

Cyclotomic ordering conjecture

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Abstract

This note describes a conjecture I made (in Aachen, Sept. 2018) and some initial thoughts towards a solution. Given positive integers m, n , the conjecture is that either $\Phi_m(q) \leq \Phi_n(q)$ or $\Phi_m(q) \geq \Phi_n(q)$ holds for all integers $q \geq 2$. Pomerance and Rubinstein-Salzedo proved the conjecture in [2].

We define a partial ordering \preceq on the set \mathcal{P} of positive integers. Recall that $t^n - 1 = \prod_{d|n} \Phi_d(t)$ where the roots of the d th cyclotomic polynomial $\Phi_d(t)$ are primitive roots of order d . Hence $\deg(\Phi_d(t)) = \phi(d)$. For $m, n \in \mathcal{P}$ write $m \preceq n$ if $\Phi_m(q) \leq \Phi_n(q)$ for all integers $q \geq 2$, and write $m \prec n$ if $m \preceq n$ and $m \neq n$. (Clearly $a \preceq a$; $a \preceq b$ and $b \preceq a$ implies $a = b$; and $a \preceq b$ and $b \preceq c$ implies $a \preceq c$.) Since

$$q - 1 < q + 1 < q^2 - q + 1 \leq q^2 + 1 < q^2 + q + 1 < q^4 - q^3 + q^2 - q + 1$$

holds for all $q \geq 2$, we have $1 \prec 2 \prec 6 \prec 4 \prec 3 \prec 10$. Similarly, one can show $10 \prec 12 \prec 8 \prec 5 \prec 14 \prec 18 \prec 9 \prec 7 \prec 15 \prec 20 \prec 24 \prec 16 \prec 30 \prec 22 \prec 11$.

Conjecture 1. *The set \mathcal{P} of positive integers is totally ordered by \prec .*

Say that m *precedes* n (or n *succeeds* m) if $m \prec n$ and there is no x with $m \prec x \prec n$.

Conjecture 2. *$2 \cdot 3^i$ precedes 3^i for $i \geq 2$. For $i = 1$, we have $6 \prec 4 \prec 3$.*

Proving $\Phi_m(q) < \Phi_n(q)$ for all q is the same as proving $\Phi_n(q) - \Phi_m(q) > 0$. After canceling any equal terms, this inequality can be written $A(q) > B(q)$ where $A(t)$ and $B(t)$ are integer polynomials whose nonzero coefficients are all positive. If the largest nonzero coefficient is c , then $A(q) > B(q)$ holds for all $q > c$ provided the leading monomial of A is greater than the corresponding

monomial of B . (The base- q expansion of $A(q)$ is greater than $B(q)$.) The conjecture asserts that the inequality also holds for $2 \leq q \leq c$.

This reasoning will determine a putative total ordering of \mathcal{P} working for sufficiently large q but maybe not for small q . I wrote a program in Magma that proved that the integers $\{1, 2, \dots, 2 \cdot 10^4\}$ can be totally ordered. Since the coefficients of $\Phi_n(t)$ are unbounded as $n \rightarrow \infty$, and their maximum absolute value grows slowly, one might suspect that the conjecture is false and the smallest incomparable pair (m, n) is large. What is positive evidence?

Lemma 1. *If $m, n \in \mathcal{P}$ and $\phi(m) < \phi(n)$, then $m \prec n$.*

Proof. It follows from [1, Theorem 3.6] that $cq^{\phi(n)} < \Phi_n(q) < c^{-1}q^{\phi(n)}$ holds for all $q \geq 2$ where $c = 1 - q^{-1}$. Clearly $\frac{1}{2} \leq c$ and $c^{-1} \leq 2$. For $n \geq 3$ we know that $\phi(n)$ is even, so if $m, n \geq 3$, then $\phi(m) \leq \phi(n) - 2$. Therefore

$$\Phi_m(q) < c^{-1}q^{\phi(m)} \leq c^{-1}q^{\phi(n)-2} \leq cq^{\phi(n)} < \Phi_n(q).$$

The cases when $m < 3$ or $n < 3$ are easily handled. \square

Thus it suffices to consider whether distinct $m, n \in \mathcal{P}$ with $\phi(m) = \phi(n)$ are comparable, i.e. $m \prec n$ or $n \prec m$. Clearly $\phi(m) = \phi(2m)$ if m is odd.

Lemma 2. *If $m \in \mathcal{P}$ is odd, then $m \prec 2m$ or $2m \prec m$.*

Proof. Let m_0 be the radical (square-free part) of m . If $\mu(m_0) = 1$, i.e. m_0 is a product of an even number of primes, then [1, Theorem 3.6] implies that

$$cq^{\phi(m)} < \Phi_m(q) < q^{\phi(m)} < \Phi_{2m}(q) < c^{-1}q^{\phi(m)}$$

where $c = 1 - q^{-1}$. Similar inequalities (with $m \leftrightarrow 2m$) hold if $\mu(m_0) = -1$, i.e. m_0 is a product of an odd number of primes. \square

Remark 3. The sequence 1, 2, 6, 4, 3, 10, 12, 8, 5, 14, ... is A206225 in the OEIS. It tacitly assumes (without proof) that \prec is a total ordering.

Remark 4. If $m \neq n$ and $\phi(m) = \phi(n)$, then $\Phi_m(t) - \Phi_n(t)$ is a power of t times a self-reciprocal polynomial. Hence $\Phi_m(t) - \Phi_n(t) > 0$ for $t \geq 2$ implies $\Phi_m(t) - \Phi_n(t) > 0$ for $0 < t \leq \frac{1}{2}$.

References

- [1] Christoph Hering, *Transitive linear groups and linear groups which contain irreducible subgroups of prime order. II*, J. Algebra **93** (1985), no. 1, 151–164.
- [2] Carl Pomerance and Simon Rubinstein-Salzedo, *Cyclotomic coincidences*.
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