CALIFORNIA STATE UNIVERSITY, NORTHRIDGE

RECTILINEAR CROSSING NUMBERS OF COMPLETE GRAPHS WITH SPECIFIC NESTED SEQUENCE OF CONVEX HULLS

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science in Mathematics

by

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August 2013

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Dedication

For Papito, Mamita and Mami because their endless love keeps me moving forward, and arduous struggles have allowed me to dream.

Acknowledgements

A very special thank you to Dr. Silvia Fernández for her time, dedication, patience, and encouragement allowed for the completion of this thesis. She is an incredible mentor, and tremendous role model. Her passion for mathematics is eminent, and it was an honor to work with some one as brilliant as she. I am grateful for this. I would also like to thank Dr. Bernardo Ábrego and Dr. Csaba Tóth for taking the time to be in my thesis committee, and for improving this work. I thank you deeply for your effort and commitment.

I would like to acknowledge and thank everyone I have encountered at CSUN. Each person has made a difference and has taught me a lot. Karen Abramowitz was an amazing first boss – I would like to thank her for trusting me, and allowing me to learn. I am thankful for all the opportunities Dr. Werner Horn offered me – for the opportunities as a TA, for the math seminars, and for allowing me to participate in numerous conferences, which broaden my passion for math. A thank you to all of my professors at the math department because it has been a pleasure to lean a beautiful subject from amazingly passionate professionals. To the staff of the math department office – thank you for welcoming me with open arms when I began my graduate work.

Every member of my family has made a small, but important difference in my life. Their love, support and advice have been with me every step of the way. I thank and love each and every one of them. To the late Javier A. Dúbon, to whom I am eternally grateful.

To the incredible, Eddie T. Thank you for guiding me through my last steps at CSUN, and for being the great human being only you know how to be; your guidance and support made the finish line a lot visible.

Preface

Around the summer prior to the start of my second year of college, I received an email asking if I was interested in working year-round project – signed Professor Fernández. I was not sure why she had emailed me, or whether the offer was only for me. I had not heard of this professor prior to the email, and I was hesitant to reply because I was scared of my lack of potential. I did not understand what the project was about nor any other detail about the execution - I decided it was not for me. Unbeknownst to me, the project became my thesis, and the email was from my advisor.

I continued to be hesitant about the project until one of my good friends was telling me about a project he started to work on and suggested I join the group. I decided to abide and attended his meeting; to my surprise he was meeting with Professor Fernández. I started working and getting involved with the project. I now realize that the email was meant for me because it was the beginning of what became my thesis.

This project has motivated me to attend conferences and presentations in which I have participated. I have had the pleasure of talking with professors at various universities about the project, and I have enjoyed all the conversations. With this, I feel proud of what I have accomplished at this moment in in my life.

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ABSTRACT

RECTILINEAR CROSSING NUMBERS OF COMPLETE GRAPHS WITH SPECIFIC NESTED

SEQUENCE OF CONVEX HULLS

By

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Master of Science in Mathematics

Let P be a set of *n* points in the plane. Draw all segments joining pairs of points in P. We are interested in the number of segment-intersections, or *crossings*, in such a drawing. For a fixed n, the problem of minimizing the number of crossings over all sets of *n* points in the plane is a famous unsolved problem in Combinatorial Geometry. Paul Turán posed the problem for *complete bipartite graph* because as a prisoner in a concentration camp, Turán's job was to transport bricks using a railway system - the rail crossings made it extremely ineffective because the bricks would often fall. We consider a variation of Turán's problem. We classify all sets of *n* points into classes according to the sizes of their *convex layers* and consider the minimum number of crossings over sets within the same class. We bound the minimum number of crossings for every class with two convex layers finding its exact value when the inner layer has one or two points and a full classification of the optimal sets. We also give exact values of the minimum number of crossings for all classes with up to 8 points, and bounds for classes with 9 points.

Chapter 1

Introduction

In this thesis we explore the rectilinear crossing number of a drawing. This is a variation of an open problem in combinatorics, which originated a long time ago, and although no precise date is given, we know mathematicians have worked on the crossing number of graphs for a long time. In the original problem, given a set of points, or vertices, in the plane, you are interested in finding the amount of times the edges, joining each pair of points, intersect each other. There is no restriction as to the amount of vertices, their arrangement nor the types of edges. There are many variations of this problem; some explore the crossing number of a graph on a box when only certain vertices are joined, graphs whose vertices are along the crest of a book with edges on the pages of that book, and others investigate the crossings of graphs on a plane with different restrictions. It is invigorating to attempt to understand the different types of graphs and drawings of a graph because they open another world of mathematics. In addition to the beauty and wonder, this has several applications – some of the more common applications are found in VLSI.

In this thesis, we focus on specific drawings in which every pair of vertices is joined by an edge represented by a straight line segment. We also only explore certain arrangements the vertices - we ask that the vertices are in convex layers, and we bound the crossing number for any amount of points for the specified convex layers. Ideal results would be to find the exact crossing number for every fixed sequence of convex layers; however, it is difficult to achieve this. Consequently, we find the exact crossing number for all possible arrangements of up to 8 points, and provide general bounds for arrangements with two convex layers.

The focus of Chapter 2 is to provide a fundamental knowledge of the concepts used throughout the rest of the thesis. Definitions, examples, and results are provided with regards to what was known about some crossing numbers. The topological crossing and rectilinear crossing numbers of the complete graph and complete bipartite graph are defined. We also briefly mention known conjectures, these crossing numbers, and its applications.

In Chapter 3, we focus on a few more topics before explaining the path this thesis will take. We illustrate the ideas we will use for achieving our results, and how the sets of points will be considered all throughout.

Furthermore, once the ideas are established in the previous chapter, Chapter 4 will illustrate the results. The focus of the thesis is rectilinear crossing number (RCN) of the complete graph; consequently, results for this will be illustrated. The first section is trivial, but necessary for the proofs that will follow. The second and third sections specify and prove the general lower and upper bound on the RCN of classes with two layers, respectively. The following section describes an implication of the theorems from the previous two sections, which provides the exact crossing number for any arrangement with 1 point in the second layer.

This chapter is broken down into sections, and each is directed to present the exact RCN for some classes. In this chapter, we focus on detailing the exact crossing number for arrangement with at most 8 points, and bounds for classes with 9 points. Tables illustrate the exact values, constructions are provided, and some proofs are included to bound the remaining cases.

Chapter 2

Crossing Numbers

2.1 Graphs

A *graph* G is a nonempty ordered pair $(V(G), E(G))$ where $V(G)$ is a set of *vertices* and $E(G)$ is a set of *edges* in which each edge is an unordered pair of vertices. The following

$$V(G) = \{1, 2, 3, 4, 5, 6\} \text{ and } E(G) = \{\{1, 4\}, \{1, 6\}, \{2, 5\}, \{3, 5\}, \{4, 6\}, \{5, 6\}, \{1, 1\}\}$$

are examples of a set of vertices and a set of edges, respectively. This is an example of a graph $G = (V(G), E(G))$ with 6 vertices and 7 edges. [5]

A *simple graph* is a graph having no loops – edges connecting a vertex to itself – or multiple edges (an edge that appears multiple times in the same graph). Graphs are usually represented by drawings in the plane: the vertices are points and edges are lines or curves joining the vertices. (A more precise definition of drawings is given below.) In Figure 2.1 (left), G is not a simple graph because it has the loop $\{1,1\}$, however, $H = (V(H), E(H))$ where

$$V(H) = \{A, B, C, D, E\} \text{ and } E(H) = \{\{A, C\}, \{A, E\}, \{B, D\}, \{B, E\}, \{C, D\}, \{D, E\}\}$$

is a simple graph.

The *complete graph* on n vertices K_n is a graph in which each pair of distinct vertices is joined by an edge. A graph where the set of vertices is partitioned into two classes and only edges from one class to the other are included is known as a *bipartite graph.*

A *complete bipartite graph* is a graph in which the set of vertices is partitioned into two classes and all edges from one class to the other are included – edges between the points of one of the sets of vertices are not part of the graph. If the classes have sizes n and m, then the complete bipartite graph is called $K_{n,m}$.

2.2 Drawings of a graph

Graphs are usually visualized by drawings in the plane. More precisely, a *drawing* D of a graph G, in the plane, is a representation of G such that the vertices are represented by distinct points and the edges are represented by Jordan arcs that connect the corresponding two vertices. Figure 2.1 shows a drawing of the graphs G and H above.

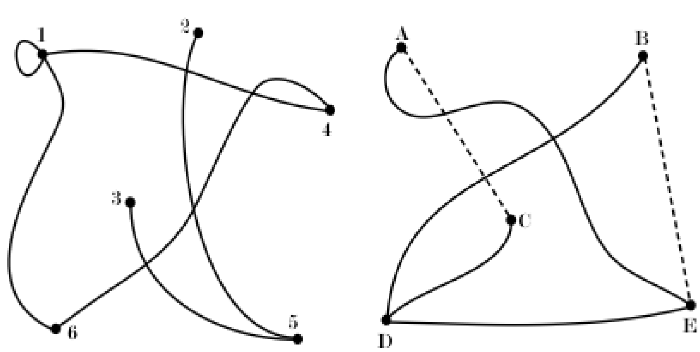


Figure 2.1: A drawing D of G (*left*). A drawing D of H (*right*).

Graphs could be used to describe many situations – telephones could be depicted as points, and telephone lines as lines joining telephones together; or points could be representations of mathematicians with edges joining pairs of mathematicians that have published together. Drawings ease the understanding of the structure of a graph. Yet, any graph has infinitely many drawings and some drawings are more helpful than others in a given situation. Topological graph theory studies the drawings of a fixed graph G. We ask that in any drawing the following conditions hold. (a) Two edges intersect each other at most a finite number of times, (b) no three edges have a common point other than a vertex, (c) any edge intersects exactly two vertices, namely the endpoints it connects, and (d) two edges can not have partial overlap. See Figure 2.2.

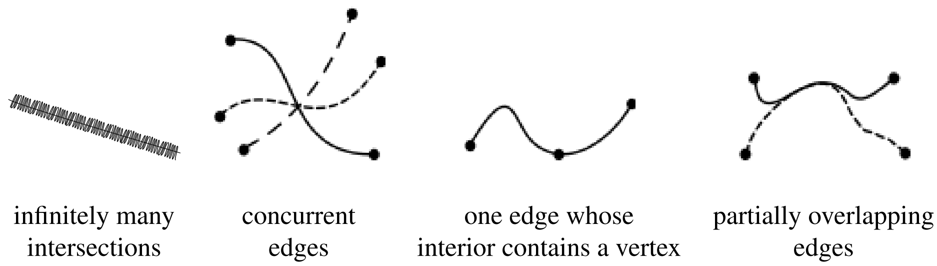


Figure 2.2: Forbidden configurations for a drawing.

A drawing of a graph in which only straight line segments are used is called a *rectilinear drawing*. (See Figure 2.3.) Figure 2.4 shows a drawing of a complete rectilinear bipartite graph $K_{3,4}$.

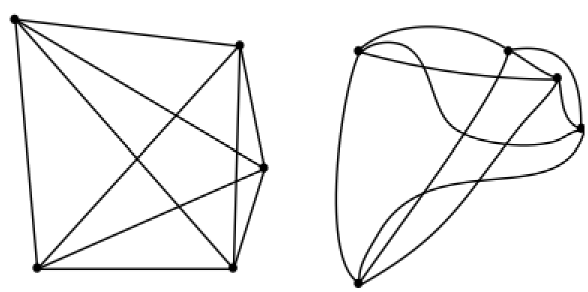


Figure 2.3: Complete rectilinear (and topological) drawing of K_5 (*left*). Complete topological (non rectilinear) drawing of K_5 (*right*).

