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On Generalized Lucas Pseudoprimality of Level k

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Abstract: We investigate the Fibonacci pseudoprimes of level k, and we disprove a statement concerning the relationship between the sets of different levels, and also discuss a counterpart of this result for the Lucas pseudoprimes of level k. We then use some recent arithmetic properties of the generalized Lucas, and generalized Pell–Lucas sequences, to define some new types of pseudoprimes of levels k^+ and k^- and parameter a. For these novel pseudoprime sequences we investigate some basic properties and calculate numerous associated integer sequences which we have added to the Online Encyclopedia of Integer Sequences.

Keywords: generalized lucas sequences; legendre symbol; jacobi symbol; pseudoprimality

MSC: 11A51; 11B39; 11B50

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1. Introduction

Let a and b be integers. The **generalized Lucas** sequence $\{U_n(a,b)\}_{n\geq 0}$ and its companion, the **generalized Pell–Lucas** sequence $\{V_n(a,b)\}_{n\geq 0}$, denoted by U_n and V_n for simplicity, are defined by

$$U_{n+2} = aU_{n+1} - bU_n, \qquad U_0 = 0,\ U_1 = 1, \quad n = 0,1,\dots \tag{1}$$

$$V_{n+2} = aV_{n+1} - bV_n, \qquad V_0 = 2,\ V_1 = a, \quad n = 0,1,\dots. \tag{2}$$

The general term of these sequences is given by the following Binet-type formulae

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{1}{\sqrt{D}}(\alpha^n - \beta^n), \quad V_n = \alpha^n + \beta^n, \quad n = 0,1,\dots, \tag{3}$$

where $D = a^2 - 4b \neq 0$ and $\alpha = \frac{a+\sqrt{D}}{2}$, $\beta = \frac{a-\sqrt{D}}{2}$ are the roots of the quadratic $z^2 - az + b = 0$. By Viéte's relations, one has $\alpha + \beta = a$ and $\alpha\beta = b$, while $\alpha - \beta = \sqrt{D}$.

Using bivariate cyclotomic polynomials, the relations (3) can be written [1] (p. 99) in terms of α and β, as

$$U_n = \prod_{d|n, d\geq 2} \Phi_d(\alpha,\beta),$$

where

$$\Phi_d(\alpha,\beta) = \prod_{j=1,\, \gcd(j,n)=1}^{n} (\alpha - \zeta^j\beta),$$

and ζ is a primitive n-th root of unity. It can be checked that $\Phi_d(\alpha,\beta)$ is an integer for any $d \geq 2$, and this feature can highlight arithmetic properties of the integers U_n.

If ω is an n-th root of -1, the following formula for can be written for V_n as

$$V_n = \prod_{d|n} \Phi_d(\alpha, \omega\beta),$$

However, the use of this formula is limited since $\Phi_d(\alpha, \omega\beta)$ is not always an integer.

The formulae (3) also extend naturally to negative indices. For any integer $n \geq 0$ one has

$$U_{-n} = \frac{1}{\sqrt{D}}(\alpha^{-n} - \beta^{-n}) = -\frac{1}{b^n}U_n, \quad V_{-n} = \alpha^{-n} + \beta^{-n} = \frac{1}{b^n}V_n.$$

Clearly, U_n and V_n are integers for all $n \in \mathbb{Z}$ if and only if $b = \pm 1$, and for this reason we shall focus on this case.

For $b = -1$, if k is a positive real number, then the k-Fibonacci and k-Lucas numbers are obtained for $F_{k,n} = U_n(k, -1)$ and $L_{k,n} = V_n(k, -1)$, in which case $D = k^2 + 4$ [2]. Clearly, for $k = 1$ we get the Fibonacci and Lucas numbers $F_n = U_n(1, -1)$ and $L_n = V_n(1, -1)$ with $D = 5$, and for $k = 2$ the Pell and Pell–Lucas numbers $P_n = U_n(2, -1)$ and $Q_n = V_n(2, -1)$, where $D = 8$.

When $b = 1$, the sequences $U_n(a, 1)$ have interesting combinatorial interpretations, while the terms $V_n(a, 1)$ can be linked to the number of solutions for certain Diophantine equations (see [3]) and to important classes of polynomials (see [4] (Chapter 2.2)).

The following results have been recently proved by the authors in [3].

Theorem 1 (Theorem 3.1, [3])**.** *Let p be an odd prime, k a non-negative integer, and r an arbitrary integer. If $b = \pm 1$ and a is an integer such that $D = a^2 - 4b > 0$ is not a perfect square, then the sequences U_n and V_n defined by* (1) *and* (2) *satisfy the following relations*

$$(1) \quad 2U_{kp+r} \equiv \left(\frac{D}{p}\right) U_k V_r + V_k U_r \pmod{p};$$

$$(2) \quad 2V_{kp+r} \equiv D\left(\frac{D}{p}\right) U_k U_r + V_k V_r \pmod{p},$$

where $\left(\frac{D}{p}\right)$ is the Legendre symbol (see, e.g., [5]).

Theorem 2 (Theorem 3.5, [3])**.** *Let p be an odd prime, and let $k > 0$ and a be integers so that $D = a^2 + 4 > 0$ is not a perfect square. If $U_n = U_n(a, -1)$ and $V_n = V_n(a, -1)$, then we have*

(1) $U_{kp - \left(\frac{D}{p}\right)} \equiv U_{k-1} \pmod{p}$;

(2) $V_{kp - \left(\frac{D}{p}\right)} \equiv \left(\frac{D}{p}\right) V_{k-1} \pmod{p}$.

Theorem 3 (Theorem 3.7, [3])**.** *Let p be an odd prime, and let $k > 0$ and a be integers so that $D = a^2 - 4 > 0$ is not a perfect square. If $U_n = U_n(a, 1)$ and $V_n = V_n(a, 1)$, then we have*

(1) $U_{kp - \left(\frac{D}{p}\right)} \equiv \left(\frac{D}{p}\right) U_{k-1} \pmod{p}$;

(2) $V_{kp - \left(\frac{D}{p}\right)} \equiv V_{k-1} \pmod{p}$.

Applying Theorem 1 for $k = 1$ and $r = 0$, we obtain the well known relations

$$U_p \equiv \left(\frac{D}{p}\right) \pmod{p}; \tag{4}$$

$$V_p \equiv a \pmod{p}. \tag{5}$$

Taking $k = 1$ in Theorems 2 and 3, and since $U_0 = 0$ and $V_0 = 2$, one has

$$U_{p-\left(\frac{D}{p}\right)} \equiv 0 \pmod{p}; \tag{6}$$

$$V_{p-\left(\frac{D}{p}\right)} \equiv 2\left(\frac{D}{p}\right)^{\frac{1-b}{2}}. \tag{7}$$

Pseudoprimes are those composite numbers that, under certain conditions, behave similarly to the prime numbers. These have numerous applications in the factorization of large integers, primality testing, and cryptography. Some important notions of pseudoprimality are linked to the generalized Lucas sequences $\{U_n(a,b)\}_{n\geq 0}$ and $\{V_n(a,b)\}_{n\geq 0}$ given by (1) and (2), based on the relations (4), (5), (6) and (7), which were known even to Lucas (see [6]).

Definition 1. *An odd composite integer n is said to be a* ***generalized Lucas pseudoprime of parameters*** a ***and*** b *if* $\gcd(n,b) = 1$ *and n divides* $U_{n-\left(\frac{D}{n}\right)}$*, where* $\left(\frac{D}{n}\right)$ *is the Jacobi symbol.*

By relation (4), we deduce that $U_p^2 \equiv 1 \pmod{p}$. Using this, in our paper [7] we have defined a weak pseudoprimality notion for generalized Lucas sequences $U_n(a,b)$.

