Reminder If $f(x) = x^n$ then $F(x) = \frac{1}{n+1}x^{n+1} + C$ is antiderivative of f $(n \neq -1)$

5.1 Lecture 06/26/2019

1) Hw 1 due tomorrow at 11:00 pm
2) office hours T: 12-1, Th: 1-2 in MSC

Entry task

If $f''(x) = 5\sqrt{x} + x$, $f(0) = 3$, $f(1) = 4$, Find f.

$f''(x) = 5x^{1/2} + x^1$ so $f'(x) = \frac{5}{3/2}x^{3/2} + \frac{1}{2}x^2 + C$

so $f'(x) = \frac{10}{3}x^{3/2} + \frac{1}{2}x^2 + C$

So $f(x) = \frac{10/3}{5/2}x^{5/2} + \frac{1/2}{3}x^3 + Cx + D$

We know $3 = f(0) = D$ so $D = 3$.

$$4 = f(1) = \frac{10/3}{5/2} + \frac{1}{6} + C + 3$$

$$= \frac{20}{15} + \frac{1}{6} + \frac{9}{3} + C$$

$$= \frac{4}{3} + \frac{1}{6} + \frac{18}{6} + C$$

$$= \frac{8+1+18}{6} + C$$

$$= \frac{27}{6} + C$$

So $C = 4 - \frac{27}{6} = \frac{24-27}{6} = \frac{-3}{6} = -\frac{1}{2}$

So $f(X) = 4 \ldots$

Geometry of antiderivatives

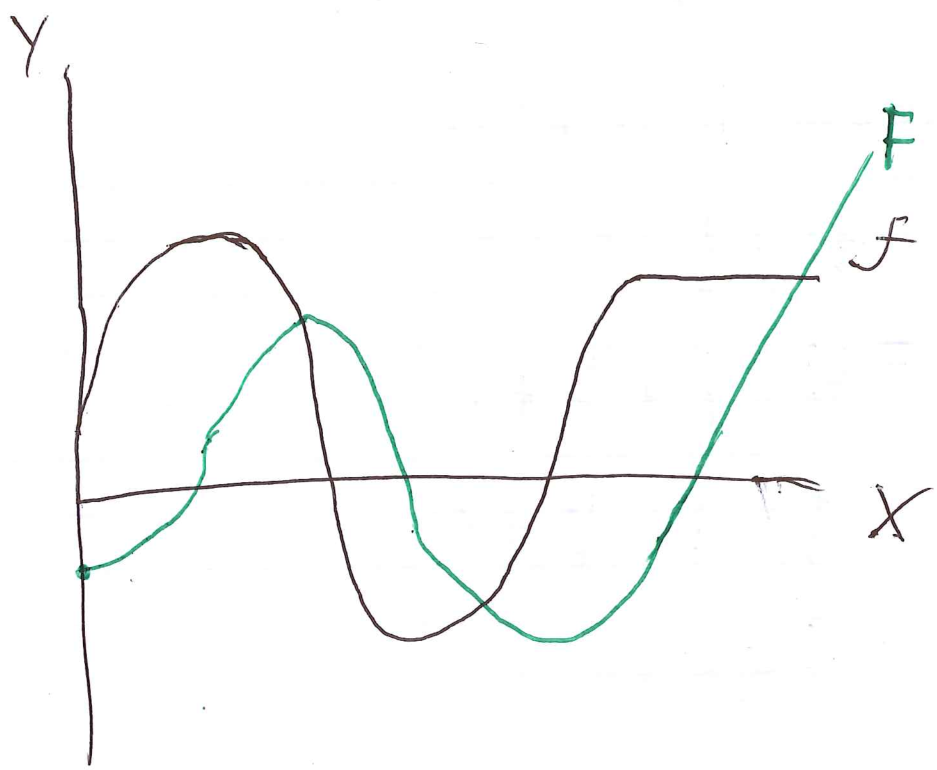
If F is an antiderivative of f, then

$F' = f$. So F increasing $\leftrightarrow$ f positive

F decreasing $\leftrightarrow$ f negative

Also $F'' = f'$ so F concave up $\leftrightarrow$ f increasing

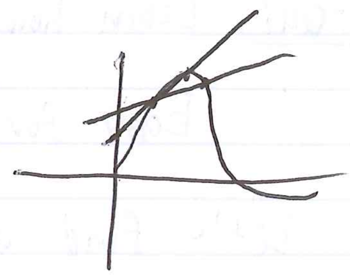
F concave down $\leftrightarrow$ f decreasing