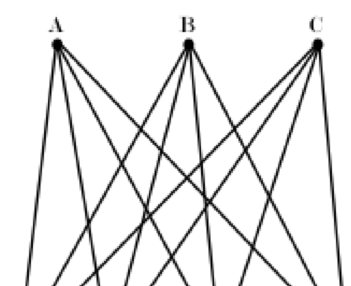


Figure 2.4: A complete rectilinear bipartite graph, $K_{3,4}$.

2.3 Crossing number of a graph

Any two edges in a graph satisfy at least one of the following (See Figure 2.5): (a) they are *incident* (they share a vertex), (b) they are *disjoint* (they have nothing in common), (c) they are *tangent* (they touch at exactly one point), or (d) they *cross* (they intersect at a point) and then we say they form a *crossing*.

(a) incident edges (b) disjoint edges (c) tangent edges (d) intersect at a point

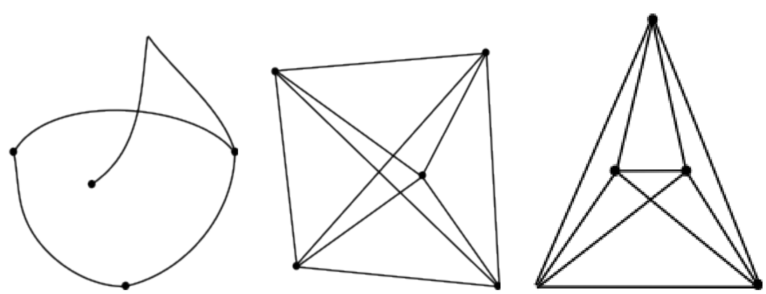
Figure 2.5: Drawings of a graph and crossings.

Among all drawings of a graph G, we are interested in the ones with the least number of crossings. We define the *crossing number* of a graph G as

$$cr(G) = \min\{cr(D) : D \text{ is a drawing of } G\}.$$

We abuse notation (commonly done) and use $cr(D)$ to denote the crossings in a drawing, and use $cr(G)$ to denote the minimum number of crossings over all drawings of a graph G.

Figure 2.3 shows drawings of K_5. Kuratowski's Theorem implies that any drawing of K_5 has at least one crossing, namely $cr(K_5) \geq 1$. Figure 2.6c shows a drawing with one crossing so $cr(K_5) \leq cr(D_3) = 1$ therefore $cr(K_5) = 1$. Figure 2.7 shows some drawings of graphs with 5 points and the number of crossings for each drawing. Note that in Figure 2.7, the second drawing has no crossings, and so we know that the minimum number of crossings for three points is 0 thus $cr(G) = 0$.



(a) $cr(D_1) = 1$ (b) $cr(D_2) = 3$ (c) $cr(D_3) = 1$

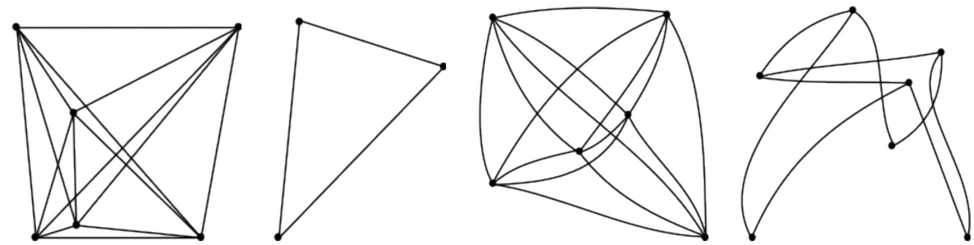
Figure 2.6: Number of crossings $Cr(D_i)$ for different drawings D_i of graphs with 5 points.

In the first two drawings of Figure 2.7, all edges are drawn as straight line segments. In 1963, Hill and Harary defined the *rectilinear crossing number* (RCN) of a graph G, denoted $\overline{cr}(G)$ as the minimum number of crossings among all drawings of a graph G where every edge is a straight line segment [7].

We concentrate on $cr(K_n)$, $\overline{cr}(K_n)$, $cr(K_{n,m})$, $\overline{cr}(K_{n,m})$.

2.3.1 The topological crossing number of the complete graph, $cr(K_n)$

Anthony Hill, a British artist who considered himself a "constructivist working as a geometric formalist," explored different types of geometrical and combinatorial objects, which lead him to write dozens of papers in graph theory. Hill started by drawing random sets of points in the plane, joined each pair with a curve, and



$cr(D_1) = 8$ $cr(D_2) = 0$ $cr(D_3) = 8$ $cr(D_4) = 7$

Figure 2.7: Number of crossings in a drawing of a graph.

investigated the number of times the curves intersected [3]. Essentially, Hill was interested in the crossing number of the complete graph [7].

Hill conjectured that the exact value of the crossing number of K_n is

$$cr(K_n) = Z(n) := \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor = \begin{cases} \frac{1}{64}(n-1)^2(n-3)^2, & n \text{ odd} \\ \frac{1}{64}n(n-4)(n-2)^2, & n \text{ even} \end{cases}$$

Hill constructed drawings of K_n with exactly $Z(n)$ crossings giving the upper bound of $Z(n)$ on $cr(n)$. Proving that $Z(n)$ is a lower bound is a very famous problem in combinatorics. The only known exact values of $cr(K_n)$ for $n \geq 5$ are [3]

n	5	6	7	8	9	10	11	12
$cr(K_n)$	1	3	9	18	36	60	100	150

and the best known lower bound is $0.8594Z(n) \leq cr(n)$, De Klerck et al.

Simulations between interconnections can be represented as a graph embedding problem in which one can determine the amount of crossings in a VLSI (Very Large Scale Integration) chip. In these problems you are studying the connections between a set of points and another. In a book embedding, the nodes of the graph G are mapped to the spine of a book and the edges of G are mapped onto pages so the edges on the same page do not cross [4]. A k-page book drawing is a drawing where all the vertices are on a line ℓ (the spine) and each edge is fully contained in one of the k-half planes (pages) defined by ℓ. Similarly to the definition of $\overline{cr}(n)$, the k-page crossing number, denoted by $\nu_k(K_n)$ is the minimum number of crossings determined by a k-page book drawing of K_n.

2.3.2 The topological crossing number of the complete bipartite graph, $cr(K_{n,m})$

The problem of determining the crossing number of the complete bipartite graph can be traced back to World War II when Paul Turán's job, as a prisoner in a concentration camp, was to transport bricks from the kilns to a storage connected by a railway system, which crossed in multiple places. The transportation was not difficult; however, when the carts would go through these intersection, some carts would jump, fall and conceivably some bricks break. This created problems because each prisoner had to load a certain amount of carts, and their day would get delayed if any of the bricks would break. Turán realized that there should, ideally, be fewer crossings, but the solution seemed to be very difficult [3]. Formally, he had to minimize the number of crossings in $K_{n,m}$, where n was the number of kilns and m was the number of storage yards. Later, after the war ended, Turán presented the problem to mathematicians in Poland. The topologist, Kazimirez Zarankiewicz conjectured that

$$cr(K_{n,m}) = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor . \tag{2.1}$$

but his conjecture still remains open [3].

2.3.3 The RCN of the complete graph, $\overline{cr}(K_n)$

In his paper, Hill defined the rectilinear crossing number, and compared the known values for the minimum crossing number of a complete graph and the rectilinear crossing number of a complete graph. He found a formula for $\bar(cr)(K_n)$, shown below, which is an upper bound. [7].

n	2	3	4	5	6	7	8	9	10
$cr(K_n)$	0	0	0	1	3	9	18	36	60
$\overline{cr}(K_n)$	0	0	0	1	3	9	19	36	63

Hill conjectured that the rectilinear crossing number $\overline{cr}(K_n)$ exceeds $cr(K_n)$ for $n = 8$ and all $n \geq 10$. These problems have shown to be extremely difficult. As an example, the exact values of $\overline{cr}(K_n)$ are only known for $n \leq 27$ and the upper third of these values were discovered only within the last couple of years [1].

2.3.4 The RCN of the complete bipartite graph $\overline{cr}(K_{n,m})$

Turán's brick factory problem is an example of a bipartite graph, but now, we only look at graphs in which every pair of vertices is joined by an edge represented by a straight line segment. These graphs are the same as the topological bipartite graph in the sense of its properties. Consequently, the results are the same, and Equation 2.1 still holds.

Chapter 3

Problem Statement

We are interested in drawings with few edge-crossings. In this case, it is enough to consider good drawings of a graph, that is, drawings that satisfy the following: 1) Any two disjoint edges intersect at most once, 2) an edge contains a vertex if and only if it is one of its endpoints, and 3) any two incident edges do not intersect at a point other than their common end point.

In a non-good drawing, it is conceivable to have two edges that intersect two or more times. If this is true then small perturbations can eliminate crossings without affecting the rest of the drawing. (See [8] for examples on small perturbations and [9] for a formal definition of good drawings.) Thus the original drawing did not achieve the minimum number of crossings as it could have more crossings than the new drawing; consequently, the original drawing may be disregarded. Furthermore, with these modifications, it is proven that the drawings that achieve the minimum number of crossings are good drawings and our attention can be restricted to only good drawings.

We are interested in the minimum number of crossings among a special class of rectilinear drawings of the complete graph.

All sets of points considered throughout this thesis are finite, in general position, and in the plane.

3.1 Convex Hull

Let P be a set of points. A point $x \in P$ is called an *extreme point* of P if there is a line m through x that leaves the rest of P on the same side of m. The *convex hull* of P is the smallest convex set that contains P; consequently, the set of external points is the set of vertices of the convex hull. Partition the set P into *convex hull layers*, or simply *layers*, as follows: The first layer, denoted by $L_1(P)$, is the convex hull of P; and for $j > 1$ the j^{th} layer of P, denoted by $L_j(P)$, is the convex hull of $P - \bigcup_{i=1}^{j-1} L_i(P)$. Note that the layers are disjoint subsets of P and each nonempty layer consists of at least three points, except perhaps for the most inner one which may have 1 or 2 points. Thus if P has n points, then there are at most $\lceil n/3 \rceil$ nonempty layers. We say that P belongs to the *class* $[|L_1(P)|, |L_2(P)|, \ldots, |L_k(P)|]$, where k is the largest index such that $L_k(P)$ is nonempty and $|L_1(P)| + |L_2(P)| + \ldots + |L_k(P)| = |P|$. We are interested in finding the minimum number of crossings over all sets P in the same class. We use the notation $\overline{cr}[C] = \overline{cr}[n_1, n_2, \ldots, n_k]$ for the minimum crossing number over all sets P is the class $[C] = [n_1, n_2, \ldots, n_k]$. Note that

$$\overline{cr}(n) = \min \{\overline{cr}[n_1, n_2, \ldots, n_k] : n_1, n_2, \ldots, n_{k-1} \geq 3,\ n_k \geq 1,\ n_1 + n_2 + \ldots + n_k = n\}. \quad (3.1)$$

We partition the collection of all n-points sets into a finite number of classes according to their convex layers and restrict the function $\overline{cr}$ to each class as previously described.

In chapter 4, we consider classes with two nonempty layers, that is, classes of the form $[p, q]$ with $p \geq 3$ and $q \geq 1$. We present explicit constructions to provide upper bounds for $\overline{cr}[p, q]$, and prove that for $q = 1$ and 2 these constructions are optimal, thus finding the exact value of $\overline{cr}[p, 1]$ and $\overline{cr}[p, 2]$. We also bound $\overline{cr}[p, q]$ for all $q \geq 3$. Finally, in chapter 5, we find the exact value of $\overline{cr}[C]$, for any class $[C]$ with at most 8 points, and include the drawings of optimal constructions.

