# Szemerédi's Proof Nonstandardized and Simplified

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$\Delta_{13}$–Workshop On Logic, Zoom, May 16, 2020

# Introduction

**Theorem (van der Waerden, 1927)**

*Given any $k, n \in \mathbb{N}$, there exists a $W(k,n) \in \mathbb{N}$ such that if $\{V_1, V_2, \ldots, V_n\}$ is a partition of $\{1, 2, \ldots, W(k,n)\}$, then there is an $i \leq n$ such that $V_i$ contains a $k$–term arithmetic progression.*

**Conjecture (P. Erdős and P. Turán, 1936)**

*If $X \subseteq \mathbb{N}$ has a positive upper density, then $X$ contains a $k$–term arithmetic progression for every $k \in \mathbb{N}$.*

In 1953 K. F. Roth gave a proof of the conjecture for $k = 3$ using harmonic analysis. In 1975, E. Szemerédi gave a combinatorial proof of the full conjecture. Hence the conjecture is now called Szemerédi's Theorem. In 1977 H. Furstenberg gave an ergodic proof of Szemerédi's Theorem. In 2001 T. Gowers gave a harmonic proof of Szemerédi's Theorem with numerical information. All of these proofs are long and complicated.

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# Tao's Efforts

In a workshop *Nonstandard methods in combinatorial number theory* sponsored by American Institute of Mathematics in San Jose, CA, August 2017, T. Tao gave a series of talks to explain the Szemerédi's original combinatorial proof and hope to simplify it so that it can be better understood. He believed that Szemerédi's combinatorial method should have a greater impact in combinatorics.

In the talks T. Tao mentioned that he had tried to present Szemerédi's original combinatorial proof using nonstandard analysis but failed to make the proof simpler than the standard proof. He challenged the audience to produce a nonstandard proof of Szemerédi's Theorem which is noticeably simpler and more transparent than Szemerédi's original proof.

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# Roth's Theorem

Let $X \subseteq \mathbb{N}$. The upper Banach density of $X$ is defined by

$$BD(X) = \lim_{n\to\infty} \sup_{k\in\mathbb{N}} \frac{|(X \cap [k, k+n-1])|}{n}.$$

A $k$–a.p. ($k$-term arithmetic progression), denoted by $p$, is a set of the form

$$p := \{a + id \mid i = 0, 1, \ldots, k-1\}$$

for some integers $a$ and $d$. We also use the term $p(i) := a + (i-1)d$, i.e., the $i$-th term of the $k$–a.p. for each $i \leq k$. We often use $P$ to denote an infinitely long a.p.

**Theorem (K. F. Roth, 1953)**

*If $X \subseteq \mathbb{N}$ and $BD(X) > 0$, then $X$ contains a 3–a.p.*

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# Nonstandard Analysis Crash Course I

**Definition (Standard Universe $(V;\in)$)**

*Fix a sufficiently large finite positive integer $\mathfrak{m}$. Let*

$$V_0 = \mathbb{R}, \quad V_{n+1} = V_n \cup \mathcal{P}(V_n), \quad \textit{and } V = \bigcup_{n=0}^{\mathfrak{m}} V_n.$$

**Definition (Nonstandard Universe $({}^*V;{}^*\in)$)**

*Consider $(V;\in)$ as a structure with one binary relation $\in$. A* **nonstandard universe** $({}^*V;{}^*\in)$ *is a countably saturated elementary extension of $(V;\in)$. Let $* : A \mapsto {}^*A$ be the elementary embedding from $V$ to ${}^*V$.*

We can view ${}^*\mathbb{R}$ as a set of urelements and ${}^*\in$ as a real membership relation $\in$. For convenience we write $a \in \mathbb{R}$ for ${}^*a$ and $\leq$ for ${}^*\leq$ as the natural order of ${}^*\mathbb{R}$. Note that in general ${}^*A \neq A$.

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# Nonstandard Analysis Crash Course II

**Definition (Internal and External Sets)**

*Note that the power set operator $\mathcal{P}$ from $V_{\mathfrak{m}-4}$ to $V_{\mathfrak{m}-3}$ is an element in $V$. A set $X$ is internal if $X \in {}^*\mathcal{P}(V_n)$ for some $n \leq \mathfrak{m} - 3$, and external if $X$ is not internal.*

All sets considered in the rest of the talk are either standard subset of $\mathbb{N}$ or internal subsets of ${}^*\mathbb{N}$. Note that an infinite subset of $\mathbb{N}$ is an external subset of ${}^*\mathbb{N}$. An integer in ${}^*\mathbb{N} \setminus \mathbb{N}$ is called hyperfinite. An internal set with a hyperfinite cardinality is called a hyperfinite set.