Definition 2. *A composite integer n for which $n \mid U_n^2 - 1$ is called a* ***weak generalized Lucas pseudoprime*** *of parameters a and b.*

This notion plays a key role in the present paper. Another weak pseudoprimality concept for generalized Pell–Lucas sequences inspired by (5) is also defined in [7].

Definition 3. *A composite integer n is said to be a* ***generalized Bruckman–Lucas pseudoprime of parameters*** a ***and*** b *if $n \mid V_n(a,b) - a$.*

Historical details and various pseudoprimality tests for generalized Lucas sequences are given in the papers by Brillhart, Lehmer, and Selfridge [8], and by Baillie and Wagstaff in [9]. Grantham [10] unified many pseudoprimality notions under the name of Frobenius pseudoprimes and several examples are listed in Rotkiewics [11]. Various strong concepts like super-pseudoprimes [12], or extensions of recurrences to more general contexts like abelian groups have been proposed [13].

Interesting divisibility results for U_n and V_n are stated in [9] (Section 2).

Proposition 1. *If n is an odd composite number such that* $\gcd(n, 2abD) = 1$*, then any two of the following statements imply the other two.*

(1) $U_n \equiv \left(\frac{D}{n}\right) \pmod{n}$;
(2) $V_n \equiv V_1 = a \pmod{n}$;
(3) $U_{n-\left(\frac{D}{n}\right)} \equiv U_0 = 0 \pmod{n}$;
(4) $V_{n-\left(\frac{D}{n}\right)} \equiv 2b^{\frac{1-\left(\frac{D}{n}\right)}{2}} \pmod{n}$ *(valid whenever* $\gcd(n,D) = 1$*).*

The structure of this paper is as follows. In Section 2 we review the notion of Fibonacci pseudoprime of level k, and propose a counterpart defined for Lucas sequences. We also disprove a statement formulated in [14] for Fibonacci numbers, which shows that the relationship between the pseudoprimes of different levels is not trivial. In Section 3 we define the generalized Lucas and Pell–Lucas pseudoprimality of level k, which involves the Jacobi symbol. For these notions we study some new related integer sequences indexed in the Online Encyclopedia of Integer Sequences (OEIS). Finally, in Section 4 we summarize the findings and suggest future directions of investigation.

The numerical simulations in this paper have been performed with specialist Matlab libraries and Wolfram Alpha (explicit formulae are indicated in OEIS). Sometimes we have provided more terms than in the OEIS (which has a limit of 260 characters), so that the readers can check the numerical examples and counterexamples.

2. Fibonacci and Lucas Pseudoprimes of Level k

In this section we present the Fibonacci pseudoprimes of level k and give a counterexample to a result from [14], about the connection between the sets of pseudoprimes on different levels. We then define the Lucas pseudoprimes of level k, for which we also explore connections between the pseudoprimes on different levels.

2.1. Fibonacci Pseudoprimes of Level k

For a prime p, the following relations follow from (4) and (6) for $a = 1$ and $b = -1$.

$$F_p \equiv \left(\frac{p}{5}\right) \pmod{p}; \tag{8}$$

$$F_{p-\left(\frac{p}{5}\right)} \equiv 0 \pmod{p}. \tag{9}$$

A composite number n is called a **Fibonacci pseudoprime** if $n \mid F_{n-\left(\frac{n}{5}\right)}$. The even Fibonacci pseudoprimes are indexed as A141137 in the OEIS [15], while the odd Fibonacci pseudoprimes indexed as A081264 start with the terms

$$323, 377, 1891, 3827, 4181, 5777, 6601, 6721, 8149, 10877, 11663, 13201, 13981, 15251, 17119,$$
$$17711, 18407, 19043, 23407, 25877, 27323, 30889, 34561, 34943, 35207, 39203, 40501, \ldots.$$

In [14], the authors introduced the following notion. Let k be a fixed positive integer. A composite number n is called a **Fibonacci pseudoprime of level** k if it satisfies

$$n \mid F_{kn-\left(\frac{n}{5}\right)} - F_{k-1}.$$

The set of all the Fibonacci pseudoprimes of level k is denoted by $\mathcal{F}_k$. Notice that for $k = 1$ we obtain the classical Fibonacci pseudoprimes. We now state a corrected version of Proposition 1 in [14], and then discuss why the original version does not hold.

Proposition 2. *Let n be a positive integer that is coprime with* 10*. If $n \in \mathcal{F}_1$, then $n \in \mathcal{F}_2$ if and only if $n \mid F_n^2 - 1$.*

Proof. Notice that the conditions in the hypothesis relate to Equations (8) and (9). Clearly, $n \in \mathcal{F}_1$ is equivalent to $n \mid F_{n-\left(\frac{n}{5}\right)}$, while $n \in \mathcal{F}_2$ is equivalent to $n \mid F_{2n-\left(\frac{n}{5}\right)} - F_1$.

For all integers $m \geq r \geq 0$, Catalan's identity $F_m^2 - F_{m+r}F_{m-r} = (-1)^{m-r}F_r^2$, is valid. Using this identity for $m = n - \left(\frac{n}{5}\right)$ and $r = n$ and since $\gcd(5, n) = 1$, one has

$$F^2_{n-\left(\frac{n}{5}\right)} + (-1)^{\left(\frac{n}{5}\right)} F_{2n-\left(\frac{n}{5}\right)} = (-1)^{-\left(\frac{n}{5}\right)} F_n^2.$$

Since $\left(\frac{n}{5}\right)$ is odd, this can be rewritten as

$$F^2_{n-\left(\frac{n}{5}\right)} + \left(F_n^2 - 1\right) = F_{2n-\left(\frac{n}{5}\right)} - F_1. \tag{10}$$

Clearly, if $n \in \mathcal{F}_1$, then by taking the relation (10) modulo n, one obtains that $n \mid F_n^2 - 1$ is equivalent to $n \in \mathcal{F}_2$. □

Remark 1. *Notice that if $n \mid F_n^2 - 1$ and $n \in \mathcal{F}_2$, then by (10) it follows that $n \mid F^2_{n-\left(\frac{n}{5}\right)}$. This may not always indicate that $n \in \mathcal{F}_1$. However, this assertion holds whenever n is square-free. We have confirmed that the numbers satisfying both $n \mid F_n^2 - 1$ and $n \in \mathcal{F}_2$ with $n \leq 39500$ are*

$$323, 377, 1891, 3827, 4181, 5777, 6601, 6721, 8149, 10877, 11663, 13201, 13981, 15251,$$
$$17119, 17711, 18407, 19043, 23407, 25877, 27323, 30889, 34561, 34943, 35207, 39203,$$

which are all square-free and satisfy $n \in \mathcal{F}_1$.

We now recall Proposition 1 in [14], which states that if $n \in \mathbb{N}$ is coprime with 10, then $n \in \mathcal{F}_k$ for all $k \geq 1$ if and only if $n \in \mathcal{F}_1$ and $n \mid F_n^2 - 1$. In particular, if $n \mid F_{n-\left(\frac{n}{5}\right)}$ and $n | F_n - \left(\frac{n}{5}\right)$, then $n \in \mathcal{F}_k$ for all $k \geq 1$. The following example gives an integer n for which $n \in \mathcal{F}_1$ and $n \mid F_n^2 - 1$, (hence in $\mathcal{F}_2$), but which is not in $\mathcal{F}_3$. This shows that Proposition 1 in [14] does not generally hold.