5.1 Estimating areas & distance

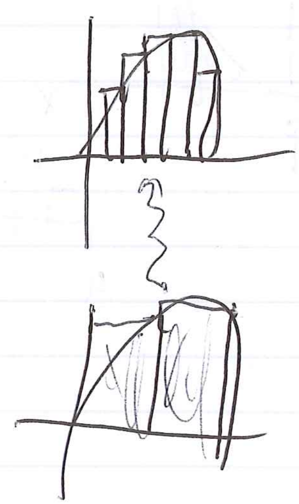
Calc 1 Derivative = Slope of tangent

$$= \lim_{h\to 0} \underbrace{\frac{f(x+h)-f(x)}{h}}_{\text{Slope of secant}}$$



Calc 2 Antiderivatives are related to

Area under a graph $= \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{f(x_i)\,\Delta x}_{\text{area of rectangle}}$



Aside: Sigma Notation

end

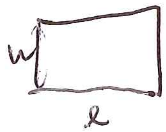
add up consecutively $\rightarrow \sum_{i=a}^{b} \underbrace{f(i)}_{\text{formula}}$ Plug in a, then a+1, a+2, ..., b, and add up

start

Ex: $\sum_{i=1}^{3} i^2 = 1^2 + 2^2 + 3^2 = 1+4+9 = 14$

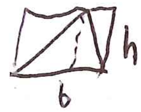
$\sum_{i=50}^{52} \frac{1}{i} = \frac{1}{50} + \frac{1}{51} + \frac{1}{52}$

You do: Compute $\sum_{i=1}^{3} \left(\frac{i}{3}\right)^2 = \left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{3}{3}\right)^2 = \frac{1}{9} + \frac{4}{9} + \frac{9}{9} = \boxed{\frac{14}{9}}$



$A = wl$

$A = \frac{1}{2} bh$

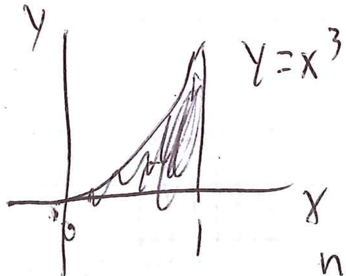


Goal: Learn how to compute area under curved graphs

Easy for rectangles! Like slope was easy for lines.

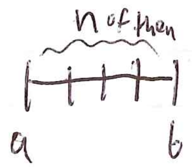
Let's find area under $f(x) = x^3$ from $a=0$ to $b=1$.

eft most
→
oard



Step 1 The break down

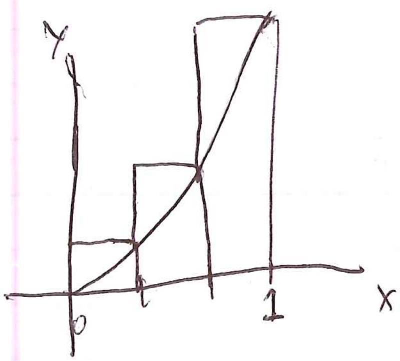
Divide region on x-axis into n equal subdivisions. $\text{width} = \frac{b-a}{n} = \Delta x$



Step 2 Rectangle heights

Use function to build a rectangle over each subdivision (left, right, or middle heights)

i^{th} $\text{height} = f(x_i^*)$ $\quad x_i^*$ some point in i^{th} subdivision



Step 3 Add up

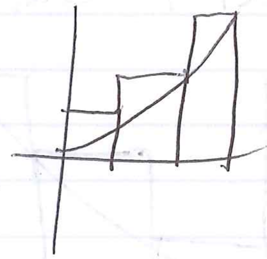
Add up all areas

Notation. Let x_i be right endpoint of i^{th} subdivision

so x_{i-1} is left endpoint

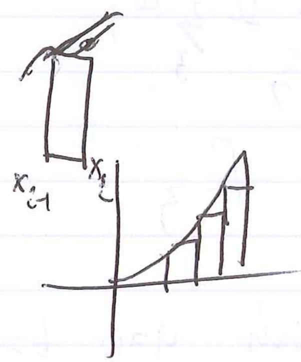
R_n = height from right end Points

$$= \sum_{i=1}^{n} \underbrace{f(x_i)}_{\text{height}} \underbrace{\Delta x}_{\text{width}}$$



L_n = height from left end Points

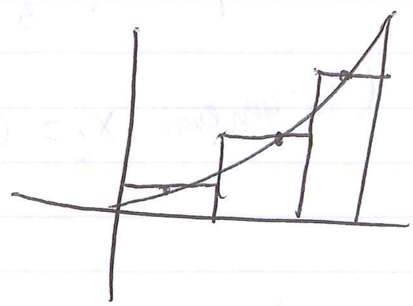
$$= \sum_{i=1}^{n} f(x_{i-1}) \Delta x$$