For internal set $A$ and positive integer $n$ let $\delta_n(A) := |A|/n$ and $\mu_n(A) := st\,(\delta_n(A))$, where $st$ is the standard part map which maps a real number $r \in (-n, n)$ for some $n \in \mathbb{N}$ in the nonstandard universe to a standard real $\alpha$ such that $r \approx \alpha$, i.e., $r$ is infinitesimally close to $\alpha$.

Note that if $A \subseteq \Omega$ and $|\Omega| = H$, then $\mu_H(A)$ is the Loeb measure of $A$ in $\Omega$.

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# Propositions I

The letter $A, B, C, R, S, T, U, V, \ldots$ represent sets and $H, K, L, M, N$ represent hyperfinite integers. Let $[n] := \{0, 1, \ldots, n-1\}$ for any positive integer $n$.

**Proposition (1)**

*Let $X \subseteq \mathbb{N}$. Then $BD(X) \geq \alpha$ if and only if there exists a hyperfinite interval $a + [N]$ in ${}^*\mathbb{N}$ such that $\mu_N({}^*X \cap (a + [N])) \geq \alpha$.*

**Proposition (2)**

*Let $X \subseteq \mathbb{N}$. Then $X$ contains a $k$–a.p. if and only if ${}^*X$ contains a $k$–a.p.*

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# Propositions II

## Proposition (3)

*Let $\alpha > 0$ be standard positive real number. Then the following are equivalent:*

1. *For any $X \subseteq \mathbb{N}$, $BD(X) > \alpha$ implies that $X$ contains a k–a.p.*
2. *For any hyperfinite $N$ and any $A \subseteq [N]$, $\mu_N(A) > \alpha$ implies that $A$ contains a k–a.p.*
3. *For any $K$–a.p. $P$ for a hyperfinite $K$ and any $A \subseteq P$, $\mu_K(A) > \alpha$ implies that $A$ contains a k–a.p.*

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We assume from now on that Roth's Theorem is not true.

To prove Roth's Theorem, it suffices to derive a contradiction.

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# Propositions III

**Definition**

*The standard real number $\alpha$ is defined by*

$$\alpha := \sup\{\mu_N(A) \mid \exists N, A\,(A \subseteq [N],\ A \text{ contains no 3–a.p.})\}.$$

**Proposition (4)**

$$0 < \alpha < 1.$$

**Proposition (5)**

*There exists $A \subseteq [N]$ such that $\mu_N(A) = \alpha$, $A$ contains no 3–a.p., and $\mu_K(A \cap P) \leq \alpha$ for every $K$–a.p. $P \subseteq [N]$.*

Fix $N$ and $A \subseteq [N]$ such that $\mu_N(A) = \alpha$ and $A$ contains no 3–a.p. We will derive a contradiction by locating a 3–a.p. in $A$.

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# Proof of Roth's Theorem

**Lemma (1)**

*Let* $H = \lfloor N/6 \rfloor$. *Then* $\mu_H(A \cap (x + [H])) = \alpha$ *for each* $x \in [N - H]$.

Note that $\{0\} \cup (H + [H]) \cup (2H + 2[H]) \subseteq [N - H]$ where $2[H] := \{2x \mid x \in [H]\}$.

For each $w \in [H]$ let

$$\mathcal{P}_w = \{p \mid p(1) \in A \cap [H] \text{ and } p(3) = w\}$$

$$\text{and } E_w := \{p(2) \in [H] \mid p \in \mathcal{P}_w\}.$$

**Lemma (2)**

$\mu_H(E_w) = \alpha/2$ *for every* $w \in [H]$.

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*$\mu_H(E_w) = \alpha/2$ for every $w \in [H]$.*

# Proof of Roth's Theorem

**Lemma (1)**

*Let $H = \lfloor N/6 \rfloor$. Then $\mu_H(A \cap (x + [H])) = \alpha$ for each $x \in [N - H]$.*

Note that $\{0\} \cup (H + [H]) \cup (2H + 2[H]) \subseteq [N - H]$ where $2[H] := \{2x \mid x \in [H]\}$.

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**Lemma (2)**

*$\mu_H(E_w) = \alpha/2$ for every $w \in [H]$.*

Let $V, W$ be finite sets and $E_w \subseteq V$ for each $w \in W$. Given any $\epsilon > 0$, there exist a partition $V = V_1 \cup V_2 \cup \cdots \cup V_n$ with $n = O(b^{1/\epsilon})$, and real numbers $0 \leq c_{i,w} \leq 1$ for $i \leq n$ such that for any set $F \subseteq V$, one can find a $T \subseteq W$ with $|W \setminus T| \leq \epsilon|W|$ and

$$\left| |F \cap E_w| - \sum_{i=1}^{n} c_{i,w} |F \cap V_i| \right| \leq \epsilon |V|$$

for all $w \in T$.

