Trigonometry Lecture Notes

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Fall 2019 edition

Contents

Chapter 1

Background material

1.1 What is allowed? What isn't allowed?

EXERCISE

Classify the following statements as true or false (in math, for a statement to be true means that it is always true):

TRUE	FALSE	
☐	☐	$5(x+2) = 5x+2.$
☐	☐	$5(x+2) = 5x+10.$
☐	☐	$(a+b)(x+y) = ax+by.$
☐	☐	$3x = x+x+x.$
☐	☐	$x^2 = x+x.$
☐	☐	$3 \cdot \frac{1}{5} = \frac{3}{5}.$
☐	☐	$\frac{a}{b} = a \cdot \frac{1}{b}.$
☐	☐	$\frac{a+b}{c} = \frac{a}{c} + b.$
☐	☐	$\frac{a+3}{2} = \frac{a}{2} + \frac{3}{2}.$
☐	☐	$\frac{a}{b+2} = \frac{a}{b} + \frac{a}{2}.$

TRUE	FALSE	
☐	☐	$\frac{2}{3} \cdot \frac{8}{7} = \frac{2 \cdot 8}{3 \cdot 7}$.
☐	☐	$\frac{2}{3} \div \frac{8}{7} = \frac{2 \div 8}{3 \div 7}$.
☐	☐	$x^2 = x \cdot x$.
☐	☐	$(xy)^2 = x^2y^2$.
☐	☐	$(x-3)^2 = x^2 - 3$.
☐	☐	$(x-3)^2 = x^2 - 3^2$.
☐	☐	$\left(\frac{x}{5}\right)^2 = \frac{x^2}{5}$.
☐	☐	$\left(\frac{x}{y}\right)^2 = \frac{x^2}{y^2}$.
☐	☐	$\sqrt{2x} = 2\sqrt{x}$.
☐	☐	$\sqrt{xy} = \sqrt{x}\sqrt{y}$.
☐	☐	$(x+2)^2 = x^2 + 2$.
☐	☐	$(x+2)^2 = x^2 + 4$.
☐	☐	$(x+y)^2 = x^2 + y^2$.
☐	☐	$\sqrt{x+y} = \sqrt{x} + \sqrt{y}$.
☐	☐	$\sqrt{x^2+y^2} = x + y$.
☐	☐	$\frac{1}{2/3} = \frac{3}{2}$.
☐	☐	$\frac{1}{\left(\frac{5}{\sqrt{3}}\right)} = \frac{\sqrt{3}}{5}$.
☐	☐	$\frac{a}{\left(\frac{b}{c}\right)} = \frac{\left(\frac{a}{b}\right)}{c}$.

1.2 Solving basic equations

Linear equations

A **linear equation** is an equation which, after some valid algebra, can be written as $ax = b$ for some numbers a and b. To solve a linear equation, you combine all the x-terms on one side of the equation and combine all the non-x-terms on the other side.

EXAMPLE 1

Solve for x, if $8x + 7 = 79$.

EXAMPLE 2

Solve for x, if $3(5 - 2x) = 7(x + 2)$.

Solving linear equations for expressions, rather than variables

Often, you have to solve for something more complicated than just "x". The principle is the same as what we've done before, except you should think of the thing you are solving for as its own letter.

Be careful not to combine unlike terms! ($3 + 5x \neq 8x$, etc.)

EXAMPLE 3

Solve for $\sqrt{x+1}$:

$$18\sqrt{x+1} = 36$$

EXAMPLE 4

Solve for x^4:

$$5 + 3x^4 = 2(7 - 5x^4)$$

EXAMPLE 5

Solve for $\cos x$ (whatever "$\cos x$" means):

$$40^2 = 30^2 + 35^2 - 2(30)(35)\cos x$$

Solving proportions

A **proportion** consists of two fractions which are equal:

$$\frac{a}{b} = \frac{c}{d}.$$

To solve a proportion involving variables, you **cross-multiply** the proportion, applying the theoretical fact that

$$\frac{a}{b} = \frac{c}{d} \text{ if and only if } ad = bc.$$

EXAMPLE 6

Solve for x:

$$\frac{x}{3} = \frac{2.78}{8.44}$$

EXAMPLE 7

Solve for x:

$$\frac{x}{x-2} = \frac{7}{5}$$

EXAMPLE 8

Solve for $\sin x$ (whatever "$\sin x$" means):

$$\frac{\sin x}{7.25} = \frac{.358}{10.459}$$

Solution: First, cross-multiply. Then, divide both sides by 10.459:

$$\begin{aligned} 10.459 \sin x &= .358(7.25) \\ 10.459 \sin x &= 2.5955 \\ \sin x &= \frac{2.5955}{10.459} \\ \sin x &= .24816. \end{aligned}$$

Quadratic equations

A **quadratic equation** has a variable squared (like x^2) in it.

How to solve a quadratic equation

If there is an x^2 term but no x term: solve for the x^2 term. Then take the square root of both sides (be sure to include both the positive and negative square root as answers for x).

If there is both an x^2 term and an x term: move all the terms of the equation to one side (i.e. make one side equal to zero). Then either factor or use the quadratic formula.

