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On Triangular Numbers Which are Sums of Consecutive Squares

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In a recent issue of the *American Mathematical Monthly*, H. E. Thomas, Jr. proposed the following

Problem. Find all integer solutions (n, r) of the equation

$$\sum_{i=1}^{n} i = \sum_{i=1}^{r} i^2.$$

In the present paper we solve Thomas's problem by proving the following:

Theorem. The only integer solutions of the equation

$$\sum_{i=1}^{n} i = \sum_{i=1}^{r} i^2$$

are

$$n = 1, r = 1; \quad n = 10, r = 5; \quad n = 13, r = 6; \quad \text{and} \quad n = 645, r = 85.$$

In a recent issue of the American Mathematical Monthly [4] H. E. Thomas Jr. proposed the following.

PROBLEM. *Find all integer solutions (n, r) of the equation*

$$\sum_{i=1}^{n} i = \sum_{i=1}^{r} i^2. \tag{1}$$

In the present paper we solve Thomas's problem by proving the following

THEOREM. *The only integer solutions of Eq.* (1) *are* $(n, r) = (1, 1)$, $(10, 5)$, $(13, 6)$ and $(645, 85)$.

Proof. Equation (1) may be written

$$3(n^2 + n) = r(r + 1)(2r + 1),$$

and, on setting $X = 2n + 1$, $Y = 2r + 1$, it reduces to solving

$$Y^3 - Y + 3 = 3x^2, \tag{2}$$

with X and Y positive and odd. We now prove that the only integer solutions of (2) with $X > 0$ and X and Y are $(X, Y) = (1, -1)$, $(1, 1)$, $(3, 3)$, $(21, 11)$, $(27, 13)$ and $(1291, 171)$, which will prove our result.

To this end, let us consider the cubic field $Q(\lambda)$ given by

$$\lambda^3 - \lambda + 3 = 0. \tag{3}$$

From a table of cubic fields [3, p. 141] we get the following data on $Q(\lambda)$:

(i) An integral basis is $[1, \lambda, \lambda^2]$;

(ii) The class number is 1;

(iii) A fundamental unit is given by

$$\epsilon_0 = \lambda^2 + \lambda - 1;$$

(iv) The factorization of 3 is given by

$$3 = -\lambda(\lambda + 1)(\lambda - 1).$$

We now write (2) as

$$(Y - \lambda)(Y^2 + \lambda Y + \lambda^2 - 1) = 3Y^2.$$

Any common factor of $Y - \lambda$ and $Y^2 + \lambda Y + \lambda^2 - 1$ must divide

$$[Y^2 + \lambda Y + \lambda^2 - 1] - [(Y + 2\lambda)(Y - \lambda)] = 3\lambda^2 - 1.$$

Since the norm of $3\lambda^2 - 1$ is 239, $3\lambda^2 - 1$ is a prime in $Z[\lambda]$. Thus we conclude that

$$Y - \lambda = \pm(\epsilon_0)^{q_1} (\lambda)^{q_2} (\lambda - 1)^{q_3} \cdot (\lambda - 1)^{q_4} \cdot (3\lambda^2 - 1)^{q_5} \cdot (a + b\lambda + c\lambda^2)^2, \tag{4}$$

where $a, b, c \in Z$ and we may assume that each q_i is 0 or 1.

On taking norms of both sides of (4), we get

$$Y^3 - Y + 3 = \pm(3)^{q_2+q_3+q_4} \cdot (239)^{q_5} \cdot z^2,$$

where $z \in Z$.