Example 1. *The first composite integer n for which $n \mid F_{n-\left(\frac{n}{5}\right)}$ and $n \mid F_n^2 - 1$ is $n = 323$. For this integer one can check that $n \mid F_{2n-\left(\frac{n}{5}\right)} - F_1$, but we have $F_{3n-\left(\frac{n}{5}\right)} - F_2 \equiv 321 \pmod{n}$, where $\left(\frac{n}{5}\right) = -1$. The calculations involving the large numbers below are implemented with the* vpi *(variable precision integer) library in Matlab®. We have*

$$
\begin{aligned}
F_{n-\left(\frac{n}{5}\right)} &= 2304148358552416826222090648964201807510161746678049679057369028996 8 \equiv 0 \pmod{n} \\
F_{2n-\left(\frac{n}{5}\right)} &= 7336995277999309135280786247013754464564049243092710404349906900145 8 \\
&\quad 4668246528603476477043108568806527592562210693671820824200536283473 \equiv 1 = F_1 \pmod{n} \\
F_{3n-\left(\frac{n}{5}\right)} &= 2336286181815299653746750781129919541766943951168971092522786214227 5 \\
&\quad 5237533996389677833107817045296765338979711721919480043169346318420450 65 \\
&\quad 7716380889475584245156876241901131223573192092275600598593453 35 \equiv 322 \pmod{n}.
\end{aligned}
$$

We now discuss why the proof of Proposition 1 in [14] fails, but we mention that the error in the proof is not trivial as we can notice in the previous numerical example.

Remark 2. *The problems appear at the induction step. When applying Catalan's identity for $m = kn - \left(\frac{n}{5}\right)$ and $r = n$ one obtains the identity*

$$F^2_{kn-\left(\frac{n}{5}\right)} - F_{(k+1)n-\left(\frac{n}{5}\right)} F_{(k-1)n-\left(\frac{n}{5}\right)} = (-1)^{(k-1)n-\left(\frac{n}{5}\right)} F_n^2.$$

Assuming $n \in \mathcal{F}_k$ and taking this relation modulo n one obtains after some steps

$$
\begin{aligned}
F_{(k+1)n-\left(\frac{n}{5}\right)} F_{k-2} &\equiv F^2_{k-1} + (-1)^k \pmod{n} \\
F_k F_{k-2} &\equiv F^2_{k-1} + (-1)^k \pmod{n},
\end{aligned}
$$

from where the authors (incorrectly) claim $n \mid F_{(k+1)n-\left(\frac{n}{5}\right)} - F_k$. In fact, we only have

$$\left[F_{(k+1)n-\left(\frac{n}{5}\right)} - F_k\right] F_{k-2} \equiv 0 \pmod{n}.$$

This holds when n is coprime with F_{k-2}, but this cannot be guaranteed in general.

2.2. Lucas Pseudoprimes of Level k

From the relations (5) and (7) applied for $a = 1$ and $b = -1$ one obtains

$$L_p \equiv 1 \pmod{p}; \tag{11}$$

$$L_{p-\left(\frac{p}{5}\right)} \equiv 2\left(\frac{p}{5}\right) \pmod{p}. \tag{12}$$

A composite integer n satisfying the property $n \mid L_n - 1$ is called a **Bruckman–Lucas pseudoprime**. The sequence is indexed in the OEIS [15] as A005845, and begins with

$$705, 2465, 2737, 3745, 4181, 5777, 6721, 10877, 13201, 15251, 24465, 29281, 34561, 35785, 51841,$$
$$54705, 64079, 64681, 67861, 68251, 75077, 80189, 90061, 96049, 97921, 100065, 100127, \ldots.$$

In 1964 Lehmer [16] proved that Fibonacci pseudoprimes are infinite, while in 1994 Bruckman showed that the Bruckman–Lucas pseudoprimes are odd [17], and also he proved that these numbers are infinitely many [18].

For a positive integer k we define the **Lucas pseudoprimes of level** k as the composite integers n satisfying the relation

$$n \mid L_{kn-\left(\frac{n}{5}\right)} - \left(\frac{n}{5}\right) L_{k-1}.$$

The set of all the Lucas pseudoprimes of level k is denoted by $\mathcal{L}_k$.

For $k = 1$ the integers $n \in \mathcal{L}_1$ satisfy $n \mid L_{n-\left(\frac{n}{5}\right)} - 2\left(\frac{n}{5}\right)$ and define the sequence A339125 added by us to OEIS, which starts with the terms

$$9, 49, 121, 169, 289, 361, 529, 841, 961, 1127, 1369, 1681, 1849, 2209, 2809, 3481, 3721,$$
$$3751, 4181, 4489, 4901, 4961, 5041, 5329, 5777, 6241, 6721, 6889, 7381, 7921, 9409, \ldots.$$

For $k = 2$ the integers $n \in \mathcal{L}_2$ satisfy the relation $n \mid L_{2n-\left(\frac{n}{5}\right)} - \left(\frac{n}{5}\right)$, and recover a sequence we have indexed as A339517, whose first elements are

$$323, 377, 609, 1891, 3081, 3827, 4181, 5777, 5887, 6601, 6721, 8149, 8841, 10877, 11663, 13201,$$
$$13981, 15251, 17119, 17711, 18407, 19043, 23407, 25877, 26011, 27323, 30889, 34561, \ldots.$$

The following result highlights a connection between the Lucas pseudoprimes of levels 1 and 2 via the positive integers with the property $n \mid F_n^2 - 1$.

Proposition 3. *Let n be a positive integer that is coprime with* 10*. If $n \in \mathcal{L}_1$, then $n \in \mathcal{L}_2$ if and only if $n \mid F_n^2 - 1$.*

Proof. One can easily check (see Lemma 2.4 [19]) that for any integers m and r we have

$$L_m^2 - L_{m+r}L_{m-r} = -5(-1)^{m-r}F_r^2, \quad L_{-m} = (-1)^m L_m.$$

Using this identity for $m = n - \left(\frac{n}{5}\right)$ and $r = n$, we get

$$L^2_{n-\left(\frac{n}{5}\right)} - L_{2n-\left(\frac{n}{5}\right)}L_{-\left(\frac{n}{5}\right)} = -5(-1)^{-\left(\frac{n}{5}\right)}F_n^2.$$

As n and 5 are coprime, we have $L_{-\left(\frac{n}{5}\right)} = (-1)^{-\left(\frac{n}{5}\right)}L_{\left(\frac{n}{5}\right)}$ and $L_{\left(\frac{n}{5}\right)} = \left(\frac{n}{5}\right)$, while since $\left(\frac{n}{5}\right) = \pm 1$, it follows that $(-1)^{-\left(\frac{n}{5}\right)} = -1$. Therefore

$$L^2_{n-\left(\frac{n}{5}\right)} + \left(\frac{n}{5}\right) L_{2n-\left(\frac{n}{5}\right)} = 5F_n^2. \tag{13}$$

This identity can be further written as

$$\left(L^2_{n-\left(\frac{n}{5}\right)} - 4\right) + \left(\frac{n}{5}\right)\left(L_{2n-\left(\frac{n}{5}\right)} - \left(\frac{n}{5}\right)\right) = 5\left(F_n^2 - 1\right). \tag{14}$$

Now we take this relation modulo n. Clearly, from $n \in \mathcal{L}_1$ we $L_{n-\left(\frac{n}{5}\right)} \equiv 2\left(\frac{n}{5}\right) \pmod{n}$, hence the first bracket vanishes. Notice that if any of the other two brackets in (14) vanish, then the third vanishes as well, hence $n \in \mathcal{L}_2$ if and only if $n \mid F_n^2 - 1$. $\square$

One could check that if $n \mid F_n^2 - 1$ and $n \in \mathcal{L}_2$, then it does not follow that $n \in \mathcal{L}_1$. We give an example below.