EXAMPLE 9

Solve for x:

$$x^2 + 3x = 40$$

EXAMPLE 10

Solve for x:

$$x^2 = 36$$

EXAMPLE 11

Solve for x:

$$x^2 + (.275)^2 = 1$$

EXAMPLE 12

Solve for x:

$$x^2 + (1.275)^2 = 8.81$$

Solution: First, square out the $(3.275)^2$ and move it to the other side:

$$\begin{aligned} x^2 + (1.275)^2 &= 8.81 \\ x^2 + 1.6256 &= 8.81 \\ x^2 &= 8.81 - 1.6256 \\ x^2 &= 7.1844 \\ x &= \pm\sqrt{7.1844} \\ x &= \pm 2.68. \end{aligned}$$

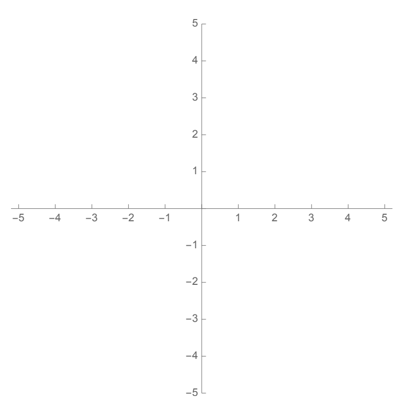
1.3 The coordinate plane

Just as real numbers can be thought of as points on a number line, **ordered pairs** (x, y) can be thought of as points in a plane. The first number in an ordered pair is called the x**-coordinate** and measures the horizontal distance the point (x, y) is from the origin; the second number in the pair is called the y**-coordinate** and measures the vertical distance that (x, y) is from the origin.

EXAMPLE 13

Graph the following points on the provided axes:

$$(0,3) \qquad (-3,4) \qquad (2,-3) \qquad (-4,0) \qquad (-1,-2)$$



EXAMPLE 14

Start at $(3, 8)$. Go left seven units, then up five units, then right three units. Where are you?

EXAMPLE 15

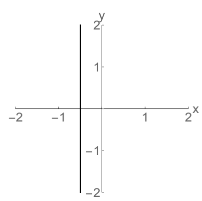
What is the distance from $(2, 7)$ to the x-axis? What is the distance from $(2, 7)$ to the y-axis?

EXAMPLE 16

Describe the set of points in the coordinate plane whose y-coordinate is equal to 2.

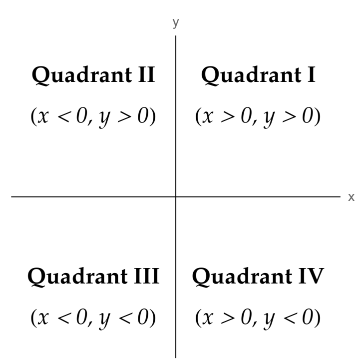
EXAMPLE 17

What do all the points on the indicated line have in common?



Quadrants

The x- and y-axes divide the coordinate plane into four **quadrants**, numbered by Roman numerals:



1.4 Functions

Question: What is a function? Don't peek at the next page. (My past experience is that many Math 120 students don't know what a function is.)

EXAMPLE 17

Suppose you buy soup at Meijer. Each can of soup costs 2 dollars, with the catch that cans of soup are "buy one, get one free". Let's create a "function" which models this situation; we'll call this function "price".

Definition 1.1 *A* **function** *is a rule of assignment which produces outputs from inputs, in such a way that each input leads to* ***one and only one*** *output. The set of inputs to a function is called the* **domain** *of the function, and the set of outputs is called the* **range** *of the function.*

We usually name functions after letters (capital or lowercase), but we also name them after words or phrases, and occasionally use a symbol to name a function.

A good way to think about a function is to think of an "arrow diagram":

Definition 1.2 *If f is a function, then $f(x)$ is* the output associated to input x, *not "f times x". Sometimes the parenthesis in the $f(x)$ is omitted and we just write fx, especially if the function is named after a word or phrase, rather than a single letter. The formula defining $f(x)$ for an arbitrary input x is called the* **rule** *or* **formula** *for f.*

EXAMPLE 18

Let f be the function which takes its input, squares it, then subtracts 8 to produce the output. Find a formula for f, and compute $f(2)$ and $f(-5)$.

IMPORTANT: To be a valid function, each input must have only one possible output.

EXAMPLE 18

Determine whether or not each given formula is the formula of a valid function:

1. $f(x) = \pm x$
2. $f(x) = (\pm x)^2$
3. $f(x) = (\pm x)^3$

EXAMPLE 19

Let T be the function which takes its input, multiplies the input by 4 and then subtracts 3 to produce the output.

1. Find a formula for $T(x)$.

2. Compute $2T(3)$.

3. Compute $T(2 \cdot 3)$.

4. Compute $T(2+1)$.

5. Compute $T(2)+T(1)$.

6. Find a formula for $T(3x)$.

7. Find a formula for $3T(x)$.

EXAMPLE 20

Let funny be the function defined as follows: start with input x. If $x = 0$, then funny $x = 5$. If x is positive, then funny $x = 1 + x$. If x is negative, then funny $x = 3 + 2x$.

1. Compute funny 6.
2. Compute funny -3.
3. Compute $-$ funny 3.
4. Compute funny 2 + funny (-2).
5. Compute funny $(2 + (-2))$.
6. Compute funny $(3 + 2)$.
7. Compute funny 3 + funny 2.
8. Compute funny $3 + 2$.
9. Compute funny $(2) \cdot 3$.
10. Compute funny $(2 \cdot 3)$.
11. Compute funny $2 \cdot 3$.
12. Compute 3 funny 2.
13. Compute funny $3 \cdot 2$.