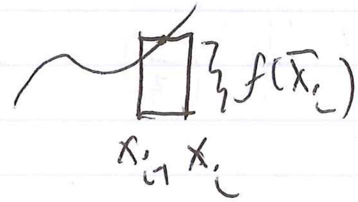


M_n = height from midPoints

$$= \sum_{i=1}^{n} f(\bar{x}_i) \Delta x$$

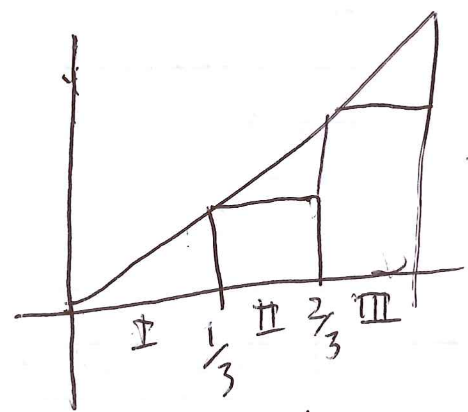
i^{th} mid Point





Let's compute L_3, R_3 togeth

R_3



L_3

① Break down $a=0$ $b=1$ $n=3$ so Δx = width each rectangle $= \frac{b-a}{3} = \frac{1-0}{3} = \frac{1}{3}$

$x_0 = 0$ $x_1 = 0+\frac{1}{3} = \frac{1}{3}$ $x_2 = \frac{1}{3}+\frac{1}{3} = \frac{2}{3}$ $x_3 = \frac{2}{3}+\frac{1}{3} = 1$

(in general $x_i = a + i\Delta x$)

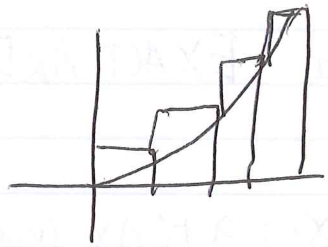
2&3

R_3 $\begin{cases} f(\cancel{1}/3)\Delta x + f(\frac{2}{3})\Delta x + f(1)\Delta x \\ \left(\frac{1}{3}\right)^3 \cdot \left(\frac{1}{3}\right) + \left(\frac{2}{3}\right)^3 \cdot \frac{1}{3} + \left(\frac{3}{3}\right)^3 \cdot \frac{1}{3} = \frac{36}{81} = 0.\overline{4} \end{cases}$

over estimate

L_3 $\begin{cases} f(0)\Delta x + f(\frac{1}{3})\Delta x + f(\frac{2}{3})\Delta x \\ \left(0\right)^3 \frac{1}{3} + \left(\frac{1}{3}\right)^3 \cdot \frac{1}{3} + \left(\frac{2}{3}\right)^2 \cdot \frac{1}{3} = \frac{1}{9} = 0.\overline{1} \end{cases}$

L_3 an underestimate.



<u>You do:</u> R_4 for $f(x) = x^3$ on $[0,1]$

(on HW)

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4} \qquad x_0 = 0 \quad x_1 = \frac{1}{4} \quad x_2 = \frac{2}{4} \quad x_3 = \frac{3}{4} \quad x_4 = \frac{4}{4}$$

$$\overset{\mathrm{I}}{f\left(\tfrac{1}{4}\right)\Delta x} + \overset{\mathrm{II}}{f\left(\tfrac{2}{4}\right)\Delta x} + \overset{\mathrm{III}}{f\left(\tfrac{3}{4}\right)\Delta x} + \overset{\mathrm{IV}}{f\left(\tfrac{4}{4}\right)\Delta x}$$

$$\left(\frac{1}{4}\right)^3 \frac{1}{4} + \left(\frac{2}{4}\right)^3 \frac{1}{4} + \left(\frac{3}{4}\right)^3 \frac{1}{4} + \left(\frac{4}{4}\right)^3 \frac{1}{4} = 0.390625$$

There's a pattern

$$R_4 = \sum_{i=1}^{4} \left(\frac{i}{4}\right)^3 \frac{1}{4}$$

$$R_{10} = \sum_{i=1}^{10} \left(\frac{i}{10}\right)^3 \frac{1}{4}$$

$$R_n = \sum_{i=1}^{n} \underbrace{\left(\frac{i}{n}\right)^3}_{x_i} \underbrace{\frac{1}{n}}_{\Delta x}$$

Riemann sum convergence picture

$$\boxed{\text{Exact area} = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 \frac{1}{n}}$$

Answer to HW

In general EXACT AREA $= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*) \Delta x$

$\Delta x = \frac{b-a}{n}$

Eg. $x_i^* = x_i = a + i\Delta x$ (right end point)

$x_i^* = x_{i-1} = a + (i-1)\Delta x$ (left end point)

doesn't matter what x_i^* is.

Write down the expression for the exact area under $f(x) = \sqrt{x}$ from $x=5$ to $x=7$

It's $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i) \Delta x$

$$\Delta x = \frac{b-a}{n} = \frac{7-5}{n} = \frac{2}{n}$$

$$x_i = a + i\Delta x = 5 + \frac{2i}{n}$$

So it's $\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\underbrace{5+\frac{2i}{n}}_{x_i}} \cdot \underbrace{\frac{2}{n}}_{\Delta x}$