Example 2. *From Example 1, we know that for $n = 323$ we have $n \mid F_n^2 - 1$ and $\left(\frac{n}{5}\right) = -1$. One can check numerically that $n \mid L_{2n-\left(\frac{n}{5}\right)} - \left(\frac{n}{5}\right)L_1$, but $L_{n-\left(\frac{n}{5}\right)} \equiv 2 \neq 2\left(\frac{n}{5}\right) \pmod{n}$.*

$$L_{n-\left(\frac{n}{5}\right)} = 51522323599677629496737990329528638956583548304378053615581043535682 \equiv 2 \pmod{n}$$

$$L_{2n-\left(\frac{n}{5}\right)} - \left(\frac{n}{5}\right)L_1 = 164060201922014224280717425066245025764781124891271838619807402041846262156924015992064406990410128506570256616706671043831467653299288 0 \equiv 0 \pmod{n}.$$

It can be checked that $n = 323$ is the smallest odd composite number for which $n \mid F_n^2 - 1$ and $n \in \mathcal{L}_2$ but $n \notin \mathcal{L}_1$, but as we will see later, there are (possibly infinitely) many numbers that satisfy this property.

3. Generalized Lucas Pseudoprimes of Level k

In this section we use Theorems 2 and 3 to extend the notions presented in Section 2 for generalized Lucas and Pell–Lucas sequences. We calculate the terms of the integer sequences obtained for a few particular parameter values and we formulate some conjectures.

3.1. Jacobi's Symbol

Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime factorization of an odd integer n. The Jacobi symbol is defined as

$$\left(\frac{a}{n}\right) = \left(\frac{a}{p_1}\right)^{\alpha_1} \left(\frac{a}{p_2}\right)^{\alpha_2} \cdots \left(\frac{a}{p_k}\right)^{\alpha_k},$$

where a is an integer. When n is a prime this recovers the Legendre symbol.

Jacobi's symbol is completely multiplicative in both the numerator and denominator, i.e., for m, n, m_1, m_2, n_1, n_2 integers, we have

$$\left(\frac{m_1 m_2}{n}\right) = \left(\frac{m_1}{n}\right)\left(\frac{m_2}{n}\right), \quad \text{so } \left(\frac{m^2}{n}\right) = \left(\frac{m}{n}\right)^2 = 1 \text{ or } 0;$$
$$\left(\frac{m}{n_1 n_2}\right) = \left(\frac{m}{n_1}\right)\left(\frac{m}{n_2}\right), \quad \text{so } \left(\frac{m}{n^2}\right) = \left(\frac{m}{n}\right)^2 = 1 \text{ or } 0.$$

The Jacobi symbol also satisfies the quadratic reciprocity law. This states that if m and n are odd positive coprime integers, then the following identity holds

$$\left(\frac{m}{n}\right)\left(\frac{n}{m}\right) = (-1)^{\frac{m-1}{2} \cdot \frac{n-1}{2}} = \begin{cases} 1 & \text{if } n \equiv 1 \pmod{4} \text{ or } m \equiv 1 \pmod{4}, \\ -1 & \text{if } n \equiv m \equiv 3 \pmod{4}. \end{cases}$$

3.2. Results for $b = -1$

We shortly denote $U_n = U_n(a, -1)$ and $V_n = V_n(a, -1)$. If p is prime number and a is an odd integer, then by the law of quadratic reciprocity for the Jacobi symbol with $D = a^2 + 4$ one has

$$\left(\frac{D}{p}\right)\left(\frac{p}{D}\right) = (-1)^{\frac{p-1}{2} \cdot \frac{D-1}{2}} = 1. \tag{15}$$

This implies $\left(\frac{D}{p}\right) = \left(\frac{p}{D}\right)$, hence the results in Theorem 2 can be written as

(1) $U_{kp-\left(\frac{p}{D}\right)} \equiv U_{k-1} \pmod{p}$;
(2) $V_{kp-\left(\frac{p}{D}\right)} \equiv \left(\frac{p}{D}\right) V_{k-1} \pmod{p}$.

We now investigate similar relations modulo a composite number n, where $\left(\frac{n}{D}\right)$ is the Jacobi symbol, which is well-defined for any odd composite integers n and D. These allow us to define new concepts of pseudoprimality.

Definition 4. *Let a, k, and n be non-negative integers, where a is odd. We say that the composite number n is a*

(1) ***generalized Lucas pseudoprime of level** k^- **and parameter** a if*

$$n \mid U_{kn-\left(\frac{n}{D}\right)} - U_{k-1}.$$

The set of all such numbers is denoted by $\mathcal{U}_k^-(a)$.

(2) ***generalized Pell–Lucas pseudoprime of level** k^- **and parameter** a if*

$$n \mid V_{kn-\left(\frac{n}{D}\right)} - \left(\frac{n}{D}\right) V_{k-1}.$$

The set of all such numbers is denoted by $\mathcal{V}_k^-(a)$.

In [19] we proved connections between the sets of generalized Lucas and Pell–Lucas pseudoprimes of levels 1^- and 2^-, which are linked through the property $n \mid U_n^2 - 1$ (see Definition 2). Integers having this property were called weak generalized Lucas pseudoprimes of parameters a and b and present interest in their own right. Some of their properties, associated integer sequences and conjectures have been discussed in [7].

Theorem 4. *Let $a, n > 0$ be odd integers with $\gcd(D, n) = 1$. The following statements hold*

(1) *Reference [19] (Theorem 4.3). If $n \in \mathcal{U}_1^-(a)$, then $n \in \mathcal{U}_2^-(a)$ if and only if $n \mid U_n^2 - 1$.*
(2) *Reference [19] (Theorem 4.6). If $n \in \mathcal{V}_1^-(a)$ and $\gcd(a, n) = 1$, then $n \in \mathcal{V}_2^-(a)$ if and only if $n \mid U_n^2 - 1$.*

We now present the integer sequences $\mathcal{U}_k^-(a)$, $\mathcal{V}_k^-(a)$ calculated for the values $a = 1, 3, 5, 7$ and $k = 1, 2, 3$. Most of these were added by the authors to OEIS [15]. For these values we show that the reciprocal statements in Theorem 4 do not hold, and also, the results cannot be extended directly to superior levels.

To begin with, we provide some details on weak generalized Lucas pseudoprimes.

Remark 3. *For $b = -1$, the odd integers n satisfying the property $n \mid U_n^2 - 1$ recover the weak Fibonacci pseudoprimes indexed as A337231 for $a = 1$, A337234 for $a = 3$, A337237 for $a = 5$, and A338081 for $a = 7$. The reader can use these to check the numerical examples.*

Remark 4. *As seen in Example 2, even when $n \mid U_n^2 - 1$, and $n \in \mathcal{U}_2^-(a)$ (or $n \in \mathcal{V}_2^-(a)$), it does not mean that $n \in \mathcal{U}_1^-(a)$ (or $n \in \mathcal{V}_1^-(a)$). For U_n we have the following examples:*

- *$a = 1$: None found for $n \leq 50000$ (see also, Remark 1);*
- *$a = 3$: 9, 63, 99, 153, 1071, 1881, 1953, 9999, 13833, 16191;*
- *$a = 5$: None found for $n \leq 15000$;*
- *$a = 7$: 49, 147, 245, 637, 833, 1127, 1225, 2499, 3185, 3479, 4753, 5537, 15925.*

For V_n we have

- *$a = 1$: 323, 377, 1891, 3827, 6601, 8149, 11663, 13981, 17119, 17711, 18407, 19043;*
- *$a = 3$: 1763, 3599, 5559, 6681, 12095, 12403, 12685, 14279, 15051, 19043;*
- *$a = 5$: 15, 45, 91, 135, 143, 1547, 1573, 1935, 2015, 6543, 8099, 10403, 10905;*
- *$a = 7$: 35, 65, 175, 391, 455, 575, 1247, 1295, 1763, 1775, 2275, 2407, 3367, 4199, 4579.*

Also the connections between the levels 2^- and 3^- are non-trivial.