Chapter 4

Results: RCN for classes with one or two layers

We start with some definitions. Let P be a set of n points. Any line l that divides the set P in almost half is called a *halving line* of P, that is, there are $\lfloor \frac{n}{2} \rfloor$ point of P on one side of l and $\lceil \frac{n}{2} \rceil$ point on the other. (See Figure 4.1)

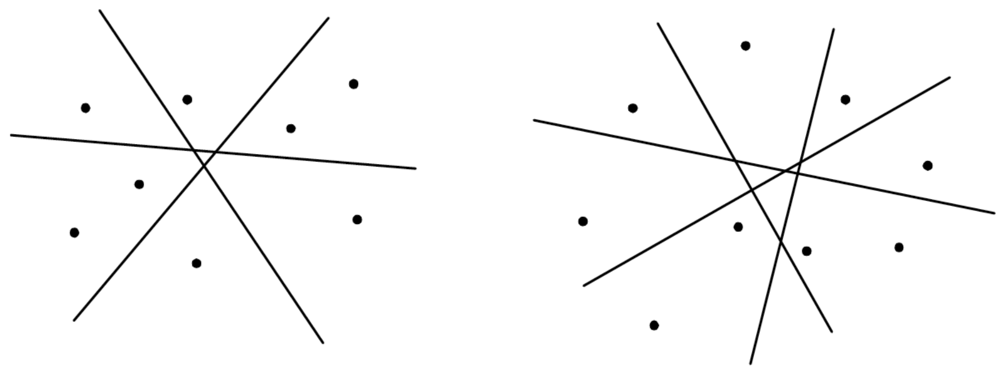


Figure 4.1: *Left*. A set with 8 points, so each line halves the set leaving 4 points on each side. *Right*. A set with 9 points, so each line halves the set leaving 3 points on one side and 5 on the other.

The previous definition is the typical definitions of a halving line. However, a special case of halving lines is when the line passes through two points of P. A line l passing through two points of P that divides the rest of the set P in half is a halving line of P, and there are $\lfloor \frac{n}{2} \rfloor - 1$ points of P on one side of l and $\lceil \frac{n}{2} \rceil - 1$ points on the other. (See Figure 4.2.)

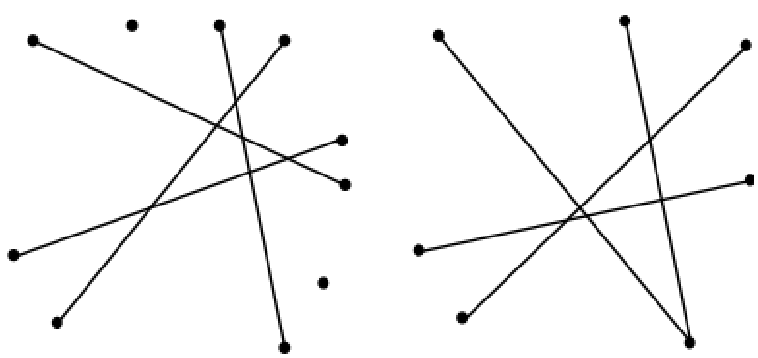


Figure 4.2: Some halving lines of two sets. *Left*. A set with 10 points, so each halving line leaves 4 points on each side. *Right*. A set with 7 points, so each halving line halves 3 points on one side and 2 on the other.

Let B be a finite set of points in convex position. The *interior* of B is the interior of the convex hull of B. Suppose now that B has exactly p points. Label the vertices of B from 1 to p in counter-clockwise order. For odd p and for every $1 \leq i \leq p$ consider the triangle T_i with vertices i, $i + \frac{p+1}{2}(\mathrm{mod} p)$, and $i + \frac{p-1}{2}(\mathrm{mod} p)$. (See Figure 4.3.) Let C_B be the intersection of $T_1, T_2, T_3, \ldots, T_p$. Similarly, for even p and for every $1 \leq i \leq p$ consider the quadrilateral Q_i with vertices $i, i + \frac{p}{2} - 1(\mathrm{mod} p), i + \frac{p}{2}(\mathrm{mod} p)$, and $i + \frac{p}{2} + 1(\mathrm{mod} p)$. Let C_B be the intersection of all $Q_1, Q_2, Q_3, \ldots, Q_p$. In both cases, the region C_B is called the *center region* of B. The open regions obtained by drawing all lines joining two points of B are called the *cells* of B. While the center region of a set B may be empty (See Figure 4.4), it is never empty when B is the set of vertices of a regular polygon. In fact, in this case C_B has a nonempty interior.

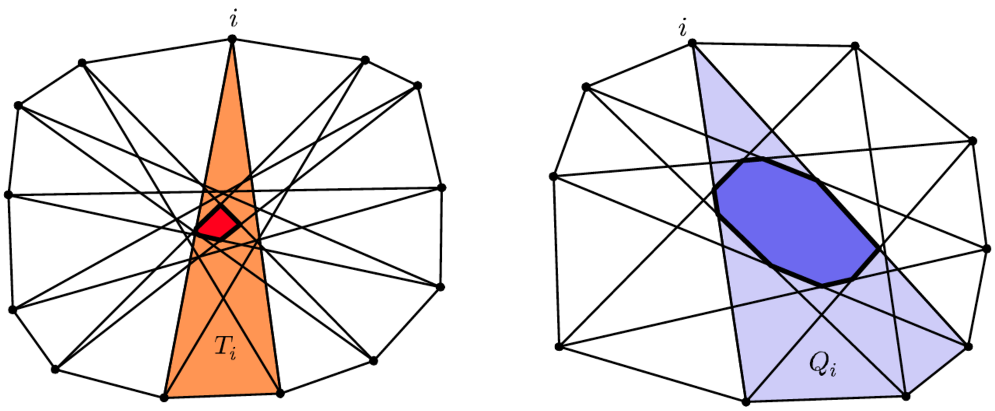


Figure 4.3: The center regions of two sets. *Left.* p odd. *Right.* p even.

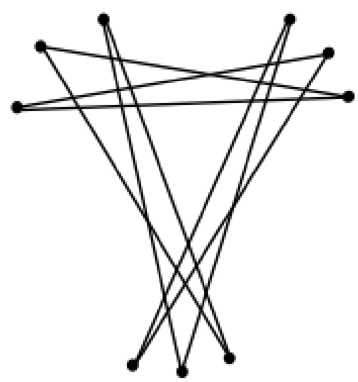


Figure 4.4: Points in convex position with an empty center region.

4.1 One Layer

A set P has exactly one nonempty layer if and only if it is in convex position. In this case, any subset of 4 points generates exactly one crossing. Then

$$\overline{cr}\,[n] = \binom{n}{4}. \tag{4.1}$$

4.2 Lower bound on the RCN of point sets with two layers

In order to achieve a lower bound for $\overline{cr}(C)$, we considered all possible locations of the points in the set.

We consider the class $[p, q]$ with $p \geq 3$ and $q \geq 1$, that is, sets with exactly two nonempty layers, and give a general lower bound for the class $[p, q]$.

Theorem 1 *For any $p \geq 3$ and $q \geq 1$,*

$$\overline{cr}\,[p,q] \geq \binom{p}{4} + \binom{q}{4} + \frac{p}{2}\left\lfloor \frac{p}{2} - 1 \right\rfloor \left\lceil \frac{p}{2} - 1 \right\rceil q + \left\lfloor \frac{p-1}{2} \right\rfloor \left\lceil \frac{p-1}{2} \right\rceil \binom{q}{2}. \tag{4.2}$$

Proof. Let P be a set in the class $[p,q]$, $B = L_1(P)$, and $I = P - B$. Then B and I are in convex position and I is contained in the interior of B. As four points create a crossing if and only if they are in convex position, we classify all subsets of four points S of P into four types, and for each type we check under which conditions S is in convex position. (See Figure 4.5.) We then give a lower bound on the number of sets S, or equivalently, on the number of crossings, of each type.

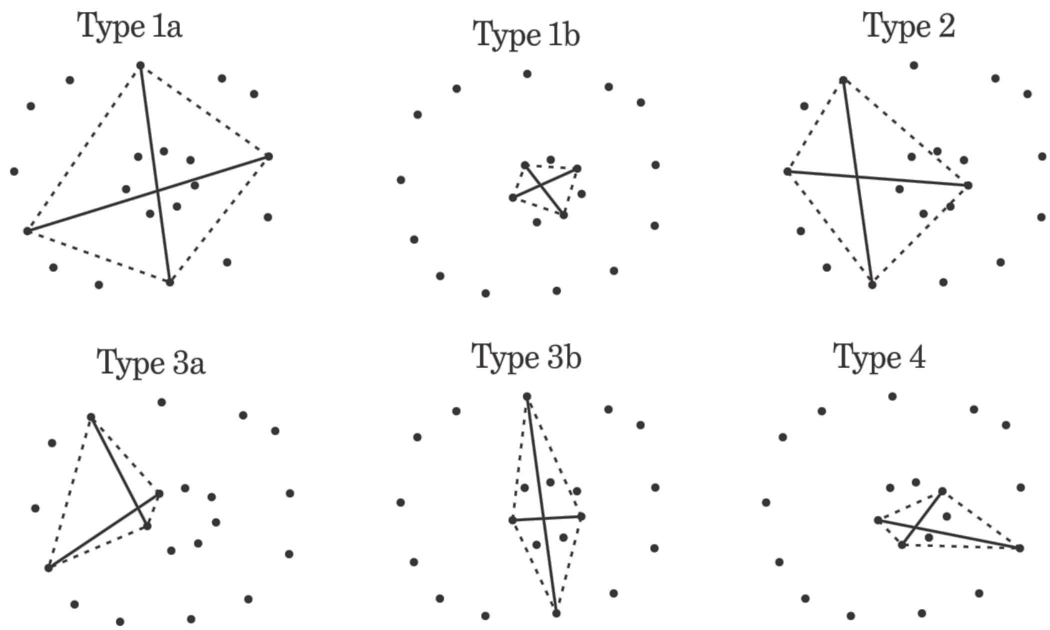


Figure 4.5: Crossing types.

Type 1. $S \subseteq B$ (Type 1a) or $S \subseteq I$ (Type 1b). Any set of four points $p_1, p_2, p_3, p_4 \in B$ or $q_1, q_2, q_3, q_4 \in I$ is in convex position as B and I are both convex. Hence, there are $\binom{p}{4}$ crossings corresponding to sets $S \subseteq B$ and $\binom{q}{4}$ corresponding to sets $S \subseteq I$. This gives a total of exactly

$$\binom{p}{4} + \binom{q}{4} \tag{4.3}$$

crossings of Type 1.

Type 2. Three points of S are in B and the other is in I. For each point $z \in I$, this reduces to counting the number of crossings given by (4.12) – found in the proof of Theorem 3 – that are determined by three points of B and z, as done for the class $[p,1]$ in Theorem 3 . That is, there are at least

$$\frac{p}{2}\left\lfloor \frac{p}{2} - 1 \right\rfloor \left\lceil \frac{p}{2} - 1 \right\rceil q \tag{4.4}$$

crossings of this type, with equality if and only if I is contained in the center region of B.

Type 3. Two points of S are in B and the other two are in I. Suppose $S = \{p_1, p_2, q_1, q_2\}$ with $p_1, p_2 \in B$

and $q_1, q_2 \in I$. Let R_i and L_i be the sets of points of B to the right and left, respectively, of the directed line q_1q_2. We have two possible types of crossings generated by S. (a) Type 3a: the segments p_1q_i and p_2q_{i+1} intersect, where the indices are taken $\bmod 2$; and (b) Type 3b: the segments q_1q_2 and p_1p_2 intersect.