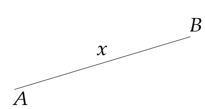
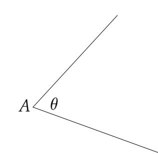
Chapter 2

Angles

2.1 Introduction to trigonometry

First question: What is "trigonometry"?

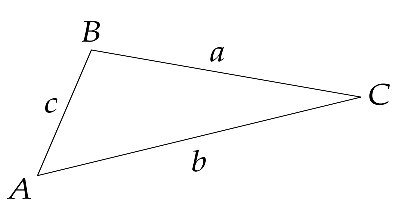
Question: What is a triangle? Put another way, what are the "parts" of a triangle?

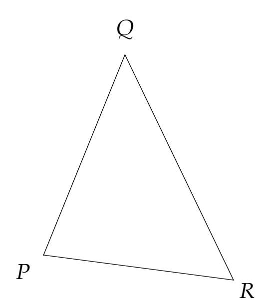
PART OF TRIANGLE	SIDE	ANGLE
HOW MANY IN A TRIANGLE	3	3
WHAT DISTINGUISHES "BIG" FROM "SMALL"	length	measure
MEASURING TOOL	ruler	protractor
TYPICAL UNIT(S) OF MEASUREMENT	inches, feet, yards, m, km, cm, ...	degrees, radians
NOTATION	Usually denoted by a lower-case English letter or with two capital letters $a, j, k, x, \ldots$ AB, XY, BC, ...	Usually denoted by a capital English or Greek letter $A, B, M, N, \ldots$ θ, ϕ, α, β, ...
PICTURES		

Some Greek letters:

α	alpha	θ	theta	ϕ or φ	phi
β	beta	ψ	psi		
γ	gamma	π	pi		

Important Convention: In a triangle, the side length labelled with a lower-case English letter should always be opposite the angle with the corresponding capital letter.





2.2 Angles

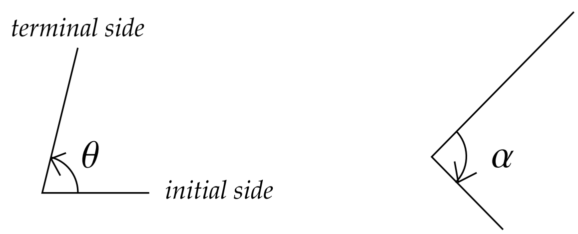
Degree measure

The measure of an angle is how much it "opens". We need to choose a unit for this measurement. The oldest unit of measurement for angles comes from the ancient Babylonians: they estimated that there were 360 days in a year, so they set up their unit of measurement so that an angle which measures "all the way around" once is 360 units. That unit is called a **degree** and is denoted with a $^\circ$.

The more an angle "opens", the greater its measure.

We measure the degrees in an angle using a **protractor**.

Important: angle measurement is **oriented**; that is, we think of the measure as starting on one side of the angle (the **initial side**) and ending on the other side (the **terminal side**). If the angle is measured counterclockwise, then the measure is positive; if the angle is measured clockwise, then the measure is negative.

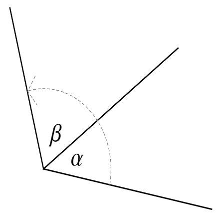


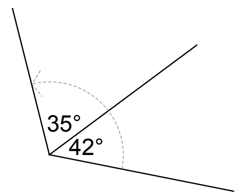
> **Theorem 2.1** *If an angle measures θ when measured counterclockwise, then it measures $-\theta$ when measured clockwise.*

Absent any other indication: always assume all angles are measured counterclockwise and that the angle under consideration is the smaller of the two options. To indicate an angle which is measured clockwise or an angle which is "large", draw an arrow inside the angle:

Angle addition

If the terminal side of angle α is the same as the initial side of angle β, then the measures of the angles add as in these pictures:

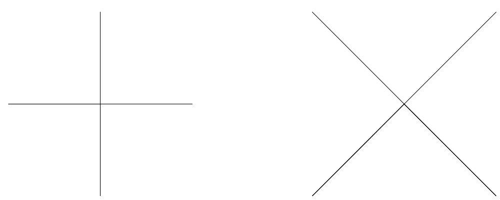




Angle vocabulary

Perpendicular lines and complementary angles

We classify angles based on their measure. First, suppose you have two perpendicular lines that meet:

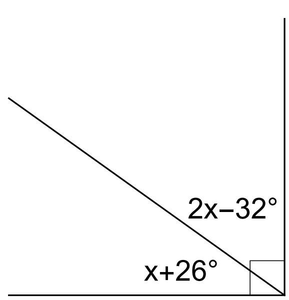


The measures of all four angles generated by this intersection are equal. Since they must add to $360°$, they must each have measure

Theorem 2.2 *Perpendicular lines meet at $90°$ angles.*

EXAMPLE 1

Find the measure of each angle in this picture:



Two angles whose measures add to 90° (like the two angles in Example 1) are called **complementary**.

EXAMPLE 2

An angle has measure equal to two-and-a-half times the measure of its complement. What is the measure of the angle?

