

## An Effective Seven Cube Theorem

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It is proved that every integer exceeding  $\exp(\exp(13.97))$  is a sum of seven positive integral cubes. Some conjectures concerning the representation of integers as a sum of nonnegative integral cubes are also discussed. © 1984 Academic Press, Inc.

### 1. INTRODUCTION

It was proved by Linnik [3] in 1943 that every sufficiently large integer is a sum of seven positive integral cubes. In 1951 Watson [9] gave a much simpler proof for this theorem. Both proofs were ineffective in the sense that they did not yield an estimate for the largest integer that requires eight cubes. The purpose of this paper is to show that by modifying Watson's proof we can prove the following.

**THEOREM.** *Every integer exceeding  $\exp(\exp(13.97))$  is a sum of seven positive integral cubes.*

Note that for eight cubes we have the theorem of Dickson [2], who proved that every integer except 23 and 239 is a sum of eight nonnegative integral cubes.

Let  $G(k)$  be defined as the least positive integer such that every sufficiently large integer is a sum of  $G(k)$  nonnegative integral  $k$ th powers. The theorem of Linnik implies that  $G(3) \leq 7$ . The best known lower bound is  $G(3) \geq 4$ . This follows from the fact that every cube is congruent to 0, 1, or  $-1$  modulo 9. Thus any number of the form  $9m \pm 4$  requires at least four cubes.

Before proving our theorem, it is interesting to take note of some numerical evidence concerning the value of  $G(3)$  and the largest integer requiring eight cubes. Let  $C_k$  denote the set of positive integers that can be expressed as a sum of  $k$  but not fewer cubes. Thus Dickson's result is that  $C_9$  contains only the integers 23 and 239. In 1926, Western [10] published

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numerical data to suggest that  $G(3)$  is four or five, and he also conjectured that the largest element of  $C_6$  does not exceed  $2 \cdot 10^6$ . In 1981, Bohman and Fröberg [1] calculated all elements of  $C_6$ ,  $C_7$ , and  $C_8$  not exceeding 2,685,000 and conjectured among other things that  $C_8$  contains 15 elements, the largest of which is 454,  $C_7$  contains 121 elements, the largest of which is 8042, and  $C_6$  contains 3922 elements, the largest of which is 1,290,740. These calculations have been extended to  $10^7$  by the author [4] and Romani [8], independently. No further elements of  $C_6$ ,  $C_7$ , or  $C_8$  were found.

## 2. PRIMES IN ARITHMETIC PROGRESSIONS

The proofs of the seven cube theorem given by Linnik and Watson both make use of an estimate for the number of primes in an arithmetic progression. If  $k$  and  $l$  are positive integers, we define

$$\theta(x; k, l) = \sum_{\substack{p \leq x \\ p \equiv l \pmod{k}}} \log p,$$

where the sum extends over primes  $p$ . Using a result of Siegel, Walfisz proved that for any positive  $\varepsilon$  there exist constants  $C_1$  and  $C_2$  depending on  $\varepsilon$  such that

$$|\theta(x; k, l) - x/\varphi(k)| < C_1 x \exp(-C_2 \sqrt{\log x}),$$

provided  $(k, l) = 1$  and  $x \geq \exp(k^\varepsilon)$ . From a computational point of view, this result suffers from the defect that for  $\varepsilon < \frac{1}{2}$  there is no known algorithm for the calculation of  $C_2$ . Linnik and Watson both used the Siegel–Walfisz theorem with  $\varepsilon < \frac{1}{2}$  in their proofs and their results were therefore ineffective. For certain values of  $k$  this difficulty can be eliminated, and the method of this paper is to show that Watson’s proof can be modified to take advantage of the “good” moduli.

As in [5], let  $R = 9.645908801$ . We shall refer to  $k$  as an exceptional modulus if there exists a real zero  $\beta$  with  $\beta > 1 - 1/(R \log k)$  for an  $L$ -function formed with a real nonprincipal character modulo  $k$ . The zero  $\beta$  will also be referred to as an exceptional zero. In place of the Siegel–Walfisz theorem, we shall use explicit estimates of the author [6, 7] which we restate here for completeness.

**LEMMA 1.** *If  $k$  is not an exceptional modulus,  $k \geq 10^{25}$ ,  $(k, l) = 1$ , and  $x \geq \exp(12.2 \log^2 k)$ , then*

$$\left| \theta(x; k, l) - \frac{x}{\varphi(k)} \right| < \frac{x}{100 \varphi(k)}.$$

LEMMA 2. If  $x > 0$ , then  $\theta(x; 3, 2) < 0.50933118x$ .

LEMMA 3. If  $x \geq 21317$ , then  $\theta(x; 3, 2) > 0.49595x$ .