Remark 5. *As seen in Example 1, even when* $n \mid U_n^2 - 1$, *if* $n \in \mathcal{U}_1^-(a)$ *(hence* $n \in \mathcal{U}_2^-(a)$*), it does not mean that* $n \in \mathcal{U}_3^-(a)$*. The following values have been found:*

- $a = 1$: 323, 377, 1891, 3827, 6601, 8149, 11663, 13981, 17119, 17711, 18407, 19043;
- $a = 3$: 1763, 3599, 5559, 6681, 12095, 12403, 12685, 14279, 15051, 19043;
- $a = 5$: 15, 45, 91, 135, 143, 1547, 1573, 1935, 2015, 6543, 8099, 10403, 10905;
- $a = 7$: 35, 65, 175, 391, 455, 575, 1247, 1295, 1763, 1775, 2275, 2407, 3367, 4199, 4579.

The following n *with* $n \mid U_n^2 - 1$ *and* $n \in \mathcal{V}_1^-(a)$, $n \in \mathcal{V}_2^-(a)$, *but* $n \notin \mathcal{V}_3^-(a)$ *were found:*

- $a = 1$: *None found for* $n \leq 50000$;
- $a = 3$: *None found for* $n \leq 20000$;
- $a = 5$: 18901, 19601, 19951;
- $a = 7$: *None found for* $n \leq 17000$.

The numerical results in Remarks 4 and 5 suggest the following conjecture.

Conjecture 1. *If* $n \mid U_n^2 - 1$, *then* $n \in \mathcal{U}_1^-(a) \setminus \mathcal{U}_3^-(a)$ *if and only if* $n \in \mathcal{V}_2^-(a) \setminus \mathcal{V}_1^-(a)$.

Example 3. *If* $b = -1$, $a = 1$, $D = 5$, *we obtain the classical Fibonacci and Lucas numbers.*

- The set $\mathcal{U}_1^-(1)$ recovers the odd Fibonacci pseudoprimes A081264 in [15].
- The set $\mathcal{U}_2^-(1)$ gives A340118 and its first elements are

$$323, 377, 609, 1891, 3081, 3827, 4181, 5777, 5887, 6601, 6721, 8149, 10877, 11663, 13201, 13601, 13981, 15251, 17119, 17711, 18407, 19043, 23407, 25877, 27323, 28441, 28623, 30889, 32509, 34561, 34943, 35207, 39203, 40501, \ldots.$$

- The set $\mathcal{U}_3^-(1)$ is A340235 and its first elements are

$$9, 27, 161, 341, 901, 1107, 1281, 1853, 2241, 2529, 4181, 5473, 5611, 5777, 6119, 6721, 7587, 8307, 9729, 10877, 11041, 12209, 13201, 13277, 14981, 15251, 16771, 17567, \ldots.$$

- The set $\mathcal{V}_1^-(1)$ recovers A339125, seen in Section 2.2.
- The set $\mathcal{V}_2^-(1)$ is A339517, seen in Section 2.2.
- The sequence $\mathcal{V}_3^-(1)$ is given by A339724 and starts with the elements

$$9, 21, 161, 341, 901, 1281, 1853, 3201, 4181, 5473, 5611, 5777, 6119, 6721, 9729, 10877, 11041, 12209, 12441, 13201, 14981, 15251, 16771, 17941, 20591, 20769, 20801, \ldots.$$

Example 4. $b = -1$, $a = 3$, $D = 13$.

- The set $\mathcal{U}_1^-(3)$ recovers pseudoprimes indexed as A327653 in [15], starting with

$$119, 649, 1189, 1763, 3599, 4187, 5559, 6681, 12095, 12403, 12685, 12871, 12970, 14041, 14279, 15051, 16109, 19043, 22847, 23479, 24769, 26795, 28421, 30743, 30889, \ldots.$$

- The set $\mathcal{U}_2^-(3)$ gives A340119 and its first elements are

$$9, 27, 63, 81, 99, 119, 153, 243, 567, 649, 729, 759, 891, 903, 1071, 1189, 1377, 1431, 1539, 1763, 1881, 1953, 2133, 2187, 3599, 3897, 4187, 4585, 5103, 5313, 5559, 5589, 5819, 6561, 6681, 6831, 6993, 8019, 8127, 8829, 8855, 9639, 9999, 10611, 11135, \ldots.$$

- The set $\mathcal{U}_3^-(3)$ is indexed as A340236 and its first elements are

$$9, 119, 121, 187, 327, 345, 649, 705, 1003, 1089, 1121, 1189, 1881, 2091, 2299, 3553, 4187, 5461, 5565, 5841, 6165, 6485, 7107, 7139, 7145, 7467, 7991, 8321, 8449, 11041, \ldots.$$

- The set $\mathcal{V}_1^-(3)$ recovers A339126, and starts with

$$9, 25, 49, 119, 121, 289, 361, 529, 649, 833, 841, 961, 1089, 1189, 1369, 1681, 1849, 1881, 2023, 2209, 2299, 2809, 3025, 3481, 3721, 4187, 4489, 5041, 5329, 6241, 6889, 7139, \ldots.$$

- The set $\mathcal{V}_2^-(3)$ giving A339518, has the first elements

$$15, 75, 105, 119, 165, 255, 375, 649, 1189, 1635, 1763, 1785, 1875, 2233, 2625, 3599, 3815, 4125, 4187, 5475, 5559, 5887, 6375, 6601, 6681, 7905, 8175, 9265, 9375, 9471, 11175, \ldots.$$

- The set $\mathcal{V}_3^-(3)$ is given by A339725 and starts with the elements

$$9, 27, 119, 133, 145, 165, 205, 261, 341, 393, 649, 693, 705, 901, 945, 1121, 1173, 1189, 1353, 1431, 1485, 1881, 2133, 2805, 3201, 3605, 3745, 4187, 5173, 5461, 5841, 5945, \ldots.$$

Example 5. $b = -1$, $a = 5$, $D = 29$.

- The set $\mathcal{U}_1^-(5)$ recovers the entry A340095 in [15], starting with

$$9, 15, 27, 45, 91, 121, 135, 143, 1547, 1573, 1935, 2015, 6543, 6721, 8099, 10403, 10877, 10905, 13319, 13741, 13747, 14399, 14705, 16109, 16471, 18901, 19043, 19109, \ldots.$$

- The set $\mathcal{U}_2^-(5)$ gives A340120 and its first elements are

$$9, 15, 25, 27, 45, 75, 91, 121, 125, 135, 143, 147, 175, 225, 275, 325, 375, 441, 483, 625, 675, 735, 755, 1125, 1323, 1547, 1573, 1875, 1935, 2015, 2205, 2275, 2485, \ldots.$$

- The set $\mathcal{U}_3^-(5)$ is indexed as A340237 and its first elements are

$$9, 27, 33, 35, 65, 81, 99, 121, 221, 243, 297, 363, 513, 585, 627, 705, 729, 891, 1089, 1539, 1541, 1881, 2145, 2187, 2299, 2673, 3267, 3605, 4181, 4573, 4579, 5265, 5633, 6721, \ldots.$$

- The set $\mathcal{V}_1^-(5)$ recovers A339127, and starts with

$$9, 25, 27, 49, 81, 121, 169, 175, 225, 243, 289, 325, 361, 529, 637, 729, 961, 1225, 1331, 1369, 1539, 1681, 1849, 2025, 2209, 2809, 3025, 3481, 3721, 4225, 4489, 5041, 5329, \ldots.$$

- The set $\mathcal{V}_2^-(5)$ giving A339519, has the first elements

$$9, 15, 27, 39, 45, 91, 117, 121, 135, 143, 195, 287, 351, 507, 585, 741, 1521, 1547, 1573, 1755, 1935, 2015, 2067, 2535, 2601, 3157, 3227, 3445, 3505, 3519, 3731, 4563, \ldots.$$

- The set $\mathcal{V}_3^-(5)$ is given by A339726 and starts with the elements

$$9, 25, 27, 33, 35, 45, 65, 81, 99, 117, 121, 161, 175, 221, 225, 297, 325, 363, 585, 645, 705, 825, 891, 1089, 1281, 1539, 1541, 1881, 2025, 2133, 2145, 2181, 2299, 2325, 2925, \ldots.$$

Example 6. $b = -1$, $a = 7$, $D = 53$.

