CHAPTER

5

Applications of the Exponential and Natural Logarithm Functions

In Chapter 4, we introduced the exponential function $y = e^x$ and the natural logarithm function $y = \ln x$, and we studied their most important properties. It is by no means clear that these functions have any substantial connection with the physical world. However, as this chapter will demonstrate, the exponential and natural logarithm functions are involved in the study of many physical problems, often in a very curious and unexpected way.

5.1 Exponential Growth and Decay

Exponential Growth

You walk into your kitchen one day and you notice that the overripe bananas that you left on the counter invited unwanted guests: fruit flies. To take advantage of this pesky situation, you decide to study the growth of the fruit flies colony. It didn't take you too long to make your first observation: The colony is increasing at a *rate that is proportional to its size*. That is, the more fruit flies, the faster their number grows.

FOR REVIEW

"The derivative is a rate of change." See Sec. 1.7, p. 107.

To help us model this population growth, we introduce some notation. Let $P(t)$ denote the number of fruit flies in your kitchen, t days from the moment you first noticed them. A very important fact that we learned about derivatives tells us that

$$\textit{the rate of change of } P(t) \textit{ is } P'(t).$$

Translating into mathematical language our observation that the rate of change of $P(t)$ is proportional to $P(t)$, we obtain the equation

$$\overbrace{P'(t)}^{\text{rate of change}} = \overbrace{k\,P(t)}^{\text{proportional to } P(t)},$$

where k is a positive constant of proportionality. If we let $y = P(t)$, the equation becomes

$$y' = ky. \tag{1}$$

Equation (1) expresses a relationship between the function y and its derivative y'. Any equation expressing a relationship between a function and its derivatives is called a **differential equation**. Differential equations will be discussed in greater detail in Chapter 10.

A **solution** of (1) is any *function* whose derivative is equal to k times itself. This is clearly a new type of equation unlike any algebraic equation that we have encountered earlier. To find a solution, we recall a useful derivative from Section 4.3.

FOR REVIEW

$\frac{d}{dx}(e^{g(x)}) = e^{g(x)}g'(x)$

See Sec. 4.3, p. 233.

Let C and k be any constants, and let $y = Ce^{kt}$. Then, $y' = Cke^{kt}$.

(We use the independent variable t instead of x throughout this chapter. The reason is that, in most applications, the variable of our exponential function is time.) Note that

$$y' = Cke^{kt} = k \cdot \overbrace{Ce^{kt}}^{y} = ky.$$

Hence, $y = Ce^{kt}$ is a solution of (1). The converse is also true in the following sense.

Theorem 1 Exponential Function Solution of a Differential Equation The function $y = Ce^{kt}$ satisfies the differential equation

$$y' = ky.$$

Conversely, if $y = f(t)$ satisfies the differential equation $y' = ky$, then, $y = Ce^{kt}$ for some constant C.

In biology, chemistry, and economics, if, at every instant, the rate of increase of a quantity is proportional to the quantity at that instant, as expressed by (1) (with $k > 0$), then the quantity is said to be *growing exponentially* or is *exhibiting exponential growth*. Theorem 1 justifies this terminology, since, in this case, the quantity is an exponential function. The proportionality constant k is also called the **growth constant**.

EXAMPLE 1 **Solving a Differential Equation** Determine all functions $y = f(t)$ such that $y' = .3\,y$.

SOLUTION The equation $y' = .3y$ has the form $y' = ky$ with $k = .3$. Therefore, according to Theorem 1, any solution of the equation has the form

$$y = Ce^{0.3t},$$

where C is a constant.

>> Now Try Exercise 3

Note that even if the constant k is known in Example 1 ($k = .3$), the equation $y' = .3y$ has *infinitely many solutions* of the form $y = Ce^{0.3t}$, one for each choice of the *arbitrary constant* C. For example, the functions $y = e^{0.3t}$, $y = 3e^{0.3t}$, and $y = 6e^{0.3t}$ are all solutions that correspond to the choices $C = 1, 3$, and 6, respectively. (See Fig. 1.) The fact that a differential equation has infinitely many solutions allows us to select a *particular* solution that fits the situation under study. We illustrate this important fact by returning to our fruit flies problem.

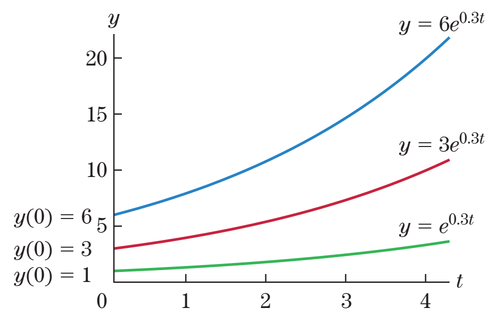

Figure 1 Some solutions of $y' = .3y$.

EXAMPLE 2 **Exponential Growth of a Fruit Flies Population** Let $y = P(t)$ denote the number of fruit flies in the kitchen, t days since you first observed them. It is known that this species of flies exhibits exponential growth with growth constant $k = .3$. Suppose that the initial number of fruit flies is 6.

(a) Find $P(t)$.

(b) Estimate the number of fruit flies after 7 days.

SOLUTION **(a)** Because $y = P(t)$ exhibits exponential growth with growth constant $k = .3$, it satisfies the differential equation

$$y' = .3y.$$

We are also told that the initial number of fruit flies is 6. That is, at time $t = 0$, there are 6 fruit flies. Hence, $P(0) = 6$. From Example 1, the differential equation $y' = .3y$ has infinitely many solutions of the form $y = Ce^{0.3t}$, but only one of these solutions will satisfy the condition $P(0) = 6$. Indeed, taking $t = 0$ in $P(t) = Ce^{0.3t}$, we get

$$6 = P(0) = Ce^{(0.3)(0)} = Ce^0 = C.$$

Thus, $C = 6$ and so $P(t) = 6e^{0.3t}$.

(b) After 7 days, we have

$$P(7) = 6e^{0.3(7)} = 6e^{2.1} \approx 48.997.$$

Thus after 7 days, there are approximately 49 fruit flies in the kitchen.

» ***Now Try Exercise 19***

The condition $P(0) = 6$ in Example 2 is called an **initial condition**. The initial condition describes the initial size of the population, which, in turn, can be used to determine a unique solution of the differential equation. Fig. 1 shows many solutions of the differential equation $y' = .3y$ but only one goes through the point $(0, 6)$ and so satisfies the initial condition $y(0) = 6$. For future reference, we state the following useful result.

Theorem 2 Solution of a Differential Equation with Initial Condition

The *unique* solution, $y = P(t)$, of the differential equation with initial condition,

$$y' = ky, \qquad y(0) = P_0,$$

is $y = P(t) = P_0e^{kt}$.

EXAMPLE 3 **A Differential Equation with Initial Condition** Solve $y' = 3y$, $y(0) = 2$.

SOLUTION Here, $k = 3$ and $P_0 = 2$. By Theorem 2, the (unique) solution is $y = 2e^{3t}$. Note the initial condition on the graph of the solution in Fig. 2.

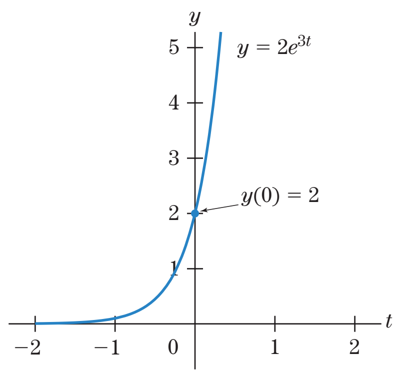

Figure 2 Unique solution of $y' = 3y$, $y(0) = 2$.

» Now Try Exercise 13

The following examples illustrate different ways to determine the constants C and k in exponential growth problems arising in real-world situations.

EXAMPLE 4 **Exponential Growth** A colony of fruit flies grows at a rate proportional to its size. At time $t = 0$, approximately 20 fruit flies are present. In 5 days there are 400 fruit flies. Determine a function that expresses the size of the colony as a function of time, measured in days.

SOLUTION Let $P(t)$ be the number of fruit flies present at time t. By assumption, $P(t)$ satisfies a differential equation of the form $y' = ky$, so $P(t)$ has the form

$$P(t) = P_0 e^{kt},$$

where the constants P_0 and k must be determined. The values of P_0 and k can be obtained from the data that give the population size at two different times. We are told that

$$P(0) = 20, \qquad P(5) = 400.$$

The first condition immediately implies that $P_0 = 20$, so

$$P(t) = 20e^{kt}.$$

FOR REVIEW

e^x and $\ln x$ are inverse of each other: $\ln e^x = x$ and $e^{\ln x} = x$. See Sec. 4.4.

Using the second condition, we have

$$P(5) = 20e^{k(5)} = 400$$

$$e^{5k} = 20 \qquad \text{Divide by 20.}$$

$$5k = \ln 20 \qquad \text{Take ln of each side.}$$

$$k = \frac{\ln 20}{5} \approx .60.$$

So, using the values of P and k, we get

$$P(t) = 20e^{0.6t}.$$

This function is a mathematical model of the growth of the colony of flies. (See Fig. 3.)

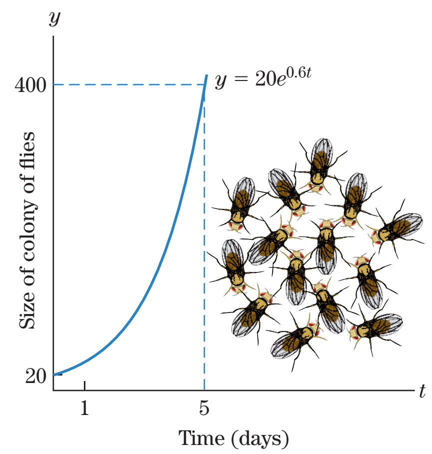

Figure 3 A model for a colony of fruit flies as a function of time $t \geq 0$.

>> *Now Try Exercise 20*

EXAMPLE 5 **Determining the Growth Constant** A colony of fruit flies is growing according to the exponential law $P(t) = P_0e^{kt}$, and the size of the colony doubles in 9 days. Determine the growth constant k.

SOLUTION We do not know the initial size of the population at $t = 0$. However, we are told that the colony doubles in 9 days. Mathematically this is represented by $P(9) = 2P(0)$; that is,

$$P_0e^{k(9)} = 2P_0$$

$$e^{9k} = 2 \qquad \text{Divide by } P_0 \neq 0.$$

$$9k = \ln 2 \qquad \text{Take ln of each side.}$$

$$k = \frac{\ln 2}{9} \approx .077. \qquad \text{Solve for } k.$$

>> *Now Try Exercise 21*

The initial size P_0 of the population was not given in Example 5. But we were able to determine the growth constant because we were told the amount of time required for the colony to double in size. Thus, the growth constant does not depend on the initial size of the population. This property is characteristic of exponential growth.

EXAMPLE 6 **Working with a Differential Equation** The initial size of the colony in Example 5 was 100.

(a) How large will the colony be after 41 days?
(b) How fast will the colony be growing at that time?
(c) At what time will the colony contain 800 fruit flies?
(d) How large is the colony when it is growing at the rate of 200 fruit flies per day?

SOLUTION **(a)** From Example 5, we have $P(t) = P_0e^{0.077t}$. Since $P(0) = 100$, we conclude that

$$P(t) = 100e^{0.077t}.$$

Therefore, after 41 days, the size of the colony is

$$P(41) = 100e^{0.077(41)} = 100e^{3.157} \approx 2350 \text{ fruit flies.}$$

(b) Recall from Example 5 that $k = .077$. Since the function $P(t)$ satisfies the differential equation $y' = .077y$,

$$P'(t) = .077P(t).$$

In particular, when $t = 41$,

$$P'(41) = .077P(41) = (.077)(2350) \approx 181.$$

Therefore, after 41 days, the colony is growing at the rate of about 181 fruit flies per day.

(c)

$$\begin{aligned} 100e^{0.077t} &= 800 && \text{Set } P(t) = 800. \\ e^{0.077t} &= 8 && \text{Divide by 100.} \\ .077t &= \ln 8 && \text{Take ln of each side.} \\ t &= \frac{\ln 8}{.077} \approx 27 \text{ days.} && \text{Solve for } t. \end{aligned}$$

(d) When the colony is growing at the rate of 200 fruit flies per day, $P'(t) = 200$. As in (b), we use the differential equation $P'(t) = .077P(t)$ and set $P'(t) = 200$. Then,

$$200 = .077P(t),$$

$$P(t) = \frac{200}{.077} \approx 2597.$$

Therefore, there are 2597 fruit flies in the colony when it is growing at the rate of 200 fruit flies per day. **» Now Try Exercise 23**

Exponential Decay

To solve the differential equation $y' = ky$ when the constant k is negative we can still appeal to Theorem 1 and obtain the solution $y = Ce^{kt}$, where k is negative and C is an arbitrary constant. In this case, we are dealing with a negative exponential growth, or **exponential decay**. An example of exponential decay is given by the disintegration of a radioactive element such as uranium 235. It is known that, at any instant, the rate at which a radioactive substance is decaying is proportional to the amount of the substance that has not yet disintegrated. If $P(t)$ is the quantity present at time t, then $P'(t)$ is the rate of decay. Since $P(t)$ is decreasing, $P'(t)$ must be negative. Thus, we may write $P'(t) = kP(t)$ for some negative constant k. To emphasize the fact that the constant is negative, k is often replaced by $-\lambda$, where λ is a positive constant (λ is the Greek lowercase letter lambda). Then, $P(t)$ satisfies the differential equation

$$P'(t) = -\lambda P(t).$$

By Theorem 1, the solution has the form

$$P(t) = P_0 e^{-\lambda t}$$

for some positive number P_0. We call such a function an **exponential decay function**. The constant λ is called the **decay constant**.

EXAMPLE 7 **Exponential Decay** The decay constant for the radioactive element **strontium** 90 is $\lambda = .0244$, where time is measured in years. How long will it take for a quantity P_0 of strontium 90 to decay to one-half its original mass?

SOLUTION Since $\lambda = .0244$, we have

$$P(t) = P_0 e^{-0.0244t}.$$

Next, set $P(t)$ equal to $\frac{1}{2}P_0$ and solve for t:

$$\begin{aligned} P_0 e^{-0.0244t} &= \tfrac{1}{2}P_0 \\ e^{-0.0244t} &= \tfrac{1}{2} = .5 && \text{Divide by } P_0. \\ -.0244t &= \ln .5 && \text{Take ln of each side.} \\ t &= \frac{\ln .5}{-.0244} \approx 28 \text{ years.} && \text{Solve for } t. \end{aligned}$$

» Now Try Exercise 35

The **half-life** of a radioactive element is the length of time required for a given quantity of that element to decay to one-half its original mass. Thus, strontium 90 has a half-life of about 28 years. It takes 28 years for it to decay to half its original mass and another 28 years for it to decay to $\frac{1}{4}$ its original mass, another 28 years to decay to $\frac{1}{8}$, and so forth. (See Fig. 4.) Notice from Example 7 that the half-life does not depend on the initial amount P_0.

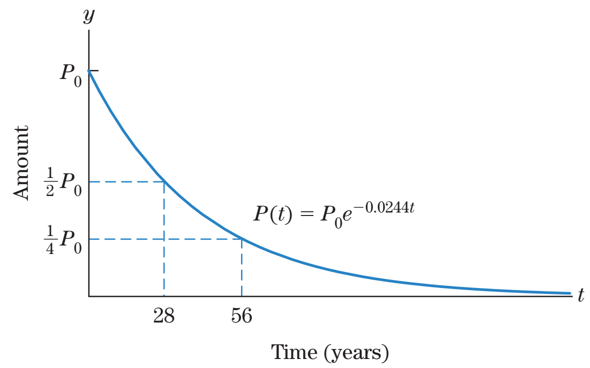

Figure 4 Half-life of radioactive strontium 90.