Type 3a may happen only if p_1 and p_2 are on the same side of the line q_1q_2. Note that both sets $R_i \cup \{q_1, q_2\}$ and $L_i \cup \{q_1, q_2\}$ are in convex position (with q_1 and q_2 consecutive points along the boundary) and thus S generates a crossing of Type 3a whenever p_1 and p_2 are on the same side of q_1q_2, namely, whenever $\{p_1, p_2\}$ is contained in R_i or in L_i. This gives a total of exactly

$$\binom{|R_i|}{2} + \binom{|L_i|}{2} \tag{4.5}$$

crossings of Type 3a. Since $|R_i| + |L_i| = p$,

$$\binom{|R_i|}{2} + \binom{|L_i|}{2} \geq \binom{\lfloor p/2 \rfloor}{2} + \binom{\lceil p/2 \rceil}{2} = \left\lfloor \frac{p-1}{2} \right\rfloor \left\lceil \frac{p-1}{2} \right\rceil, \tag{4.6}$$

with equality if and only if the line through any pair of points in I halves B.

Note that there are sets P for which there are no crossings of Type 3b. Namely, whenever I is contained in a single cell of B. Therefore the number of crossings of Type 3 is at least

$$\left\lfloor \frac{p-1}{2} \right\rfloor \left\lceil \frac{p-1}{2} \right\rceil \binom{q}{2}, \tag{4.7}$$

and this lower bound is achieved if and only if the line through any pair of points in I is a halving line of B and I is contained in a single cell of B.

Type 4. One point of S is in B and the other three are in I. Since there are sets P that do not generate crossings of this type, we just bound the crossings of this type by 0. For example, if $P = B \cup I$ is a set of points in the class $[10, 3]$ (See Figure 4.6), with B fully contained in the shaded regions, then this set will have no crossings of Type 4. Namely, any subset of 4 points S will not create a convex quadrilateral. Recall that any four points in convex position creates a crossing. Consequently, there will not be any convex quadrilateral that would create a crossing. In general for any set of points, if the first layer is fully contained in the shaded regions, there will be no crossings of Type 4.

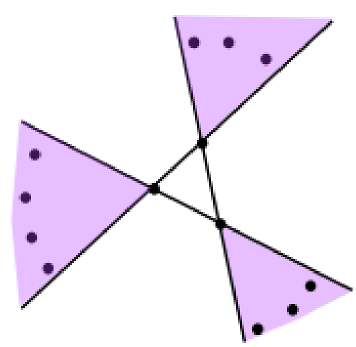


Figure 4.6: A set P of points in the class $[10, 3]$ first layer P fully contained within the shaded regions generated by second layer Q.

Adding the lower bounds (4.3), (4.4), and (4.7) for types 1, 2, and 3, respectively, yields the result. ■

4.3 Upper bound on the RCN of point sets with two layers

Now we give an upper bound for $\overline{cr}\,[p,q]$ by providing an explicit construction that achieves the lower bounds on the number of crossings of Types 1-3 in the previous proof.

Theorem 2 *For any $p \geq 3$ and $q \geq 1$,*

$$\overline{cr}\,[p,q] \leq \binom{p}{4} + \binom{q}{4} + \frac{p}{2}\left\lfloor \frac{p}{2} - 1 \right\rfloor \left\lceil \frac{p}{2} - 1 \right\rceil q + \left\lfloor \frac{p-1}{2} \right\rfloor \left\lceil \frac{p-1}{2} \right\rceil \binom{q}{2} + \left\lceil \frac{p}{2} \right\rceil \binom{q}{3}. \tag{4.8}$$

Proof. We present an explicit construction achieving the corresponding bound. Let P be the set in the class $[p,q]$ such that $P = B \cup I$, where B is the set of vertices of a regular p-gon and I is a set of q points in convex position in the center region C_B of B as described below. We label the vertices of B from 1 to p in counter-clockwise order.

Partition B into the two almost-equal-size sets

$$X = \left\{ x \in B : x = 1 \text{ or } \left\lceil \frac{p+3}{2} \right\rceil \leq x \leq p \right\} \text{ and } Y = \left\{ y \in B : 2 \leq y \leq \left\lceil \frac{p+1}{2} \right\rceil \right\}. \tag{4.9}$$

Consider an arc of circle A passing through 1 and $\lceil \frac{p+3}{2} \rceil$, contained in the interior of the triangle with vertices 1, $\lceil \frac{p+1}{2} \rceil$, and $\lceil \frac{p+3}{2} \rceil$, such that A intersects C_B and flat enough so that the line through any two points in $A \cap C_B$ separates X and Y. (See Figure 4.7.) Finally, let I be any q points on $A \cap C_B$ contained on a single cell of B. Note that, since the line through any two points of I separates X and Y, $|X| = \lceil p/2 \rceil$, $|Y| = \lfloor p/2 \rfloor$, and $X \cup Y = B$, any line passing through two points of I halves B.

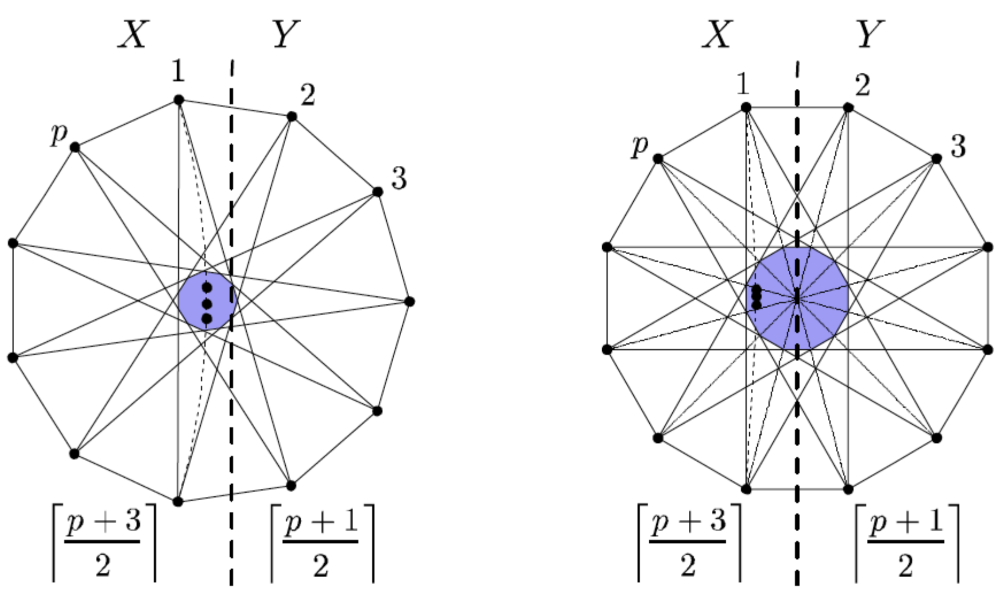


Figure 4.7: Upper bound construction for a class with two layers. *Left*. p odd. *Right*. p even. All points of I must be in the same cell of B.

We count the crossings generated by P, or equivalently, the number of subsets S of P of size four in convex position. Expression (4.3) always represents the number of crossings of Type 1. The bound (4.4) on type 2 crossings is achieved because I is contained in C_B. Because the line through any two points in I halves B and I is contained in a single cell of B, the bound (4.7) on the Type 3 crossings is achieved by P. Finally, to bound the crossings of Type 4, suppose that $S = \{p_1, q_1, q_2, q_3\}$ with $p_1 \in B$ and $q_1, q_2, q_3 \in I$. Note that $X \cup \{q_1, q_2, q_3\}$ is itself in convex position and so there are at least

$$\left\lceil \frac{p}{2} \right\rceil \binom{q}{3} \tag{4.10}$$

crossings of Type 4. (Note that this is the exact number of crossings of Type 4. Indeed, if $p_1 \in Y$ and q_2 is between q_1 and q_3 along the arc A, then q_2 is in the interior of $\triangle p_1 q_1 q_3$ and thus S is not in convex position.) Adding (4.3), (4.4), (4.7), and (4.10), and noting that $\overline{cr}\,[p,q] \le \overline{cr}(P)$ by definition of $\overline{cr}\,[p,q]$, yields the result. ■

The bounds in Theorems 1 and 2 match when $q = 1$ and $q = 2$. The exact value of $\overline{cr}\,[p,2]$, together with the sets achieving it, are included in Theorem 4.

4.4 Exact RCN for classes with two layers and at most 2 points in the second layer

We first determine the exact value of $\overline{cr}[p,1]$, that is, sets with two nonempty layers having 1 point in their second layer, because we prove that each consecutive convex layer, starting with the second one, must be contained within the center region of the first layer. Furthermore, we determine the exact value of $\overline{cr}[p,2]$. We then investigate what the bounds for configurations of points in $[p,3]$; we do this in order to improve the previous bounds.

Theorem 3 *For any $p \ge 3$,*

$$\overline{cr}\,[p,1] = \binom{p}{4} + \frac{p}{2} \left\lfloor \frac{p}{2} - 1 \right\rfloor \left\lceil \frac{p}{2} - 1 \right\rceil. \tag{4.11}$$

Moreover, $P \in [p,1]$ is an optimal configuration if and only if $L_2(P)$ is contained in the center region of P.

Proof. Let $P = B \cup \{z\}$ be a set in the class $[p,1]$ where B is a set of p points in convex position and z is a point in the interior of B that is not collinear with any pair of vertices in B. Below, we bound the number of crossings in P. Note that, as in the case of one layer (convex position, see Equation 4.1) there are exactly $\binom{p}{4}$ crossings generated by four points in B. It remains to count the number of crossings generated by three points in B and the point z. We claim that there are at most

$$\frac{p}{2} \left\lfloor \frac{p}{2} - 1 \right\rfloor \left\lceil \frac{p}{2} - 1 \right\rceil \tag{4.12}$$

such crossings. Label the vertices of B from 1 to p in counter-clockwise order. Let i be a fixed vertex of B and let r_i be the number of points in B to the right of the ray iz. (See Figure 4.8.) For all $1 \le k \le p$ color the segment zk blue (dashed), and if $k \ne i$, color the segments ik red (solid). We count the blue-red crossings for each $1 \le i \le p$.

First we consider the blue-red crossings that occur on the right of $\overrightarrow{iz}$. Each blue segment $(z, i + r_i - t)$ where $0 \le t \le r_i - 1$, intersects the t red segments $(i, i + r_i - t')$ where $0 \le t' \le t - 1$. This gives a total of

$$\sum_{t=0}^{r_i - 1} t = \binom{r_i}{2} \tag{4.13}$$

blue-red crossings on the right of $\overrightarrow{iz}$. Similarly, but considering that there are $p - r_i - 1$ points of B on the left of $\overrightarrow{iz}$, each blue segment $(z, i + r_i + t + 1)$, where $0 \le t \le p - r_i - 2$, intersects the t red segments

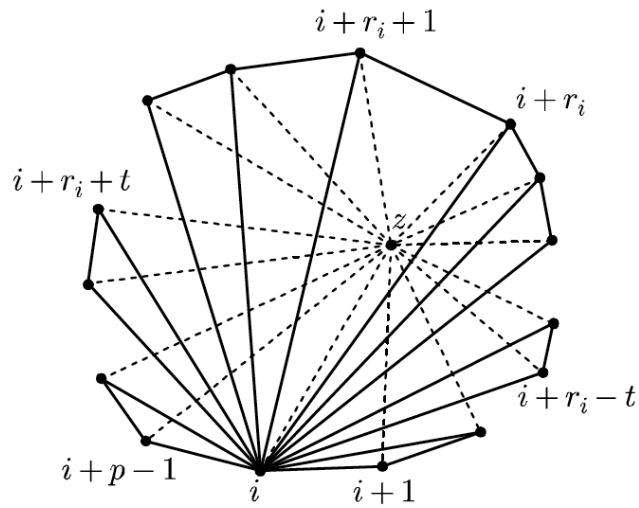


Figure 4.8: Counting blue-red crossings for the proof of Theorem 3. Blue = dashed, red = solid.