- The set $\mathcal{U}_1^-(7)$ recovers the entry A340096 in [15], starting with

$$25, 35, 51, 65, 91, 175, 325, 391, 455, 575, 1247, 1295, 1633, 1763, 1775, 1921, 2275, 2407, 2599, 2651, 3367, 4199, 4579, 4623, 5629, 6441, 9959, 10465, 10825, 10877, \ldots.$$

- The set $\mathcal{U}_2^-(7)$ gives A340121 and its first elements are

$$25, 35, 39, 49, 51, 65, 91, 147, 175, 245, 301, 325, 343, 391, 455, 507, 575, 605, 637, 663, 741, 833, 897, 903, 935, 1127, 1205, 1225, 1247, 1295, 1505, 1595, 1633, 1715, 1763, \ldots.$$

- The set $\mathcal{U}_3^-(7)$ is indexed as A340238 and its first elements are

$$9, 25, 27, 51, 91, 105, 153, 185, 225, 289, 325, 425, 459, 481, 513, 747, 867, 897, 925, 945, 1001, 1189, 1299, 1469, 1633, 1785, 1921, 2241, 2245, 2599, 2601, 2651, 2769, \ldots.$$

- The set $\mathcal{V}_1^-(7)$ recovers A339128, and starts with

$$9, 25, 49, 51, 91, 121, 125, 153, 169, 289, 325, 361, 441, 529, 625, 637, 833, 841, 867, 961, 1183, 1225, 1369, 1633, 1681, 1849, 1921, 2209, 2599, 2601, 2651, 3481, 3721, 4225, \ldots.$$

- The set $\mathcal{V}_2^-(7)$ giving A339520, has the first elements

$$25, 35, 51, 65, 75, 91, 105, 175, 203, 325, 391, 455, 575, 645, 861, 1247, 1275, 1295, 1633, 1763, 1775, 1785, 1875, 1921, 2275, 2407, 2415, 2599, 2625, 2651, 3045, 3367, \ldots.$$

- The set $\mathcal{V}_3^-(7)$ is given by A339727 and starts with the elements

$$9, 25, 49, 51, 69, 91, 105, 143, 145, 153, 185, 221, 225, 325, 339, 391, 425, 441, 481, 637, 645, 705, 805, 833, 897, 925, 1001, 1173, 1189, 1207, 1225, 1281, 1299, 1365, \ldots.$$

In 1964, E. Lehmer [16] proved that the sequence $\mathcal{U}_1^-(1)$ is infinite.

Conjecture 2. *If a and k are positive integers with a odd, then $\mathcal{U}_k^-(a)$ and $\mathcal{V}_k^-(a)$ are infinite.*

3.3. Results for $b = 1$

We shortly denote $U_n = U_n(a, 1)$ and $V_n = V_n(a, 1)$. If p is prime and a odd, then we have $D = a^2 - 4$, and by the law of quadratic reciprocity for the Jacobi symbol (15) we get $\left(\frac{D}{p}\right) = \left(\frac{p}{D}\right)$, hence the results in Theorem 3 can be rewritten as

(1) $U_{kp-\left(\frac{p}{D}\right)} \equiv \left(\frac{p}{D}\right)U_{k-1} \pmod{p}$;

(2) $V_{kp-\left(\frac{p}{D}\right)} \equiv V_{k-1} \pmod{p}$.

We investigate similar relations modulo a composite number n, where $\left(\frac{n}{D}\right)$ is the Jacobi symbol, which is well-defined for any odd composite integers n and D, which allow us to naturally define new pseudoprimality notions.

Definition 5. *Let a, k and n be non-negative integers, with a odd. We say that the composite number n is a*

(1) ***generalized Lucas pseudoprime of level k^+ and parameter a** if*

$$n \mid U_{kn-\left(\frac{n}{D}\right)} - \left(\frac{n}{D}\right)U_{k-1}.$$

The set of all such numbers is denoted by $\mathcal{U}_k^+(a)$.

(2) ***generalized Pell–Lucas pseudoprime of level*** k^+ ***and parameter*** a *if*

$$n \mid V_{kn-\left(\frac{n}{D}\right)} - V_{k-1}.$$

The set of all such numbers is denoted by $\mathcal{V}_k^+(a)$.

In [19] we have proved connections between the sets of generalized Lucas and Pell–Lucas pseudoprimes of levels 1^+ and 2^+, linked through the property $n \mid U_n^2 - 1$ (similarly to Theorem 4).

Theorem 5. *Let* $a, n > 0$ *be odd integers with* $\gcd(D, n) = 1$. *We have:*

(1) *Reference [19] (Theorem 4.9). If* $n \in \mathcal{U}_1^+(a)$, *then* $n \in \mathcal{U}_2^+(a)$ *if and only if* $n \mid U_n^2 - 1$.
(2) *Reference [19] (Theorem 4.12). If* $n \in \mathcal{V}_1^+(a)$ *and* $\gcd(a, n) = 1$, *then* $n \in \mathcal{V}_2^+(a)$ *if and only if* $n \mid U_n^2 - 1$.

We now present the integer sequences $\mathcal{U}_k^+(a)$, $\mathcal{V}_k^+(a)$ calculated for the values $a = 3, 5, 7$ and $k = 1, 2, 3$. Most of these have been added by the authors to OEIS [15]. For these values we show that the reciprocal statements in Theorem 5 do not hold, and also, the results cannot be extended directly to superior levels.

We first provide some details on weak generalized Lucas pseudoprimes.

Remark 6. *For* $b = 1$, *the odd integers* n *satisfying the property* $n \mid U_n^2 - 1$ *recover the sequences A338007 for* $a = 3$, *A338009 for* $a = 5$, *and A338011 for* $a = 7$. *The reader can use these links to check the numerical examples given below.*

We now show that the reciprocals of Theorem 5 do not hold.

Remark 7. *(1) If* $n \mid U_n^2 - 1$ *with* $n \in \mathcal{U}_2^+(a)$, *does not imply* $n \in \mathcal{U}_1^+(a)$. *A counterexample is given by* $U_n = U_n(3, 1)$ *(bisection of Fibonacci numbers), where* $D = 5$. *For* $n = 9$ *we have* $U_n = 2584$, $\left(\frac{n}{5}\right) = 1$, $n \mid U_{2n-\left(\frac{n}{5}\right)} - U_1$ *and* $n \mid U_n^2 - 1$, *but* $U_{n-\left(\frac{n}{5}\right)} \equiv 6 \neq 0 \pmod{n}$.

$$\begin{aligned} U_{n-\left(\frac{n}{5}\right)} &= 987 \equiv 6 \pmod{n}; \\ U_{2n-\left(\frac{n}{5}\right)} - U_1 &= 5702886 \equiv 0 \pmod{n}; \\ U_n^2 - 1 &= 6677055 \equiv 0 \pmod{n}. \end{aligned}$$

(2) *When* $n \mid U_n^2 - 1$ *with* $n \in \mathcal{V}_2^+(a)$, *it does not imply* $n \in \mathcal{V}_1^+(a)$. *A counterexample is given by* $V_n = V_n(3, 1) = L_{2n}$ *(bisection of Lucas numbers), where* $D = 5$. *For* $n = 21$ *we get* $V_n = 599074578$, *one has* $\left(\frac{n}{5}\right) = 1$ *and*

$$\begin{aligned} V_{n-\left(\frac{n}{5}\right)} &= 228826125 \equiv 5 \neq 2 \pmod{n}; \\ V_{2n-\left(\frac{n}{5}\right)} - V_1 &= 137083915467899400 \equiv 0 \pmod{n}; \\ U_n^2 - 1 &= 71778070001175615 \equiv 0 \pmod{n}. \end{aligned}$$

For the calculations we have used the vpi *(variable precision integer) library in Matlab.*

For each value $a = 3, 5, 7$ there might be infinitely many such integers n.