One problem connected with above ground nuclear explosions is the radioactive debris, or fallout, that contaminates plants and grass, the food supply of animals. Strontium 90 is one of the most dangerous components of radioactive debris because it has a relatively long half-life. Also, it is chemically similar to calcium and is absorbed into the bone structure of animals (including humans) who eat contaminated food. Iodine 131 is also produced by nuclear explosions, but it presents less of a hazard because it has a half-life of 8 days. See Exercise 41.

EXAMPLE 8 **Half-Life and Decay Constant** Radioactive carbon 14 has a half-life of about 5730 years. Find its decay constant.

SOLUTION If P_0 denotes the initial amount of carbon 14, the amount after t years will be

$$P(t) = P_0e^{-\lambda t}.$$

After 5730 years, $P(t)$ will equal $\frac{1}{2}P_0$. That is,

$$P_0e^{-\lambda(5730)} = P(5730) = \tfrac{1}{2}P_0 = .5P_0.$$

Solving for λ gives

$$P_0e^{-\lambda(5730)} = .5P_0$$

$$e^{-5730\lambda} = .5 \qquad \text{Divide by } P_0.$$

$$-5730\lambda = \ln .5 \qquad \text{Take ln of each side.}$$

$$\lambda = \frac{\ln .5}{-5730} \approx .00012. \qquad \text{Solve for } \lambda.$$

» *Now Try Exercise 37*

FOR REVIEW

- $\ln x < 0$ if $0 < x < 1$, so $\ln(.5) < 0$.
- $\ln \frac{1}{x} = -\ln x$ so $\ln(.5) = \ln\left(\frac{1}{2}\right) = -\ln 2$
- $\ln 2 \approx .69$.

See the graph of $y = \ln x$, p. 237.

Radiocarbon Dating

Knowledge about radioactive decay is valuable to archaeologists and anthropologists who want to estimate the age of objects belonging to ancient civilizations. Several different substances are useful for radioactive-dating techniques; the most common is radiocarbon, ^{14}C. Carbon 14 is produced in the upper atmosphere when cosmic rays react with atmospheric nitrogen. Because the ^{14}C eventually decays, the concentration of ^{14}C cannot rise above certain levels. An equilibrium is reached where ^{14}C is produced at the same rate as it decays. Scientists usually assume that the total amount of ^{14}C in the biosphere has remained constant over the past 50,000 years. Consequently, it is assumed that the *ratio* of ^{14}C to ordinary nonradioactive carbon 12, ^{12}C, has been constant during this same period. (The ratio is about one part ^{14}C to 10^{12} parts of ^{12}C.) Both ^{14}C and ^{12}C are in the atmosphere as constituents of carbon dioxide. All living vegetation and most forms of animal life contain ^{14}C and ^{12}C in the same proportion as the atmosphere because plants absorb carbon dioxide through photosynthesis. The ^{14}C and ^{12}C in plants are distributed through the food chain to almost all animal life.

When an organism dies, it stops replacing its carbon; therefore, the amount of ^{14}C begins to decrease through radioactive decay, but the ^{12}C in the dead organism remains constant. The ratio of ^{14}C to ^{12}C can be later measured to determine when the organism died.

EXAMPLE 9 **Carbon Dating** A parchment fragment made from animal skin was discovered that had about 80% of the ^{14}C level found today in living matter. Estimate the age of the parchment.

SOLUTION We assume that the original ^{14}C level in the parchment was the same as the level in living organisms today. Consequently, about eight-tenths of the original ^{14}C remains. From Example 8 we obtain the formula for the amount of ^{14}C present t years after the parchment was made from an animal skin:

$$P(t) = P_0 e^{-0.00012t},$$

where $P_0 =$ initial amount. We want to find t such that $P(t) = .8P_0$:

$$P_0 e^{-0.00012t} = .8P_0$$

$$e^{-0.00012t} = .8 \qquad \text{Divide by } P_0.$$

$$-.00012t = \ln .8 \qquad \text{Take ln of each side.}$$

$$t = \frac{\ln .8}{-.00012} \approx 1860 \text{ years old.} \qquad \text{Solve for } t.$$

>> Now Try Exercise 47

Thus the parchment is about 1860 years old.

The Time Constant

Consider an exponential decay function $y = Ce^{-\lambda t}$. Figure 5 shows the tangent line to the decay curve when $t = 0$. The slope there is the initial rate of decay. If the decay process were to continue at this rate, the decay curve would follow the tangent line, and y would be zero at some time T. This time is called the **time constant** of the decay curve. It can be shown (see Exercise 52) that $T = 1/\lambda$ for the curve $y = Ce^{-\lambda t}$. Thus, $\lambda = 1/T$ and the decay curve can be written in the form

$$y = Ce^{-t/T}.$$

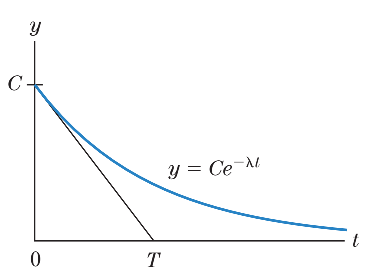

Figure 5 The time constant T in exponential decay: $T = 1/\lambda$.

If we have experimental data that tend to lie along an exponential decay curve, the numerical constants for the curve may be obtained from Fig. 5. First, sketch the curve and estimate the y-intercept, C. Then, sketch an approximate tangent line, and from this, estimate the time constant, T. This procedure is sometimes used in biology and medicine.

Check Your Understanding 5.1

Solutions can be found following the section exercises.

1. **(a)** Solve the differential equation $P'(t) = -.6P(t)$, $P(0) = 50$.
   (b) Solve the differential equation $P'(t) = kP(t)$, $P(0) = 4000$, where k is some constant.
   (c) Interpret the meaning of $P(2) = 100P(0)$, where t is in hours.
   (d) Find the value of k in part (b) for which $P(2) = 100P(0)$.
2. Under ideal conditions a colony of *Escherichia coli* bacteria can grow by a factor of 100 every 2 hours. If 4000 bacteria are present initially, how long will it take before there are 1 million bacteria?

EXERCISES 5.1

In Exercises 1–10, determine the growth constant k, then find all solutions of the given differential equation.

1. $y' = y$

2. $y' = .4y$

3. $y' = 1.7y$

4. $y' = \frac{y}{4}$

5. $y' - \frac{y}{2} = 0$

6. $y' - 6y = 0$

7. $2y' - \frac{y}{2} = 0$

8. $y = 1.6y'$

9. $\frac{y}{3} = 4y'$

10. $5y' - 6y = 0$

In Exercises 11–18, solve the given differential equation with initial condition.

11. $y' = 3y$, $y(0) = 1$

12. $y' = 4y$, $y(0) = 0$

13. $y' = 2y$, $y(0) = 2$

14. $y' = y$, $y(0) = 4$

15. $y' - .6y = 0$, $y(0) = 5$

16. $y' - \frac{y}{7} = 0$, $y(0) = 6$

17. $6y' = y$, $y(0) = 12$

18. $5y = 3y'$, $y(0) = 7$

19. Population with Exponential Growth Let $P(t)$ be the population (in millions) of a certain city t years after 2015, and suppose that $P(t)$ satisfies the differential equation

$$P'(t) = .01P(t),\ P(0) = 2.$$

(a) Find a formula for $P(t)$.
(b) What was the initial population, that is, the population in 2015?
(c) Estimate the population in 2019.

20. Growth of a Colony of Fruit Flies A colony of fruit flies exhibits exponential growth. Suppose that 500 fruit flies are present. Let $P(t)$ denote the number of fruit flies t days later, and let $k = .08$ denote the growth constant.
(a) Write a differential equation and initial condition that model the growth of this colony.
(b) Find a formula for $P(t)$.
(c) Estimate the size of the colony 5 days later.

21. Growth Constant for a Bacteria Culture A bacteria culture that exhibits exponential growth quadruples in size in 2 days.
(a) Find the growth constant if time is measured in days.
(b) If the initial size of the bacteria culture was 20,000, what is its size after just 12 hours?

22. Growth of a Bacteria Culture The initial size of a bacteria culture that grows exponentially was 10,000. After 1 day, there are 15,000 bacteria.
(a) Find the growth constant if time is measured in days.
(b) How long will it take for the culture to double in size?

23. Using the Differential Equation Let $P(t)$ be the population (in millions) of a certain city t years after 2015, and suppose that $P(t)$ satisfies the differential equation

$$P'(t) = .03P(t),\ P(0) = 4.$$

(a) Use the differential equation to determine how fast the population is growing when it reaches 5 million people.
(b) Use the differential equation to determine the population size when it is growing at the rate of 400,000 people per year.
(c) Find a formula for $P(t)$.

24. Growth of Bacteria Approximately 10,000 bacteria are placed in a culture. Let $P(t)$ be the number of bacteria present in the culture after t hours, and suppose that $P(t)$ satisfies the differential equation

$$P'(t) = .55P(t).$$

(a) What is $P(0)$?
(b) Find the formula for $P(t)$.
(c) How many bacteria are there after 5 hours?
(d) What is the growth constant?
(e) Use the differential equation to determine how fast the bacteria culture is growing when it reaches 100,000.
(f) What is the size of the bacteria culture when it is growing at a rate of 34,000 bacteria per hour?

25. Growth of Cells After t hours there are $P(t)$ cells present in a culture, where $P(t) = 5000e^{0.2t}$.
(a) How many cells were present initially?
(b) Give a differential equation satisfied by $P(t)$.
(c) When will the initial number of cells double?
(d) When will 20,000 cells be present?

26. Insect Population The size of a certain insect population is given by $P(t) = 300e^{0.01t}$, where t is measured in days.
(a) How many insects were present initially?
(b) Give a differential equation satisfied by $P(t)$.
(c) At what time will the initial population double?
(d) At what time will the population equal 1200?

27. Population Growth Determine the growth constant of a population that is growing at a rate proportional to its size, where the population doubles in size every 40 days and time is measured in days.

28. Time to Triple Determine the growth constant of a population that is growing at a rate proportional to its size, where the population triples in size every 10 years and time is measured in years.

29. Exponential Growth A population is growing exponentially with growth constant .05. In how many years will the current population triple?

30. Time to Double A population is growing exponentially with growth constant .04. In how many years will the current population double?

31. Exponential Growth The rate of growth of a certain cell culture is proportional to its size. In 10 hours a population of 1 million cells grew to 9 million. How large will the cell culture be after 15 hours?

32. World's Population The world's population was 5.51 billion on January 1, 1993, and 5.88 billion on January 1, 1998. Assume that, at any time, the population grows at a rate proportional to the population at that time. In what year will the world's population reach 7 billion?

33. Population of Mexico City At the beginning of 1990, 20.2 million people lived in the metropolitan area of Mexico City, and the population was growing exponentially. The 1995 population was 23 million. (Part of the growth is due to immigration.) If this trend continues, how large will the population be in the year 2010?

34. A Population Model The population (in millions) of a state t years after 2010 is given by the graph of the exponential function $y = P(t)$ with growth constant .025 in Fig. 6. [In parts (c) and (d) use the differential equation satisfied by $P(t)$.]

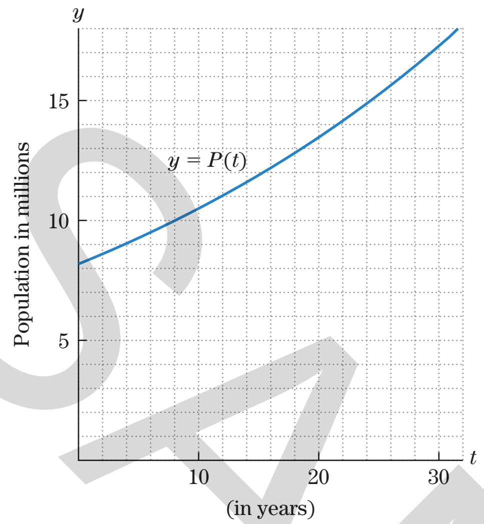

Figure 6

(a) What is the population in 2020?
(b) When is the population 10 million?
(c) How fast is the population growing in 2020?
(d) When is the population growing at the rate of 275,000 people per year?

35. Radioactive Decay A sample of 8 grams of radioactive material is placed in a vault. Let $P(t)$ be the amount remaining after t years, and let $P(t)$ satisfy the differential equation $P'(t) = -.021P(t)$.
(a) Find the formula for $P(t)$
(b) What is $P(0)$?
(c) What is the decay constant?
(d) How much of the material will remain after 10 years?
(e) Use the differential equation to determine how fast the sample is disintegrating when just 1 gram remains.
(f) What amount of radioactive material remains when it is disintegrating at the rate of .105 gram per year?
(g) The radioactive material has a half-life of 33 years. How much will remain after 33 years? 66 years? 99 years?

36. Radioactive Decay Radium 226 is used in cancer radiotherapy. Let $P(t)$ be the number of grams of radium 226 in a sample remaining after t years, and let $P(t)$ satisfy the differential equation

$$P'(t) = -.00043P(t), \qquad P(0) = 12.$$

(a) Find the formula for $P(t)$.
(b) What was the initial amount?
(c) What is the decay constant?
(d) Approximately how much of the radium will remain after 943 years?
(e) How fast is the sample disintegrating when just 1 gram remains? Use the differential equation.
(f) What is the weight of the sample when it is disintegrating at the rate of .004 gram per year?
(g) The radioactive material has a half-life of about 1612 years. How much will remain after 1612 years? 3224 years? 4836 years?

37. Decay of Penicillin in the Bloodstream A person is given an injection of 300 milligrams of penicillin at time $t = 0$. Let $f(t)$ be the amount (in milligrams) of penicillin present in the person's bloodstream t hours after the injection. Then, the amount of penicillin decays exponentially, and a typical formula is $f(t) = 300e^{-0.6t}$.
(a) Give the differential equation satisfied by $f(t)$
(b) How much will remain at time $t = 5$ hours?
(c) What is the biological half-life of the penicillin (that is, the time required for half of a given amount to decompose) in this case?

38. Radioactive Decay Ten grams of a radioactive substance with decay constant .04 is stored in a vault. Assume that time is measured in days, and let $P(t)$ be the amount remaining at time t.
(a) Give the formula for $P(t)$
(b) Give the differential equation satisfied by $P(t)$.
(c) How much will remain after 5 days?
(d) What is the half-life of this radioactive substance?

39. Radioactive Decay The decay constant for the radioactive element cesium 137 is .023 when time is measured in years. Find its half-life.

40. Drug Constant Radioactive cobalt 60 has a half-life of 5.3 years. Find its decay constant.

41. Iodine Level in Dairy Products If dairy cows eat hay containing too much iodine 131, their milk will be unfit to drink. Iodine 131 has half-life of 8 days. If the hay contains 10 times the maximum allowable level of iodine 131, how many days should the hay be stored before it is fed to dairy cows?

42. Half-Life Ten grams of a radioactive material disintegrates to 3 grams in 5 years. What is the half-life of the radioactive material?

43. Decay of Sulfate in the Bloodstream In an animal hospital, 8 units of a sulfate were injected into a dog. After 50 minutes, only 4 units remained in the dog. Let $f(t)$ be the amount of sulfate present after t minutes. At any time, the rate of change of $f(t)$ is proportional to the value of $f(t)$. Find the formula for $f(t)$.

44. Radioactive Decay Forty grams of a certain radioactive material disintegrates to 16 grams in 220 years. How much of this material is left after 300 years?