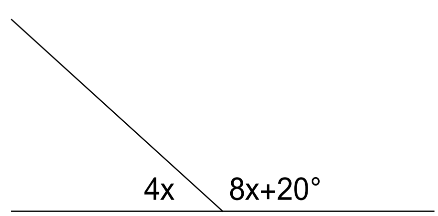
Straight lines and supplementary angles

If you measure the angle of a single straight line, it must be equal to the sum of two 90° angles, so it must have measure

Theorem 2.3 *A straight line has an angle measure of* 180°.

EXAMPLE 3

Find the measure of each angle in this picture:



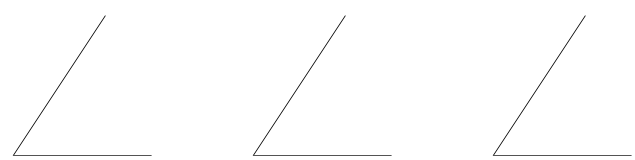
Two angles whose measures sum to $180°$ (like in Example 3) are called **supplementary**.

EXAMPLE 4

An angle has measure $40°$ greater than its supplement. What is the measure of the angle? What is the measure of the complement of the angle?

Coterminal angles

Two angles are called **coterminal** if the difference between their measures is a multiple of $360°$.



EXAMPLE 5

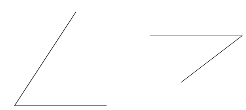
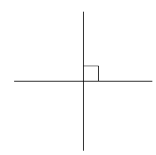
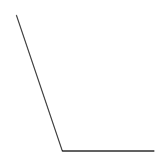
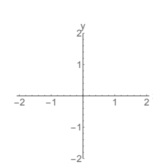
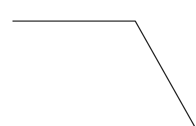
Find three angles which are coterminal with $58°$.

EXAMPLE 6

Find an angle between $0°$ and $360°$ which is coterminal with $2057°$.

Classification of angles

We classify angles based on whether they are greater than or less than the special angles described above:

TYPE OF ANGLE	DESCRIPTION IN ENGLISH	CONDITION ON MEASURE	PICTURE
Zero angle	Angle has zero measure	$\theta = 0^\circ$	
Acute angle	Angle is less than a right angle	$0^\circ < \theta < 90^\circ$	
Right angle	Angle formed by 2 perpendicular lines	$\theta = 90^\circ$	
Obtuse angle	Angle measures more a right angle, but less than a straight line	$90^\circ < \theta < 180^\circ$	
Straight angle	Angle is equal to the sum of 2 right angles (i.e. the angle formed by a straight line)	$\theta = 180^\circ$	
Quadrantal angle	Angle is a multiple of 90°	$\theta = 0^\circ, \pm 90^\circ, \pm 180^\circ, \pm 270^\circ, \pm 360^\circ \ldots$	
Reflex angle	Angle is more than the sum of 2 right angles	$\theta > 180^\circ$	

2.3 Radian measure

Preliminary material needed to understand radians

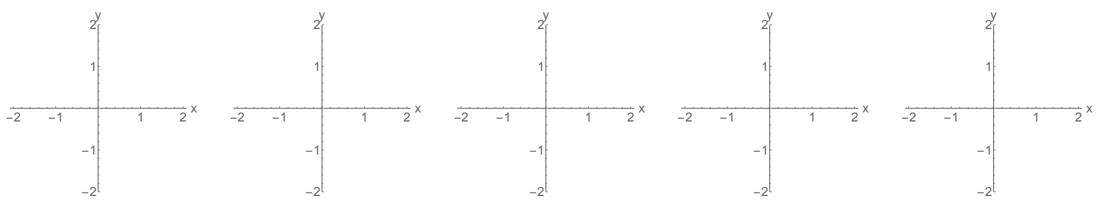
Standard position of an angle

To draw an angle in **standard position** means to draw it on an xy-plane so that its vertex is at $(0,0)$ and its initial side is on the positive x-axis.

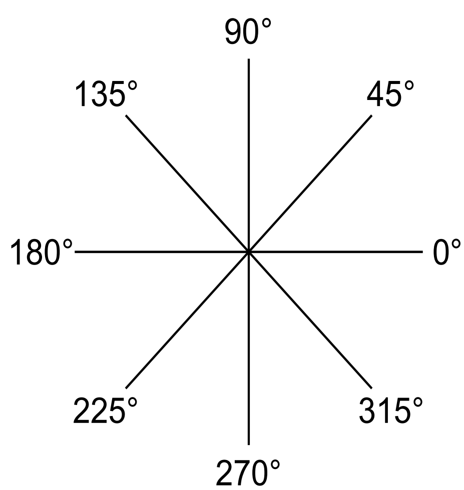
EXAMPLE 7

Draw each of these angles in standard position:

$45°$ $\qquad 270°$ $\qquad 115°$ $\qquad -20°$ $\qquad 400°$

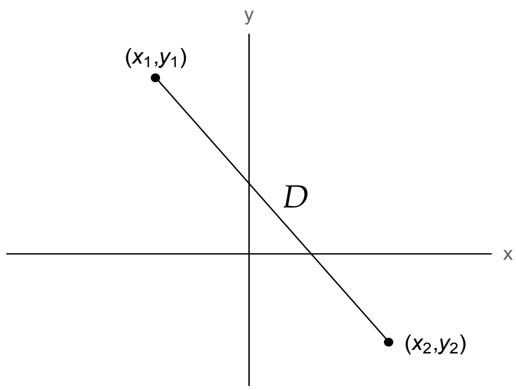


How accurate should you be when drawing an angle in standard position?



