Diagonalization of Quadratic Forms

Recall in days past when you were given an equation which looked like $x^2 + y + y^2 = 1$ and you were asked to sketch the set of points which satisfy this equation. It was necessary to complete the square so that the equation looked like the (h,k) form of an ellipse. That is,

$$\begin{aligned} & x^2 + y + y^2 = 1 \\ & \Rightarrow x^2 + \left(\left(\tfrac{1}{2}\right)^2 + y + y^2\right) = 1 + \left(\tfrac{1}{2}\right)^2 \\ & \Rightarrow x^2 + \left(y + \tfrac{1}{2}\right)^2 = \tfrac{5}{4} \end{aligned}$$

Then, now that we have rewritten the equation into a form we recognize, we can see that this is a circle (which is an ellipse) of radius $\sqrt{\tfrac{5}{4}}$ centered at the point $\left(0,-\tfrac{1}{2}\right)$. But we had to do some work to get to this point. Similarly, we need to be able to rewrite $ax^2 + 2bxy + cy^2 = 1$ into a new form so that we can read off all of the information necessary to sketch the set.

Suppose that we are given an equation of the following form,

$$ax^2 + 2bxy + cy^2 = 1.$$

Then, as it turns out, we can rewrite this equation using matrix notation. Indeed,

$$ax^2 + 2bxy + cy^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

The nice thing about this representation is that the matrix $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ has a property called *symmetry*. Specifically the matrix is symmetric about the diagonal. Formerly, $\begin{bmatrix} a & b \\ b & c \end{bmatrix}^T = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$. There are many nice results about symmetric matrices. One of them is the fact that any symmetric matrix is *unitarily diagonalizable*. Okay, so I'm throwing out a lot of fancy words, so let me try to elaborate on exactly what this means.

<u>Definition</u> : (Unitary Matrix) An nxn matrix U is said to be <u>unitary</u> if the columns of U form an orthonormal basis for $\Re^n$.

<u>Definition</u> : (Unitarily Diagonalizable) An nxn matrix A is said to be <u>unitarily diagonalizable</u> if there exists a unitary matrix U so that $A = U\Lambda U^{-1}$ where Λ is a diagonal matrix.

<u>Recall</u> : A set of vectors $\{e_1, e_2, \ldots, e_n\}$ is an <u>orthonormal basis</u> for $\Re^n$ if and only if any vector $\vec{x} \in \Re^n$ has a (unique) representation $\vec{x} = \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n$ and

$$e_i \cdot e_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}.$$

Then with these facts it should become clear as mud that $U^{-1} = U^T$. Care to see the proof? Too bad, I'm doing it anyway.

<u>Proposition</u> : If U is an nxn unitary matrix then, $U^T U = I_n$, where I_n denotes the nxn identity matrix. Thus, $U^T = U^{-1}$.

Proof : Let U be an nxn unitary matrix. Then observe the following computation.

$$U^T U = \begin{bmatrix} - & \vec{u}_1 & - \\ - & \vec{u}_2 & - \\ - & \vdots & - \\ - & \vec{u}_n & - \end{bmatrix} \begin{bmatrix} | & | & | & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \\ | & | & | & | \end{bmatrix} = \begin{bmatrix} \vec{u}_1 \cdot \vec{u}_1 & \vec{u}_1 \cdot \vec{u}_2 & \cdots & \vec{u}_1 \cdot \vec{u}_n \\ \vec{u}_2 \cdot \vec{u}_1 & \vec{u}_2 \cdot \vec{u}_2 & \cdots & \vec{u}_2 \cdot \vec{u}_n \\ \vdots & \vdots & \ddots & \vdots \\ \vec{u}_n \cdot \vec{u}_1 & \vec{u}_n \cdot \vec{u}_2 & \cdots & \vec{u}_n \cdot \vec{u}_n \end{bmatrix}$$

Then since the $\vec{u}_i$'s form an orthonormal basis we have,

$$\begin{bmatrix} \vec{u}_1 \cdot \vec{u}_1 & \vec{u}_1 \cdot \vec{u}_2 & \cdots & \vec{u}_1 \cdot \vec{u}_n \\ \vec{u}_2 \cdot \vec{u}_1 & \vec{u}_2 \cdot \vec{u}_2 & \cdots & \vec{u}_2 \cdot \vec{u}_n \\ \vdots & \vdots & \ddots & \vdots \\ \vec{u}_n \cdot \vec{u}_1 & \vec{u}_n \cdot \vec{u}_2 & \cdots & \vec{u}_n \cdot \vec{u}_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = \mathrm{I}_n \text{ as desired.}$$

So, to translate this into slightly more understandable language, this means that a unitary matrix is a matrix which moves around the coordinate axes to a new set of orthogonal axes (i.e. a change of basis). Or, we could say that the axes have been rotated (or reflected) to a new set of perpendicular axes.