Remark 8. *As seen in Example 2, even when* $n \mid U_n^2 - 1$, *and* $n \in \mathcal{U}_2^+(a)$ *(or* $n \in \mathcal{V}_2^+(a)$*), it does not mean that* $n \in \mathcal{U}_1^+(a)$ *(or* $n \in \mathcal{V}_1^+(a)$*). For* U_n *we have:*

- $a = 3$: $9, 63, 423, 2871, 2961, 8001$;
- $a = 5$: $25, 275, 425, 575, 775, 6325, 6575, 9775, 13175, 17825$;
- $a = 7$: $49, 1127, 2303$

For V_n we have

- $a = 3$: $21, 329, 451, 861, 1081, 1819, 2033, 2211, 3653, 4089, 5671, 8557, 11309, 13861, 14701, 17513, 17941, 19951, 20473$;
- $a = 5$: $115, 253, 391, 713, 715, 779, 935, 1705, 2627, 2893, 2929, 3281, 4141, 5191, 5671, 7739, 8695, 11815, 12121, 17963$;
- $a = 7$: $1771, 7471, 7931, 15449$.

We show that for U_n one cannot make the jump from levels 1^+ and 2^+ to level 3^+, even under the extra condition $n \mid U_n^2 - 1$.

Example 7. *When $b = 1$ and $a = 3$ we have $D = 5$. The first composite integer n for which $n \mid U_{n-\left(\frac{n}{5}\right)}$ and $n \mid U_n^2 - 1$ is $n = 21$. For this integer one can check that $n \mid U_{2n-\left(\frac{n}{5}\right)} - U_1$, but we have $U_{3n-\left(\frac{n}{5}\right)} - U_2 \equiv 15 \pmod{n}$, where $\left(\frac{n}{5}\right) = 1$. The calculations with large integers are implemented with the* vpi *library in Matlab®. We have*

$$
\begin{aligned}
U_{n-\left(\frac{n}{5}\right)} &= 102334155 \equiv 0 \pmod{n}; \\
U_{2n-\left(\frac{n}{5}\right)} &= 61305790721611591 \equiv \left(\frac{n}{5}\right) U_1 = 1 \pmod{n}; \\
U_{3n-\left(\frac{n}{5}\right)} &= 36726740705505779255899443 \equiv 18 \neq \left(\frac{n}{5}\right) U_2 = 3 \pmod{n}; \\
U_n^2 - 1 &= 71778070001175615 \equiv 0 \pmod{n}.
\end{aligned}
$$

We now find multiple such integers for U_n, as in Remark 5.

Remark 9. *Below we present some integers n which satisfy the properties $n \mid U_n^2 - 1$ and $n \in \mathcal{U}_1^+(a) \cap \mathcal{U}_2^+(a)$, but $n \notin \mathcal{U}_3^+(a)$.*

- $a = 3$: $21, 329, 451, 861, 1081, 1819, 2033, 2211, 3653, 4089, 5671, 8557, 11309, 13861, 14701, 17513, 17941, 19951, 20473$;
- $a = 5$: $115, 253, 391, 713, 715, 779, 935, 1705, 2627, 2893, 2929, 3281, 4141, 5191, 5671, 7739, 11815, 12121, 17963$;
- $a = 7$: $1771, 7471, 7931, 15449$.

We conjecture that these sequences exist and are infinite for all odd integers a.

By Theorem 5 we have that whenever $n \mid U_n^2 - 1$ we have $\mathcal{V}_1^+(a) \subseteq \mathcal{V}_2^+(a)$. The following property for V_n is suggested by numerical simulations for $a = 3, 5, 7$ and $n \leq 17000$, but we do not currently have a proof.

Conjecture 3. *If $a, n \geq 3$ are odd integers such that n is composite and $n \mid U_n^2 - 1$, then we have $\mathcal{V}_2^+(a) \subseteq \mathcal{V}_3^+(a)$.*

Example 8. $b = 1$, $a = 3$, $D = 5$ *(bisection of Fibonacci and Lucas numbers).*

- The set $\mathcal{U}_1^+(3)$ recovers the entry A340097 in [15], starting with

$$21, 323, 329, 377, 451, 861, 1081, 1819, 1891, 2033, 2211, 3653, 3827, 4089, 4181, 5671, 5777, 6601, 6721, 8149, 8557, 10877, 11309, 11663, 13201, 13861, 13981, \ldots.$$

- The set $\mathcal{U}_2^+(3)$ recovers A340122 and its first elements are

$$9, 21, 27, 63, 81, 189, 243, 323, 329, 351, 377, 423, 451, 567, 729, 783, 861, 891, 963, 1081, 1701, 1743, 1819, 1891, 1967, 2033, 2187, 2211, 2871, 2889, 2961, 3321, 3653, \ldots.$$

- The set $\mathcal{U}_3^+(3)$ is indexed as A340239 and its first elements are

$$9, 49, 63, 141, 161, 207, 323, 341, 377, 441, 671, 901, 1007, 1127, 1281, 1449, 1853,$$
$$1891, 2071, 2303, 2407, 2501, 2743, 2961, 3827, 4181, 4623, 5473, 5611, 5777, 6119, \ldots.$$

- The set $\mathcal{V}_1^+(3)$ recovers A339129, and starts with

$$9, 49, 63, 121, 169, 289, 323, 361, 377, 441, 529, 841, 961, 1127, 1369, 1681, 1849, 1891,$$
$$2209, 2303, 2809, 2961, 3481, 3721, 3751, 3827, 4181, 4489, 4901, 4961, 5041, 5329, 5491,$$
$$5777, 6137, 6241, 6601, 6721, 6889, 7381, 7921, 8149, 9409, 10201, 10609, 10877, 10933,$$
$$11449, 11663, 11881, 12769, 13201, 13981, 14027, 15251, 16129, 17119, 17161, \ldots.$$

- The set $\mathcal{V}_2^+(3)$ giving A339521, has the first elements

$$21, 203, 323, 329, 377, 451, 609, 861, 1001, 1081, 1183, 1547, 1729, 1819, 1891, 2033,$$
$$2211, 2821, 3081, 3549, 3653, 3827, 4089, 4181, 4669, 5671, 5777, 5887, 6601, \ldots.$$

- The set $\mathcal{V}_3^+(3)$ is given by A339728 and starts with the elements

$$9, 21, 27, 63, 161, 189, 207, 261, 287, 323, 341, 377, 671, 783, 861, 901, 987, 1007,$$
$$1107, 1269, 1281, 1287, 1449, 1853, 1891, 2071, 2241, 2407, 2431, 2501, 2529, 2567,$$
$$2743, 2961, 3201, 3827, 4181, 4623, 5029, 5473, 5611, 5777, 5781, 6119, 6601, \ldots.$$

Recall that $U_n(1,-1) = F_n$ and $V_n(1,-1) = L_n$, while $U_n(3,1) = F_{2n}$ (A001906) and $V_n(3,1) = L_{2n}$ (A001906) represent the bisection of Fibonacci and Lucas sequences, respectively. The numerical results suggest the following two conjectures.