45. Radioactive Decay A sample of radioactive material decays over time (measured in hours) with decay constant .2. The graph of the exponential function $y = P(t)$ in Fig. 7 gives the number of grams remaining after t hours. [*Hint:* In parts (c) and (d) use the differential equation satisfied by $P(t)$.]

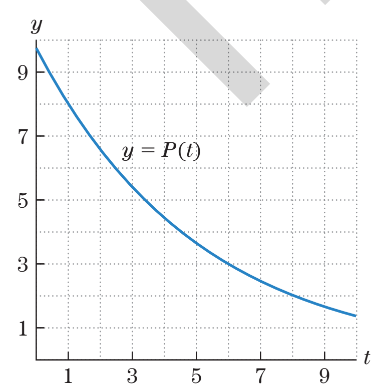

Figure 7

(a) How much was remaining after 1 hour?
(b) Approximate the half-life of the material.
(c) How fast was the sample decaying after 6 hours?
(d) When was the sample decaying at the rate of .4 grams per hour?

46. **Rate of Decay** A sample of radioactive material has decay constant .25, where time is measured in hours. How fast will the sample be disintegrating when the sample size is 8 grams? For what sample size will the sample size be decreasing at the rate of 2 grams per day?

47. **Carbon Dating** In 1947, a cave with beautiful prehistoric wall paintings was discovered in Lascaux, France. Some charcoal found in the cave contained 20% of the ^{14}C expected in living trees. How old are the Lascaux cave paintings? (Recall that the decay constant for ^{14}C is .00012.)

48. **King Arthur's Round Table** According to legend, in the fifth century King Arthur and his knights sat at a huge round table. A round table alleged to have belonged to King Arthur was found at Winchester Castle in England. In 1976, carbon dating revealed the amount of radiocarbon in the table to be 91% of the radiocarbon present in living wood. Could the table possibly have belonged to King Arthur? Why? (Recall that the decay constant for ^{14}C is .00012.)

49. **Radioactive Decay** A 4500-year-old wooden chest was found in the tomb of the twenty-fifth century B.C. Chaldean king Meskalumdug of Ur. What percentage of the original ^{14}C would you expect to find in the wooden chest?

50. **Population of the Pacific Northwest** In 1938, sandals woven from strands of tree bark were found in Fort Rock Creek Cave in Oregon. The bark contained 34% of the level of ^{14}C found in living bark. Approximately how old were the sandals? [*Note:* This discovery by University of Oregon anthropologist Luther Cressman forced scientists to double their estimate of how long ago people came to the Pacific Northwest.]

51. **Time of the Fourth Ice Age** Many scientists believe there have been four ice ages in the past 1 million years. Before the technique of carbon dating was known, geologists erroneously believed that the retreat of the Fourth Ice Age began about 25,000 years ago. In 1950, logs from ancient spruce trees were found under glacial debris near Two Creeks, Wisconsin. Geologists determined that these trees had been crushed by the advance of ice during the Fourth Ice Age. Wood from the spruce trees contained 27% of the level of ^{14}C found in living trees. Approximately how long ago did the Fourth Ice Age actually occur?

52. **Time Constant** Let T be the time constant of the curve $y = Ce^{-\lambda t}$ as defined in Fig. 5. Show that $T = 1/\lambda$. [*Hint:* Express the slope of the tangent line in Fig. 5 in terms of C and T. Then, set this slope equal to the slope of the curve $y = Ce^{-\lambda t}$ at $t = 0$.]

53. **Differential Equation and Decay** The amount in grams of a certain radioactive material present after t years is given by the function $P(t)$. Match each of the following answers with its corresponding question.

Answers

a. Solve $P(t) = .5P(0)$ for t.
b. Solve $P(t) = .5$ for t.
c. $P(.5)$
d. $P'(.5)$
e. $P(0)$
f. Solve $P'(t) = -.5$ for t.
g. $y' = ky$
h. P_0e^{kt}, $k < 0$

Questions

A. Give a differential equation satisfied by $P(t)$.
B. How fast will the radioactive material be disintegrating in $\frac{1}{2}$ year?
C. Give the general form of the function $P(t)$.
D. Find the half-life of the radioactive material.
E. How many grams of the material will remain after $\frac{1}{2}$ year?
F. When will the radioactive material be disintegrating at the rate of $\frac{1}{2}$ gram per year?
G. When will there be $\frac{1}{2}$ gram remaining?
H. How much radioactive material was present initially?

54. **Time Constant and Half-life** Consider an exponential decay function $P(t) = P_0e^{-\lambda t}$, and let T denote its time constant. Show that, at $t = T$, the function $P(t)$ decays to about one-third of its initial size. Conclude that the time constant is always larger than the half-life.

55. **An Initial Value Problem** Suppose that the function $P(t)$ satisfies the differential equation

$$y'(t) = -.5y(t), \quad y(0) = 10.$$

(a) Find an equation of the tangent line to the graph of $y = P(t)$ at $t = 0$. [*Hint:* What are $P'(0)$ and $P(0)$?]
(b) Find $P(t)$.
(c) What is the time constant of the decay curve $y = P(t)$?

56. **Time to Finish** Consider the exponential decay function $y = P_0e^{-\lambda t}$, with time constant T. We define the time to finish to be the time it takes for the function to decay to about 1% of its initial value P_0. Show that the time to finish is about four times the time constant T.

Solutions to Check Your Understanding 5.1

1. (a) Answer: $P(t) = 50e^{-0.6t}$. Differential equations of the type $y' = ky$ have as their solution $P(t) = Ce^{kt}$, where C is $P(0)$.

(b) Answer: $P(t) = 4000e^{kt}$. This problem is like part (a) except that the constant is not specified. Additional information is needed if we want to determine a specific value for k.

(c) After 2 hours, the initial population increased 100-fold.

(d) Answer: $P(t) = 4000e^{2.3t}$. From the solution to part (b), we know that $P(t) = 4000e^{kt}$. We are given that $P(2) = 100P(0) = 100(4000) = 400{,}000$. So,

$$P(2) = 4000e^{k(2)} = 400{,}000$$
$$e^{2k} = 100$$
$$2k = \ln 100$$
$$k = \frac{\ln 100}{2} \approx 2.3.$$

2. Let $P(t)$ be the number of bacteria present after t hours. We must first find an expression for $P(t)$ and then determine the value of t for which $P(t) = 1{,}000{,}000$. From the discussion at the beginning of the section, we know that $P'(t) = k \cdot P(t)$. Also, we are given that $P(2)$ (the population after 2 hours) is $100P(0)$ (100 times the initial population). From part (d) of the previous problem, we have an expression for $P(t)$:

$$P(t) = 4000e^{2.3t}.$$

Now we must solve $P(t) = 1{,}000{,}000$ for t:

$$\begin{aligned} 4000e^{2.3t} &= 1{,}000{,}000 \\ e^{2.3t} &= 250 \\ 2.3t &= \ln 250 \\ t &= \frac{\ln 250}{2.3} \approx 2.4. \end{aligned}$$

Therefore, after 2.4 hours, there will be 1,000,000 bacteria.

5.2 Compound Interest

FOR REVIEW

Compound interest was introduced in Sec. 0.5, pp. 35–38.

Continuous Compounding

The subject of compound interest was introduced in Section 0.5, where we derived a formula for the compound amount in a savings account if interest is compounded at stated intervals of time per year (compound periods). In an era of online banking, it is possible to compound interest every month, or every day, or every hour, or perhaps even more frequently. We say that interest is **compounded continuously** if the number of compound periods per year is increased indefinitely. As we will show momentarily, if interest is compounded continuously, your savings account will grow exponentially, much like the fruit flies in your kitchen, or the bacteria in a petri dish (Sec. 5.1).

Let P_0 denote your initial deposit in dollars (also called the principal amount), r the annual rate of interest, and $y = A(t)$ the compound amount or balance in your savings account at the end of time t, where t is measured in years. Even though it is difficult to describe $A(t)$, it is not difficult to describe its *rate of change* at any time t. Indeed, since the interest rate is r, if at time t you have $A(t)$ dollars in the account, then the account is growing at a rate of r times $A(t)$ dollars per year. Since the rate of change is $A'(t)$, we get

$$\overbrace{A'(t)}^{\text{[rate of change]}} = r \times A(t).$$

Thus, the balance in your savings account satisfies the differential equation and initial condition

$$A'(t) = rA(t), \qquad A(0) = P_0. \tag{1}$$

The solution of this equation follows from Theorem 2 of the previous section:

$$A(t) = P_0e^{rt}. \tag{2}$$

This is the **continuous compound interest formula**. It gives the balance of your savings account or the compound amount after t years, when interest is compounded continuously.

The formula $A(t) = P_0e^{rt}$ contains four variables. (Remember that the letter e represents a specific constant, $e = 2.718\ldots$.) In a typical problem, we are given values for three of these variables and must solve for the remaining variable.

EXAMPLE 1 **Continuous Compound Interest** One thousand dollars is invested at 5% interest compounded continuously.

(a) Give the formula for $A(t)$, the compound amount after t years.
(b) How much will be in the account after 6 years?
(c) After 6 years, at what rate will $A(t)$ be growing?
(d) How long is required for the initial investment to double?

SOLUTION

(a) $P_0 = 1000$ and $r = .05$. By the continuous compound formula (2), $A(t) = 1000e^{0.05t}$.

(b) $A(6) = 1000e^{0.05(6)} = 1000e^{0.3} \approx \$1349.86$.

(c) Rate of growth is different from interest rate. Interest rate is fixed at 5% and does not change with time. However, the rate of growth $A'(t)$ is always changing. Since $A(t) = 1000e^{0.05t}$, $A'(t) = (1000)\cdot(.05)e^{0.05t} = 50e^{0.05t}$. So, after 6 years,

$$A'(6) = 50e^{0.05(6)} = 50e^{0.3} \approx 67.49 \text{ dollars per year.}$$

After 6 years, the investment is growing at the rate of \$67.49 per year.

There is an easier way to answer part (c), given that we have already calculated $A(6)$. Since $A(t)$ satisfies the differential equation $A'(t) = rA(t)$,

$$A'(6) = .05A(6) = .05 \cdot 1349.86 \approx \$67.49 \text{ per year.}$$

(d) We must find t such that $A(t) = \$2000$. So we set $1000e^{0.05t} = 2000$ and solve for t.

$$1000e^{0.05t} = 2000 \qquad \text{Given equation.}$$

$$e^{0.05t} = 2 \qquad \text{Divide by 1000.}$$

$$\ln e^{0.05t} = \ln 2 \qquad \text{Take ln of each side.}$$

$$.05t = \ln 2 \qquad \ln e^{0.05t} = .05t.$$

$$t = \frac{\ln 2}{.05} \approx 13.86 \text{ years} \qquad \text{Solve for } t.$$

» Now Try Exercise 1

NOTE The calculations in Example 1(d) would be essentially unchanged after the first step if the initial amount of the investment were changed from \$1000 to any arbitrary amount P. When this investment doubles, the compound amount will be $2P$. So, we set $2P = Pe^{0.05t}$ and solve for t as we did previously to conclude that, at 5% interest compounded continuously, any amount doubles in about 13.86 years. «

EXAMPLE 2 **Appreciation of a Painting** Pablo Picasso's *The Dream* was purchased in 1941 for a war-distressed price of \$7000. The painting was sold in 1997 for \$48.4 million, the second highest price ever paid for a Picasso painting at auction. What rate of interest compounded continuously did this investment earn?

SOLUTION Let P_0e^{rt} be the value (in millions) of the painting t years after 1941. Since the initial value is .007 million, $P_0 = .007$. Since the value after 56 years is 48.4 million dollars, $.007e^{r(56)} = 48.4$. Now solve for r:

$$.007e^{r(56)} = 48.4 \qquad \text{Given equation.}$$

$$e^{r(56)} = \frac{48.4}{.007} \approx 6914.29 \qquad \text{Divide by .007.}$$

$$r(56) = \ln(6914.29) \qquad \text{Take ln of each side.}$$

$$r = \frac{\ln(6914.29)}{56} \approx .158 \qquad \text{Solve for } r.$$

Therefore, as an investment, the painting earned an interest rate of about 15.8%.

» Now Try Exercise 17

Ordinary Versus Continuous Compounding

How much is gained from continuous compounding as opposed to ordinary compounding? Our intuition tells us that if we compound interest frequently enough, then the compound amount should be close to the compound amount from continuous compounding. Let $A(t)$ denote the continuous compound amount that we derived in (2), and let B denote the compound amount if we use m compound periods per year. In Section 0.5, we derived the formula

$$B = P_0\left(1 + \frac{r}{m}\right)^{mt}. \tag{3}$$

For instance, suppose that \$1000 is invested at 6% interest for 1 year, and that interest is compounded once per year. In formula (3), this corresponds to $P_0 = \$1000$, $r = .06$, $m = 1$, and $t = 1$ year. The amount at the end of one year is

$$B = P_0(1 + r)^1 = 1000(1 + .06) = \$1060.$$

If interest is compounded quarterly ($m = 4$),

$$B = P_0\left(1 + \frac{r}{4}\right)^{4t} = 1000\left(1 + \frac{.06}{4}\right)^4 \approx \$1061.36.$$

If interest is compounded monthly ($m = 12$),

$$B = P_0\left(1 + \frac{r}{12}\right)^{12t} = 1000\left(1 + \frac{.06}{12}\right)^{12} \approx \$1061.68.$$

If interest is compounded continuously, we use (2) and get

$$A = P_0e^{rt} = 1000e^{0.06(1)} \approx \$1061.84.$$

Table 1 contains these results along with the one for daily compounding for 1 year.

Table 5.1 Effect of Increased Compounding Periods

Frequency of Compounding	Annually	Quarterly	Monthly	Daily	Continuous
m	1	4	12	365	
Balance after 1 year (\$)	1060.00	1061.36	1061.68	1061.83	1061.84

Comparing the results from Table 1, we note that continuous compounding yields only one cent more than the result of the daily compounding. Consequently, frequent compounding (such as every hour or every second) will produce, at most, 1 cent more in our case.

In many computations, it is simpler to use the formula for interest compounded continuously than the formula for ordinary compound interest. In these instances, it is commonplace to use interest compounded continuously as an approximation to ordinary compound interest.

Negative Interest Rates

In 2015, some European banks began "paying" negative interest on short-term deposits to encourage customers to invest their savings. Let $-r$ denote the negative interest rate, P_0 the principal amount, and $A(t)$ the compound amount after time t in years. To describe the account in this case, we modify equation (1) as follows:

$$A'(t) = -rA(t), \qquad A(0) = P_0, \tag{4}$$

where the minus sign indicates that $A'(t)$, the rate of change of the compound amount $A(t)$, is negative. Using Theorem 2, Sec. 5.1, we see that the solution is an exponentially decaying function

$$A(t) = P_0e^{-rt}. \tag{5}$$

EXAMPLE 3 **Negative Interest Rate** In 2015, The Swiss National Bank used a negative interest rate of −.75% on savings deposits. A customer makes an initial deposit of 10,000 Swiss Francs (SFr).

(a) What is the formula for $A(t)$, the balance in SFr after t years?
(b) How much money is in the account after 2 years?

SOLUTION **(a)** Use (5) with $r = 0.0075$ and $P_0 = 10{,}000$:

$$A(t) = 10{,}000e^{-0.0075t}.$$

(b) After 2 years, the balance is

$$A(2) = 10{,}000e^{-0.0075(2)} \approx 9851.12 \text{ SFr}.$$

Thus, the balance decreased by about 49 Swiss Francs in 2 years. «

Present Value

If P dollars are invested today, the formula $A = Pe^{rt}$ gives the value of this investment after t years (assuming continuously compounded interest). We say that P is the **present value** of the amount A to be received in t years. If we solve for P in terms of A, we obtain

$$P = Ae^{-rt}. \tag{6}$$

The concept of the present value of money is an important theoretical tool in business and economics. Problems involving depreciation of equipment, for example, may be analyzed by calculus techniques when the present value of money is computed from (2), using continuously compounded interest.