The unit circle

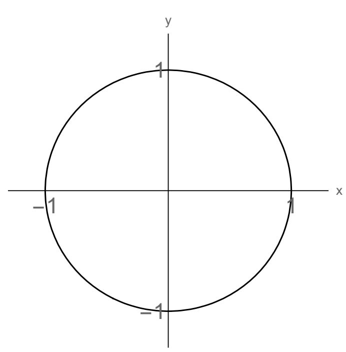
Recall the **distance formula**, which tells you that the distance between points (x_1, y_1) and (x_2, y_2) in the xy-plane is given by



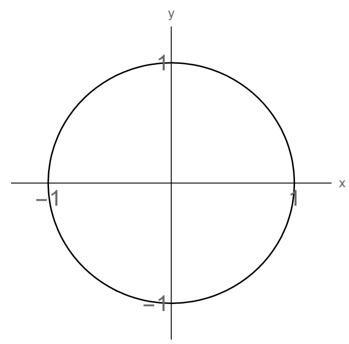
EXAMPLE 8

Find the distance between the points $(4, 6)$ and $(7, 2)$.

The **unit circle** is a circle of radius 1 centered at $(0, 0)$. In other words, the unit circle is the set of points (x, y) such that the distance from (x, y) to $(0, 0)$ is 1. Here is a picture of the unit circle:



Question: What is the equation of the unit circle?

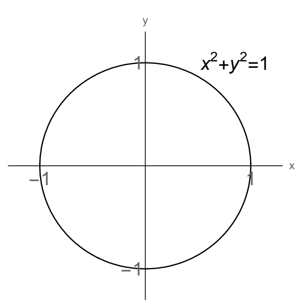


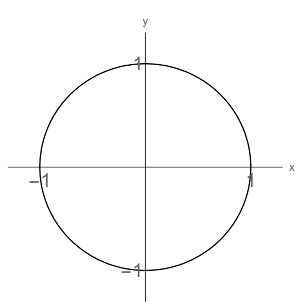
$$\begin{aligned} D &= 1 \\ \text{Distance formula:} \qquad \sqrt{(x-0)^2+(y-0)^2} &= 1 \\ \sqrt{x^2+y^2} &= 1 \\ \text{Square both sides:} \qquad x^2+y^2 &= 1^2 \\ x^2+y^2 &= 1. \end{aligned}$$

Theorem 2.4 *Every point (x, y) on the unit circle satisfies $x^2 + y^2 = 1$.*

EXAMPLE 9

1. Find all y such that the point $(\frac{2}{3}, y)$ is on the unit circle.
2. Find the point where the terminal side of a $135°$ angle, drawn in standard position, intersects the unit circle.



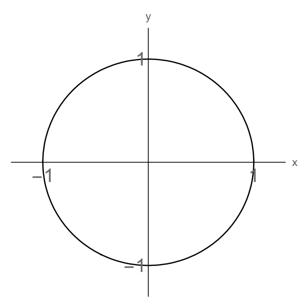


Length of a curve

The **length** of a line segment in the xy-plane is the distance between its endpoints (given by the distance formula in the previous section).

To define the length of a *curve* (theoretically), imagine the curve is made of string. Take the string and pull it until it is taut. The length of the taut string is the length of the curve.

In general, it is very, very hard to find the exact length of curves, and most curves do not have lengths which are nice (rational) numbers like 3 or $\frac{8}{3}$ or $\frac{127}{37}$ or 175.6. There are formulas for lengths of curves you learn in calculus, but even these are hard to implement in practice.



In particular, the circumference (i.e. length of one revolution) of the unit circle is not a whole number (it is between 6 and 7). It is not a terminating decimal (although it is about 6.28), or a fraction whose numerator and denominator are whole numbers (although it is about $\frac{44}{7}$). Because this number is useful, we give it a name:

Definition 2.5 *The number τ (pronounced "tau") is the circumference of the unit circle, i.e. the circumference of any circle of radius 1 unit.*

Math would be easier if formulas were evaluated in terms of τ. However, the ancient Greeks, whose geometric and trigonometric language persists to this day, decided to work with exactly half of τ, which is the distance around half of the unit circle. They gave that number its name, which we still use today:

Definition 2.6 *The number π (pronounced "pi") is $\frac{1}{2}\tau$, i.e. π is half of the circumference of any circle of radius 1 unit.*

$\pi \approx 3.14$ and $\pi \approx \frac{22}{7}$, but most calculators have a π button which you should use.

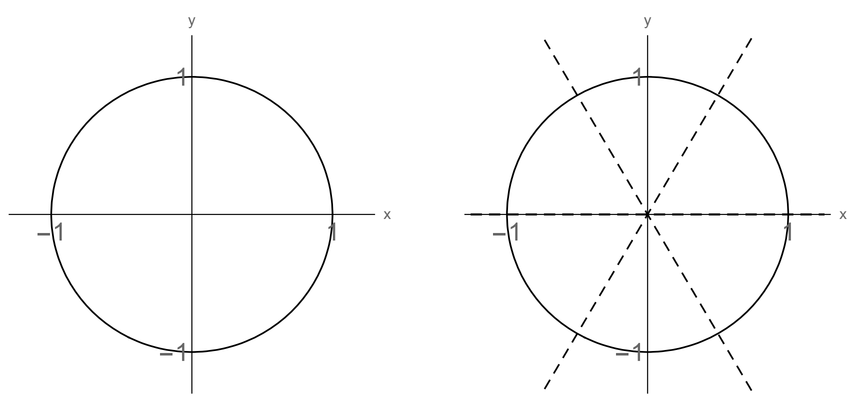
Theorem 2.7 *The circumference of the unit circle is 2π.*
The circumference of a circle of radius r units is

$$C = 2\pi r.$$

Definition of a radian

A radian is an extremely useful unit of measurement for angle measure. The definition is a little abstract:

Definition 2.8 *A* **radian** *is a unit of angle measure such that the radian measure of any angle equals the length of the arc on the unit circle subtended by the angle, when drawn in standard position.*



Based on the definition of a radian, we know that one complete revolution is equal to ...

Therefore a right angle, in radians, measures ...

Converting between degrees and radians

From above, we know that one complete revolution is 2π radians.
We also know (from earlier) that one revolution is equal to $360°$. Therefore

This gives us formulas for converting between degrees and radians:

Theorem 2.9 (Converting between degrees and radians)

1. *Multiply an angle measure by* $\frac{\pi}{180^\circ}$ *to convert degrees to radians.*
2. *Multiply an angle measure by* $\frac{180^\circ}{\pi}$ *to convert radians to degrees.*

EXAMPLE 10

1. Convert 124° to radians.

   Solution: $124^\circ \cdot \frac{\pi}{180^\circ} = 124^\circ \cdot \frac{3.14159}{180^\circ} = 2.164.$

2. Convert -240° to radians.

3. Convert 4.25 radians to degrees.

   Solution: $4.25 \cdot \frac{180^\circ}{\pi} = 243.507^\circ.$

4. Convert $\frac{17}{10}\pi$ radians to degrees.

5. Convert 3.75 revolutions to degrees and radians.

Special angles in radians

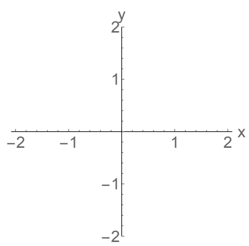
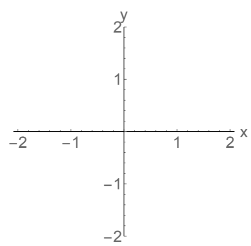
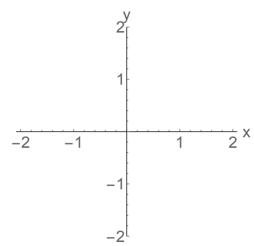
There are certain angles (in radians) that you should just "know" how many degrees they correspond to:

$$\pi = 180^\circ \qquad \frac{\pi}{2} = 90^\circ \qquad \frac{\pi}{3} = 60^\circ \qquad \frac{\pi}{4} = 45^\circ \qquad \frac{\pi}{6} = 30^\circ$$

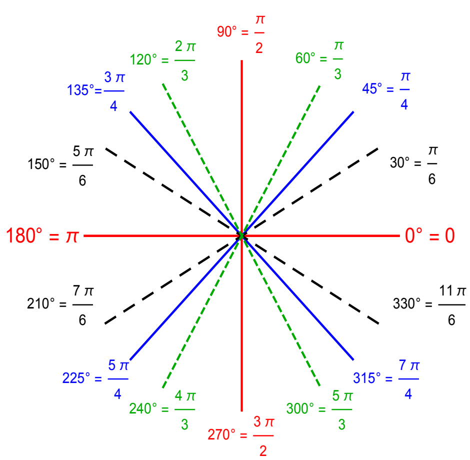
EXAMPLE 11

Convert the following angles to degrees, and draw each in standard position.

$$\frac{3\pi}{2} \qquad \frac{-3\pi}{4} \qquad \frac{5\pi}{6}$$

A useful diagram



<u>IF IN LOWEST TERMS:</u>

(whole #) $\pi \longleftrightarrow$

$\frac{(\text{odd \#})\,\pi}{2} \longleftrightarrow$

$\frac{(\text{whole \#})\,\pi}{4} \longleftrightarrow$

$\frac{(\text{whole \#})\,\pi}{3} \longleftrightarrow$

$\frac{(\text{whole \#})\,\pi}{6} \longleftrightarrow$

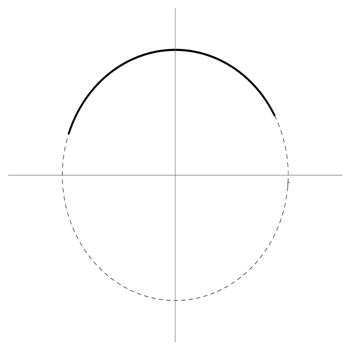
IMPORTANT: <u>The default unit of measurement for angles is radians!</u> That means that unless you write a $\circ$, the measure of an angle is understood to be in radians. So an angle measuring "90" measures 90 radians, not 90°.

2.4 Applications of radian measure

Radians are a superior unit of angle measure to degrees. One reason is that radians can be used to convert angle measures to lengths of curves and areas of sectors without any additional work.

Arc length

Take a circle of radius r, and take an arc on that circle subtended by angle θ. Call the length of the arc s.



