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# The truncated pentagonal number theorem

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A new expansion is given for partial sums of Euler's pentagonal number series. As a corollary we derive an infinite family of inequalities for the partition function, $p(n)$.



## 1. Introduction

In [3], the second author produced the fastest known algorithm for the generation of the partitions of $n$. The work required a proof of the following inequality: For $n > 0$

$$p(n) - p(n-1) - p(n-2) + p(n-5) \leqq 0, \tag{1.1}$$

where $p(n)$ is the number of partitions of $n$ [2].

Upon reflection, one expects that there might be an infinite family of such inequalities where (1.1) is the second entry, and the trivial inequality

$$p(n) - p(n-1) \geqq 0 \tag{1.2}$$

is the first.

In this paper, we shall prove:

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**Theorem 1.1.** *For $n > 0$, $k \geqq 1$,*

$$(-1)^{k-1}\sum_{j=0}^{k-1}(-1)^j\big(p\big(n-j(3j+1)/2\big)-p\big(n-j(3j+5)/2-1\big)\big)=M_k(n),$$

*where $M_k(n)$ is the number of partitions of n in which k is the least integer that is not a part and there are more parts $> k$ than there are $< k$.*

**Example.** If $n = 18$, and $k = 3$,

$$p(18)-p(17)-p(16)+p(13)+p(11)-p(6)=3,$$

and $M_3(18)$ equals 3 because the three partitions in questions are

$$5+5+5+2+1, \qquad 6+5+4+2+1, \quad \text{and} \quad 7+4+4+2+1.$$

This result follows directly from

**Lemma 1.2.**

$$\frac{1}{(q;q)_\infty}\sum_{j=0}^{k-1}(-1)^j q^{j(3j+1)/2}\big(1-q^{2j+1}\big)=1+(-1)^{k-1}\sum_{n=1}^{\infty}\frac{q^{\binom{k}{2}+(k+1)n}}{(q;q)_n}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}, \tag{1.3}$$

*where*

$$(A;q)_n=\prod_{j=0}^{\infty}\frac{(1-Aq^j)}{(1-Aq^{j+n})}$$
$$=\big((1-A)(1-Aq)\cdots\big(1-Aq^{n-1}\big) \text{ if } n \text{ is a positive integer}\big)$$

*and*

$$\begin{bmatrix} A \\ B \end{bmatrix}=\begin{cases} 0, & \text{if } B<0 \text{ or } B>A, \\ \frac{(q;q)_A}{(q;q)_B(q;q)_{A-B}}, & \text{otherwise.} \end{cases}$$

**Corollary 1.3.** *For $n > 0$, $k \geqq 1$*

$$(-1)^{k-1}\sum_{j=0}^{k-1}(-1)^j\big(p\big(n-j(3j+1)/2\big)-p\big(n-j(3j+5)/2-1\big)\big)\geqq 0 \tag{1.4}$$

*with strict inequality if $n \geqq k(3k+1)/2$.*

This is immediate from Theorem 1.1 because the smallest number that has a partition counted by $M_k(n)$ is

$$k(3k+1)/2=1+2+\cdots+(k-1)+(k+1)+(k+1)+\cdots+(k+1)$$

and any larger number $N$ has at least one such partition, namely

$$1+2+\cdots+(k-1)+(k+1)+(k+1)+\cdots+\big(k+1+N-k(3k+1)/2\big).$$

We note that (1.1) is the case $k = 2$ and (1.2) is the case $k = 1$. In the final section of the paper, we note the relationship of this result to D. Shanks's formula for the truncated pentagonal number series [4].

## 2. Proof of Lemma 1.2

Denote the left side of (1.3) by $L_k$ and the right side by $R_k$.
Clearly

$$L_1 = \frac{1-q}{(q;q)_\infty} = \frac{1}{(q^2;q)_\infty} = \sum_{n=0}^{\infty} \frac{q^{2n}}{(q;q)_n} = R_1,$$

where we have invoked [2, p. 19, Eq. (2.2.5)]. Thus Lemma 1.2 is true when $k = 1$.
It is immediate from the definition that