EXAMPLE 4 **Present Value** Suppose you want to invest some amount of money now (i.e., the present value) so that you have \$5000 in 2 years. Assume you can earn 12% compounded continuously. How much would you need to invest? Find the present value of \$5000 to be received in 2 years if money can be invested at 12% compounded continuously.

SOLUTION Use formula (6) with $A = 5000$, $r = .12$, and $t = 2$.

$$P = 5000e^{-(0.12)(2)} = 5000e^{-0.24}$$
$$\approx \$3933.14$$

» *Now Try Exercise 19*

A Limit Formula for *e*

We have defined continuous compounding as the limit of ordinary compounding if the number of compound periods per year increases indefinitely. So, if we let m tend to infinity in (3), the formula for ordinary compounding, we should get (2), the formula for continuous compounding. In other words,

$$\lim_{m\to\infty} P_0\left(1 + \frac{r}{m}\right)^{mt} = P_0e^{rt}.$$

In this formula, take $P_0 = 1$, $r = 1$, $t = 1$, and get

$$\lim_{m\to\infty} \left(1 + \frac{1}{m}\right)^{m} = e^1 = e.$$

This gives a limit formula for the number e that can be used to approximate the number e, and can be verified with the help of a calculator.

Check Your Understanding 5.2

Solutions can be found following the section exercises.

1. One thousand dollars is to be invested in a bank for 4 years. Would 8% interest compounded semiannually be better than $7\frac{3}{4}\%$ interest compounded continuously?

2. A building was bought for \$150,000 and sold 10 years later for \$400,000. What interest rate (compounded continuously) was earned on the investment?

EXERCISES 5.2

1. **Savings Account** Let $A(t) = 5000e^{0.04t}$ be the balance in a savings account after t years.
   (a) How much money was originally deposited?
   (b) What is the interest rate?
   (c) How much money will be in the account after 10 years?
   (d) What differential equation is satisfied by $y = A(t)$?
   (e) Use the results of parts (c) and (d) to determine how fast the balance is growing after 10 years.
   (f) How large will the balance be when it is growing at the rate of \$280 per year?

2. **Savings Account** Let $A(t)$ be the balance in a savings account after t years, and suppose that $A(t)$ satisfies the differential equation

$$A'(t) = .045A(t), \qquad A(0) = 3000.$$

   (a) How much money was originally deposited in the account?
   (b) What interest rate is being earned?
   (c) Find the formula for $A(t)$.
   (d) What is the balance after 5 years?
   (e) Use part (d) and the differential equation to determine how fast the balance is growing after 5 years.
   (f) How large will the balance be when it is growing at the rate of \$270 per year?

3. **Savings Account** Four thousand dollars is deposited in a savings account at 3.5% yearly interest compounded continuously.
   (a) What is the formula for $A(t)$, the balance after t years?
   (b) What differential equation is satisfied by $A(t)$, the balance after t years?
   (c) How much money will be in the account after 2 years?
   (d) When will the balance reach \$5000?
   (e) How fast is the balance growing when it reaches \$5000?

4. **Savings Account** Ten thousand dollars is deposited in a savings account at 4.6% yearly interest compounded continuously.
   (a) What differential equation is satisfied by $A(t)$, the balance after t years?
   (b) What is the formula for $A(t)$?
   (c) How much money will be in the account after 3 years?
   (d) When will the balance triple?
   (e) How fast is the balance growing when it triples?

5. **Investment Analysis** An investment earns 4.2% yearly interest compounded continuously. How fast is the investment growing when its value is \$9000?

6. **Investment Analysis** An investment earns 5.1% yearly interest compounded continuously and is currently growing at the rate of \$765 per year. What is the current value of the investment?

7. **Continuous Compound** One thousand dollars is deposited in a savings account at 6% yearly interest compounded continuously. How many years are required for the balance in the account to reach \$2500?

8. **Continuous Compound** Ten thousand dollars is invested at 6.5% interest compounded continuously. When will the investment be worth \$41,787?

9. **Technology Stock** One hundred shares of a technology stock were purchased on January 2, 1990, for \$1200 and sold on January 2, 1998, for \$12,500. What rate of interest compounded continuously did this investment earn?

10. **Appreciation of Art Work** Pablo Picasso's *Angel Fernandez de Soto* was acquired in 1946 for a postwar splurge of \$22,220. The painting was sold in 1995 for \$29.1 million. What yearly rate of interest compounded continuously did this investment earn?

11. **Investment Analysis** How many years are required for an investment to double in value if it is appreciating at the yearly rate of 4% compounded continuously?

12. **Doubling an Investment** What yearly interest rate (compounded continuously) is earned by an investment that doubles in 10 years?

13. **Tripling an Investment** If an investment triples in 15 years, what yearly interest rate (compounded continuously) does the investment earn?

14. **Real Estate Investment** If real estate in a certain city appreciates at the yearly rate of 15% compounded continuously, when will a building purchased in 2010 triple in value?

15. **Negative Interest Rates** Suppose that the bank in Example 3 increased its fees by charging a negative annual interest rate of $-.9\%$. Find the balance after two years in a savings account if $P_0 = 10,000$ SFr.

16. **Negative Interest Rates** How is the account in Exercise 15 changing when the balance is 9,500 SFr?

17. **Real Estate Investment** A farm purchased in 2000 for \$1 million was valued at \$3 million in 2010. If the farm continues to appreciate at the same rate (with continuous compounding), when will it be worth \$10 million?

18. **Real Estate Investment** A parcel of land bought in 1990 for \$10,000 was worth \$16,000 in 1995. If the land continues to appreciate at this rate, in what year will it be worth \$45,000?

19. **Present Value** Find the present value of \$1000 payable at the end of 3 years, if money may be invested at 8% with interest compounded continuously.

20. **Present Value** Find the present value of \$2000 to be received in 10 years, if money may be invested at 8% with interest compounded continuously.

21. **Present Value** How much money must you invest now at 4.5% interest compounded continuously to have \$10,000 at the end of 5 years?

22. **Present Value** If the present value of \$1000 to be received in 5 years is \$559.90, what rate of interest, compounded continuously, was used to compute this present value?

23. Comparing Two Investments Investment A is currently worth \$70,200 and is growing at the rate of 13% per year compounded continuously. Investment B is currently worth \$60,000 and is growing at the rate of 14% per year compounded continuously. After how many years will the two investments have the same value?

24. Compound Interest Ten thousand dollars is deposited in a money market fund paying 8% interest compounded continuously. How much interest will be earned during the second year of the investment?

25. Differential Equation and Interest A small amount of money is deposited in a savings account with interest compounded continuously. Let $A(t)$ be the balance in the account after t years. Match each of the following answers with its corresponding question.

Answers

a. Pe^{rt} **b.** $A(3)$ **c.** $A(0)$ **d.** $A'(3)$
e. Solve $A'(t) = 3$ for t.
f. Solve $A(t) = 3$ for t.
g. $y' = ry$
h. Solve $A(t) = 3A(0)$ for t.

Questions

A. How fast will the balance be growing in 3 years?
B. Give the general form of the function $A(t)$.
C. How long will it take for the initial deposit to triple?
D. Find the balance after 3 years.
E. When will the balance be 3 dollars?
F. When will the balance be growing at the rate of 3 dollars per year?
G. What was the principal amount?
H. Give a differential equation satisfied by $A(t)$.

26. Growth of a Savings Account The curve in Fig. 1 shows the growth of money in a savings account with interest compounded continuously.
(a) What is the balance after 20 years?
(b) At what rate is the money growing after 20 years?
(c) Use the answers to parts (a) and (b) to determine the interest rate.

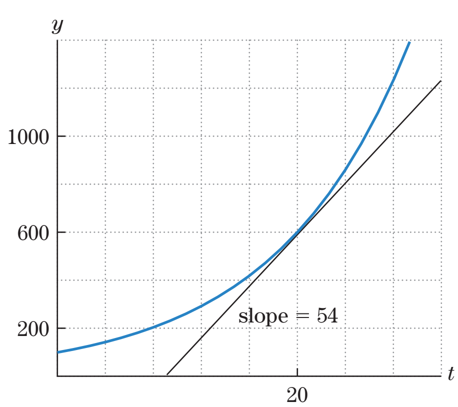

Figure 1 Growth of money in a savings account.

27. Savings Account The function $A(t)$ in Fig. 2(a) gives the balance in a savings account after t years with interest compounded continuously. Figure 2(b) shows the derivative of $A(t)$.
(a) What is the balance after 20 years?
(b) How fast is the balance increasing after 20 years?

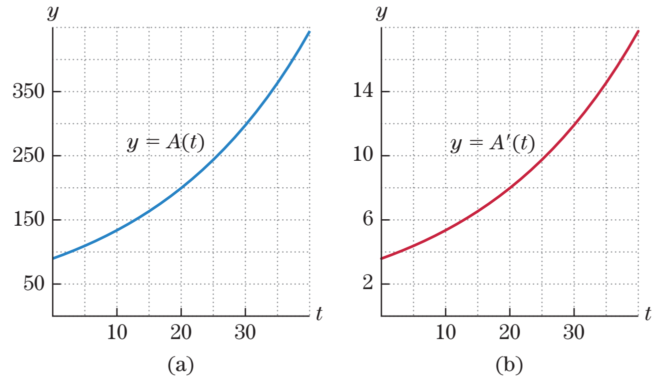

Figure 2

(c) Use the answers to parts (a) and (b) to determine the interest rate.
(d) When is the balance \$300?
(e) When is the balance increasing at the rate of \$12 per year?
(f) Why do the graphs of $A(t)$ and $A'(t)$ look the same?

28. When \$1000 is invested at r% interest (compounded continuously) for 10 years, the balance is $f(r)$ dollars, where f is the function shown in Fig. 3.
(a) What will the balance be at 7% interest?
(b) For what interest rate will the balance be \$3000?
(c) If the interest rate is 9%, what is the growth rate of the balance with respect to a unit increase in interest?

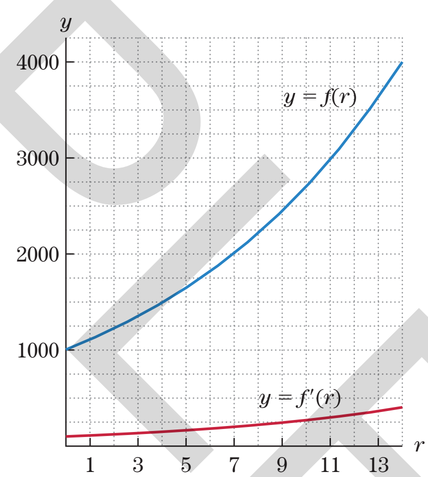

Figure 3 Effect of interest rate on balance.

TECHNOLOGY EXERCISES

29. Verify that $\lim_{m\to\infty}\left(1 + \frac{1}{m}\right)^m = e$ by taking m increasingly large and noticing that $\left(1 + \frac{1}{m}\right)^m$ approaches 2.718.

30. Verify that daily compounding is nearly the same as continuous compounding by graphing $y = 100[1 + (.05/360)]^{360x}$, together with $y = 100e^{0.05x}$ in the window $[0, 64]$ by $[250, 2500]$. The two graphs should appear the same on the screen. Approximately how far apart are they when $x = 32$? When $x = 64$?

31. **Internal Rate of Return** An investment of \$2000 yields payments of \$1200 in 3 years, \$800 in 4 years, and \$500 in 5 years. Thereafter, the investment is worthless. What constant rate of return r would the investment need to produce to yield the payments specified? The number r is called the *internal rate of return* on the investment. We can consider the investment as consisting of three parts, each part yielding one payment. The sum of the present values of the three parts must total \$2000. This yields the equation

$$2000 = 1200e^{-3r} + 800e^{-4r} + 500e^{-5r}.$$

Solve this equation to find the value of r.

Solutions to Check Your Understanding 5.2

1. Let us compute the balance after 4 years for each type of interest.

   8% *compounded semiannually*: Use the formula at the beginning of this section. Here $P = 1000$, $r = .08$, $m = 2$ (semiannually means there are two interest periods per year), and $t = 4$. Therefore,

   $$A = 1000\left(1 + \frac{.08}{2}\right)^{2\cdot 4} = 1000(1.04)^8 \approx \$1368.57.$$

   $7\frac{3}{4}\%$ *compounded continuously*: Use the formula $A = Pe^{rt}$, where $P = 1000$, $r = .0775$, and $t = 4$. Then,

   $$A = 1000e^{(0.0775)\cdot 4} = 1000e^{0.31} \approx \$1363.43.$$

   Therefore, 8% compounded semiannually is better.

2. If the \$150,000 had been compounded continuously for 10 years at interest rate r, the balance would be $150{,}000e^{r\cdot 10}$. The question is at what value of r will the balance be \$400,000? We need to solve an equation for r.

   $$150{,}000e^{r\cdot 10} = 400{,}000$$
   $$e^{r\cdot 10} \approx 2.67$$
   $$r\cdot 10 = \ln 2.67$$
   $$r = \frac{\ln 2.67}{10} \approx .098$$

   Therefore, the investment earned 9.8% interest per year.

5.3 Applications of the Natural Logarithm Function to Economics

In this section, we consider two applications of the natural logarithm to the field of economics. Our first application is concerned with relative rates of change and the second with elasticity of demand.

In 2015, the price of eggs began to climb due to a shortage of supply caused by the Avian flu, commonly known as bird flu. The average price of one dozen eggs rose to \$2.80 and was increasing at the rate of \$1.20 per year. At the same time, the price of a new compact car rose to \$12,500 and was increasing at the rate of \$1100 per year. As a consumer, you want to know which price is increasing more quickly. It is not meaningful to say that the car price is increasing faster simply because \$1100 is larger than \$1.20. We must take into account the vast difference between the actual cost of a car and the cost of one dozen eggs. A more meaningful basis of comparison of price increase is the **percentage rate of increase**, which compares the rate of change with the actual price. We can say that, in 2015, the price of one dozen eggs is increasing at the percentage rate

$$\frac{[\text{rate of increase}]}{[\text{actual price}]} = \frac{1.2}{2.8} \approx .43 = 43\% \text{ per year.}$$

At the same time, the price of a new compact car is increasing at the percentage rate

$$\frac{[\text{rate of increase}]}{[\text{actual price}]} = \frac{1100}{12{,}500} \approx .09 = 9\% \text{ per year.}$$

Thus, the price of one dozen eggs is increasing at a faster percentage rate than the price of a new compact car.

The concept of percentage rate of change is useful to economists. To state a general definition, let us recall that the rate of change of a function is given by its derivative. We can now introduce the following useful concept.

DEFINITION For a given function $f(t)$, the **relative rate of change** of $f(t)$ per unit change of t is defined to be

$$\frac{[\text{rate of change}]}{[\text{actual value}]} = \frac{f'(t)}{f(t)}. \tag{1}$$

The **percentage rate of change** is the relative rate of change of $f(t)$ expressed as a percentage.

The relative rate of change of $f(t)$ is also called the **logarithmic derivative** of $f(t)$, because of the derivative formula you saw in Section 4.5:

$$\frac{d}{dt}\ln[f(t)] = \frac{f'(t)}{f(t)} \tag{2}$$

EXAMPLE 1 **Log Derivative and Relative Rate of Change** Find the logarithmic derivative and then compute the relative rate of change and percentage rate of change at the given value of t.