If the arc went one complete revolution, it would have length $2\pi r$. That corresponds to 2π radians of angle measure. If you have θ radians of angle measure, then by proportionality the length satisfies

Theorem 2.10 (Arc length formula) *The length of an arc is*

$$s = r\theta$$

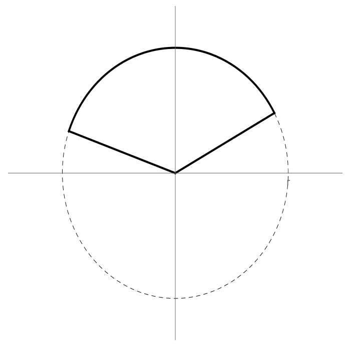
where r is the radius of the circle and θ is the angle <u>**measured in radians**</u>.

EXAMPLE 12

A circle has radius 35 in. Find the length of the arc intercepted by a central angle of $62°$.

Area of a sector

A **sector** is the mathematical name for a shape that looks like a pizza wedge:



Suppose you have a sector subtended by angle θ in a circle of radius r. If the angle went one complete revolution, the area of the sector would be the area of the circle, which is

That area corresponds to 2π radians of angle measure. So by proportionality, if you have θ radians of angle measure, the area A of the sector satisfies

$$\frac{\pi r^2}{2\pi} = \frac{A}{\theta} \Rightarrow$$

Theorem 2.11 (Sector area formula) *The area of a sector is*

$$A = \frac{1}{2} r^2 \theta$$

where r is the radius of the circle and θ is the angle **measured in radians**.

EXAMPLE 13

A pizza has diameter 14 inches. If a slice of pizza has an angle of $\frac{\pi}{8}$ radians, find the area of the pizza slice.

Angular and linear velocity

Suppose an object is moving along a circular path (think of something spinning around on the edge of a wheel). We want to describe how "fast" the object is moving. There are two ways to do this:

- The **linear velocity (linear speed)** v of the object is its arc length traveled divided by the elapsed time:

$$v = \frac{s}{t} = \frac{r\theta}{t}$$

- The **angular velocity (angular speed)** ω (omega) of the object is the change in its angle divided by the elapsed time:

$$\omega = \frac{\theta}{t}$$

Theorem 2.12 (Converting between angular and linear velocity) *Suppose an object is moving along a circular path of radius r. If its angular velocity is ω (measured in radians per unit of time) and its linear velocity is v, then*

$$v = r\omega \quad \text{and } \omega = \frac{v}{r}.$$

EXAMPLE 14

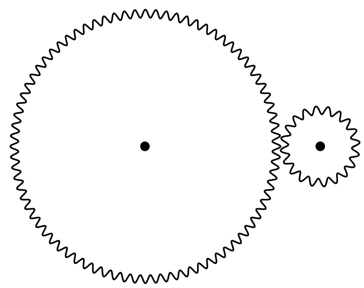
The Earth is $93,000,000$ miles from the sun. What is the linear velocity of Earth, in miles per hour? (How about miles per second?)

Why do we need to understand this?

By controlling the power supplied to a gear or pulley system, you control the angular velocity of that gear or pulley.

EXAMPLE 15

Two gears are interlocked so that they rotate together (see picture below). One gear has a radius of 4 in, and the other gear has a radius of 1 in.

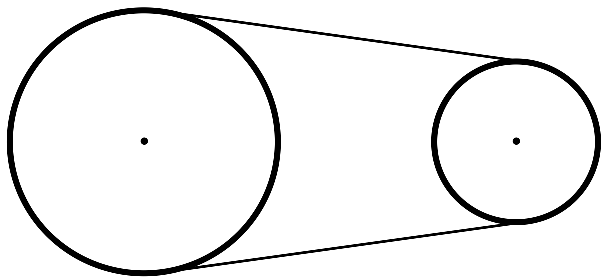


1. If the larger gear is rotated at 400 rpm (revolutions per minute), what is the angular velocity of the smaller gear (in rpm)?
2. If you want the larger gear to rotate at 120 radians per second, what angular velocity (in radians per second) should the smaller gear be rotated at?

EXAMPLE 16

Two pulleys in the figure that will be drawn below have radii of 18 cm and 11 cm, respectively. If the larger pulley rotates 30 times in 12 seconds, find:

1. the angular velocity of each pulley (in radians per second);
2. the linear velocity of each pulley (in cm per second);
3. the distance a point on the belt travels in 10 seconds; and
4. the change in angle a point on the right-hand pulley experiences in 2 seconds.



2.5 Angle relationships

Angles in pictures involving parallel and intersecting lines

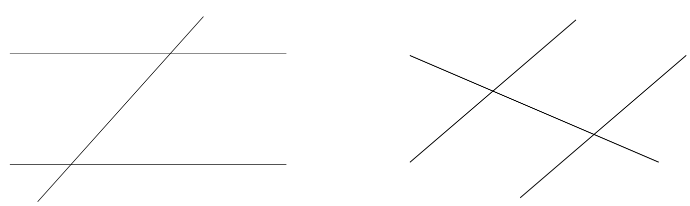
Vertical angles

When two lines cross, the "opposite" angles formed are called **vertical angles**:

> **Theorem 2.13** *Vertical angles have the same measure.*

Parallel lines and transversals

When a third line crosses a pair of parallel lines, the third line is called a **transversal**. In such a picture, eight angles are created; these angles divide into two sets of four where the angles in each set have the same measure:

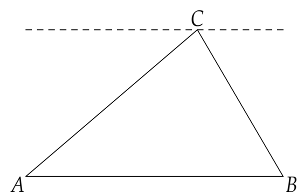


WARNING: In a picture, do not assume lines that "look" parallel are actually parallel, unless told.