$$L_{k+1} - L_k = \frac{(-1)^k q^{k(3k+1)/2}(1-q^{2k+1})}{(q;q)_\infty}.$$

On the other hand, for $k > 1$, we see by [2, p. 35, Eq. (3.3.4)], that

$$\begin{aligned}
R_{k+1} - R_k &= (-1)^k \sum_{n=1}^{\infty} \frac{q^{\binom{k+1}{2}+(k+2)n}}{(q;q)_n} \begin{bmatrix} n-1 \\ k \end{bmatrix} + (-1)^k \sum_{n=1}^{\infty} \frac{q^{\binom{k}{2}+(k+1)n}}{(q;q)_n} \left( \begin{bmatrix} n \\ k \end{bmatrix} - q^k \begin{bmatrix} n-1 \\ k \end{bmatrix} \right) \\
&= (-1)^k \sum_{n=1}^{\infty} \frac{q^{\binom{k}{2}+(k+1)n}}{(q;q)_n} \begin{bmatrix} n \\ k \end{bmatrix} - (-1)^k \sum_{n=1}^{\infty} \frac{q^{\binom{k+1}{2}+(k+1)n}}{(q;q)_{n-1}} \begin{bmatrix} n-1 \\ k \end{bmatrix} \\
&= (-1)^k \sum_{n=k}^{\infty} \frac{q^{\binom{k}{2}+(k+1)n}}{(q;q)_k (q;q)_{n-k}} - (-1)^k \sum_{n=k+1}^{\infty} \frac{q^{\binom{k+1}{2}+(k+1)n}}{(q;q)_k (q;q)_{n-k-1}} \\
&= \frac{(-1)^k q^{k(3k+1)/2}(1-q^{2k+1})}{(q;q)_k} \sum_{n=0}^{\infty} \frac{q^{n(k+1)}}{(q;q)_n} \\
&= \frac{(-1)^k q^{k(3k+1)/2}(1-q^{2k+1})}{(q;q)_\infty}.
\end{aligned}$$

Thus $L_1 = R_1$ and both sequences satisfy the same first order recurrence. So for $k \geqq 1$,

$$L_k = R_k$$

and Lemma 1.2 is proved.

## 3. Proof of Theorem 1.1

We see by Lemma 1.2 that the generating function for

$$(-1)^{k-1} \sum_{j=0}^{k-1} (-1)^j \big( p\big(n - j(3j+1)/2\big) - p\big(n - j(3j+5)/2 - 1\big)\big)$$

is

$$\begin{aligned}
(-1)^{k-1} L_k &= (-1)^{k-1} R_k \\
&= (-1)^{k-1} + \sum_{n=1}^{\infty} \frac{q^{\binom{k}{2}+(k+1)n}}{(q;q)_n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.
\end{aligned} \tag{3.1}$$

Now

$$\frac{q^{(k+1)n}}{(q;q)_n}$$

is the generating function for partitions into $n$ parts each $>k$ (cf. [2, Sec. 1.2]). In addition

$$q^{\binom{k}{2}}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$$

is the generating function for partitions into $k-1$ distinct parts each $\leqq n-1$ (cf. [2, Thm. 3.3]) and by conjugation, this is the generating function for partitions into at most $n-1$ parts with every integer $<k$ appearing as a part. Hence the series in (3.1) is the generating function for $M_k(n)$.

## 4. Shanks's formula

In [4], D. Shanks proved that

$$1+\sum_{j=1}^{k}(-1)^j\left(q^{j(3j-1)/2}+q^{j(3j+1)/2}\right)=\sum_{j=0}^{k}\frac{(-1)^j(q;q)_k q^{jk+\binom{j+1}{2}}}{(q;q)_j}. \tag{4.1}$$

We note that the left-hand side of (4.1) has $(2k+1)$ terms of the pentagonal number series while the numerator of $L_{k+1}$ has $2k$ terms. As we will see, it is possible to deduce from Theorem 1 a companion to (4.1) treating the case with an even number of terms.