In order to show that a modulus is not exceptional, we use the following result from [5].

LEMMA 4. Let  $\chi_1$  and  $\chi_2$  be distinct real primitive nonprincipal characters modulo  $k_1$  and  $k_2$ , respectively, and let  $\beta_i$  be a real zero of  $L(s, \chi_i)$ ,  $i = 1, 2$ . Then

$$\min\{\beta_1, \beta_2\} < 1 - \frac{1}{R_1 \log M_1},$$

where  $R_1 = (5 - \sqrt{5})/(15 - 10\sqrt{2})$  and  $M_1 = \max\{k_1 k_2/17, 13\}$ .

### 3. PROOF OF THE THEOREM

Let  $a = 2^{1/18}$ . We begin with a lemma.

LEMMA 5. Let  $N$  be a positive integer, and  $p, q$ , and  $r$  be primes such that

$$p \equiv q \equiv r \equiv 5 \pmod{6}, \quad (1)$$

$$r < q < ar, \quad (2)$$

$$\frac{1}{4}p^3q^{18} < N < p^3q^{18}, \quad (3)$$

$$N \equiv 3p \pmod{6p}, \quad (4)$$

$$4N \equiv r^{18}p^3 \pmod{q^6}, \quad (5)$$

and

$$2N \equiv q^{18}p^3 \pmod{r^6}. \quad (6)$$

Then  $N$  is a sum of six positive integral cubes.

This is Watson's Lemma 3, except that  $a$  replaces 1.01 in (2). The proof is the same as Watson's.

Let  $n$  be a positive integer to be expressed as a sum of seven cubes, and define

$$y = 6^{-1/12} \exp \left\{ \frac{\sqrt{2.25 + 12c \log n}}{72c} - \frac{1}{48c} \right\}, \quad (7)$$

where  $c = 12.2$ .

LEMMA 6. If  $R_1 = (5 - \sqrt{5})/(15 - 10\sqrt{2})$ ,  $R = 9.645908801$ ,  $b = R_1/(3R)$ , and  $n > \exp(\exp(13.97))$ , then there exist two pairs of primes  $r_1, q_1$ , and  $r_2, q_2$  such that

$$y^b < r_i < q_i \leq y, \quad (8)$$

$$r_i < q_i < ar_i, \quad (9)$$

$$(r_i q_i, n) = 1, \quad (10)$$

and

$$r_i \equiv q_i \equiv 5 \pmod{6}. \quad (11)$$

*Proof.* We first show that it is sufficient to prove that

$$\sum_{\substack{y^b < p \leq y \\ p \equiv 2 \pmod{3} \\ p \nmid n}} \log p \geq \left( \frac{1-b}{\log a} \log y + 4 \right) \log y. \quad (12)$$

Let  $m = [(1-b) \log y / \log a]$ , and consider the intervals  $(ya^{-i}, ya^{1-i}]$ ,  $i = 1, \dots, m$ , and the interval  $(y^b, ya^{-m}]$ . These  $m+1$  intervals are disjoint and their union is the interval  $(y^b, y]$ . From (12) it follows that in the  $m+1$  intervals there are at least  $m+4$  primes satisfying (8), (10), and (11). Hence by the box principle there is either a single interval containing at least four such primes, or two intervals containing at least two such primes. In either case we can choose two pairs of primes satisfying (9), and it then suffices to prove (12).

Note that  $y$  exceeds  $10^7$ ; hence Lemmas 2 and 3 yield

$$\begin{aligned} \sum_{\substack{y^b < p \leq y \\ p \equiv 2 \pmod{3} \\ p \nmid n}} \log p &\geq \theta(y; 3, 2) - \theta(y^b; 3, 2) - \log n \\ &> 0.49595y - 0.50933118y^b - \log n. \end{aligned}$$

The proof of the lemma is completed by verifying that

$$0.49595y - 0.50933118y^b - \log n > \frac{1-b}{\log a} \log^2 y + 4 \log y.$$

We now turn to the proof of our theorem. Let  $r_1, q_1, r_2$ , and  $q_2$  be the primes of Lemma 6, and set  $k_i = 6q_i^6 r_i^6$ ,  $i = 1, 2$ . We now show that one of  $k_1$  or  $k_2$  is not exceptional. Suppose that both are exceptional with exceptional zeros  $\beta_i$  of  $L(s, \chi_i)$ . If  $k_i^*$  is the conductor of  $\chi_i$ , it follows that  $k_i^*$  is an odd squarefree divisor of  $k_i$ , hence is one of  $3, q_i, r_i, 3q_i, 3r_i, q_i r_i$ , or  $3q_i r_i$ . Furthermore,  $k_i^*$  cannot be 3, since the  $L$  function formed with the real

nonprincipal character modulo 3 is known to have no positive zeros. The smallest primes satisfying (9) and (11) are 167 and 173, so that  $k_i^*$  is at least 167. It follows from Lemma 4 that

$$\begin{aligned}\min\{\beta_1, \beta_2\} &< 1 - \frac{1}{R_1 \log(k_1^* k_2^*/17)} \\ &< 1 - \frac{1}{4R_1 \log y}.\end{aligned}\tag{13}$$