$(i, i+r_i+t')$, where $1 \le t' \le t$. Thus the number of blue-red crossings that occur to the left of $\overrightarrow{iz}$ is

$$\sum_{t=0}^{p-r_i-2} t = \binom{p-r_i-1}{2}. \tag{4.14}$$

Adding over all points i in B, and dividing by 2 (because each crossing is counted twice, specifically, once per each endpoint of the segment joining two points in B), we obtain the total number of crossings created by z and three other points in B. Namely,

$$\frac{1}{2}\sum_{i=1}^{p}\left(\binom{r_i}{2}+\binom{p-r_i-1}{2}\right) \ge \frac{1}{2}\sum_{i=1}^{p}\left(\binom{\lfloor\frac{p-1}{2}\rfloor}{2}+\binom{\lceil\frac{p-1}{2}\rceil}{2}\right) = \frac{p}{2}\left\lfloor\frac{p}{2}-1\right\rfloor\left\lceil\frac{p}{2}-1\right\rceil. \tag{4.15}$$

To show that this bound is tight and to find all optimal sets, note that the previous inequality is an identity if and only if for each $1 \le i \le p$ the value of r_i is $\lfloor\frac{p-1}{2}\rfloor$ or $\lceil\frac{p-1}{2}\rceil$. This happens if and only if z is in the center region C_B of B. In particular, equality is achieved when B is the set of vertices of a regular p-gon and z is a point in the center region of B that is not collinear with any pair of vertices in B. ■

Theorem 4 *For each* $p \ge 3$*,*

$$\overline{cr}\,[p,2] = \binom{p}{4} + p\left\lfloor\frac{p}{2}-1\right\rfloor\left\lceil\frac{p}{2}-1\right\rceil + \left\lfloor\frac{p-1}{2}\right\rfloor\left\lceil\frac{p-1}{2}\right\rceil. \tag{4.16}$$

Moreover, $P \in [p,2]$ *with* $L_2(P) = \{x, y\}$ *is an optimal set if and only if* $L_2(P)$ *is contained in the center region of* P*,* xy *halves* P*, and the segment* xy *is contained in a single cell of* $L_1(P)$*.*

Proof. The arrangement of the points follows the restrictions implemented in Theorems 1 and 2, thus the identity holds. ■

4.5 What's next?

The following table summarizes the lower bounds used in Theorem 1 for the number of crossings of Types 3 and 4 and the required conditions to achieve the bounds. Here the set $P \cup Q$ belongs to the class $[p, q]$ with first layer P and second layer Q.

	crossings	Lower bound used in Theorem 1	Conditions to achieve bound
I	Type 3a	$\left\lfloor \frac{p-1}{2} \right\rfloor \left\lceil \frac{p-1}{2} \right\rceil$	xy halves P
II	Type 3b	0	Q is fully contained in a single cell generated by P
III	Type 4	0	P is fully contained in the shaded region generated by $\{x, y, z\} \subseteq Q$

Table 4.1: Summary of the lower bounds used in Theorem 1 for the number of crossings of Types 3 and 4.

For $q = 1$ and $q = 2$, it is possible to simultaneously achieve all these bounds as shown in Theorem 3 and 4. However, it is no loner possible for $q > 2$. We analyze the case $q = 3$, which could potentially be used to understand the general case, $q > 2$.

First we analyze what happens if II and III (but no necessarily I), in the table above, are satisfied.

Theorem 5 *If a set $P \cup Q \in [p, 3]$ with first layer P and second layer Q has no crossings of Types 3b and 4, then $P \cup Q$ has at most $\frac{1}{2}\left(\frac{p-1}{3}\right)\left[15\left(\frac{p-1}{3}\right)+1\right]$ crossings of Type 3a.*

Proof. Just as in Theorem 1, we will consider subsets of 4 points (1 point from P and 3 from Q) from the set of points. There may or may not be crossings of this type. In order to restrict this type of crossings to 0, the sum of all possible crossings generated by the points must be minimized.

We considered an arrangement of p points in the first layer as in Figure 4.6, and denoted the number of points in each of the 3 shaded regions as $x,\ y,\ z$ where $x + y + z = p$. Thus we are interested in minimizing the number of crossings

$$f(x, y) = \binom{x}{2} + \binom{p-x}{2} + \binom{y}{2} + \binom{p-y}{2} + \binom{x+y}{2} + \binom{p-x-y}{2}$$

which result in having $\frac{1}{3}p$ points in each of the shaded cells. This can only be achieved if the total amount of points is divisible by 3.

For any number of points p in the first layer, if $p \equiv 1 \bmod 3$, equivalently $p = 3k + 1$ for any positive integer k, the three closest solutions are $\left(\frac{p-1}{3}, \frac{p+2}{3}\right), \left(\frac{p+2}{3}, \frac{p-1}{3}\right), \left(\frac{p-1}{3}, \frac{p-1}{3}\right)$. All three yield the same minimum number of crossings, namely, $\frac{1}{2}\left(\frac{p-1}{3}\right)\left[15\left(\frac{p-1}{3}\right)+1\right]$.

Now if $p \equiv 2 \bmod 3$, equivalently $p = 3k + 2$ for any positive integer k, the three closest solutions are $\left(\frac{p-2}{3}, \frac{p+1}{3}\right), \left(\frac{p+1}{3}, \frac{p+1}{3}\right), \left(\frac{p+1}{3}, \frac{p-2}{3}\right)$, and the minimum number of crossings for all these tree solutions is $\frac{1}{2}\left[15\left(\frac{p-2}{3}\right)^2 + 11\left(\frac{p-2}{3}\right) + 2\right]$.

Since the total number of points is an integer, solutions are restricted to the closest possible integer solution. The three closest solutions to the real valued minimum form a triangle and the actual minimum is the varicenter of such triangle. ■

There are fewer crossings for $p \equiv 2 \bmod 3$ than for $p \equiv 1 \bmod 3$.

Now, we analyze what happens if I and II are satisfied.

Theorem 6 *If a set $P \cup Q \in [p, 3]$ with first layer P and second layer Q has at most $\left\lfloor \frac{p-1}{2} \right\rfloor \left\lceil \frac{p-1}{2} \right\rceil$ crossings of Types 3a and no crossings of Type 3b, then $P \cup Q$ has at most $\frac{p}{2}$ (for p is even) and $\frac{p-3}{2}$ (for p odd) crossings of Type 4.*

Proof. We minimize the number of crossings of Type 4 when we included points in the unshaded regions from Figure 4.6; let the number of points in those sections be $A,\ B,\ C$, respectively. Also, let the number of points in the other three unshaded regions be $D,\ E,\ F$. (See Figure 4.9.). We must ensure that the each pair of points from Q halves P. Consequently, we have the following three equations.

$$\begin{array}{rrcrcl} 1. & B+C+E & = & A+D+F & = & \frac{p}{2} \\ 2. & A+F+C & = & B+D+E & = & \frac{p}{2} \\ 3. & A+B+D & = & C+E+F & = & \frac{p}{2} \end{array}$$

where $A+B+C+D+E+F = p$. We want to minimize the sum $D+E+F$.

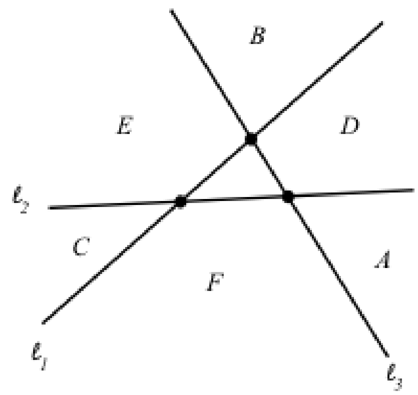


Figure 4.9: Lines ℓ_i for $i \in \{1, 2, 3\}$ are halving lines, and the letters $A, B, \ldots, F$ represent the amount of points in each of the sections generated by the lines panned by Q.

With these requirements, and provided that p is even, we get the following results. Equation 2 and 3 imply that $A = E$, Equation 1 and 3 imply that $B = F$, and Equation 1 and 2 imply that $C = D$. Moreover, together with Equation 3, $C = D$ implies that $D + E + F = \frac{p}{2}$.

Now, for p odd, each pair of points from Q must also halve P, and we still require that $A + B + C + D + E + F = p$. However, for this choice of p, there are different cases to consider. The following table summarizes the results for the different cases in which "+" and "−" represent the amount of points $\frac{p+1}{2}$ and $\frac{p-1}{2}$, respectively, on either side of the halving lines ℓ_1 and ℓ_2. (See Figure 4.9)

Case		A	B	C	$D+E+F$
I	ℓ_1: + − ℓ_2: + −	$A = E + 1$	$B = F + 1$	$C = D + 1$	$\frac{p-3}{2}$
II	ℓ_1: − + ℓ_2: + −	$A = E + 1$	$B = F$	$C = D$	$\frac{p-1}{2}$
III	ℓ_1: + − ℓ_2: − +	$A = E$	$B = F + 1$	$C = D$	$\frac{p-1}{2}$
IV	ℓ_1: − + ℓ_2: − +	$A = E$	$B = F$	$C = D - 1$	$\frac{p+1}{2}$

We must ensure that the three lines spanned by the points in Q are halving lines. Without loss of generality, we can say ℓ_3 is a halving line in the following way: $A+B+D=\frac{p+1}{2}$ and $C+E+F=\frac{p-1}{2}$, namely

$$A+B+D=C+E+F+1=\frac{p+1}{2}. \tag{4.17}$$

With this assumption, we have 4 different cases for the amount of points on either side of the halving lines ℓ_1 and ℓ_2. Similar to p even, in each case we get three different equations – including Equation 4.17.

For Case I with halving line ℓ_1, we get

$$B+C+E=A+D+F+1=\frac{p+1}{2} \tag{4.18}$$

and with halving line ℓ_2, we get

$$A+F+C=B+D+E+1=\frac{p+1}{2} \tag{4.19}$$

Finally, with Equations 4.17 and 4.19 we get $A=E+1$, with Equations 4.17 and 4.18 we get $B=F+1$ and with Equations 4.18 and 4.19 we get $C=D+1$. Moreover, using Equation 4.19 we get the desired result $D+E+F=\frac{p-3}{2}$.

In a similar manner we obtain the results stated in the table above for the remaining Cases II, III, and IV. ■

Recall that with 3 points in the second layer, Theorem 1 given the following general lower bound.

$$\overline{cr}\,[p,3]\geq\binom{p}{4}+\binom{3}{4}+\frac{p}{2}\left\lfloor\frac{p}{2}-1\right\rfloor\left\lceil\frac{p}{2}-1\right\rceil 3+\left\lfloor\frac{p-1}{2}\right\rfloor\left\lceil\frac{p-1}{2}\right\rceil\binom{3}{2}.$$

Note that Theorem 5 gives a better lower bound than Theorem 1 in the case when the maximum number of points from the first layer, p are completely contained in the shaded regions from Figure 4.6.

Also, Theorem 6 gives a better lower bound than Theorem 1 in the case the when number of halving lines of a set of points $P\cup Q$, spanned by Q, is maximized.