(a) $f(t) = t^3 + 2t^2 - 11$, $t = 1$.

(b) $f(t) = e^{\sqrt{t}}$, $t = 4$.

SOLUTION **(a)** Differentiating $f(t)$, we find $f'(t) = 3t^2 + 4t$. So, from (2), the logarithmic derivative of f is

$$\frac{f'(t)}{f(t)} = \frac{3t^2 + 4t}{t^3 + 2t^2 - 11}.$$

When $t = 1$,

$$\frac{f'(1)}{f(1)} = \frac{7}{-8} = -\frac{7}{8} = -.875.$$

Thus, when $t = 1$, the relative rate of change of $f(t)$ with respect to t is $-.875$, which corresponds to a percentage rate of change of -87.5%. (A negative percentage corresponds to a decrease of 87.5%.)

(b) Because of the exponential in $f(t)$, we can simplify our computations by using (2). The logarithmic derivative is

$$\begin{aligned}\frac{d}{dt}\ln[f(t)] &= \frac{d}{dt}\ln[e^{\sqrt{t}}] \\ &= \frac{d}{dt}(\sqrt{t}) \qquad \text{Because } \ln(e^{\sqrt{t}}) = \sqrt{t} \\ &= \frac{1}{2\sqrt{t}}.\end{aligned}$$

When $t = 4$, the logarithmic derivative is equal to

$$\frac{1}{2\sqrt{4}} = \frac{1}{4} = .25.$$

Thus, when $t = 4$, the relative rate of change of $f(t)$ is .25, and so the percentage rate of change is 25%.

» Now Try Exercise 3

Economists often use percentage rates of change when discussing the growth of various economic quantities, such as national income or national debt, because such rates of change can be meaningfully compared.

EXAMPLE 2 **Gross Domestic Product** A certain school of economists modeled the nominal gross domestic product of the United States at time t (measured in years from January 1, 2005) by the formula

$$f(t) = 13.2 + .7t - .11t^2 + .01t^3,$$

where $f(t)$ is measured in trillions of dollars. (See Fig. 1.) What was the predicted percentage rate of growth (or decline) of the economy at $t = 3$ and $t = 9$?

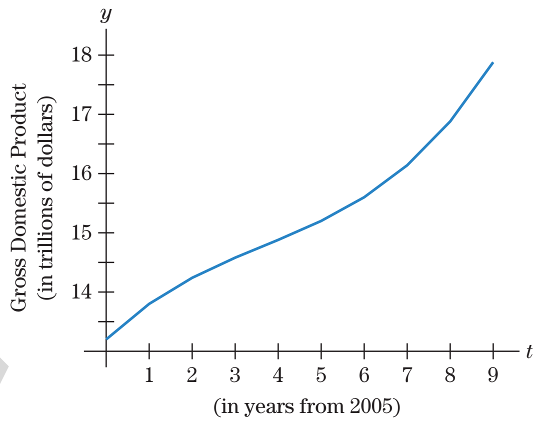

Figure 1

SOLUTION Since

$$f'(t) = .7 - .22t + .03t^2,$$

we see that

$$\frac{f'(3)}{f(3)} = \frac{.7 - .22(3) + .03(9)}{13.2 + .7(3) - .11(9) + .01(27)} = \frac{.31}{14.58} \approx .021$$

$$\frac{f'(9)}{f(9)} = \frac{.7 - .22(9) + .03(81)}{13.2 + .7(9) - .11(81) + .01(729)} = \frac{1.15}{17.88} \approx .064.$$

So on January 1, 2008 ($t = 3$), the economy is predicted to grow at a relative rate of about 2.1% per year. On January 1, 2014 ($t = 9$), the economy is predicted to be still growing, but at a relative rate of about 6.4% per year. **>> Now Try Exercise 9**

EXAMPLE 3 **Constant Relative Rate of Change** If the function $f(t)$ has a constant relative rate of change k, show that $f(t) = Ce^{kt}$ for some constant C.

SOLUTION We are given that

$$\frac{f'(t)}{f(t)} = k.$$

Hence, $f'(t) = kf(t)$. But this is just the differential equation satisfied by the exponential function (Theorem 1, Section 5.1). Therefore, we must have $f(t) = Ce^{kt}$ for some constant C. «

Elasticity of Demand

In Section 2.7, we considered demand equations for companies and for entire industries. Recall that a demand equation expresses, for each quantity x to be produced, the market price that will generate a demand of exactly x. For instance, the demand equation

$$p = 150 - .01x$$

says that, to sell x units, the price must be set at $150 - .01x$ dollars. To be specific, to sell 6000 units, the price must be set at $150 - .01(6000) = \$90$ per unit.

The demand equation may be solved for x in terms of p to yield

$$x = 100(150 - p).$$

This last equation gives quantity in terms of price. If we let the letter q represent quantity, the equation becomes

$$q = 100(150 - p).$$

This equation is of the form $q = f(p)$, where, in this case, $f(p)$ is the function $f(p) = 100(150 - p)$. In what follows, it will be convenient to always write our demand functions so that the quantity q is expressed as a function $f(p)$ of the price p.

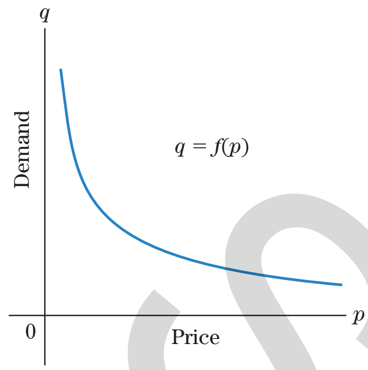

Figure 2

Usually, raising the price of a commodity lowers demand. Therefore, the typical demand function $q = f(p)$ is decreasing and has a negative slope everywhere. (See Fig. 2.) But does raising the price typically also lowers revenue? The answer is "sometimes yes, and sometimes no." But how can we predict the answer to this important question? By using a concept called elasticity of demand.

Recall that the derivative $f'(p)$ compares the change in the quantity demanded with the change in price. By way of contrast, the concept of elasticity is designed to compare the *relative* rate of change of the quantity demanded with the *relative* rate of change of price.

Let us be more explicit. Consider a particular demand function $q = f(p)$. From our interpretation of the logarithmic derivative in (1), we know that the relative rate of change of the quantity demanded with respect to p is

$$\frac{(d/dp)f(p)}{f(p)} = \frac{f'(p)}{f(p)}.$$

Similarly, the relative rate of change of price with respect to p is

$$\frac{(d/dp)p}{p} = \frac{1}{p}.$$

Hence, the ratio of the relative rate of change of the quantity demanded to the relative rate of change of price is

$$\frac{[\text{relative rate of change of quantity}]}{[\text{relative rate of change of price}]} = \frac{f'(p)/f(p)}{1/p} = \frac{pf'(p)}{f(p)}.$$

Since $f'(p)$ is always negative for a typical demand function, the quantity $pf'(p)/f(p)$ will be negative for all values of p. For convenience, economists prefer to work with positive numbers, and therefore, the *elasticity of demand* is taken to be this quantity multiplied by -1.

Elasticity of Demand The **elasticity of demand** $E(p)$ at price p for the demand function $q = f(p)$ is defined to be

$$E(p) = \frac{-pf'(p)}{f(p)}.$$

EXAMPLE 4 **Elasticity of Demand** The demand function for a certain metal is $q = 100 - 2p$, where p is the price per pound and q is the quantity demanded (in millions of pounds).

(a) What quantity can be sold at \$30 per pound?
(b) Determine the function $E(p)$.

(c) Determine and interpret the elasticity of demand at $p = 30$.
(d) Determine and interpret the elasticity of demand at $p = 20$.

SOLUTION

(a) In this case, $q = f(p)$, where $f(p) = 100 - 2p$. When $p = 30$, we have $q = f(30) = 100 - 2(30) = 40$. Therefore, 40 million pounds of the metal can be sold. We also say that the *demand* is 40 million pounds.

(b) $E(p) = \dfrac{-pf'(p)}{f(p)} = \dfrac{-p(-2)}{100 - 2p} = \dfrac{2p}{100 - 2p}$

(c) The elasticity of demand at price $p = 30$ is $E(30)$.

$$E(30) = \frac{2(30)}{100 - 2(30)} = \frac{60}{40} = \frac{3}{2}.$$

When the price is set at \$30 per pound, a small increase in price will result in a relative rate of decrease in quantity demanded of about $\frac{3}{2}$ times the relative rate of increase in price. For example, if the price is increased from \$30 by 1%, the quantity demanded will decrease by about 1.5%.

(d) When $p = 20$, we have

$$E(20) = \frac{2(20)}{100 - 2(20)} = \frac{40}{60} = \frac{2}{3}.$$

When the price is set at \$20 per pound, a small increase in price will result in a relative rate of decrease in quantity demanded of only $\frac{2}{3}$ of the relative rate of increase of price. For example, if the price is increased from \$20 by 1%, the quantity demanded will decrease by $\frac{2}{3}$ of 1%.

» ***Now Try Exercise 21***

Elasticity, Price and Revenue

The significance of the concept of elasticity may perhaps be best appreciated by a study of how revenue, $R(p)$, responds to change in price. Let's start by expressing the revenue function as a function of price:

$$R(p) = f(p) \cdot p,$$

where $f(p)$ is the demand function. Differentiate $R(p)$ using the product rule and get:

$$\begin{aligned} R'(p) &= \frac{d}{dp}[f(p) \cdot p] = f(p) \cdot 1 + p \cdot f'(p) \\ &= f(p)\left[1 + \frac{pf'(p)}{f(p)}\right] && \text{Factor } f(p). \\ &= f(p)[1 - E(p)]. && \text{Because } E(p) = -\frac{pf'(p)}{f(p)}. \end{aligned} \tag{3}$$

This equation relates the rate of change of revenue to elasticity of demand. Note that if $E(p) = 1$ then $R'(p) = 0$. The cases $E(p) < 1$ and $E(p) > 1$ have interesting implications. Let us introduce a terminology used by economists.

> **DEFINITION Elastic and Inelastic Demand** We say that demand is **elastic** at price p_0 if $E(p_0) > 1$ and **inelastic** if $E(p_0) < 1$.

Now, suppose that demand is elastic at some price p_0. Then $E(p_0) > 1$ and $1 - E(p_0) < 0$. Since $f(p)$ is always positive, we see from (3) that $R'(p_0) < 0$. Therefore, by the first derivative rule, $R(p)$ is decreasing at p_0. So, an increase in price will result in a decrease in revenue, and a decrease in price will result in an increase in revenue. In a similar way, we can show that if demand is inelastic, then $R'(p)$ will be positive. In this case, an increase in price will result in an increase in revenue, and a decrease in price will result in a decrease in revenue. This can be summarized as follows.

> **Elasticity Rule** When demand is elastic ($E(p) > 1$), the change in revenue is in the opposite direction of the change in price. And, when demand is inelastic ($E(p) < 1$), the change in revenue is in the same direction of the change in price.

As noted previously, when $E(p_0) = 1$, then, from (3), $R'(p_0) = 0$, and so p_0 is a critical value of R.

EXAMPLE 5 **Elasticity of Demand** Figure 3 shows the elasticity of demand for the metal in Example 4:

$$E(p) = \frac{2p}{100 - 2p}.$$

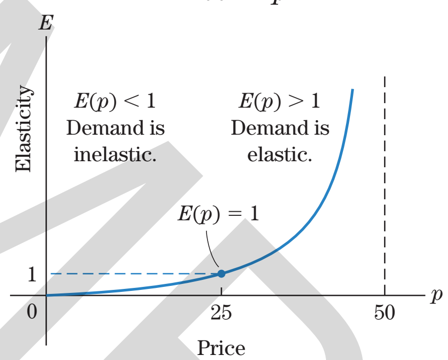

Figure 3

(a) For what values of p is demand elastic? Inelastic?
(b) Find and plot the revenue function for $0 < p < 50$.
(c) Verify the elasticity rule by analyzing how revenue responds to an increase in price when demand is elastic or, respectively, inelastic.

SOLUTION **(a)** In Example 4(b), we found the elasticity of demand to be

$$E(p) = \frac{2p}{100 - 2p}.$$

Let us solve $E(p) = 1$ for p.

$$\frac{2p}{100 - 2p} = 1 \quad \text{Given equation.}$$

$$2p = 100 - 2p \quad \text{Multiply by } 100 - 2p.$$

$$4p = 100 \quad \text{Add } 2p \text{ to both sides.}$$

$$p = 25 \quad \text{Divide by 4.}$$

By definition, demand is elastic at price p if $E(p) > 1$ and inelastic if $E(p) < 1$. From Figure 4, we see that demand is elastic if $25 < p < 50$ and inelastic if $0 < p < 25$.

(b) Recall that

$$[\text{revenue}] = [\text{quantity}] \cdot [\text{price per unit}].$$

Using the formula for demand (in millions of pounds) from Example 4, we obtain the revenue function

$$R = (100 - 2p) \cdot p = p(100 - 2p) \quad \text{(in millions of dollars).}$$

This is a parabola opening down with p-intercepts at $p = 0$ and $p = 50$. Its maximum is located at the midpoint of the p-intercepts, or $p = 25$. (See Fig. 4.)

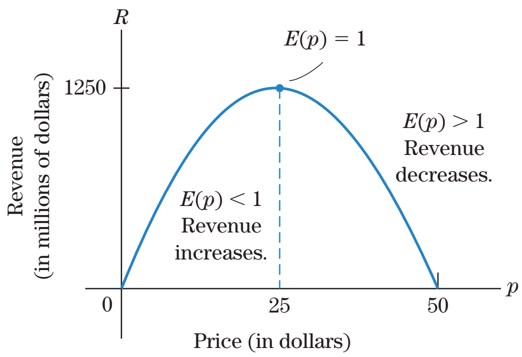

Figure 4

(c) In part (a), we determined that demand is elastic for $25 < p < 50$. For p in this price range, Figure 4 shows that an increase in price results in a decrease in revenue, and a decrease in price results in an increase in revenue. Hence, we conclude that, when demand is elastic, the change of revenue is in the opposite direction of the change in price. Similarly, when demand is inelastic ($0 < p < 25$), Figure 4 shows that the change of revenue is in the same direction as the change in price.

>> Now Try Exercise 23

Check Your Understanding 5.3

Solutions can be found following the section exercises.

The current toll for the use of a certain toll road is \$2.50. A study conducted by the state highway department determined that, with a toll of p dollars, q cars will use the road each day, where $q = 60{,}000e^{-0.5p}$.

1. Compute the elasticity of demand at $p = 2.5$.
2. Is demand elastic or inelastic at $p = 2.5$?
3. If the state increases the toll slightly, will the revenue increase or decrease?

EXERCISES 5.3

Find the logarithmic derivative and then determine the percentage rate of change of the functions at the points indicated.

1. $f(t) = t^2$ at $t = 10$ and $t = 50$
2. $f(t) = t^{10}$ at $t = 10$ and $t = 50$
3. $f(x) = e^{0.3x}$ at $x = 10$ and $x = 20$
4. $f(x) = e^{-0.05x}$ at $x = 1$ and $x = 10$
5. $f(t) = e^{0.3t^2}$ at $t = 1$ and $t = 5$
6. $G(s) = e^{-0.05s^2}$ at $s = 1$ and $s = 10$
7. $f(p) = 1/(p + 2)$ at $p = 2$ and $p = 8$
8. $g(p) = 5/(2p + 3)$ at $p = 1$ and $p = 11$
9. **Percentage Rate of Growth** The annual sales S (in dollars) of a company may be approximated by the formula

$$S = 50{,}000\sqrt{e^{\sqrt{t}}},$$

where t is the number of years beyond some fixed reference date. Use a logarithmic derivative to determine the percentage rate of growth of sales at $t = 4$.