# Mixing Lemma

1. For a set $E \subseteq [H]$ with $\mu_H(E) > 0$ and a $K$–a.p. $P \subseteq H + [H]$, there is an $x \in P$ such that
$$\mu_H(A \cap (x + E)) \geq \alpha \mu_H(E);$$
2. Given a $K$–a.p. $P \subseteq H + [H]$, let $m$ be hyperfinite such that $W(3^m, m) \leq K$. For any internal partition $\{V_i \mid i \in [m]\}$ of $[H]$ there exists a $m$–a.p. $P' \subseteq P$, a set $I \subseteq [m]$ with $\mu_H(U) = 1$ where $U = \bigcup\{V_i \mid i \in I\}$, and an infinitesimal $\epsilon > 0$ such that for all $i \in I$ and all $x \in P'$
$$|\delta_H(A \cap (x + V_i)) - \alpha \delta_H(V_i)| \leq \epsilon \delta_H(V_i);$$
3. Given a $K$–a.p. $P \subseteq H + [H]$ and an internal collection of sets $\{E_w \subseteq [H] \mid w \in [H]\}$ with $\mu_H(E_w) > 0$ for every $w \in [H]$, there exists an $x \in P$ and a set $T \subseteq [H]$ such that $\mu_H(T) = 1$ and for every $w \in T$,
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By (iii) of the mixing lemma we have a set $T \subseteq [H]$ with $\mu_H(T) = 1$ and an $I \in H + [H]$ such that

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Let $v_0 \in E_{w_0}$ be such that $H + I + v_0 \in A \cap (H + I + E_{w_0})$. By the definition of $E_{w_0}$ there is a 3–a.p. $p$ such that $p(3) = w_0$, $p(2) = v_0$, and $p(1) \in A$. Now

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**Theorem (E. Szemerédi, 1969)**

*Let $k \in \mathbb{N}$. If $U \subseteq \mathbb{N}$ has a positive upper Banach density, then $U$ contains 4–a.p.'s.*

**Sketch of Proof:** Let $U \subseteq \mathbb{N}$. Define the strong upper Banach density of $U$ by

$$SD(U) = \lim_{n \to \infty} \sup\{\mu_n(U \cap P) \mid P \text{ is an } n\text{–a.p.}\}.$$

Let $\alpha = SD(U)$. There is a $K$–a.p. $P$ such that $\mu_{|P|}({}^*U \cap P) = \alpha$. Let $A = {}^*U \cap P$. We look for a 4–a.p. in $A$. By an affine transform we can assume that $P = [N]$.

For sufficiently large $n \in \mathbb{N}$ and $j \in \mathbb{N}$ let $S_{\gamma_n} := \{x \in [N] \mid \mu_n(A \cap (x + [n]) \geq \alpha - 1/j$ and $\mu_N(S_{\gamma_n}) = \gamma_n$. Note that if $n \to \infty$, then $\gamma_n \to 1$. Choose a $\tau_n \subseteq [n]$ such that $B_{\tau_n} = \{x \in S_{\gamma_n} \mid (x + [n]) \cap A = x + \tau_n\}$ with $\mu_N(B_{\tau_n}) > 0$.

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# Proof continued

By an argument similar to the proof of Roth's Theorem we can find a hyperfinite interval $x_n + [H_n] \subseteq [N]$, a set $T_n \subseteq (x_n + [H_n]) \cap S_{\gamma_n}$ with $\mu_H(T_n) \geq 3\gamma_n - 2$, and a collection $\mathcal{P}_n := \{p_x \mid x \in T_n\}$ of 4–a.p.'s such that $p_x(1), p_x(2) \in B_{\tau_n}$, $p_x(3) = x$, and $p_x(4) \in S_{\gamma_n}$ for every $x \in T_n$.

By countable saturation we can assume that $n$ is hyperfinite. with $\gamma := \gamma_n \approx 1$, $x' + [H] := x_n + [H_n]$, $\mu_N(S_\gamma) = 1$, $\mu_H(T) := \mu_H(T_n) = 1$, and $\mu_n((x+[n]) \cap A) = \alpha$ for every $x \in S_\gamma$. Since $\mu_H(T) = 1$, $T$ contains a $K$–a.p. $P$ of consecutive integers.

By an argument similar to the proof of Roth's Theorem again, we can find an interval $y_0 + [h] \subseteq [n]$ with $h = \lfloor n/8 \rfloor$, a set $T' \subseteq y_0 + [h]$ with $\mu_h(T') = 1$, a collection $\mathcal{Q}_w$ of 4–a.p.'s in $[n]$ for each $w \in T'$ such that $q(1), q(2) \in \tau$, $q(4) = w$, and $E_w = \{q(3) \mid q \in \mathcal{Q}_w\}$ satisfying $\mu_n(E_w) > 0$.

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The End

Thank you for your attention