e} - \frac{1}{e\log n} + \frac{\log\log n - 0.345}{e\log^2 n} + r_n,$$

where

$$r_n = -\frac{0.155\log\log n + 0.7675}{e\log^2 n(\log n + \log\log n - \frac{3}{2})} < 0.$$

This implies the desired result for every integer $n \geq n_1$. For the remaining cases of n we use a computer. This completes the proof. $\square$

We get the following two corollaries.

Corollary 1. *For every integer $n \geq 43$, we have*

$$C_n < \frac{1}{e} - \frac{1}{e\log n} + \frac{\log\log n}{e\log^2 n}.$$

Proof. If $n \geq 55$, this is a consequence of Theorem 2. We conclude by direct computation. $\square$

Corollary 2. *For every integer $n \geq 14$, we have*

$$C_n < \frac{1}{e}.$$

Proof. From Corollary 1 follows that the required inequality holds for every integer $n \geq 43$. A computer check for smaller values of n completes the proof. $\square$

4. Proof of Theorem 3

We use Lemma 2 and the identities given in Equation (3.4) and Equation (3.5) combined with an explicit lower bound for the prime counting function $\pi(x)$ due to Dusart [10] to give the following proof of Theorem 3.

Proof of Theorem 3. We start with the proof of the left-hand side inequality of Equation (1.4) and first consider the case where $n \geq r = 8597$. Using [4, Proposition 5.1], we get

$$\sum_{k=1}^{n} G_k^{-1} > \sum_{k=1}^{r-1} G_k^{-1} + e \sum_{k=r}^{n} \frac{\exp\left(\frac{1}{\log p_k} + \frac{2.7}{\log^2 p_k}\right)}{p_k}.$$

Now we apply the inequality $e^x \geq 1 + x$, which holds for every $x \in \mathbb{R}$, to obtain

$$\sum_{k=1}^{n} G_k^{-1} > \delta_0 + e \sum_{k=1}^{n} \frac{1}{p_k} + e \sum_{k=r}^{n} \frac{1}{p_k \log p_k} + 2.7e \sum_{k=r}^{n} \frac{1}{p_k \log^2 p_k}, \tag{4.1}$$

where the constant δ_0 is given by $\delta_0 = \sum_{k=1}^{r-1} G_k^{-1} - e \sum_{k=1}^{r-1} p_k^{-1}$. Let $x_0 = p_r = 88789$. By [10, Corollary 5.2], for $x \geq x_0$ we have

$$\pi(x) \geq \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2x}{\log^3 x}. \tag{4.2}$$

Applying Equation (4.2) to Equation (3.2), we obtain

$$\sum_{k=r}^{n} \frac{1}{p_k \log p_k} = \sum_{k=1}^{n} \frac{1}{p_k \log p_k} - \sum_{k=1}^{r-1} \frac{1}{p_k \log p_k} \geq \alpha_0 - \frac{1}{\log p_n} + \frac{3}{2 \log^4 p_n}$$

for $n \geq r$, where

$$\alpha_0 = -\sum_{k=1}^{r-1} \frac{1}{p_k \log p_k} + \int_2^{x_0} \pi(t) \left(\frac{1}{t^2 \log t} + \frac{1}{t^2 \log^2 t} \right) \mathrm{d}t + \frac{1}{\log x_0} + \frac{1}{\log^2 x_0} + \frac{1}{\log^3 x_0} + \frac{1}{2 \log^4 x_0}.$$

A direct computation shows that $\alpha_0 \geq 0.087676913224$. Hence

$$\sum_{k=r}^{n} \frac{1}{p_k \log p_k} \geq 0.087676913224 - \frac{1}{\log p_n}. \tag{4.3}$$

Similarly, we combine Equation (4.2) and Equation (3.3) to get

$$\sum_{k=r}^{n} \frac{1}{p_k \log^2 p_k} \geq 0.00384517884595949 - \frac{1}{2 \log^2 p_n}. \tag{4.4}$$