10. **Percentage Rate of Change** The price of wheat per bushel at time t (in months) is approximated by

$$f(t) = 4 + .001t + .01e^{-t}.$$

What is the percentage rate of change of $f(t)$ at $t = 0$? $t = 1$? $t = 2$?

11. **Price of Ground Beef** The wholesale price in dollars of one pound of ground beef is modeled by the function $f(t) = 3.08 + .57t - .1t^2 + .01t^3$, where t is measured in years from January 1, 2010.
    (a) Estimate the price in 2011 and find the rate in dollars per year at which the price was rising in 2011.
    (b) What is the percentage rate of increase of the price of one pound of beef in 2011?
    (c) Answer parts (a) and (b) for the year 2016.
12. **Price of Pork** The wholesale price in dollars of one pound of pork is modeled by the function $f(t) = 1.4 + .26t - .1t^2 + .01t^3$, where t is measured in years from January 1, 2010.
    (a) Estimate the price in 2012 and find the percentage rate of increase of the price in 2012?
    (b) Answer part (a) for the year 2017.

For each demand function, find $E(p)$ and determine if demand is elastic or inelastic (or neither) at the indicated price.

13. $q = 700 - 5p,\ p = 80$

14. $q = 600e^{-0.2p},\ p = 10$

15. $q = 400(116 - p^2),\ p = 6$

16. $q = (77/p^2) + 3,\ p = 1$

17. $q = p^2 e^{-(p+3)},\ p = 4$

18. $q = 700/(p + 5),\ p = 15$

19. **Elasticity of Demand** Currently, 1800 people ride a certain commuter train each day and pay \$4 for a ticket. The number of people q willing to ride the train at price p is $q = 600(5 - \sqrt{p})$. The railroad would like to increase its revenue.
(a) Is demand elastic or inelastic at $p = 4$?
(b) Should the price of a ticket be raised or lowered?

20. **Elasticity of Demand** An electronic store can sell $q = 10{,}000/(p + 50) - 30$ cellular phones at a price p dollars per phone. The current price is \$150.
(a) Is demand elastic or inelastic at $p = 150$?
(b) If the price is lowered slightly, will revenue increase or decrease?

21. **Elasticity of Demand** A movie theater has a seating capacity of 3000 people. The number of people attending a show at price p dollars per ticket is $q = (18{,}000/p) - 1500$. Currently, the price is \$6 per ticket.
(a) Is demand elastic or inelastic at $p = 6$?
(b) If the price is lowered, will revenue increase or decrease?

22. **Elasticity of Demand** A subway charges 65 cents per person and has 10,000 riders each day. The demand function for the subway is $q = 2000\sqrt{90 - p}$.
(a) Is demand elastic or inelastic at $p = 65$?
(b) Should the price of a ride be raised or lowered to increase the amount of money taken in by the subway?

23. **Elasticity of Demand** A country that is the major supplier of a certain commodity wishes to improve its balance-of-trade position by lowering the price of the commodity. The demand function is $q = 1000/p^2$.
(a) Compute $E(p)$.
(b) Will the country succeed in raising its revenue?

24. Show that any demand function of the form $q = a/p^m$ has constant elasticity m.

Relative Rate of Change of Cost A cost function $C(x)$ gives the total cost of producing x units of a product. The elasticity of cost at quantity x, $E_c(x)$, is defined to be the ratio of the relative rate of change of cost (with respect to x) divided by the relative rate of change of quantity (with respect to x).

25. Show that $E_c(x) = x \cdot C'(x)/C(x)$.

26. Show that E_c is equal to the marginal cost divided by the average cost.

27. Let $C(x) = (1/10)x^2 + 5x + 300$. Show that $E_c(50) < 1$. (Hence, when 50 units are produced, a small relative increase in production results in an even smaller relative increase in total cost. Also, the average cost of producing 50 units is greater than the marginal cost at $x = 50$.)

28. Let $C(x) = 1000e^{0.02x}$. Determine and simplify the formula for $E_c(x)$. Show that $E_c(60) > 1$, and interpret this result.

TECHNOLOGY EXERCISES

29. Consider the demand function $q = 60{,}000e^{-0.5p}$ from Check Your Understanding 5.3.
(a) Determine the value of p for which the value of $E(p)$ is 1. For what values of p is demand inelastic?
(b) Graph the revenue function in the window $[0, 4]$ by $[-5000, 50{,}000]$, and determine where its maximum value occurs. For what values of p is the revenue an increasing function?

Solutions to Check Your Understanding 5.3

1. The demand function is $f(p) = 60{,}000e^{-0.5p}$.

$$f'(p) = -30{,}000e^{-0.5p}$$

$$E(p) = \frac{-pf'(p)}{f(p)} = \frac{-p(-30{,}000)e^{-0.5p}}{60{,}000e^{-0.5p}} = \frac{p}{2}$$

$$E(2.5) = \frac{2.5}{2} = 1.25$$

2. The demand is elastic, because $E(2.5) > 1$.

3. Since demand is elastic at \$2.50, a slight change in price causes revenue to change in the *opposite* direction. Hence, revenue will decrease.

5.4 Further Exponential Models

Terminal Velocity After jumping out of an airplane, a skydiver falls at an increasing rate. However, the wind rushing past the skydiver's body creates an upward force that begins to counterbalance the downward force of gravity. This air friction finally becomes so great that the skydiver's velocity reaches a limiting speed called the **terminal velocity**. If we let $v(t)$ be the downward velocity of the skydiver after t seconds of free fall, a good mathematical model for $v(t)$ is given by

$$v(t) = M(1 - e^{-kt}), \qquad (1)$$

where M is the terminal velocity and k is some positive constant. (See Fig. 1.) When t is close to zero, e^{-kt} is close to 1, and the velocity is small. As t increases, e^{-kt} becomes small, and so $v(t)$ approaches M.

FOR REVIEW

The line $y = M$ is a horizontal asymptote. See Sec. 2.1, p. 137.

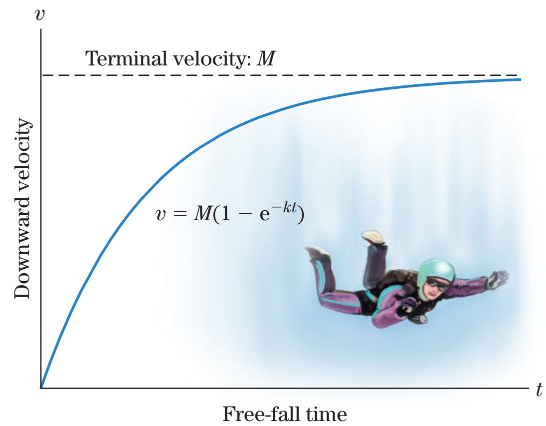

Figure 1 Downward velocity of a skydiver.

EXAMPLE 1 **Velocity of a Skydiver** Show that the velocity given in equation (1) satisfies the equations

$$v'(t) = k[M - v(t)], \quad v(0) = 0. \tag{2}$$

SOLUTION From (1) we have $v(t) = M - Me^{-kt}$. Then,

$$v'(t) = Mke^{-kt}.$$

However,

$$k[M - v(t)] = k[M - (M - Me^{-kt})] = kMe^{-kt},$$

so the differential equation $v'(t) = k[M - v(t)]$ holds. Also,

$$v(0) = M - Me^{0} = M - M = 0.$$

›› Now Try Exercise 3

The differential equation (2) says that the rate of change in v is proportional to the difference between the terminal velocity M and the actual velocity v. It can be shown that the only solution of (2) is given by the formula in (1). We summarize this useful fact as follows.

Solution of a Differential Equation The unique solution of the differential equation and initial condition

$$y'(t) = k(M - y(t)), \quad y(0) = 0 \quad \text{is}$$

$$y(t) = M(1 - e^{-kt})$$

The two equations (1) and (2) arise as mathematical models in a variety of situations. Some of these application are described next.

The Learning Curve Psychologists have found that, in many learning situations, a person's rate of learning is rapid at first and then slows down. Finally, as the task is mastered, the person's level of performance reaches a level above which it is almost impossible to rise. For example, within reasonable limits, each person seems to have a certain maximum capacity for memorizing a list of nonsense syllables. Suppose that a subject can memorize M syllables in a row if given sufficient time—say, an hour—to study the list but cannot memorize $M + 1$ syllables in a row even if allowed several hours of study. By giving the subject different lists of syllables and varying lengths of time to study the lists, the psychologist can determine an empirical relationship between the number of nonsense syllables memorized accurately and the number of minutes of study time. It turns out that a good model for this situation is

$$f(t) = M(1 - e^{-kt})$$

for some appropriate positive constant k. (See Fig. 2.)

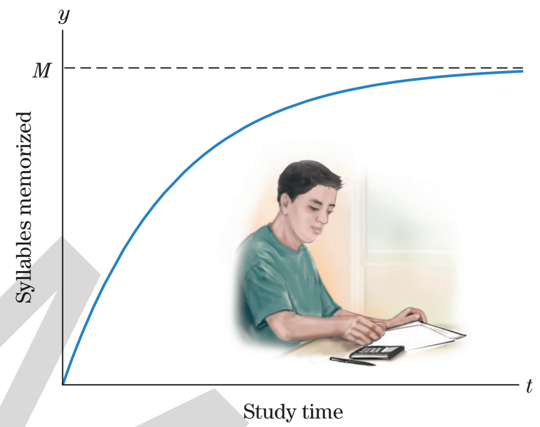

Figure 2 Learning curve, $f(t) = M(1 - e^{-kt})$.

The *slope* of this learning curve at time t is approximately the number of additional syllables that can be memorized if the subject is given 1 more minute of study time. Thus, the slope is a measure of the *rate of learning*. The differential equation satisfied by the function $y = f(t)$ is

$$y' = k(M - y), \qquad f(0) = 0.$$

This equation says that, if the subject is given a list of M nonsense syllables, the rate of memorization is proportional to the number of syllables remaining to be memorized.

Diffusion of Information by Mass Media Sociologists have found that the differential equation (2) provides a good model for the way information is spread (or "diffused") through a population when the information is being publicized constantly by mass media, such as television or online. (*Source: Introduction to Mathematical Sociology*.) Given a fixed population P, let $f(t)$ be the number of people who have already heard a certain piece of information by time t. Then, $P - f(t)$ is the number who have not yet heard the information by time t. Also, $f'(t)$ is the rate of increase of the number of people who have heard the news (the "rate of diffusion" of the information). If the information is being publicized often by some mass media, it is likely that the number of *newly informed* people per unit time is proportional to the number of people who have not yet heard the news. Therefore,

$$f'(t) = k[P - f(t)]. \tag{3}$$

Assume that $f(0) = 0$ (that is, there was a time $t = 0$ when nobody had heard the news). Then, the solution box following Example 1 shows that

$$f(t) = P(1 - e^{-kt}). \tag{4}$$

(See Fig. 3.)

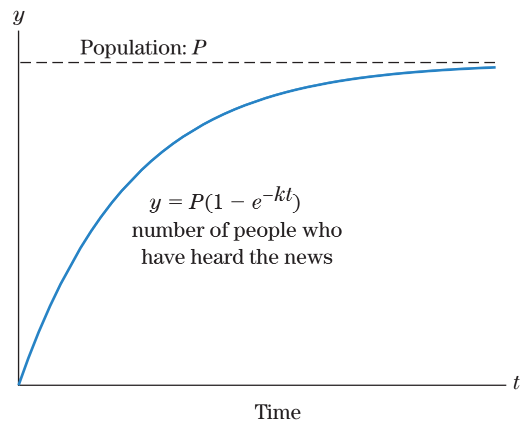

Figure 3 Diffusion of information by mass media.

EXAMPLE 2 **Diffusion of Information** The news of the resignation of a public official is broadcast frequently by internet news media and television stations. Also, one-half of the residents of a city have heard the news within 4 hours of its initial release. Use the exponential model (4) to estimate when 90% of the residents will have heard the news.

SOLUTION We must find the value of k in (4). If P is the number of residents, the number who have heard the news in the first 4 hours is given by (4), with $t = 4$. By assumption, this number is half the population. So,

$$\begin{aligned}
\tfrac{1}{2}P &= P(1 - e^{-k\cdot 4}) \\
.5 &= 1 - e^{-4k} && \text{Divide by } P. \\
e^{-4k} &= 1 - .5 = .5 && \text{Solve for } e^{-4k}. \\
\ln(e^{-4k}) &= \ln(.5) && \text{Take ln.} \\
-4k &= \ln(.5) \\
k &= -\frac{1}{4}\ln(.5) && \text{Solve for } k. \\
&\approx .173
\end{aligned}$$

Note that

$$-\frac{1}{4}\ln(.5) = -\frac{1}{4}\ln\left(\frac{1}{2}\right) = \frac{1}{4}\ln(2).$$

So the model for this particular situation is

$$f(t) = P(1 - e^{-kt}), \quad \text{where } k = \frac{\ln 2}{4}.$$

Now we want to find t such that $f(t) = .90P$. We solve for t:

$$\begin{aligned}
.90P &= P(1 - e^{-kt}) \\
.90 &= 1 - e^{-kt} \\
e^{-kt} &= 1 - .90 = .10 \\
-kt &= \ln .10 \\
t &= \frac{\ln .10}{-k} = -4\frac{\ln .10}{\ln 2} \approx 13.29.
\end{aligned}$$

Therefore, 90% of the residents will hear the news in 13.3 hours of its initial release.

>> Now Try Exercise 7

Intravenous Infusion of Glucose

The human body both manufactures and uses glucose (blood sugar). Usually, there is a balance in these two processes, so the bloodstream has a certain equilibrium level of glucose. Suppose that a patient is given a single intravenous injection of glucose, and let $A(t)$ be the amount of glucose (in milligrams) above the equilibrium level. Then, the body will start using up the excess glucose at a rate proportional to the amount of excess glucose; that is,

$$A'(t) = -\lambda A(t), \tag{5}$$

where λ is a positive constant called the **velocity constant of elimination**. This constant depends on how fast an individual patient's metabolic processes eliminate the excess glucose from the blood. Equation (5) describes a simple exponential decay process.

Now suppose that, instead of a single shot, the patient receives a continuous intravenous infusion of glucose. A bottle of glucose solution is suspended above the patient, and a small tube carries the glucose down to a needle that runs into a vein. In this case, there are two influences on the amount of excess glucose in the blood: the glucose being added steadily from the bottle and the glucose being removed from the body by metabolic processes. Let r be the rate of infusion of glucose (often, from 10 to 100 milligrams per minute). If the body did not remove any glucose, the excess glucose would increase at a constant rate of r milligrams per minute; that is,

$$A'(t) = r. \tag{6}$$

Taking into account the two influences on $A'(t)$ described by (5) and (6), we can write

$$A'(t) = r - \lambda A(t). \tag{7}$$

Define M to be r/λ, and note that initially there is no excess glucose; then,

$$A'(t) = \lambda(M - A(t)), \qquad A(0) = 0.$$

As stated in the solution box following Example 1, a solution of this differential equation is given by

$$A(t) = M(1 - e^{-\lambda t}) = \frac{r}{\lambda}(1 - e^{-\lambda t}). \tag{8}$$

Note that M is the limiting value of the glucose level. Reasoning as in Example 1, we conclude that the amount of excess glucose rises until it reaches a stable level. (See Fig. 4.)