Thus we conclude that $q_5 = 0$ and that either two of q_2, q_3, q_4 are 0 and one is 1 or all three are 1. Thus we must consider the following 8 equations:

$$Y - \lambda = \pm\lambda(a + b\lambda + c\lambda^2)^2, \tag{5}$$

$$Y - \lambda = \pm(\epsilon_0)\,\lambda(a + b\lambda + c\lambda^2)^2, \tag{6}$$

$$Y - \lambda = \pm(\lambda - 1)(a + b\lambda + c\lambda^2)^2, \tag{7}$$

$$Y - \lambda = \pm(\epsilon_0)(\lambda - 1)(a + b\lambda + c\lambda^2)^2, \tag{8}$$

$$Y - \lambda = \pm(\lambda + 1)(a + b\lambda + c\lambda^2)^2, \tag{9}$$

$$Y - \lambda = \pm(\epsilon_0)(\lambda + 1)(a + b\lambda + c\lambda^2)^2, \tag{10}$$

$$Y - \lambda = \pm 3(a + b\lambda + c\lambda^2)^2, \tag{11}$$

$$Y - \lambda = \pm 3(\epsilon_0)(a + b\lambda + c\lambda^2)^2. \tag{12}$$

Equation (11) yields

$$\mp(Y - \lambda) = 3(a^2 - 6bc) + 3(2ab + 2bc - 3c^2)\lambda + 3(b^2 + 2ac + c^2)\,\lambda^2,$$

which is impossible. Similarly, Eq. (12) is impossible. Equation (6) yields

$$\begin{aligned} -a^2 - 3b^2 + 6c^2 - 6ac - 6ab &= 0, \\ a^2 - 2b^2 - 2c^2 - 4ac - 6bc &= \mp Y, \\ a^2 - 3c^2 + 2ab - 4bc &= \pm 1. \end{aligned}$$

This yields a odd, since Y is odd. Thus b is odd, c is even and the first equation is impossible modulo 4. Similarly, Eqs. (8) and (10) are impossible.

Let us now consider Eq. (5). We write it as

$$Y\lambda - \lambda^2 = \pm(a + b\lambda + c\lambda^2)^2$$

and get

$$\begin{aligned} a^2 - 6bc &= 0, \\ b^2 + 2ac + c^2 &= \pm 1, \\ 2ab + 2bc - 3c^2 &= \mp Y. \end{aligned}$$

This implies that a is even, c is odd and b is even. Thus the upper sign must be taken, and we get

$$\begin{aligned} a^2 - 6bc &= 0, \\ b^2 + 2ac + c^2 &= 1. \end{aligned} \tag{13}$$

Thus we have two possibilities; either

$$b = 2n^2, \qquad c = 3m^2, \qquad a = 6mn \tag{14}$$

with m odd, or

$$b = 6m^2, \qquad c = n^2, \qquad a = 6mn \tag{15}$$

with n odd.

On substituting (14) into (13), we get

$$4n^4 + 36nm^3 + 9m^4 = 1,$$

which reduces to

$$u^4 - 3078u^2v^2 + 92952uv^3 - 789471v^4 = 1. \tag{16}$$

Further, on substituting (15) into (13), we get

$$n^4 + 12n^3m + 36m^4 = 1,$$

which reduces to

$$u^4 - 54u^2v^2 + 216uv^3 - 207v^4 = 1. \tag{17}$$

Equation (7) may be written

$$\mp 3(Y - \lambda) = (\lambda + \lambda^2)(a + b\lambda + c\lambda^2)^2.$$

This yields

$$b^2 - 2c^2 + 2ac + 2ab + 2bc = \pm Y,$$

$$a^2 - 2b^2 - 5c^2 - 4ac + 2ab - 4bc = \mp 3, \tag{18}$$

$$a^2 + b^2 - 2c^2 + 2ac + 2ab - 4bc = 0. \tag{19}$$

This yields b odd, a odd and c even. Thus the upper sign must be taken, and thus Eqs. (18) and (19) yield

$$b^2 + c^2 + 2ac = 1. \tag{20}$$

Equation (19) may be written

$$(a + b + c)^2 = 3c(c + 2b).$$

Thus there are two possibilities:

$$\text{(i)}\quad c = 6m^2,\ c + 2b = 2n^2,\ a + b + c = 6mn \tag{21}$$

or

$$\text{(ii)}\quad c = 2m^2,\ c + 2b = 6n^2,\ a + b + c = 2mn. \tag{22}$$

On solving (21) for a, b, c and substituting the results in (20), we get

$$n^4 - 18n^2m^2 + 72nm^3 + 9m^4 = 1. \tag{23}$$

On solving (22) for a, b, c and substituting the results in (20), we get

$$m^4 + 24m^3n - 18m^2n^2 + 9n^4 = 1,$$

which reduces to

$$u^4 - 234u^2v^2 + 1944uv^3 - 4527v^4 = 1. \tag{24}$$

Similarly, Eq. (9) reduces to solving the quartic equations

$$n^4 + 18n^2m^2 + 72nm^3 + 9m^4 = 1 \tag{25}$$

and

$$u^4 - 198u^2v^2 + 1512uv^3 - 3231v^4 = 1. \tag{26}$$

Thus the solution of (2) reduces to solving the 6 quartic equations

$$(1, 0, -3078, 92952, -789471) = 1, \tag{16}$$
$$(1, 0, -54, 216, -207) = 1, \tag{17}$$
$$(1, 0, -18, 72, 9) = 1, \tag{23}$$
$$(1, 0, -234, 1944, -4527) = 1, \tag{24}$$
$$(1, 0, 18, 72, 9) = 1, \tag{25}$$

and

$$(1, 0, -198, 1512, -3231) = 1. \tag{26}$$

We now prove the following.

LEMMA. *The only integer solutions of each of these* 6 *quartic equations are* $(\pm 1, 0)$.

Proof. Equation (16) defines the field $Q(\theta)$, where

$$\theta^4 = 3078\theta^2 + 92952\theta + 789471.$$

By the usual method of solving this equation, we find that it has two real and two complex roots, and thus, by the Dirichlet–Minkowski theorem on the group of units, the ring of integers of $Q(\theta)$, $Z(\theta)$, has two fundamental units E_1 and E_2. Since the only roots of unity in $Z(\theta)$ are ± 1, Eq. (16) yields

$$u + v\theta = \pm E_1{}^m E_2{}^n, \tag{27}$$

and we must find all the units in $Z(\theta)$ of this special type. Since all these units lie in the ring $Z[1, \theta, \theta^2, \theta^3]$, Eq. (27) may be rewritten

$$u + v\theta = \pm \epsilon_1{}^m \epsilon_2{}^n, \tag{28}$$

where ϵ_1 and ϵ_2 are fundamental units of *any* subring of $Z(\theta)$.

Since $\theta^4 \equiv 0 \pmod 9$, $\theta^2/3$ is an integer in $Z(\theta)$ and thus the integers $1, \theta, \theta^2/3, \theta^3/3$ generate such a subring of $Z(\theta)$ which we designate by $Z[1, \theta, \theta^2/3, \theta^3/3]$.

By Billevich's algorithm [2], we find that

$$\epsilon_1 = 1724768 + 126173\theta + 1099\theta^2 - 49\theta^3$$

and

$$\epsilon_2 = (2025 + 180\theta + 4\theta^2)/3$$

are fundamental units of $Z(1, \theta, \theta^2/3, \theta^3/3)$. Further, we find

$$\epsilon_1^{-1} = -34318 - 2549\theta - 23\theta^2 + \theta^3,$$
$$\epsilon_2^{-1} = (1129041 + 824766 + 716\theta^2 - 32\theta^3)/3.$$

Thus Eq. (28) may be written

$$\mp(u + v\theta) = (1724768 + 1261730 + 1099\theta^2 - 49\theta^3)^m \cdot \left(\frac{2025 + 180\theta + 4\theta^2}{3}\right)^n. \tag{29}$$