Now, we set up the general case – the case when I and II are satisfied from Table 4.5. We want to minimize the number or crossings of Types 3a and 4. Given that $A+B+C+D+E+F=p$, let

$$\begin{aligned}f(A,B,C,D,E,F)=&\binom{B+E+C}{2}+\binom{D+A+F}{2}+\binom{B+D+E}{2}+\binom{C+F+A}{2}\\&+\binom{C+E+F}{2}+\binom{A+B+D}{2}+D+E+F.\end{aligned}$$

This case is slightly more tricky, so we only provide our attempts. We first consider

$$\binom{a}{2}+\binom{p-a}{2}=a^2-ap+\frac{p^2-p}{2}$$

and so we have the following three equations.

$$(B+E+C)^2 - p(B+E+C) + \tfrac{p^2-p}{2}$$

$$(C+F+A)^2 - p(C+F+A) + \tfrac{p^2-p}{2}$$

$$(A+B+D)^2 - p(A+B+D) + \tfrac{p^2-p}{2}$$

Thus,

$$f(A,B,C,D,E,F) = (B+E+C)^2+(C+F+A)^2+(A+B+D)^2-p(B+E+C+C+F+A+A+B+D)$$
$$+3\left(\frac{p^2-p}{2}\right)+D+E+F.$$

The following attempt was to consider $p^2 = (A+B+C+D+E+F)^2$, so f becomes

$$p^2 - 2(FD+FE+DE) - 2(AE+BF+CD) + A^2 + B^2 + C^2$$
$$-p\left[2(A+B+C)+D+E+F\right] + 3\left(\frac{p^2-p}{2}\right) + D + E + F$$

We also looked at the arrangements with 4 points in Q. We determined different V_i cells, just as in Figure 5.3 (right), which would result in less Type 4 crossings. In order to maintain the maximum amount of halving lines, the set of points must be arranged in clusters in convex position as the ones in Figure 4.4, and the center region would be empty. Consequently, this arrangement does not yield the best result.

Chapter 5

Exact RCN for classes with at most 9 points

Tables 5.1 and 5.2 provide the values for sets of points some classes with at most 9 points. We use Theorems 3 and 4 to obtain some of the values; some remaining values are proved below.

5.1 Up to 5 points

3 points	4 points	5 points
$\overline{cr}\,[3] = 0$	$\overline{cr}\,[4] = 1$	$\overline{cr}\,[5] = 5$
	$\overline{cr}\,[3,1] = 0$	$\overline{cr}\,[4,1] = 3$
		$\overline{cr}\,[3,2] = 1$

Table 5.1: Exact values for sets with 3, 4 or 5 points.

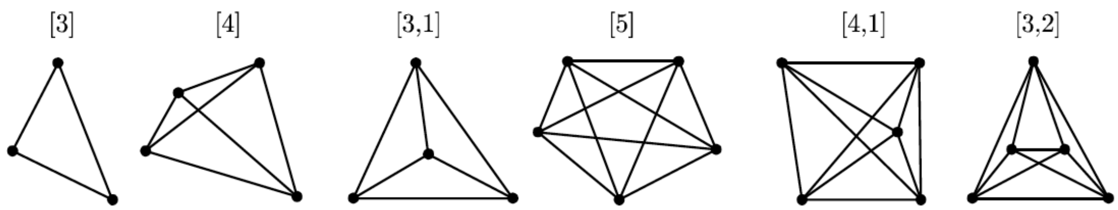


Figure 5.1: Optimal configurations for all classes with up to 5 points.

5.2 6,7, or 8 points

For classes with 6, 7, or 8 points more work is needed. The exact crossing numbers are summarized in the following table.

6 points	7 points	8 points
$\overline{cr}\,[6] = 15$	$\overline{cr}\,[7] = 35$	$\overline{cr}\,[8] = 70$
$\overline{cr}\,[5,1] = 10$	$\overline{cr}\,[6,1] = 27$	$\overline{cr}\,[7,1] = 56$
$\overline{cr}\,[4,2] = 7$	$\overline{cr}\,[5,2] = 19$	$\overline{cr}\,[6,2] = 45$
$\overline{cr}\,[3,3] = 3$	$\overline{cr}\,[4,3] = 15$	$\overline{cr}\,[5,3] = 33$
	$\overline{cr}\,[3,4] = 9$	$\overline{cr}\,[4,4] = 28$
	$\overline{cr}\,[3,3,1] = 9$	$\overline{cr}\,[3,5] = 21$
		$\overline{cr}\,[4,3,1] = 27$
		$\overline{cr}\,[3,4,1] = 20$
		$\overline{cr}\,[3,3,2] = 19$

Table 5.2: Exact values for sets with 6, 7 or 8 points.

The exact values for classes of the form $[n]$, $[p,1]$, and $[p,2]$ are given by (4.1), Theorem 3 and 4, respectively. All upper bounds are given by the optimal constructions shown in Figure 5.2.

By (3.1), $\overline{cr}\,[C_n] \geq \overline{cr}\,(n)$ for any class $[C_n]$ on n points. Using the following table of known values of $\overline{cr}\,(n)$ [6],

$$\begin{array}{c|c|c|c|c|c|c} n & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline \overline{cr}\,(n) & 0 & 0 & 1 & 3 & 9 & 19 \end{array}, \tag{5.1}$$

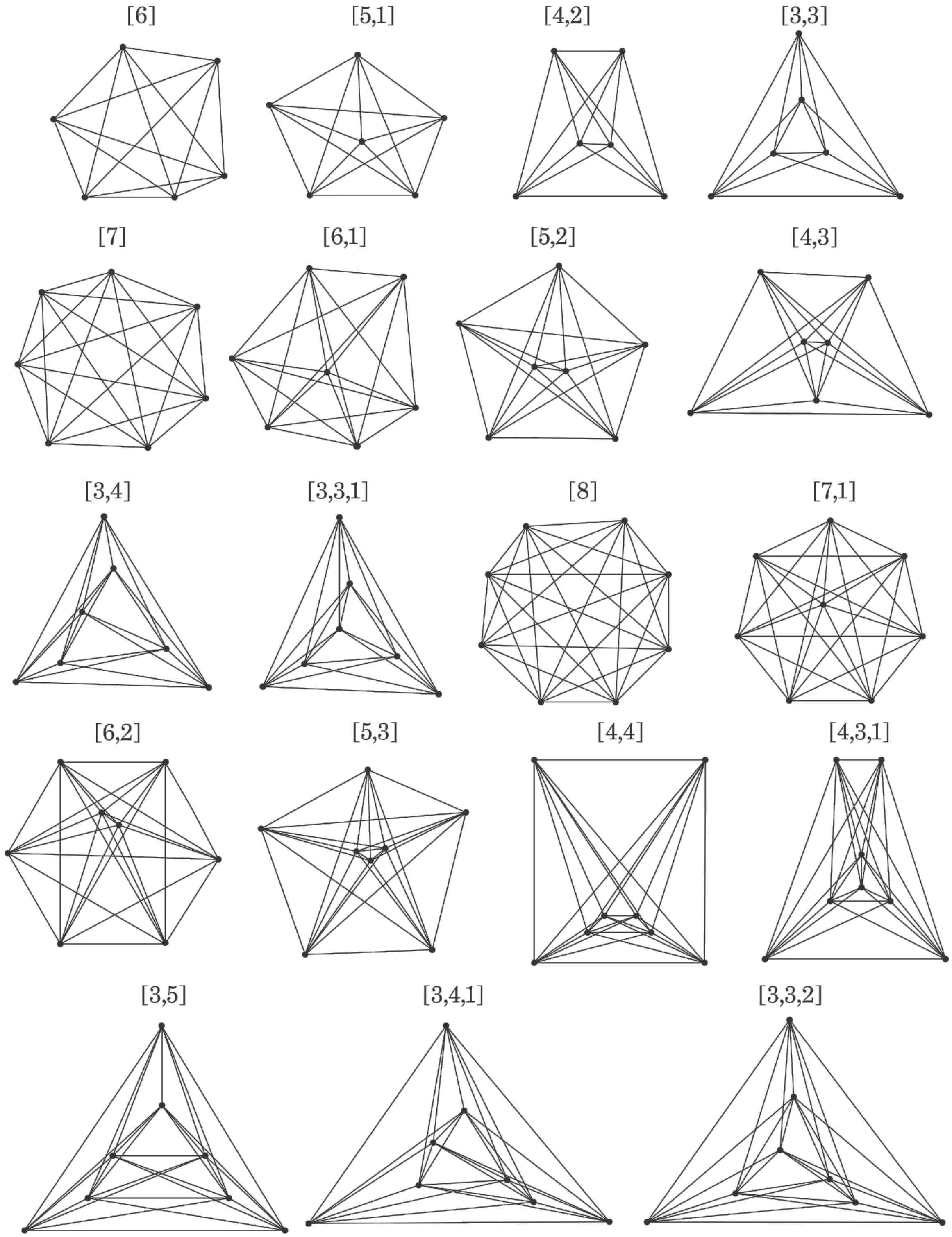


Figure 5.2: Optimal sets for all classes with 6, 7, or 8 points.

we obtain the corresponding lower bounds for the classes

$$[3]\,,[3,1]\,,[3,2]\,,[3,3]\,,[3,4]\,,[3,3,1]\,,\text{ and }[3,3,2]\,. \tag{5.2}$$

The following theorem takes care of the lower bounds for the remaining classes.

Theorem 7 *Lower bounds for classes with* $6, 7,$ *or* 8 *points not covered by (3.1), (4.1), or Theorems 3 or 4.*

$$\begin{array}{ll} 1.\ \overline{cr}\,[5,3] \geq 33. & 4.\ \overline{cr}\,[4,4] \geq 28. \\ 2.\ \overline{cr}\,[4,3,1] \geq 27. & 5.\ \overline{cr}\,[3,4,1] \geq 20. \\ 3.\ \overline{cr}\,[4,3] \geq 15. & 6.\ \overline{cr}\,[3,5] \geq 21. \end{array} \tag{5.3}$$

Proof. Let P be a finite set of points. Denote by $B = L_1(P)$ and $I = P - B$ the sets of boundary and interior points of P, respectively. For each class above, we take an arbitrary set P in the class and prove that it determines at least as many crossings as indicated by (5.3).

For the first two cases we use an average argument. We consider all subsets of P obtained by deleting one point and bound below their number of crossings. We compensate for the overcounting which results when some crossings are counted more than once, and the ceiling of the average is the resulting lower bound.