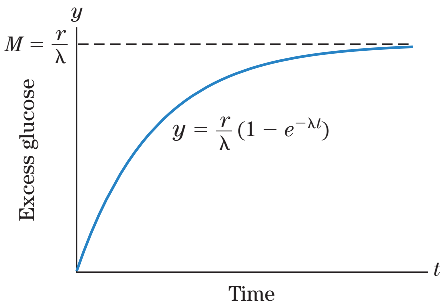

Figure 4 Continuous infusion of glucose.

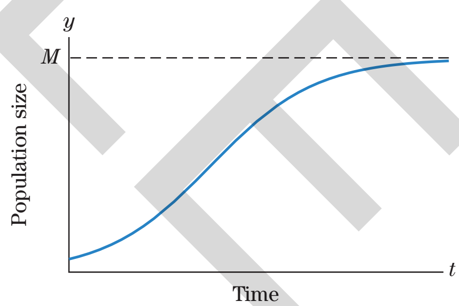

Figure 5 Logistic growth.

Logistic Growth

The model for simple exponential growth discussed in Section 5.1 is adequate for describing the growth of many types of populations, but obviously, a population cannot increase exponentially forever. The simple exponential growth model becomes inapplicable when the environment begins to inhibit the growth of the population. The logistic growth curve is an important exponential model that takes into account some of the effects of the environment on a population. (See Fig. 5.) For small values of t, the curve has the same basic shape as an exponential growth curve. Then, when the population begins to suffer from overcrowding or lack of food, the growth rate (the slope of the population curve) begins to slow down. Eventually, the growth rate tapers off to zero as the population reaches the maximum size that the environment will support. This latter part of the curve resembles the growth curves studied earlier in this section.

The equation for logistic growth has the general form

Logistic Growth

$$y = \frac{M}{1 + Be^{-Mkt}}, \quad (9)$$

where B, M, and k are positive constants. We can show that y satisfies the differential equation

$$y' = ky(M - y). \quad (10)$$

The factor y reflects the fact that the growth rate (y') depends in part on the size y of the population. The factor $M - y$ reflects the fact that the growth rate also depends on how close y is to the maximum level M.

The logistic curve is often used to fit experimental data that lie along an S-shaped curve. Examples are given by the growth of a fish population in a lake and the growth of a fruit fly population in a laboratory container. Also, certain enzyme reactions in animals follow a logistic law. One of the earliest applications of the logistic curve occurred in about 1840, when the Belgian sociologist P. Verhulst fit a logistic curve to six U.S. census figures, 1790 to 1840, and predicted the U.S. population for 1940. His prediction missed by fewer than 1 million people (an error of about 1%).

EXAMPLE 3 **Logistic Growth** A lake is stocked with 100 fish. After 3 months, there are 250 fish. A study of the ecology of the lake predicts that the lake can support 1000 fish. Find a formula for the number $P(t)$ of fish in the lake t months after it has been stocked.

SOLUTION The limiting population M is 1000. Therefore, we have

$$P(t) = \frac{1000}{1 + Be^{-1000kt}}.$$

At $t = 0$ there are 100 fish, so that

$$100 = P(0) = \frac{1000}{1 + Be^{0}} = \frac{1000}{1 + B}.$$

Thus, $1 + B = 10$, or $B = 9$. Finally, since $P(3) = 250$, we have

$$\begin{aligned} 250 &= \frac{1000}{1 + 9e^{-3000k}} \\ 1 + 9e^{-3000k} &= 4 \\ e^{-3000k} &= \frac{1}{3} \\ -3000k &= \ln\frac{1}{3} \\ k &\approx .00037. \end{aligned}$$

Therefore,

$$P(t) = \frac{1000}{1 + 9e^{-0.37t}}.$$

>> Now Try Exercise 9

An Epidemic Model

It will be instructive to actually "build" a mathematical model. Our example concerns the spread of a highly contagious disease. We begin by making several simplifying assumptions:

1. The population is a fixed number P, and each member of the population is susceptible to the disease.
2. The duration of the disease is long, so that no cures occur during the time period under study.
3. All infected individuals are contagious and circulate freely among the population.
4. During each time period (such as 1 day or 1 week), each infected person makes c contacts, and each contact with an uninfected person results in transmission of the disease.

Consider a short period of time from t to $t + h$. Each infected person makes $c \cdot h$ contacts. How many of these contacts are with uninfected persons? If $f(t)$ is the number of infected persons at time t, then $P - f(t)$ is the number of uninfected persons, and $[P - f(t)]/P$ is the fraction of the population that is uninfected. Thus, of the $c \cdot h$ contacts made,

$$\left[\frac{P - f(t)}{P}\right] \cdot c \cdot h$$

will be with uninfected persons. This is the number of new infections produced by one infected person during the time period of length h. The total number of *new* infections during this period is

$$f(t)\left[\frac{P - f(t)}{P}\right] ch.$$

But this number must equal $f(t + h) - f(t)$, where $f(t + h)$ is the total number of infected persons at time $t + h$. So,

$$f(t + h) - f(t) = f(t)\left[\frac{P - f(t)}{P}\right] ch.$$

Dividing by h, the length of the time period, we obtain the average number of new infections per unit time (during the small time period):

$$\frac{f(t + h) - f(t)}{h} = \frac{c}{P} f(t)[P - f(t)].$$

If we let h approach zero and let y stand for $f(t)$, the left-hand side approaches the rate of change in the number of infected persons, and we derive the following equation:

$$\frac{dy}{dt} = \frac{c}{P} y(P - y). \tag{11}$$

This is the same type of equation as that used in (10) for logistic growth, although the two situations leading to this model appear to be quite dissimilar.

Comparing (11) with (10), we see that the number of infected individuals at time t is described by a logistic curve with $M = P$ and $k = c/P$. Therefore, by (9), we can write

$$f(t) = \frac{P}{1 + Be^{-ct}}.$$

B and c can be determined from the characteristics of the epidemic. (See Example 4.)

The logistic curve has an inflection point at that value of t for which $f(t) = P/2$. The position of this inflection point has great significance for applications of the logistic curve. From inspecting a graph of the logistic curve, we see that the inflection point is the point at which the curve has greatest slope. In other words, the inflection point corresponds to the instant of fastest growth of the logistic curve. This means, for example, that, in the foregoing epidemic model, the disease is spreading with the greatest rapidity precisely when half the population is infected. Any attempt at disease control (through immunization, for example) must strive to reduce the incidence of the disease to as low a point as possible, but, in any case, at least below the inflection point at $P/2$, at which point the epidemic is spreading fastest.

EXAMPLE 4 **Spread of an Epidemic** The Department of Public Health monitors the spread of an epidemic of a particularly long-lasting strain of flu in a city of 500,000 people. At the beginning of the first week of monitoring, 200 cases had been reported; during the first week, 300 new cases are reported. Estimate the number of infected individuals after 6 weeks.

SOLUTION Here, $P = 500{,}000$. If $f(t)$ denotes the number of cases at the end of t weeks, then,

$$f(t) = \frac{P}{1 + Be^{-ct}} = \frac{500{,}000}{1 + Be^{-ct}}.$$

Moreover, $f(0) = 200$, so

$$\begin{aligned} 200 &= \frac{500{,}000}{1 + Be^{0}} = \frac{500{,}000}{1 + B}. \\ 200(1 + B) &= 500{,}000 \\ 1 + B &= 2500 \\ B &= 2499. \end{aligned}$$

Consequently, since $f(1) = 300 + 200 = 500$, we have

$$\begin{aligned} 500 = f(1) &= \frac{500{,}000}{1 + 2499e^{-c}}, \\ 500(1 + 2499e^{-c}) &= 500{,}000 \\ 1 + 2499\,e^{-c} &= 1000 \\ 2499\,e^{-c} &= 999 \\ e^{-c} &= \frac{999}{2499} \approx .4. \end{aligned}$$

so $-c \approx \ln(.4)$ or $c \approx .92$. Finally,

$$f(t) = \frac{500{,}000}{1 + 2499e^{-0.92t}}$$

and

$$f(6) = \frac{500{,}000}{1 + 2499e^{-0.92(6)}} \approx 45{,}000.$$

After 6 weeks, about 45,000 individuals are infected. **>> Now Try Exercise 11**

This epidemic model is used by sociologists (who still call it an epidemic model) to describe the spread of a rumor. In economics, the model is used to describe the diffusion of knowledge about a product. An "infected person" represents an individual who possesses knowledge of the product. In both cases, it is assumed that the members of the population are themselves primarily responsible for the spread of the rumor or knowledge of the product. This situation is in contrast to the model described earlier, in which information was spread through a population by external sources, such as online news outlets, radio, and television.

There are several limitations to this epidemic model. Each of the four simplifying assumptions made at the outset is unrealistic in varying degrees. More complicated models can be constructed that rectify one or more of these defects, but they require more advanced mathematical tools.

Check Your Understanding 5.4

Solutions can be found following the section exercises.

1. A sociological study was made to examine the process by which doctors decide to adopt a new drug. The doctors were divided into two groups. The doctors in group A had little interaction with other doctors and so received most of their information through mass media. The doctors in group B had extensive interaction with other doctors and so received most of their information through word of mouth. For each group, let $f(t)$ be the number who have learned about the new drug after t months. Examine the appropriate differential equations to explain why the two graphs were of the types shown in Fig. 6. (*Source: Sociometry.*)

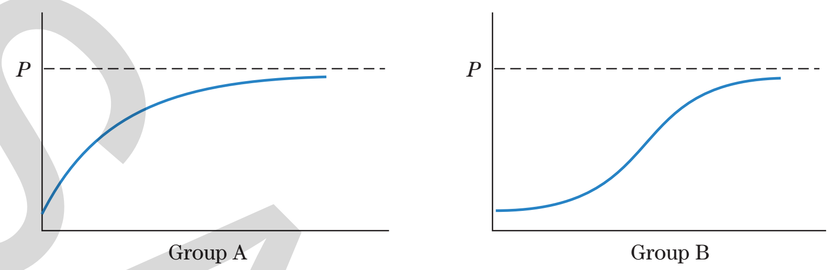

Figure 6 Results of a sociological study.

EXERCISES 5.4

1. Consider the function $f(x) = 5(1 - e^{-2x})$, $x \geq 0$.
   (a) Show that $f(x)$ is increasing and concave down for all $x \geq 0$.
   (b) Explain why $f(x)$ approaches 5 as x gets large.
   (c) Sketch the graph of $f(x)$, $x \geq 0$.

2. Consider the function $g(x) = 10 - 10e^{-0.1x}$, $x \geq 0$.
   (a) Show that $g(x)$ is increasing and concave down for $x \geq 0$.
   (b) Explain why $g(x)$ approaches 10 as x gets large.
   (c) Sketch the graph of $g(x)$, $x \geq 0$.

3. If $y = 2(1 - e^{-x})$, compute y' and show that $y' = 2 - y$.

4. If $y = 5(1 - e^{-2x})$, compute y' and show that $y' = 10 - 2y$.

5. If $f(x) = 3(1 - e^{-10x})$, show that $y = f(x)$ satisfies the differential equation

$$y' = 10(3 - y), \qquad f(0) = 0.$$

6. **Ebbinghaus Model for Forgetting** A student learns a certain amount of material for some class. Let $f(t)$ denote the percentage of the material that the student can recall t weeks later. The psychologist Hermann Ebbinghaus found that this percentage of retention can be modeled by a function of the form

$$f(t) = (100 - a)e^{-\lambda t} + a,$$

where λ and a are positive constants and $0 < a < 100$. Sketch the graph of the function $f(t) = 85e^{-0.5t} + 15$, $t \geq 0$.

7. **Spread of News** When a grand jury indicted the mayor of a certain town for accepting bribes, the newspaper, online news outlets, radio, and television immediately began to publicize the news. Within an hour, one-quarter of the citizens heard about the indictment. Estimate when three-quarters of the town heard the news.

8. Examine formula (8) for the amount $A(t)$ of excess glucose in the bloodstream of a patient at time t. Describe what would happen if the rate r of infusion of glucose were doubled.

9. **Spread of News** A news item is spread by word of mouth to a potential audience of 10,000 people. After t days,

$$f(t) = \frac{10{,}000}{1 + 50e^{-0.4t}}$$

people will have heard the news. The graph of this function is shown in Fig. 7.
   (a) Approximately how many people will have heard the news after 7 days?
   (b) At approximately what rate will the news spread after 14 days?
   (c) Approximately when will 7000 people have heard the news?
   (d) Approximately when will the news spread at the rate of 600 people per day?
   (e) When will the news spread at the greatest rate?
   (f) Use equations (9) and (10) to determine the differential equation satisfied by $f(t)$.
   (g) At what rate will the news spread when half the potential audience has heard the news?

10. **Concentration of Glucose in the Bloodstream** Physiologists usually describe the continuous intravenous infusion of glucose in terms of the excess *concentration* of glucose, $C(t) = A(t)/V$, where V is the total volume of blood in the patient. In this case, the rate of increase in the concentration of glucose due to the continuous injection is r/V. Find a differential equation that gives a model for the rate of change of the excess concentration of glucose.

11. **Spread of News** A news item is broadcast by mass media to a potential audience of 50,000 people. After t days,

$$f(t) = 50{,}000(1 - e^{-0.3t})$$

people will have heard the news. The graph of this function is shown in Fig. 8.
   (a) How many people will have heard the news after 10 days?
   (b) At what rate is the news spreading initially?
   (c) When will 22,500 people have heard the news?

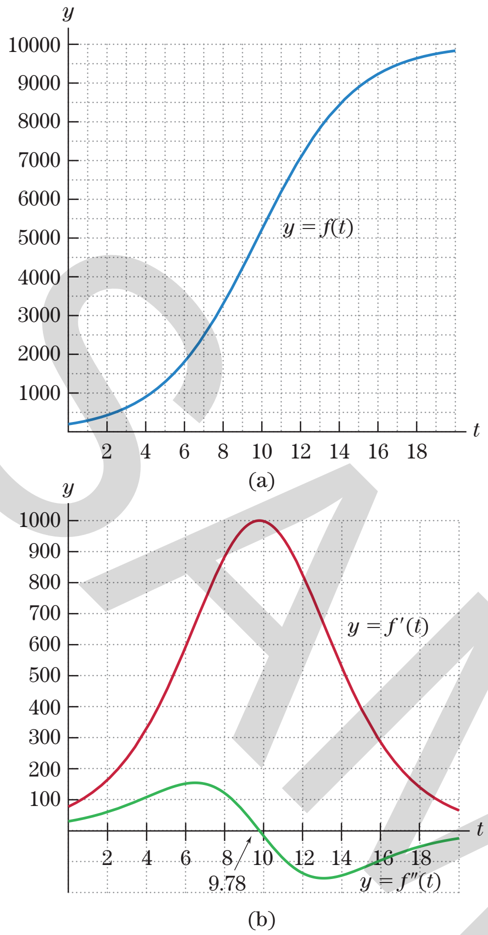

Figure 7

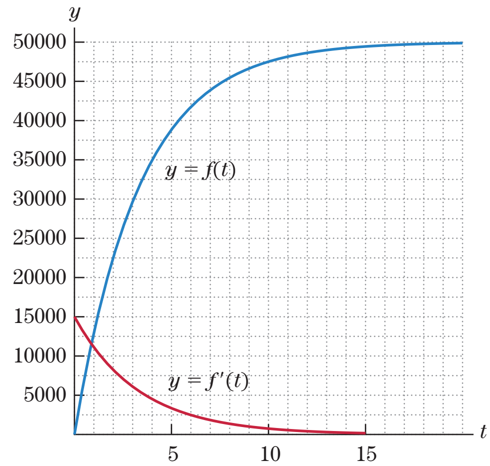

Figure 8

(d) Approximately when will the news spread at the rate of 2500 people per day?