**Theorem 4.1.**

$$\sum_{j=0}^{k}(-1)^j q^{j(3j+1)/2}\left(1-q^{2j+1}\right)=\sum_{j=0}^{k}\frac{(-1)^j(q;q)_{k+1}q^{(k+2)j+\binom{j}{2}}}{(q;q)_j}. \tag{4.2}$$

**Proof.** By Lemma 1.2 (with $k$ replaced by $k+1$),

$$\begin{aligned}
&\sum_{j=0}^{k}(-1)^j q^{j(3j+1)/2}\left(1-q^{2j+1}\right)\\
&\quad=(q;q)_\infty\left(1+(-1)^k\sum_{n=1}^{\infty}\frac{q^{\binom{k+1}{2}+(k+2)n}(q^{n-k};q)_k}{(q;q)_n(q;q)_k}\right)\\
&\quad=(q;q)_\infty(-1)^k\sum_{n=0}^{\infty}\frac{q^{\binom{k+1}{2}+(k+2)n}}{(q;q)_n}\sum_{j=0}^{k}\frac{(-1)^j q^{\binom{j}{2}+(n-k)j}}{(q;q)_j(q;q)_{k-j}}\qquad\left(\text{by [2, p. 36, Eq. (3.3.6)]}\right)\\
&\quad=(q;q)_\infty(-1)^k q^{\binom{k+1}{2}}\sum_{j=0}^{k}\frac{(-1)^j q^{\binom{j}{2}-kj}}{(q;q)_j(q;q)_{k-j}}\frac{1}{(q^{j+k+2};q)_\infty}\\
&\quad=\sum_{j=0}^{k}\frac{(-1)^{k-j}q^{\binom{j-k}{2}}(q;q)_{j+k+1}}{(q;q)_j(q;q)_{k-j}}\\
&\quad=\frac{1}{(q;q)_k}\sum_{j=0}^{k}\begin{bmatrix} k \\ j \end{bmatrix}(-1)^{k-j}q^{\binom{j-k}{2}}(q;q)_{j+k+1}\\
&\quad=(-1)^k\left(1-q^{k+1}\right)\sum_{j=0}^{k}\begin{bmatrix} k \\ j \end{bmatrix}(-1)^j q^{\binom{j-k}{2}}\left(q^{k+2};q\right)_j\\
&\quad=(-1)^k\left(1-q^{k+1}\right)q^{\binom{k+1}{2}}\sum_{j=0}^{k}\frac{(q^{-k};q)_j(q^{k+2};q)_j}{(q;q)_j}
\end{aligned}$$

$$= (q;q)_{k+1} \sum_{j=0}^{k} \frac{(-1)^j q^{\binom{j}{2}+(k+2)j}}{(q;q)_j},$$

where the last line follows from [2, p. 38, next to last line with $b = q^{-k}$, then $t = 1$ and $c \to 0$]. Thus Theorem 4 is proved. □

It is an easy exercise to deduce (4.1) from Theorem 4 and vice versa. Consequently we could prove Lemma 1.2 by starting with (4.1), then deducing Theorem 4.1, and then reversing the proof of Theorem 4.1 to obtain Lemma 1.2. We chose this way of proceeding because of the natural motivation provided by (1.1) and (1.2).

We note that (4.1) and truncated identities like it arose in important ways in [1]; indeed Section 4 of that paper was entitled Extensions of Shanks's formulas. As was noted there in Lemma 2 of [1, p. 118], a two parameter generalization of our Theorem 4.1 is an immediate specialization of Watson's $q$-analog of Whipple's theorem. Surprisingly, the relationship of Shanks's work to our very basic Theorem 1.1 went unobserved for 60 years.

There are several natural questions arising from our work:

(1) Provide a combinatorial proof of Theorem 1 hopefully characterizing the partitions remaining after a sieving process.

(2) There is a substantial amount of numerical evidence to conjecture that for $1 \leqslant S < R/2$, $k \geqslant 1$

$$\frac{\sum_{j=0}^{k-1} (-1)^j q^{R\binom{j+1}{2}-Sj}(1-q^{(2j+1)S})}{(q^S;q^R)_\infty (q^{R-S};q^R)_\infty (q^R;q^R)_\infty}$$

has nonnegative coefficients if $k$ is odd and nonpositive coefficients if $k$ is even.
Note that the corollary implies the case $R = 3$, $S = 1$.

(3) In light of the importance of Theorem 4.1 and its generalizations in [1], it is possible that there are extensions of Lemma 1.2 that might have applications to mock theta functions.

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