1. Let $P \in [5,3]$. For each $x \in B$, the set $P - \{x\}$ contains at least 4 points in its convex hull, namely, all the points in $B - \{x\}$. Then $P - \{x\}$ belongs to one of the classes $[4,3]\,,[5,2]\,,[6,1]$ or $[7]$. Hence,

$$\overline{cr}\,(P - \{x\}) \geq \min\{\overline{cr}\,[4,3]\,,\overline{cr}\,[5,2]\,,\overline{cr}\,[6,1]\,,\overline{cr}\,[7]\} = 15. \tag{5.4}$$

On the other hand, for each $x \in I$ the set $P - \{x\}$ always belongs to the class $[5,2]$ and thus

$$\overline{cr}\,(P - \{x\}) \geq \overline{cr}\,[5,2] = 19. \tag{5.5}$$

When we add the number of crossings over all subsets of P of size 7 (that is, all subsets of the form $P - \{x\}$), each crossing of P is counted exactly 4 times. This is because a crossing is generated by a set S of 4 points in P and there are exactly $\binom{8-4}{3} = 4$ subsets of P of size 7 containing S. Therefore,

$$\overline{cr}\,(P) = \frac{1}{4}\sum_{x \in P} \overline{cr}\,(P - \{x\}) \geq \frac{1}{4}\left(5\,(15) + 3\,(19)\right) = \frac{132}{4} = 33. \tag{5.6}$$

2. Let $P \in [4,3,1]$. Suppose that $Q = P - L_1(P) = \{q_1, q_2, q_3, q_4\}$. Then $P - \{q_j\} \in [4,3]$ for all $1 \leq j \leq 4$, and thus $\overline{cr}\,(P - \{q_j\}) \geq 15$. Label the points of $L_1(P)$ in clockwise order by p_1, p_2, p_3, p_4. For each $1 \leq i \leq 4$, if the $\triangle\, p_{i-1}p_ip_{i+1}$ (where the indices are taken $\mathrm{mod} 4$) contains at least one point of Q in its interior, then $P - \{p_i\}$ has at least 4 points in its convex hull, that is,

$$\overline{cr}\,(P - \{p_i\}) \geq \min\{\overline{cr}\,[7]\,,\overline{cr}\,[6,1]\,,cr\,[5,2]\,,\overline{cr}\,[4,3]\} = 15. \tag{5.7}$$

Note that at least two (out of the four) triangles of the form $p_{i-1}p_ip_{i+1}$ must contain points of Q in their interior. For the vertex p_i of the other two triangles, the most we can guarantee is that $\overline{cr}\,(P - \{p_i\}) \geq \overline{cr}\,(7) = 9$. Then

$$\overline{cr}\,(P) = \frac{1}{4}\left(\sum_{i=1}^{4} \overline{cr}\,(P - \{p_i\}) + \sum_{j=1}^{4} \overline{cr}\,(P - \{q_j\})\right) \geq \frac{1}{4}\left(2\,(15) + 2\,(9) + 4\,(15)\right) = \frac{108}{4} = 27.$$

For the last four classes, we generalize the classification of the crossings of P in the proof of Theorem 1. For a crossing $S = \{a, b, c, d\}$ of P, say $\overline{ab}$ intersects $\overline{cd}$, the classification is as follows.

S is of type	Condition
1	$\{a, b, c, d\} \subseteq B$ or $\{a, b, c, d\} \subseteq I$
2	$\{a, b, c\} \subseteq B$ and $d \in I$
3a	$\{a, c\} \subseteq B$ and $\{b, d\} \subseteq I$
3b	$\{a, b\} \subseteq B$ and $\{c, d\} \subseteq I$
4	$a \in B$ and $\{b, c, d\} \subseteq I$

(5.8)

The following table summarizes the number of crossings of each type guaranteed by the proof of Theorem 1 for any set P in the corresponding class. Whenever B is a set of 4 points, we denote by b the number of lines spanned by I leaving two points of B on each side, these lines are called *balanced*, and by u the number of lines spanned by I leaving three points of B on one side and one point on the other, these lines are called *unbalanced*. Note that $b + u = \binom{|I|}{2}$. Finally, let f denote the number of crossings of type 4.

Class	Type 1	Type 2	Type 3i	Lower bound	Need to show
$[4, 3]$	1	6	$2b + 3u = 6 + u$	$\overline{cr}(P) \geq 13 + u + f \geq 15$	$u + f \geq 2$
$[4, 4]$	2	8	$2b + 3u = 12 + u$	$\overline{cr}(P) \geq 22 + u + f \geq 28$	$u + f \geq 6$
$[3, 4, 1]$	3	0	10	$\overline{cr}(P) \geq 13 + f \geq 20$	$f \geq 7$
$[3, 5]$	5	0	10	$\overline{cr}(P) \geq 15 + f \geq 21$	$f \geq 6$

(5.9)

Now we look at each of the classes separately to prove the corresponding inequality in the last column above.

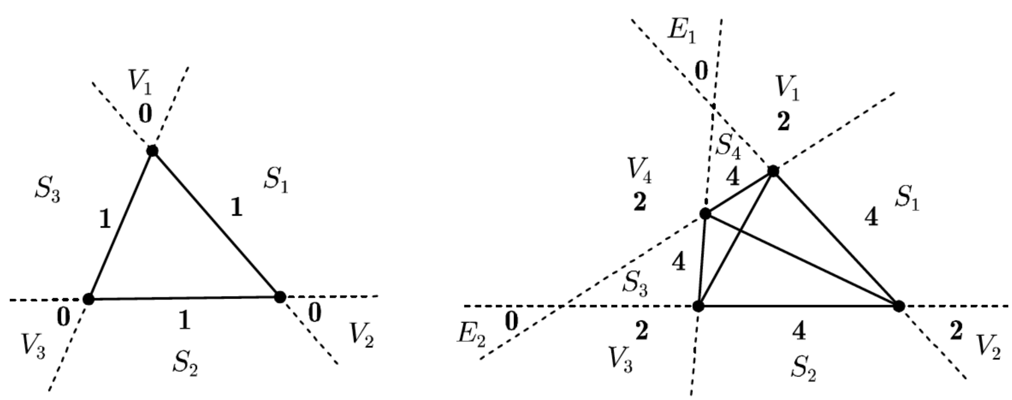


Figure 5.3: The cells of I and the least number of crossings generated by adding a point to a given cell.

3. Let $P \in [4, 3]$. We have to show that $u + f \geq 2$. Figure 5.3(left) shows the cells of I. Adding a point to $S_1 \cup S_2 \cup S_3$ creates a crossing of Type 4 (with the boundary of I), whereas adding a point to $V_1 \cup V_2 \cup V_3$ does not generate crossings of this type.

If two or more points of B are in $S_1 \cup S_2 \cup S_3$, then $f \geq 2$. Figure 5.4 shows all possible positions of B with at most one point of B in $S_1 \cup S_2 \cup S_3$. All positions have been considered because by symmetry, we can assume, without loss of generality, that S_1, S_2, S_3, as well as V_1, V_2, V_3 in Figure 5.3, play the same roll. Thus, the case of having 3 points in V_1 and one in S_2 is the same as having 3 points in V_3 and one in S_1. Now, all possible distributions of the 4 points have been considered with at most one in S_1, S_2, S_3 (and the rest in V_1, V_2, V_3) ensuring that all the points are in convex position.

In the first four cases there is one point of B in $S_1 \cup S_2 \cup S_3$ and thus $f \geq 1$, but also there is at least one unbalanced line making $u \geq 1$. In the last case $f = 0$ since $B \subset V_1 \cup V_2 \cup V_3$, however two unbalanced lines are forced (in this case, the solid lines), making $u \geq 2$. In all cases $u + f \geq 2$.

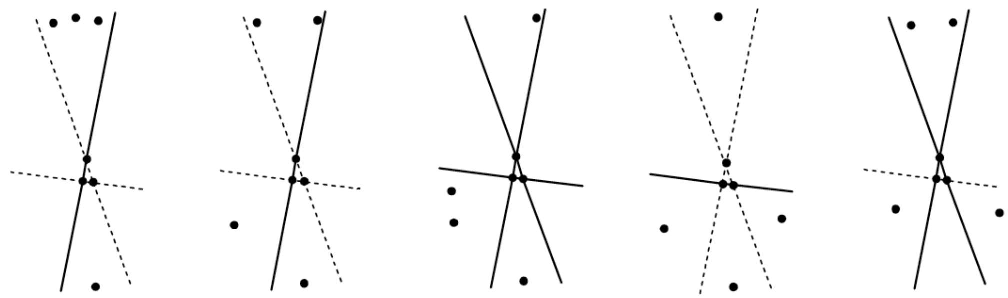


Figure 5.4: Possible arrangements when at most one point of B is in $S_1 \cup S_2 \cup S_3$. Thick lines represent unbalanced lines forced by the position of B with respect to the cells of I.

4. Let $P \in [4, 4]$. We have to show that $u + f \geq 6$. Figure 5.3(right) shows the relevant cells of I. Adding any one point from the first layer to $S_1 \cup S_2 \cup S_3$ creates 4 crossings of type 4 (with the segments generated by I), adding a point to $V_1 \cup V_2 \cup V_3$ generates 2 crossings of type 4, and adding any one point to $E_1 \cup E_2$ does not generate any crossing of type 4. So if there is at least one point of B in $S_1 \cup S_2 \cup S_3 \cup S_4$ and a different point of B in $S_1 \cup S_2 \cup S_3 \cup S_4 \cup V_1 \cup V_2 \cup V_3 \cup V_4$, then $f \geq 4 + 2 = 6$. Similarly, if there are at least 3 points in $V_1 \cup V_2 \cup V_3 \cup V_4$, then $f \geq 2 + 2 + 2 = 6$. We assume then that either there is one point of B in $S_1 \cup S_2 \cup S_3 \cup S_4$ and the other three are in $E_1 \cup E_2$, or there are at most 2 points of B in $V_1 \cup V_2 \cup V_3 \cup V_4$ and the rest are in $E_1 \cup E_2$. Figure 5.5 shows all possible distributions of B among the cells of I satisfying our assumption.

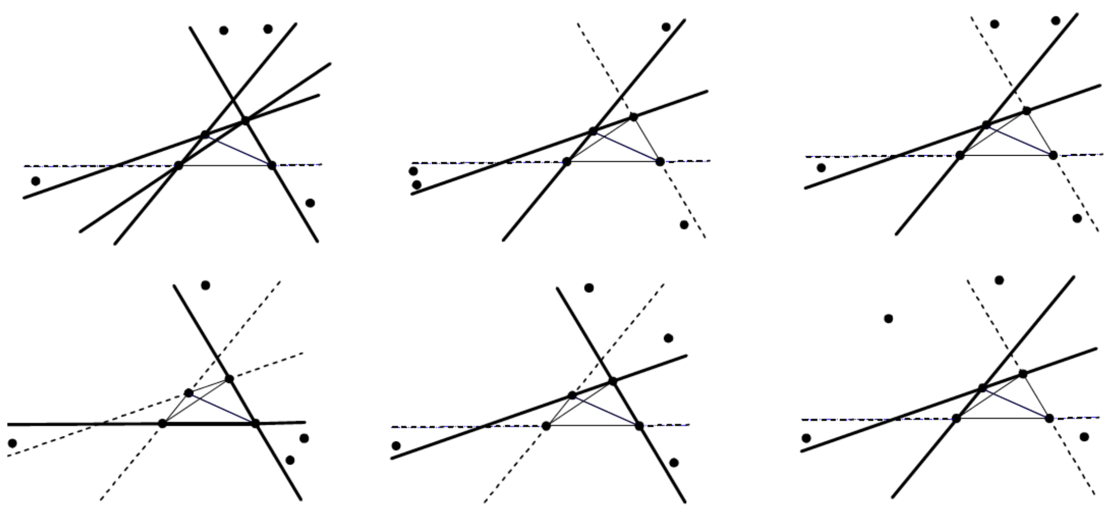


Figure 5.5: Possible positions of B with respect to the cells of I satisfying the required assumptions. Thick lines represent unbalanced lines.

In the first case, there is one point of B in $V_1 \cup V_2 \cup V_3 \cup V_4$ and three points of B in $E_1 \cup E_2$. This forces four unbalanced lines implying $u \geq 4$. In all other cases, when there is one point of B in $S_1 \cup S_2 \cup S_3 \cup S_4$ or two points of B in $V_1 \cup V_2 \cup V_3 \cup V_4$, we have $f \geq 4$. In all these cases, two unbalanced lines are forced giving $u \geq 2$. Therefore $u + f \geq 6$.