Allow me to go back to a previous example to try to lower the level of intimidation a little bit. Recall the 'ol complete the square trick to find the (h,k) form for an equation representing an ellipse. This representation corresponds to *shifting* the origin to the point (h,k) and drawing the ellipse around that point, right? So, if you feel reasonably comfortable shifting around the coordinate axes, which is exactly what you are doing when completing the square, then you should feel comfortable twirling the axes around. After all it's the same idea. The only difference is that this new tool we have developed is more computation intensive.

Let us now set up a set of procedures to perform this task.

1) Write $ax^2 + 2bxy + cy^2 = 1$ in the matrix form $\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1$.
2) Find the eigenvalues l_1 and l_2 for the matrix $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$.
3) Find the eigenvectors $\vec{v}_1$ and $\vec{v}_2$ corresponding to the eigenvalues l_1 and l_2 respectively.
4) Draw lines passing through the vectors $\vec{v}_1$ and $\vec{v}_2$. (These are the new axes.)
5) Rewrite the new equation as $l_1 w_1^2 + l_2 w_2^2 = 1$ where w_1 and w_2 are variables representing the distance from the origin in the $\vec{v}_1$ and $\vec{v}_2$ directions

respectively. Just like x and y represent the distance from the origin in the direction e_1 and e_2.

6) Draw the darn picture.

I can hear you now, “Whoa hoss! You didn’t do that unitary diagonalization thing you were talking about!” You’re right. To sketch the picture I didn’t need to go that far. However, notice that I have all the information to do so if I wish. Let $\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix}$ and $\vec{u}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|} = \begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix}$. Then,

$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} u_{11} & u_{21} \\ u_{12} & u_{22} \end{bmatrix} \begin{bmatrix} l_1 & 0 \\ 0 & l_2 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$ is the desired unitary diagonalization.

Example 1: Sketch the image of the set of points which satisfy the equation

$$27x^2 - 18xy + 3y^2 = 3.$$

Solution 1:

1) $27x^2 - 18xy + 3y^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 27 & -9 \\ -9 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

2) $\det\left(\begin{bmatrix} 27-\lambda & -9 \\ -9 & 3-\lambda \end{bmatrix} \right) = 0$

$\Rightarrow (27-\lambda)(3-\lambda) - 81 = 0$

$\Rightarrow 81 - 30\lambda + \lambda^2 - 81 = 0$

$\Rightarrow \lambda^2 - 30\lambda = 0$

$\Rightarrow \lambda(\lambda - 30) = 0$

$\Rightarrow \lambda_1 = 0,\ \lambda_2 = 30$

3) $\begin{bmatrix} 27-0 & -9 \\ -9 & 3-0 \end{bmatrix}$

$\sim \begin{bmatrix} 27 & -9 \\ -9 & 3 \end{bmatrix}$

$\sim \begin{bmatrix} 0 & 0 \\ -9 & 3 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & \frac{-1}{3} \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \vec{v}_1 = \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda_1 = 0$.

$\begin{bmatrix} 27-30 & -9 \\ -9 & 3-30 \end{bmatrix}$

$\sim \begin{bmatrix} -3 & -9 \\ -9 & -27 \end{bmatrix}$

$\sim \begin{bmatrix} -3 & -9 \\ 0 & 0 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \vec{v}_2 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda_2 = 30$.

4) Sketch on a piece of paper lines passing through $\vec{v}_1 = \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

   These represent the new axes. Notice $\vec{v}_1 \cdot \vec{v}_2 = 0$.

5) The new equation we have is then $0{w_1}^2 + 30{w_2}^2 = 3$. Notice this is a bit of a weird animal. What does it say? Well, doing some algebra we have,

   $0{w_1}^2 + 30{w_2}^2 = 3$

   $\Rightarrow 30{w_2}^2 = 3$

   $\Rightarrow w_2 = \pm\sqrt{\frac{1}{10}}$

   This gives us a set of parallel lines which intersect the w_2 axis at $\pm\sqrt{\frac{1}{10}}$.

6) Draw the darn picture.