Now we apply Equation (4.3) and Equation (4.4) to Equation (4.1). Thus,

$$\sum_{k=1}^{n} G_k^{-1} > e\sum_{k=1}^{n} \frac{1}{p_k} + \delta_0 + 0.0980588961e - \frac{e}{\log p_n} - \frac{2.7e}{2\log^2 p_n}.$$

Since $\delta_0 > -1.1492179413$ and $e/\log x_0 + 2.7e/(2\log^2 x_0) < 0.26683761711$, we obtain

$$\sum_{k=1}^{n} G_k^{-1} > e\sum_{k=1}^{n} \frac{1}{p_k} - 1.14951.$$

Using [2, Proposition 7] and the definition of C_n', we get the left-hand side inequality of Equation (1.4) for every $n \geq r$. A computer check shows that this inequality also holds for every integer n with $3 \leq n \leq r-1$.

Next, we prove that the right-hand side inequality of Equation (1.4) holds for every integer $n \geq 6$. First let $n \geq s = 406\,161$. We use [4, Proposition 5.6] to see that

$$\sum_{k=1}^{n} G_k^{-1} < \alpha_1 + e\sum_{k=1}^{n} \frac{1}{p_k} + e\sum_{k=s}^{n} \frac{1}{p_k \log p_k} + 6.83e\sum_{k=s}^{n} \frac{1}{p_k \log^2 p_k},$$

where $\alpha_1 = \sum_{k=1}^{s-1} G_k^{-1} - e\sum_{k=1}^{s-1} p_k^{-1}$. By Equation (3.4), we have

$$\sum_{k=s}^{n} \frac{1}{p_k \log p_k} \leq \sum_{p\in\mathbb{P}} \frac{1}{p\log p} - \sum_{k=1}^{s-1} \frac{1}{p_k \log p_k} \leq 0.064143634391656,$$

and from Equation (3.5) it follows that

$$\sum_{k=s}^{n} \frac{1}{p_k \log^2 p_k} \leq \sum_{p\in\mathbb{P}} \frac{1}{p\log^2 p} - \sum_{k=1}^{s-1} \frac{1}{p_k \log^2 p_k} \leq 0.002057210885594.$$

If we combine this with the fact that $\alpha_1 < -1.059057099768616$, we get

$$C_n' < e - \frac{0.8465}{\sum_{k=1}^{n} p_k^{-1}}.$$

By Dusart [9, Théorème 2], we have

$$\sum_{k=1}^{n} \frac{1}{p_k} < \log\log p_n + B + \frac{1}{10\log^2 p_n} + \frac{4}{15\log^3 p_n}. \tag{4.5}$$

Here B denotes the Mertens' constant and is defined by

$$B = \gamma + \sum_{p\in\mathbb{P}} \left(\log\left(1 - \frac{1}{p}\right) + \frac{1}{p}\right) = 0.2614972128476427837554268386\ldots,$$

where $\gamma = 0.57721\ldots$ denotes the Euler–Mascheroni constant. Rosser and Schoenfeld showed [14, p. 69] that $p_n < n(\log n + \log\log n)$. Applying this inequality to Equation (4.5), we see that

$$\sum_{k=1}^{n} \frac{1}{p_k} < \log\log n + \log\left(1 + \frac{\log(\log n + \log\log n)}{\log n}\right) + B + \frac{1}{10\log^2 p_n} + \frac{4}{15\log^3 p_n}. \tag{4.6}$$

A simple calculation shows that the right-hand side of Equation (4.6) is less than $\log\log n + 0.454329$ which gives

$$C_n' < e - \frac{0.8465}{\log\log n + 0.454329}.$$

Now it suffices to apply the inequality

$$-\frac{0.8465}{\log\log n + 0.454329} < -\frac{0.71884}{\log\log n}$$

to complete the proof of the right-hand side inequality of Equation (1.4) for every integer $n \geq s$. For smaller values of n, we check the required inequality by direct computation. $\square$

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