5. Let $P \in [3,4,1]$. We have to show that $f \geq 7$. Let x be the point in $L_3(P)$. Add x to the interior of $L_2(P)$ in Figure 5.3(right). The number of crossings of type 4 added by a new point in each of the cells of $L_2(P)$ changes according to whether x is in one of the two shaded triangles in Figure 5.6 . Note that if at least one point of B is in $S_1 \cup S_2 \cup S_3 \cup S_4$ then $f \geq 7$, and if two ore more points of B are in $V_1 \cup V_2 \cup V_3 \cup V_4$ then $f \geq 4 + 4 = 8$.

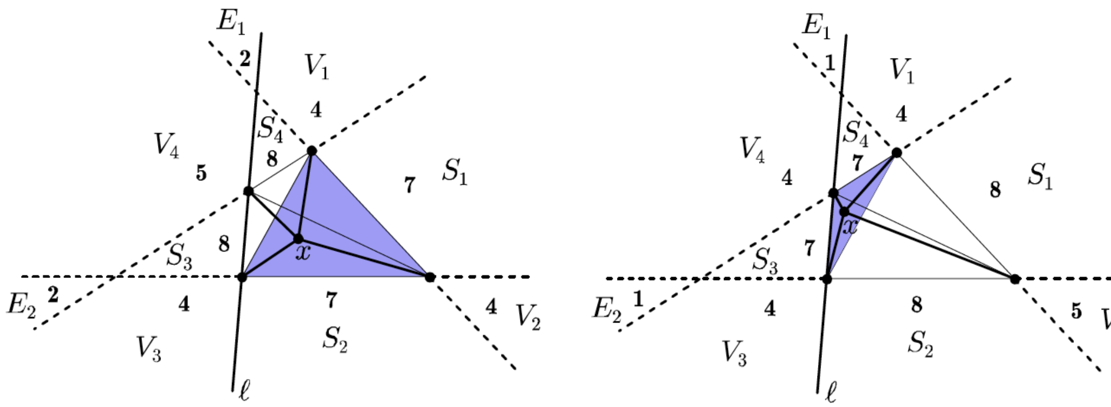


Figure 5.6: Adding the point $x \in L_3(P)$.

On the other hand, as B is the convex hull of P, B cannot be separated from the rest of P by a line. For instance, B cannot be contained in $E_1 \cup E_2$ as it would be separated from the rest of P by the line ℓ. Therefore, the only remaining possibility is that there is one point of B in each of the cells E_1, E_2, and V_2. But this means that $f \geq 2 + 2 + 4 = 8$ or $f \geq 1 + 1 + 5 = 7$.

6. Let $P \in [3,5]$. We have to show that $f \geq 6$. For an illustration, we consider Figure 5.3(right), and introduce the point $x \in L_2(P)$. Note that x must belong to $S_1 \cup S_2 \cup S_3 \cup S_4$. We divide this into two cases: $x \in S_1 \cup S_2$ or $x \in S_3 \cup S_4$. Figure 5.7 shows the changes on the number of crossings of Type 4 that a point of B generates depending on the cell it occupies.

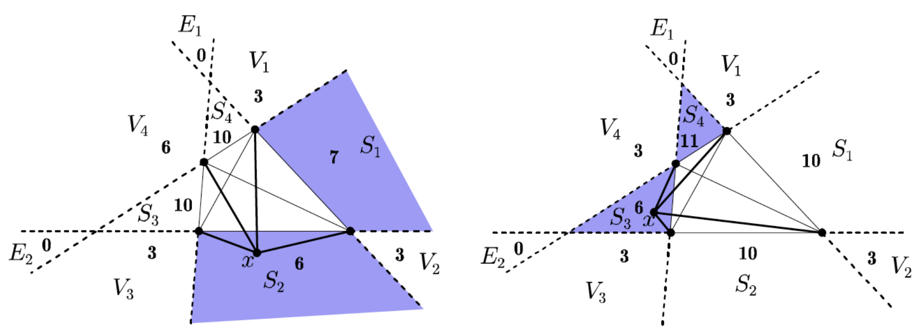


Figure 5.7: Adding the fifth point of $L_2(P)$.

If at least one point of B is in $S_1 \cup S_2 \cup S_3 \cup S_4$, then $f \geq 6$. If two or more points of B are in $V_1 \cup V_2 \cup V_3 \cup V_4$, then $f \geq 3 + 3 = 6$. Then assume that there is at most one point of B in $V_1 \cup V_2 \cup V_3 \cup V_4$. As before, since B is the convex hull of P, then it cannot be separated from other points in P by a line. Therefore, the only remaining case is when there is exactly one point of B in each of the cells V_2, E_1, and E_2. This requires a more detailed analysis of the crossings of Type 4 added by a point in one of these regions. For instance, a point $y \in V_2$ adds 3 crossings if all segments from y to the points in I do not intersect the interior of I.

Otherwise, it adds 6 crossings. The same happens when $y \in E_1$ or $y \in E_2$, adding 3 crossings instead of 0. This means that either $f = 3$ or $f \geq 6$. But $f = 3$ implies that $\overline{cr}(P) = 18$ which is impossible since it is known that $\overline{cr}(8) = 19$. Therefore $f \geq 6$. ■

5.3 9 points

From our previous constructions, we introduce an extra point in the center region so that our new set consists of 9 points. We explore all possible locations of this new point within the center region. By (3.1), $\overline{cr}[C_n] \geq \overline{cr}(n)$ for any class $[C_n]$ on n points, so $\overline{cr}[C_9] \geq \overline{cr}(9) = 36$. For each of the different values of p and q the constructions are within the bounds in Theorems 1 and 2. Table 5.3 shows the RCN, the best bounds we know for classes with 9 points.

One Layer			Two Layers			Three Layers	
$\overline{cr}[9] =$	126	(4.1)	$\overline{cr}[8,1] =$	106	Thm. 3	$\overline{cr}[5,3,1] \leq$	56
			$\overline{cr}[7,2] =$	86	Thm. 4	$\overline{cr}[4,4,1] \leq$	50
			$\overline{cr}[6,3] =$	72	Thm. 8	$\overline{cr}[4,3,2] \leq$	46
			$\overline{cr}[5,4] \leq$	56		$\overline{cr}[3,5,1] \leq$	41
			$\overline{cr}[4,5] \leq$	48		$\overline{cr}[3,4,2] \leq$	38
			$\overline{cr}[3,6] \leq$	45		$\overline{cr}[3,3,3] \leq$	36

Table 5.3: RCN for 9 points

5.3.1 Lower bound for the class $[6, 3]$

Theorem 8 $\overline{cr}[6, 3] = 72$

Proof. Figure 5.8 shows a set in the class $[6, 3]$ with 72 crossings proving that $\overline{cr}[6, 3] \leq 72$.

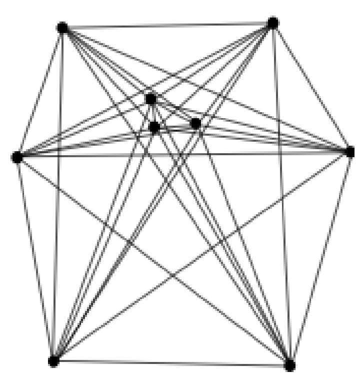


Figure 5.8: A set of points in the class $[6, 3]$ with 72 crossings.

Now, to prove the lower bound ($\overline{cr}[6, 3] \geq 72$), let D be a drawn in the class $[6, 3]$ with the first layer P and the second layer $\{a, b, c\}$. Since, $\overline{cr}[6, 2] = 45$, we have $cr(P \cup \{a, b\}) \geq 45$, $cr(P \cup \{a, c\}) \geq 45$ and $cr(P \cup \{b, c\}) \geq 45$. Also, $cr(P) = 15$ and $\overline{cr}[6, 1] = 27$, from Equation 4.1 and 5.2, respectively. Consequently, $cr(D) \geq 3(45) - 3(27 - 15) - 27 = 72$. ■

The remaining drawings in Figure 5.9 show the upper bound for all the different classes of 9 points as shown in Table 5.3.

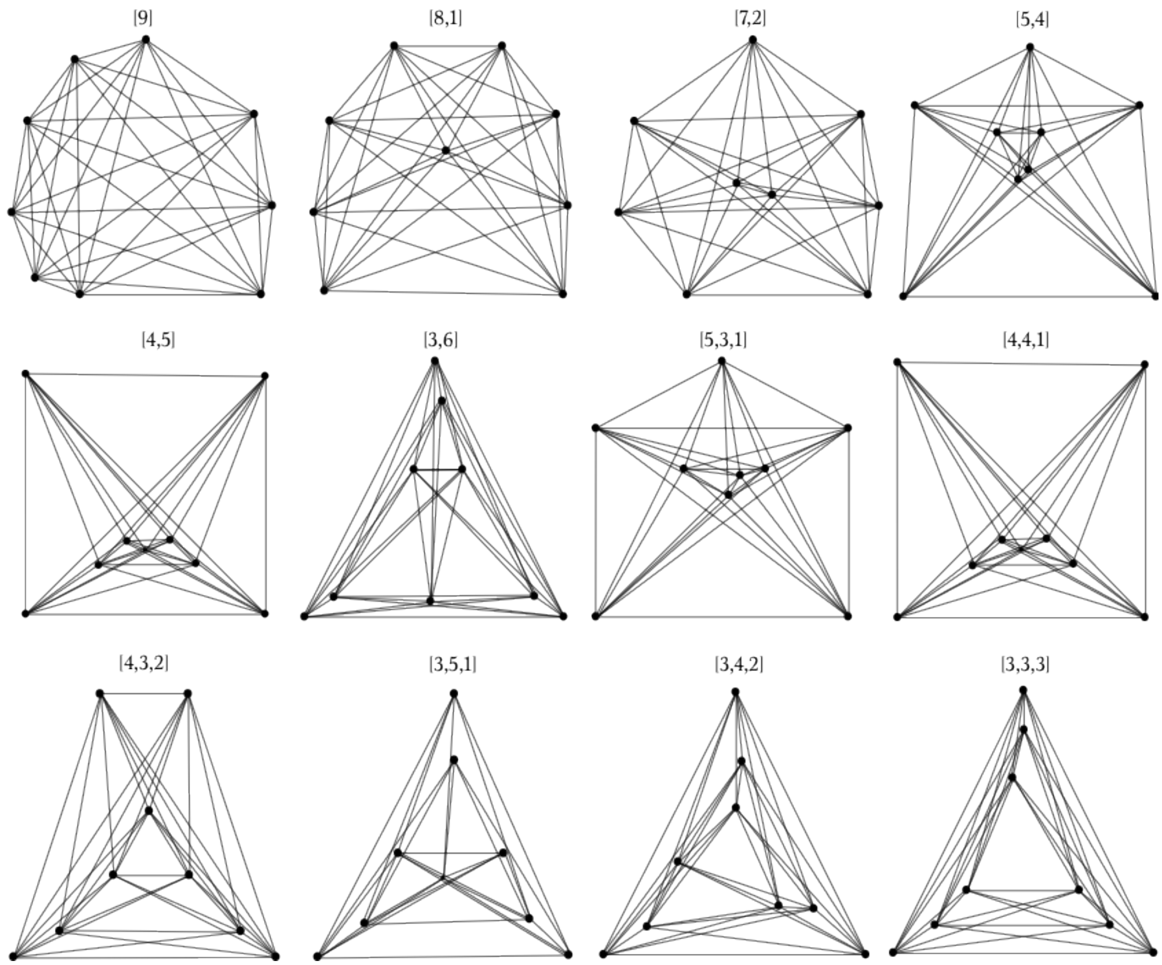


Figure 5.9: Constructions to show the upper point for classes with 9 points.

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