Suppose first that $m \geqslant 0$, $n \geqslant 0$. Then (29) reduces to

$$\mp(u + v\theta) = \alpha_1{}^m \alpha_2{}^n, \tag{30}$$

where $\alpha_1 = \epsilon_1$, $\alpha_2 = 3\epsilon_2$.
Now we find that

$$\alpha_1{}^4 = 1 + 4\xi 1,$$
$$\xi_1 = 1 + 3\theta + 3\theta^2 + \theta^3 + 4A,$$
$$\alpha_2 = 1 + 4\xi_2,$$
$$\xi_2 = 2 + \theta + \theta^2 + 4B,$$

where A and B denote integers belonging to $Z[1, \theta, \theta^2, \theta^3]$.

Thus we set

$$m = 4u + r, \qquad n = 4v + s,$$

where $r = 0, 1, 2, 3$, and $s = 0$.
Considering (30) as a congruence modulo 4, we get

$$V(r, s) \equiv \mp(u + v\theta) \equiv (\theta + 3\theta^2 + 3\theta^3)^r \pmod 4, \tag{31}$$

and computing all possible values of r modulo 4, we find that

$$\begin{aligned} V(0, 0) &\equiv 1, \\ V(1, 0) &\equiv \theta + 3\theta^2 + 3\theta^3, \\ V(2, 0) &\equiv 3 + 2\theta + 2\theta^2 + 2\theta^3, \\ V(3, 0) &\equiv 2 + 3\theta + \theta^2 + \theta^3. \end{aligned}$$

Consequently, (31) holds only for $r = 0$. Thus (30) now becomes

$$\mp(u + v\theta) = (1 + 4\xi_1)^u(1 + 4\xi_2)^v, \tag{32}$$

and since $\left|\begin{smallmatrix} 3 & 1 \\ 1 & 0 \end{smallmatrix}\right|$ is odd, it follows from a result of Avanesov [1, p. 162] that the only solution of (32) is $u = v = 0$.

Thus the only solution of (30) is $m = n = 0$.

Suppose next that $m \leqslant 0$, $n \leqslant 0$. Then we must solve

$$\mp(u + v\theta) = \beta_1^M \beta_2^N, \tag{33}$$

where $\beta_1 = \alpha_1^{-1}$, $\beta_2 = 3\epsilon_2^{-1}$, $M = -m$, $N = -n$. Since α_1 and β_1, as well as α_2 and β_2 are inverses mod 4 the values of $V(r, s)$ we compute will be the same as those computed above, except for order. Consequently, we may assume that $M = 4u$, $N = 4v$.

Now we find that

$$\beta_1^4 = 1 + 4\xi_1',$$

where

$$\xi_1' = 3 + \theta + \theta^2 + 3\theta^3 + 4A',$$

and

$$\beta_2 = 1 + 4\xi_2',$$

where

$$\xi_2' = 3\theta + \theta^2 + 4B'.$$

Since $\left|\begin{smallmatrix} 1 & 3 \\ 1 & 0 \end{smallmatrix}\right|$ is odd, the only solution of (33) is $M = N = 0$. Similarly, the cases $m \leqslant 0$, $n \geqslant 0$ and $m \geqslant 0$, $n \leqslant 0$ yield $m = n = 0$ as the only solutions. Thus the only integer solutions of Eq. (16) are $(u, v) = (\pm 1, 0)$.

In a similar fashion, we can show that the only integer solutions of each of the remaining 5 quartic equations are $(\pm 1, 0)$, and the proof of the lemma is complete.

Finally, we note that the solutions $(\pm 1, 0)$ of the 6 quartics yield

$$(X, Y) = (1291, 71), \quad (3, 3), \quad (1, 1), \quad (27, 13), \quad (1, -1) \quad \text{and} \quad (21, 11),$$

respectively, as the only integer solutions of (2) with $X > 0$ and X and Y, and this completes the proof of the main theorem of this paper.

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