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Abstract. This is a summary of Don Reble's mail to the `list.seqfan.eu` mailing list on 9 Sep 2022 discussing absence of infinite primality chains of the form $n2^k + 1$ and $n + 2^k$. I am merely acting as a secretary here, writing this down; all credit is to Don Reble. (R. J. Mathar)

1. S3 sequence

The sequence S3 in [1, A076336] is defined by the set of integers n such that $n2^k + 1$ is prime for all $k \geq 0$. This set is empty, because for all $n > 0$ there is a $k > 0$ such that $n2^k + 1$ is composite.

Proof. There are two cases

- $n + 1$ is not a power of 2. Therefore $n + 1$ has an (at least one) odd prime factor q, and $n \equiv mq - 1$. Let $k \equiv q - 1$. Then

$$n2^k + 1 = (mq - 1)2^{q-1} + 1. \tag{1}$$

  Reducing the factors of the right hand side (RHS) modulo q gives $mq-1 \equiv_q -1$ and by Fermat's little theorem $2^{q-1} \equiv_q 1$ (see e.g. [1, A177023] for all odd q):

$$n2^k + 1 \equiv_q -1 \times 1 + 1 = 0. \tag{2}$$

  A lower bound is

$$n2^k + 1 \geq (q - 1)2^{3-1} + 1 = 4q - 3 > q. \tag{3}$$

  So q is a proper factor of $n2^k + 1$, and $n2^k + 1$ is composite.
- $n + 1$ is a power of 2. So $n \equiv 2^m - 1$ with $m > 0$. Let $k \equiv m + 2$ and therefore $2^k = 4(n+1)$. In consequence $n2^k+1 = n\cdot 4(n+1)+1 = (2n+1)^2$, composite.

□

2. S4 sequence

The sequence S4 in [1, A076336] is defined by the set of integers n such that $2^k + n$ is prime for all $k \geq 0$. This set is empty, because for all $n > 0$ there is a $k > 0$ such that $n + 2^k$ is composite.

Proof. There are two cases

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- $n+1$ is not a power of 2. Therefore $n+1$ has an (at least one) odd prime factor q, and $n \equiv mq-1$. Let $k \equiv q-1$. Then

$$n + 2^k = mq - 1 + 2^{q-1}. \tag{4}$$

  Reducing the factors of the right hand side (RHS) modulo q gives with the same reasoning as in Section 1

$$n + 2^k \equiv_q -1 + 1 = 0. \tag{5}$$

  A lower bound is

$$n + 2^k \geq q - 1 + 2^{3-1} = q + 3 > q. \tag{6}$$

  So q is a proper factor of $n+2^k$, and $n+2^k$ is composite.
- $n+1$ is a power of 2. So $n \equiv 2^m - 1$ with $m > 0$. There are 3 cases:
  - m is odd, $m \equiv 2r+1$. Let $k=3$; then $n+2^k = 2^{2r+1}-1+2^3 = 2\times 4^r+7$. Reducing the RHS modulo 3 gives $2 \times 4^r \equiv_3 2$, so $n+2^k \equiv_3 0$. A lower bound is

$$n + 2^k \geq 2^1 - 1 + 2^3 = 9 > 3. \tag{7}$$

    So 3 is a proper factor of $n+2^k$, which is composite.
  - m is even with an odd proper factor q: $m \equiv qr$. Let $k=1$; then $n + 2^k = 2^{qr} - 1 + 2^1 = (2^r)^q + 1$. The divisibility properties of cyclotomic polynomials are $s+1 \mid s^q+1$ for odd q, so 2^r+1 is a proper factor of $n+2^k$, which is composite.
  - m is a power of 2, $m = 2^x$ and $n = 2^{2^x} - 1$. The small and general subcases for x are:
    * $x=0$. 13 properly divides $2^{2^x} - 1 + 2^6$.
    * $x=1$. 13 properly divides $2^{2^x} - 1 + 2^{10}$.
    * $x = 2, 4, 6, \ldots$ and even. Then $2^{2^x} \equiv_{13} 3$ and then 13 properly divides $2^{2^x} - 1 + 2^7$ assuming $k=7$.
    * $x = 3, 5, 7, 9\ldots$ and odd. Then $2^{2^x} \equiv_{13} 9$ and then 13 properly divides $2^{2^x} - 1 + 2^9$ assuming $k=9$.

    Here we used that in the sequence 2^{2^x}, $x = 2, 3, 4, 5, \ldots$ each term is the square of the previous, which shows that $2^{2^x} \equiv_{13} = 3, 9, 3, 9, 3\ldots$ with period length 2 since $9^2 \equiv_{13} 3$.

□

References

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