On the other hand, if both  $\beta_1$  and  $\beta_2$  are exceptional, then

$$\begin{aligned}\beta_i &> 1 - \frac{1}{R \log k_i} \\ &> 1 - \frac{1}{12bR \log y},\end{aligned}$$

and this is a contradiction to (13). Hence one of  $k_1$  or  $k_2$  is not exceptional; call it  $k = 6q^6 r^6$ .

The rest of the proof follows Watson's. We choose  $h$  so that

$$4n \equiv hr^{18} \pmod{q^6},\tag{14}$$

and

$$2n \equiv hq^{18} \pmod{r^6}.\tag{15}$$

Next choose  $m$  so that

$$m^3 \equiv h \pmod{q^6 r^6}.\tag{16}$$

This is possible since  $q \equiv r \equiv 5 \pmod{6}$ , and every number is a cubic residue modulo  $q^6 r^6$ . We choose  $l$  to satisfy

$$l \equiv m \pmod{q^6 r^6},$$

and

$$l \equiv 5 \pmod{6}.\tag{17}$$

Set  $x = n^{1/3} q^{-6}$ . Then (7) and (8) yield

$$\begin{aligned}\log x &> \frac{1}{3} \log n - \frac{1}{2} \log(6y^{12}) \\ &= c \log^2(6y^{12}), \\ &> c \log^2 k.\end{aligned}$$

Since  $k \geq 6(167 \cdot 173)^6$ , it follows from Lemma 1 that there exists a prime  $p$  such that

$$p \equiv l \pmod{k} \quad (18)$$

and

$$x < p \leq 1.1x.$$

Every number is a cubic residue modulo  $6p$ , so we can choose  $t$  such that

$$t^3 \equiv n - 3p \pmod{6p}, \quad (19)$$

$$t \equiv 0 \pmod{q^2 r^2}, \quad (20)$$

and

$$0 \leq t < 6pq^2 r^2.$$

Let  $N = n - t^3$ . We will be finished if we can show that  $N$  satisfies the conditions of Lemma 5. Note that

$$N \leq n < p^3 q^{18},$$

and also that

$$\begin{aligned} N &\geq n - 216p^3 q^6 r^6 \\ &\geq \frac{p^3 q^{18}}{1.331} - 216p^3 q^6 r^6 \\ &> p^3 q^{18} \left( \frac{1}{1.331} - \frac{216}{q^6} \right) \\ &> \frac{3}{4} p^3 q^{18}, \end{aligned}$$

since  $q \geq 167$ . From (19) we find that  $N \equiv 3p \pmod{6p}$ . From (20), (14), (16), (17), and (18) we obtain

$$\begin{aligned} 4N &\equiv 4n \pmod{q^6} \\ &\equiv hr^{18} \pmod{q^6} \\ &\equiv m^3 r^{18} \pmod{q^6} \\ &\equiv l^3 r^{18} \pmod{q^6} \\ &\equiv p^3 r^{18} \pmod{q^6}, \end{aligned}$$

and similarly from (20), (15), (16), (17), and (18) it follows that

$$2N \equiv p^3 q^{18} \pmod{r^6}.$$

The hypotheses of Lemma 5 are then satisfied, and  $n = t^3 + N$  is a sum of seven cubes. Thus the theorem is proved.

In conclusion we mention that our theorem can be improved slightly. In place of Lemma 1, we can prove the following.

LEMMA 1'. *If  $k$  is not an exceptional modulus,  $k \geq 3.48 \times 10^{27}$ ,  $(k, l) = 1$ , and  $x \geq \exp(11.44 \log^2 k)$ , then*

$$\left| \theta(x; k, l) - \frac{x}{\phi(k)} \right| < \frac{x}{21\phi(k)}.$$

This follows from Theorem 3.9 of [6] by a calculation carried out in a manner described in Section 4 of [6]. The values of  $\eta$  and  $\alpha$  are taken to be 0.085 and 0.071, respectively.

Using Lemma 1' instead of Lemma 1, we can replace  $c$  in (7) by 11.44, and conclude that every integer exceeding  $\exp(\exp(13.89))$  is a sum of seven cubes.

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