Example 2: Sketch the image of the set of points which satisfy the equation

$$13x^2 - 10xy + 13y^2 = 25.$$

Solution 2:

1) $13x^2 - 10xy + 13y^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 13 & -5 \\ -5 & 13 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

2) $\det\left(\begin{bmatrix} 13-\lambda & -5 \\ -5 & 13-\lambda \end{bmatrix}\right) = 0$

$\Rightarrow (13-\lambda)(13-\lambda) - 25 = 0$

$\Rightarrow 169 - 26\lambda + \lambda^2 - 25 = 0$

$\Rightarrow \lambda^2 - 26\lambda + 144 = 0$

$\Rightarrow (\lambda - 8)(\lambda - 18) = 0$

$\Rightarrow \lambda_1 = 8,\ \lambda_2 = 18$

3) $\begin{bmatrix} 13-8 & -5 \\ -5 & 13-8 \end{bmatrix}$

$\sim \begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix}$

$\sim \begin{bmatrix} 5 & -5 \\ 0 & 0 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda_1 = 8$.

$\begin{bmatrix} 13-18 & -5 \\ -5 & 13-18 \end{bmatrix}$

$\sim \begin{bmatrix} -5 & -5 \\ -5 & -5 \end{bmatrix}$

$\sim \begin{bmatrix} -5 & -5 \\ 0 & 0 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda_2 = 18$.

4) Sketch on a piece of paper lines passing through $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

These represent the new axes. Notice $\vec{v}_1 \cdot \vec{v}_2 = 0$.

5) The new equation we have is then $8{w_1}^2 + 18{w_2}^2 = 25$, which after some cleaning up looks like $\frac{{w_1}^2}{\left(\frac{5}{2\sqrt{2}}\right)^2} + \frac{{w_2}^2}{\left(\frac{5}{3\sqrt{2}}\right)^2} = 1$. This then is an ellipse that intersects the w_1 axis at $\pm\frac{2\sqrt{2}}{5}$ and intersects the w_2 axis at $\pm\frac{3\sqrt{2}}{5}$

6) Draw the darn picture.

Example 3: Sketch the image of the set of points which satisfy the equation

$$-4x^2 + 2xy - y^2 = 1.$$

Solution 3:

1) $-4x^2 + 2xy - y^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} -4 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

2) $\det\left(\begin{bmatrix} -4-\lambda & 1 \\ 1 & -1-\lambda \end{bmatrix}\right) = 0$

$\Rightarrow (-4-\lambda)(-1-\lambda) - 1 = 0$

$\Rightarrow 4 + 5\lambda + \lambda^2 - 1 = 0$

$\Rightarrow \lambda^2 + 5\lambda + 3 = 0$

$\Rightarrow \left(\lambda - \frac{-5+\sqrt{13}}{2}\right)\left(\lambda - \frac{-5-\sqrt{13}}{2}\right) = 0$

$\Rightarrow \lambda_1 = \frac{-5+\sqrt{13}}{2},\ \lambda_2 = \frac{-5-\sqrt{13}}{2}$

Ick. How can we possible find eigenvalues corresponding to these monsters? Well, you can if you want but you don't need to. Because if we skip the whole *find the eigenvectors* step notice what will happen. That is, assume for the moment that we found the eigenvectors for these eigenvalues. Then we have the new representation for the equation as follows,

$$\left(\frac{-5+\sqrt{13}}{2}\right)w_1^{\,2} + \left(\frac{-5-\sqrt{13}}{2}\right)w_2^{\,2} = 1$$

$$\Rightarrow -1 = \left(\frac{5-\sqrt{13}}{2}\right)w_1^{\,2} + \left(\frac{5+\sqrt{13}}{2}\right)w_2^{\,2}$$

Do you see anything wrong with this picture. Notice, $\left(\frac{5-\sqrt{13}}{2}\right) > 0$, $\left(\frac{5+\sqrt{13}}{2}\right) > 0$, $w_1^{\,2} > 0$, and $w_2^{\,2} > 0$ but the equation says that I can multiply them together and add them up to obtain a *negative* number. There are certainly no real numbers which can satisfy this equation. So, the sketch of the set of points which satisfy the equation $\left(\frac{-5+\sqrt{13}}{2}\right)w_1^{\,2} + \left(\frac{-5-\sqrt{13}}{2}\right)w_2^{\,2} = 1$ is empty.

To summarize the cases we have the following,

i) If $\lambda_1 > 0$ and $\lambda_2 > 0$ we have an ellipse.

ii) If $\lambda_1 \lambda_2 < 0$ (so either $\lambda_1 < 0$ or $\lambda_2 < 0$, not both) we have a hyperbola.

iii) If $\lambda_1 = 0$ or $\lambda_2 = 0$ (not both zero) we have a set of parallel lines.

iv) If $\lambda_1 = \lambda_2 = 0$ we have the empty set.

v) If $\lambda_1 < 0$ and $\lambda_2 < 0$ we have the empty set.