Conjecture 4. $\mathcal{U}_1^-(1) \subset \mathcal{U}_1^+(3)$. *Notice that the terms of* $\mathcal{U}_1^-(1)$ *(Fibonacci pseudoprimes)*

$$323, 377, 1891, 3827, 4181, 5777, 6601, 6721, 8149,$$

can be found amongst the elements of $\mathcal{U}_1^+(3)$.

Conjecture 5. $\mathcal{V}_1^-(1) \subset \mathcal{V}_1^+(3)$. *One may notice that the elements of* $\mathcal{V}_1^-(1)$ *smaller than* 10000 *also belong to the set* $\mathcal{V}_1^+(3)$.

Note that for $a = 5$ and $a = 7$, the values $D = 21$ and $D = 45$ are not prime.

Example 9. $b = 1$, $a = 5$, $D = 21$.

- The set $\mathcal{U}_1^+(5)$ recovers the entry A340098 in [15], starting with

$$115, 253, 391, 527, 551, 713, 715, 779, 935, 1705, 1807, 1919, 2627, 2893, 2929, 3281,$$
$$4033, 4141, 5191, 5671, 5777, 5983, 6049, 6479, 7645, 7739, 8695, 9361, 11663, \ldots.$$

- The set $\mathcal{U}_2^+(5)$ recovers A340123 and its first elements are

$$25, 115, 125, 253, 275, 391, 425, 505, 527, 551, 575, 625, 713, 715, 775, 779, 935, 1705,$$
$$1807, 1919, 2525, 2627, 2875, 2893, 2929, 3125, 3281, 4033, 4141, 5191, 5555, \ldots.$$

- The set $\mathcal{U}_3^+(5)$ is indexed as A340240 and its first elements are

$$55, 407, 527, 529, 551, 559, 965, 1199, 1265, 1633, 1807, 1919, 1961, 3401, 3959, 4033,$$
$$4381, 5461, 5777, 5977, 5983, 6049, 6233, 6439, 6479, 7141, 7195, 7645, 7999, \ldots.$$

- The set $\mathcal{V}_1^+(5)$ recovers A339130, and starts with

$$25, 121, 169, 275, 289, 361, 527, 529, 551, 575, 841, 961, 1369, 1681, 1807, 1849, 1919, 2209, 2783, 2809, 3025, 3481, 3721, 4033, 4489, 5041, 5329, 5777, 5983, 6049, 6241, 6479, 6575, 6889, 7267, 7645, 7921, 8959, 8993, 9361, 9409, 9775, \ldots.$$

- The set $\mathcal{V}_2^+(5)$ giving A339522, has the first elements

$$95, 115, 145, 253, 391, 527, 551, 713, 715, 779, 935, 1045, 1615, 1705, 1805, 1807, 1919, 2185, 2627, 2755, 2893, 2929, 2945, 3281, 4033, 4141, 4205, \ldots.$$

- The set $\mathcal{V}_3^+(5)$ is given by A339729 and starts with the elements

$$25, 55, 85, 115, 155, 187, 253, 275, 341, 407, 527, 551, 559, 575, 851, 925, 1199, 1265, 1633, 1775, 1807, 1919, 1961, 2123, 2507, 2635, 2641, 2725, \ldots.$$

Example 10. *$b = 1$, $a = 7$, $D = 45$.*

The following sequences of pseudoprimes are obtained.

- The set $\mathcal{U}_1^+(7)$ recovers the entry A340099 in [15], starting with

$$323, 329, 377, 451, 1081, 1771, 1819, 1891, 2033, 3653, 3827, 4181, 5671, 5777, 6601, 6721, 7471, 7931, 8149, 8557, 10877, 11309, 11663, 13201, 13861, 13981, 14701, \ldots.$$

- The set $\mathcal{U}_2^+(7)$ recovers A340124 and its first elements are

$$49, 323, 329, 343, 377, 451, 1081, 1127, 1771, 1819, 1891, 2033, 2303, 2401, 3653, 3827, 4181, 5671, 5777, 6601, 6721, 7471, 7931, 8149, 8557, 9691, 10877, 11309, \ldots.$$

- The set $\mathcal{U}_3^+(7)$ is indexed as A340241 and its first elements are

$$161, 323, 329, 341, 377, 451, 671, 901, 1007, 1079, 1081, 1271, 1819, 1853, 1891, 2033, 2071, 2209, 2407, 2461, 2501, 2743, 3653, 3827, 4181, 4843, 5473, 5611, 5671, \ldots.$$

- The set $\mathcal{V}_1^+(7)$ recovers A339131, and starts with

$$49, 121, 169, 289, 323, 329, 361, 377, 451, 529, 841, 961, 1081, 1127, 1369, 1681, 1819, 1849, 1891, 2033, 2209, 2303, 2809, 3481, 3653, 3721, 3751, 3827, 4181, 4489, 4901, 4961, 5041, 5329, 5491, 5671, 5777, 6137, 6241, 6601, 6721, 6889, 7381, 7921, \ldots.$$

- The set $\mathcal{V}_2^+(7)$ giving A339523, has the first elements

$$91, 203, 323, 329, 377, 451, 1001, 1081, 1183, 1547, 1729, 1771, 1819, 1891, 1967, 2033, 2093, 2639, 2821, 3197, 3311, 3653, 3731, 3827, 4181, 4669, \ldots.$$

- The set $\mathcal{V}_3^+(7)$ is given by A339730 and starts with the elements

$$49, 161, 287, 323, 329, 341, 377, 451, 671, 737, 901, 1007, 1079, 1081, 1127, 1271, 1363, 1541, 1819, 1853, 1891, 1927, 2033, 2071, 2303, 2407, 2431, 2461, 2501, 2567, 2743, \ldots.$$

Conjecture 6. *If a and k are positive integers with a odd, then $\mathcal{U}_k^+(a)$ and $\mathcal{V}_k^+(a)$ are infinite.*

4. Conclusions and Further Work

In this paper we have analyzed the Fibonacci pseudoprimes of level k, and we have formulated an analogous version of this concept for the Lucas numbers (Section 2.2).

In Section 3 we have generalized these notions for Lucas $\{U_n(a,b)\}_{n\geq 0}$, and generalized Pell–Lucas sequences $\{V_n(a,b)\}_{n\geq 0}$, obtaining the generalized Lucas and Pell–Lucas pseudoprimes of levels k^- (for $b=-1$) and k^+ (for $b=1$) and parameter a. For these concepts, it was known from [19], that under the supplementary condition $n \mid U_n^2 - 1$, the pseudoprimes of levels 1^- and 2^-, and 1^+ and 2^+, respectively, coincide.

The purpose of this paper has been threefold. First, to calculate the explicit values of these pseudoprimes for levels $k = 1,2,3$, for $b=-1$ with $a = 1,3,5,7$ and for $b=1$ with $a = 3,5,7$. This effort led to numerous new additions to OEIS. Second, we have shown that reciprocal statements for Theorems 4 and 5 do not hold, providing a range of counterexamples (Remark 4 and Remarks 7 and 8, respectively). Thirdly, we have shown that the transition from levels 1^- and 2^- to level 3^- (and from levels 1^+ and 2^+ to 3^+, respectively) cannot be guaranteed in general, even under the supplementary condition $n \mid U_n^2 - 1$ (Remarks 5 and 7, respectively).

An interesting problem for further investigation is the connection between the generalized Lucas and Pell–Lucas pseudoprimes of levels k^- and k^+ and parameter a, and the weak pseudoprimality concepts defined in [7].

Numerous open problems remain to be solved, as seen from Conjectures 1, 2, 3, 4, 5, or 6. Another interesting direction for further study, suggested by one of the referees, was to explore whether any odd composite integer could be a pseudoprime of a given level, or to find the smallest such integer that cannot be a pseudoprime at all. We invite the readers to join us in trying to solve these problems.

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