(e) Use equations (3) and (4) to determine the differential equation satisfied by $f(t)$

(f) At what rate will the news spread when half the potential audience has heard the news?

12. **Glucose Elimination** Describe an experiment that a doctor could perform to determine the velocity constant of elimination of glucose for a particular patient.

TECHNOLOGY EXERCISES

13. **Amount of a Drug in the Bloodstream** After a drug is taken orally, the amount of the drug in the bloodstream after t hours is $f(t) = 122(e^{-0.2t} - e^{-t})$ units.
   (a) Graph $f(t)$, $f'(t)$, and $f''(t)$ in the window $[0, 12]$ by $[-20, 75]$
   (b) How many units of the drug are in the bloodstream after 7 hours?
   (c) At what rate is the level of drug in the bloodstream increasing after 1 hour?
   (d) While the level is decreasing, when is the level of drug in the bloodstream 20 units?
   (e) What is the greatest level of drug in the bloodstream, and when is this level reached?
   (f) When is the level of drug in the bloodstream decreasing the fastest?

14. **Growth with Restriction** A model incorporating growth restrictions for the number of bacteria in a culture after t days is given by $f(t) = 5000(20 + te^{-0.04t})$.
   (a) Graph $f'(t)$ and $f''(t)$ in the window $[0, 100]$ by $[-700, 300]$.
   (b) How fast is the culture changing after 100 days?
   (c) Approximately when is the culture growing at the rate of 76.6 bacteria per day?
   (d) When is the size of the culture greatest?
   (e) When is the size of the culture decreasing the fastest?

Solutions to Check Your Understanding 5.4

1. The difference between transmission of information by mass media (Group A) and by word of mouth (Group B) is that in Group B the rate of transmission depends not only on the number of people who have not yet received the information, but also on the number of people who know the information and therefore are capable of spreading it. Therefore, for Group A, $f'(t) = k[P - f(t)]$, and for Group B, $f'(t) = kf(t)[P - f(t)]$. Note that the spread of information by word of mouth follows the same pattern as the spread of an epidemic.

CHAPTER 5 Summary

KEY TERMS AND CONCEPTS / EXAMPLES

5.1 Exponential Growth and Decay

- The solutions of the differential equation $y' = ky$ are all of the form $y = Ce^{kt}$, where C is an arbitrary constant.
- If $k > 0$, the solution is an exponential growth function, and k is called the growth constant.
- If $k < 0$, write $k = -\lambda$, then the solution, $y = Ce^{-\lambda t}$, is an exponential decay function, and the constant λ is called the decay constant.
- The unique solution of the differential equation $y' = ky$ with initial condition $y(0) = P_0$ is the function $y = P_0 e^{kt}$.

Examples

- The differential equation $y' = .02y$ has infinitely many solutions. They are all of the form $y = Ce^{0.02t}$, where C is an arbitrary constant.
- The unique solution of the differential equation with initial condition, $y' = 3y$, $y(0) = 413$, is $y(t) = 413e^{3t}$.
- Radioactive cobalt has a decay constant $\lambda = .13$. If 2 grams of cobalt is present, let $P(t)$ be the number of grams remaining after t years. Then $y = P(t)$ satisfies $y' = -.13y$, $y(0) = 2$. The formula for $P(t)$ is $P(t) = 2e^{-0.13t}$.

5.2 Compound Interest

When the interest rate is compounded continuously in a savings account, the account grows exponentially. If r is the annual interest rate, P_0 the initial deposit, and $y = A(t)$ the balance in the account after t years, then y satisfies the equation

$$y' = ry, \quad y(0) = P_0$$

and is given by

$$y = A(t) = P_0 e^{rt}.$$

Present Value If P dollars are invested today, then $A = Pe^{rt}$ gives the value of this investment after t years. We say that P is the present value of the amount A to be received in t years. We have $P = Ae^{-rt}$.

Example

An amount of \$2000 dollars is deposited in an account that earns 6% annual interest rate, compounded continuously.

(a) Give a formula for $A(t)$, the compound amount in the account after t years.

(b) How long is required for the amount to reach \$3000?

To answer (a), we appeal to the solution of the differential equation $y' = .06y$, $y(0) = 2000$. Then, $A(t) = 2000e^{0.06t}$.
(b) To find t, we set $A(t) = 3000$ and solve for t:

$$2000e^{0.06t} = 3000$$
$$e^{0.06t} = \frac{3000}{2000} = \frac{3}{2}$$
$$\ln[e^{0.06t}] = \ln\left(\frac{3}{2}\right)$$
$$.06t = \ln\left(\frac{3}{2}\right)$$
$$t = \frac{\ln(\frac{3}{2})}{.06} \approx 6.8 \text{ years.}$$

5.3 Applications of the Natural Logarithm Function to Economics

The **relative rate of change** of a function is

$$\frac{f'(t)}{f(t)}.$$

We note that this is also $\frac{d}{dt}[\ln(f(t))]$.

The **percentage rate of change** is the relative rate of change expressed as a percentage.

Example

The value of an investment t years later is approximated by the formula

$$f(t) = 10{,}000e^{\frac{t}{t+1}}.$$

Determine the percentage rate of change of the investment when $t = 1$ and when $t = 5$.

The percentage rate of change is given by $\frac{f'(t)}{f(t)} = \frac{d}{dt}[\ln(f(t))]$. In our case,

$$\ln(f(t)) = \ln\left[10{,}000e^{\frac{t}{t+1}}\right]$$
$$= \ln(10{,}000) + \ln\left[e^{\frac{t}{t+1}}\right] = \ln(10{,}000) + \frac{t}{t+1}$$
$$\frac{d}{dt}[\ln(f(t))] = \frac{d}{dt}\left[\ln(10{,}000) + \frac{t}{t+1}\right] = 0 + \frac{d}{dt}\left[\frac{t}{t+1}\right],$$

because $\ln(10{,}000)$ is a constant, its derivative is 0. Now, by the quotient rule,

$$\frac{d}{dt}\left[\frac{t}{t+1}\right] = \frac{(t+1) - t}{(t+1)^2} = \frac{1}{(t+1)^2}.$$

KEY TERMS AND CONCEPTS	EXAMPLES
	So, $$\frac{f'(t)}{f(t)} = \frac{d}{dt}[\ln(f(t))] = \frac{1}{(t+1)^2}.$$ When $t = 1$, $$\frac{f'(1)}{f(1)} = \frac{1}{(1+1)^2} = \frac{1}{4} = .25 = 25\%.$$ When $t = 5$, $$\frac{f'(5)}{f(5)} = \frac{1}{(1+5)^2} = \frac{1}{36} \approx .03 = 3\%.$$
5.4 Further Exponential Models Several applications are discussed in this section, where the exponential function plays a central role: the velocity of a skydiver; the learning curve; exponential growth applications with a limiting capacity, such as a population of fish in a lake with a maximum capacity; the spread of an epidemic in a limited environment; the diffusion of information by mass media. These applications are modeled by one of the two following differential equation: $y' = k(M - y)$ and $y' = ky(M - y)$.	In a model of diffusion of information by mass media, the number of people who have heard a certain piece of information by time t is denoted by $y = f(t)$, where $f(t)$ satisfies the differential equation $y' = .2(1 - y)$, $y(0) = 0$. The solution of this differential equation is given following Example 1 of Sec. 5.4. It is $f(t) = 1000(1 - e^{-0.2t})$.

CHAPTER 5 Fundamental Concept Check Exercises

1. What differential equation is key to solving exponential growth and decay problems? State a result about the solution to this differential equation.
2. What is a growth constant? A decay constant?
3. What is meant by the half-life of a radioactive element?
4. Explain how radiocarbon dating works.
5. State the formula for each of the following quantities:
   (a) The compound amount of P dollars in t years at interest rate r, compounded continuously
   (b) The present value of A dollars in n years at interest rate r, compounded continuously
6. What is the difference between a relative rate of change and a percentage rate of change?
7. Define the elasticity of demand, $E(p)$, for a demand function. How is $E(p)$ used?
8. Describe an application of the differential equation $y' = k(M - y)$.
9. Describe an application of the differential equation $y' = ky(M - y)$.

CHAPTER 5 Review Exercises

1. **Atmospheric Pressure** The atmospheric pressure $P(x)$ (measured in inches of mercury) at height x miles above sea level satisfies the differential equation $P'(x) = -.2P(x)$. Find the formula for $P(x)$ if the atmospheric pressure at sea level is 29.92.
2. **Population Model** The herring gull population in North America has been doubling every 13 years since 1900. Give a differential equation satisfied by $P(t)$, the population t years after 1900.
3. **Present Value** Find the present value of \$10,000 payable at the end of 5 years if money can be invested at 12% with interest compounded continuously.
4. **Compound Interest** One thousand dollars is deposited in a savings account at 10% interest compounded continuously. How many years are required for the balance in the account to reach \$3000?
5. **Half-Life** The half-life of the radioactive element tritium is 12 years. Find its decay constant.
6. **Carbon Dating** A piece of charcoal found at Stonehenge contained 63% of the level of ^{14}C found in living trees. Approximately how old is the charcoal?
7. **Population Model** From January 1, 2010, to January 1, 2017, the population of a state grew from 17 million to 19.3 million.
   (a) Give the formula for the population t years after 2010.
   (b) If this growth continues, how large will the population be in 2020?
   (c) In what year will the population reach 25 million?

8. **Compound Interest** A stock portfolio increased in value from \$100,000 to \$117,000 in 2 years. What rate of interest, compounded continuously, did this investment earn?

9. **Comparing Investments** An investor initially invests \$10,000 in a risky venture. Suppose that the investment earns 20% interest, compounded continuously, for 5 years and then 6% interest, compounded continuously, for 5 years thereafter.
   (a) How much does the \$10,000 grow to after 10 years?
   (b) The investor has the alternative of an investment paying 14% interest compounded continuously. Which investment is superior over a 10-year period, and by how much?

10. **Bacteria Growth** Two different bacteria colonies are growing near a pool of stagnant water. The first colony initially has 1000 bacteria and doubles every 21 minutes. The second colony has 710,000 bacteria and doubles every 33 minutes. How much time will elapse before the first colony becomes as large as the second?

11. **Population Model** The population of a city t years after 1990 satisfies the differential equation $y' = .02y$. What is the growth constant? How fast will the population be growing when the population reaches 3 million people? At what level of population will the population be growing at the rate of 100,000 people per year?

12. **Bacteria Growth** A colony of bacteria is growing exponentially with growth constant .4, with time measured in hours. Determine the size of the colony when the colony is growing at the rate of 200,000 bacteria per hour. Determine the rate at which the colony will be growing when its size is 1 million.

13. **Population Model** The population of a certain country is growing exponentially. The total population (in millions) in t years is given by the function $P(t)$. Match each of the following answers with its corresponding question.

Answers

a. Solve $P(t) = 2$ for t.
b. $P(2)$
c. $P'(2)$
d. Solve $P'(t) = 2$ for t.
e. $y' = ky$
f. Solve $P(t) = 2P(0)$ for t
g. P_0e^{kt}, $k > 0$
h. $P(0)$

Questions

A. How fast will the population be growing in 2 years?
B. Give the general form of the function $P(t)$.
C. How long will it take for the current population to double?
D. What will be the size of the population in 2 years?
E. What is the initial size of the population?
F. When will the size of the population be 2 million?
G. When will the population be growing at the rate of 2 million people per year?
H. Give a differential equation satisfied by $P(t)$.

14. **Radioactive Decay** You have 80 grams of a certain radioactive material, and the amount remaining after t years is given by the function $f(t)$ shown in Fig. 1.
   (a) How much will remain after 5 years?
   (b) When will 10 grams remain?
   (c) What is the half-life of this radioactive material?
   (d) At what rate will the radioactive material be disintegrating after 1 year?
   (e) After how many years will the radioactive material be disintegrating at the rate of about 5 grams per year?

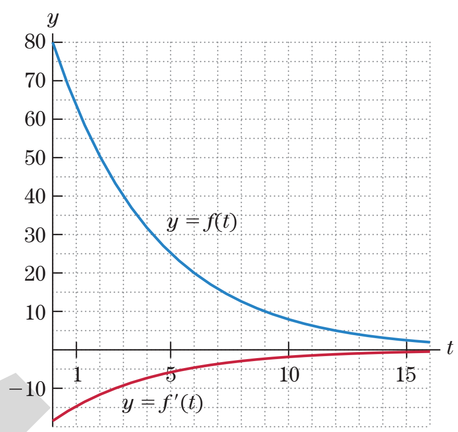

Figure 1

15. **Compound Interest** A few years after money is deposited into a bank, the compound amount is \$1000, and it is growing at the rate of \$60 per year. What interest rate (compounded continuously) is the money earning?

16. **Compound Interest** The current balance in a savings account is \$1230, and the interest rate is 4.5%. At what rate is the compound amount currently growing?

17. Find the percentage rate of change of the function $f(t) = 50e^{0.2t^2}$ at $t = 10$.

18. Find $E(p)$ for the demand function $q = 4000 - 40p^2$, and determine if demand is elastic or inelastic at $p = 5$.

19. **Elasticity of Demand** For a certain demand function, $E(8) = 1.5$. If the price is increased to \$8.16, estimate the percentage decrease in the quantity demanded. Will the revenue increase or decrease?

20. Find the percentage rate of change of the function $f(p) = \dfrac{1}{3p + 1}$ at $p = 1$.

21. **Elasticity of Demand** A company can sell $q = 1000p^2e^{-0.02(p+5)}$ calculators at a price of p dollars per calculator. The current price is \$200. If the price is decreased, will the revenue increase or decrease?

22. **Elasticity of Demand** Consider a demand function of the form $q = ae^{-bp}$, where a and b are positive numbers. Find $E(p)$, and show that the elasticity equals 1 when $p = 1/b$.

23. Refer to Check Your Understanding 5.4. Out of 100 doctors in Group A, none knew about the drug at time $t = 0$, but 66 of them were familiar with the drug after 13 months. Find the formula for $f(t)$.

24. Height of a Weed The growth of the yellow nutsedge weed is described by a logistic growth formula $f(t)$ of type (9) in Section 5.4. A typical weed has length 8 centimeters after 9 days and length 48 centimeters after 25 days and reaches length 55 centimeters at maturity. Find the formula for $f(t)$.

25. Temperature of a Rod When a rod of molten steel with a temperature of 1800°F is placed in a large vat of water at temperature 60°F, the temperature of the rod after t seconds is

$$f(t) = 60(1 + 29e^{-0.15t})°\text{F}.$$

The graph of this function is shown in Fig. 2.

(a) What is the temperature of the rod after 11 seconds?

(b) At what rate is the temperature of the rod changing after 6 seconds?

(c) Approximately when is the temperature of the rod 200°F?

(d) Approximately when is the rod cooling at the rate of 200°F per second?

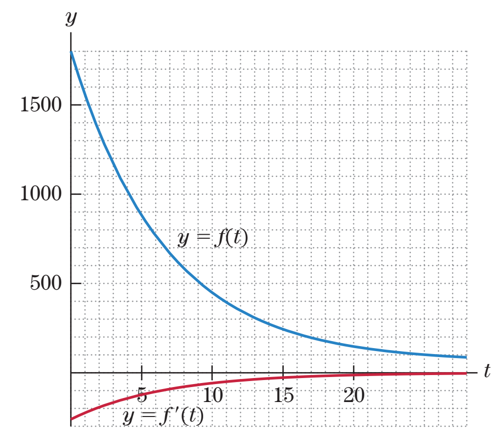

Figure 2

26. Bacteria Growth A certain bacteria culture grows at a rate proportional to its size. If 10,000 bacteria grow at the rate of 500 bacteria per day, how fast is the culture growing when it reaches 15,000 bacteria?