Angles in polygons

Theorem 2.14 *The sum of the measures of the three angles of any triangle is* $180°$.

Why is this true?

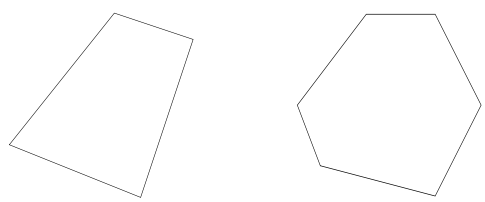


EXAMPLE 17

If two angles of a triangle measure $47°$ and $78°$, what is the measure of the third angle of the triangle?

Theorem 2.15 *The sum of the measures of the angles in any polygon with* n *sides is* $180°(n-2)$.

Why is this true?



EXAMPLE 18

1. Three angles of a pentagon have the same measure. The other two angles are both twice as large as the first three. Find the measure of each angle of the pentagon.

2. $ABCD$ is a quadrilateral such that angle A and angle B have the same measure. Angle C has one-half the measure of angle A, and angle D has measure which is $40°$ less than the measure of angle A. Find the measure of angle C.

Classes of triangles

ADJECTIVE	WHAT THE ADJECTIVE MEANS	EXAMPLE(S)
Right	One of the angles of the triangle is a right angle	
Obtuse	One of the angles of the the triangle is obtuse	
Acute	All 3 angles of the triangle are acute	
Equilateral (a.k.a. regular)	All three sides of the triangle have the same length (equivalently, each angle measures 60°)	
Isoceles	Two sides of the triangle have the same length (equivalently, the two angles opposite those sides have the same measure)	
Scalene	All three sides of the triangle have different lengths (equivalently, all three angles of the triangle have different measures)	

EXAMPLE 19

A right triangle has a 41° angle in it. Find the measures of all three angles of the triangle.

Pythagorean Theorem

In a right triangle, the **hypotenuse** is the longest side (which must be the side opposite the right angle). The other two sides are called **legs**. Right triangles are special because of the following property:

Theorem 2.16 (Pythagorean Theorem) *Suppose ABC is a right triangle, where the right angle is C (and therefore the hypotenuse is c). Then*

$$a^2 + b^2 = c^2.$$

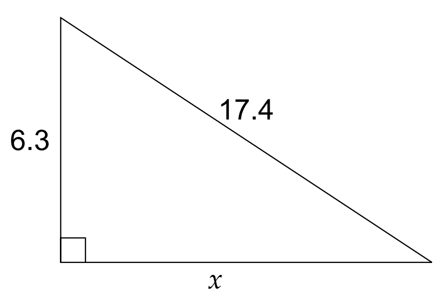
The Pythagorean Theorem does <u>not</u> hold for triangles that are not right triangles.

EXAMPLE 20

If the two legs of a right triangle have lengths 15 and 13, what is the length of the hypotenuse?

EXAMPLE 21

Find x:



2.6 Review material for Exam 1

NOTE: This section (and other sections containing review material) is not meant to be an exact representation of exam material. I always reserve the right to ask (a small number of) questions that use the course material in a creative way.

Exam 1 content

What I will ask you to do without a calculator:

- Tell me whether a given angle is acute, right, obtuse, straight, quadrantal or reflex
- Draw an angle in standard position
- Convert angles from radians to degrees, if the radian measure is $\frac{A\pi}{B}$ with $B = 1, 2, 3, 4, 6$
- Answer questions that use function notation

What I will ask you to do with a calculator:

- Anything you should be able to do without a calculator
- Solve simple equations like those in Chapter 1 of the lecture notes
- Find angles which are coterminal with a given angle
- Find an angle between $0°$ and $360°$ which is coterminal with a given angle
- Find the distance between two points
- Convert between degrees and radians
- Find circumferences, arc lengths and sector areas
- Solve "angle pictures" and story problems that involve angle addition, right and straight angles, complementary and supplementary angles, parallel lines and transversals, angles in a polygon, etc.
- Solve problems involving linear and angular velocity (gears and pulleys, etc.)
- Find a missing side length in a right triangle using the Pythagorean Theorem

Practice problems (no calculator allowed):

1. Draw a picture of an obtuse angle.

2. Is an angle that measures $180°$ an acute, right, obtuse or straight angle?

3. Draw the following angles in standard position:

$$200° \qquad -120° \qquad 270° \qquad \frac{5\pi}{3} \qquad 80° \qquad \pi \qquad \frac{-\pi}{6}$$

4. Convert the following angles from radians to degrees:

$$\frac{3\pi}{4} \qquad 2\pi \qquad \frac{7\pi}{6} \qquad \frac{-\pi}{4} \qquad \frac{-5\pi}{6} \qquad \frac{-\pi}{3} \qquad \frac{\pi}{2}$$

5. Let g be the function defined by $g(x) = x^2$. Compute each quantity:

$$g(2{+}4) \qquad g(2){+}g(4) \qquad g(2){+}4 \qquad 2{+}g(4) \qquad 3g(2) \qquad g(3{\cdot}2) \qquad 2g(3{\cdot}2)$$

6. Let abs denote the function which is defined like this: if $x \geq 0$, then abs $x = 3x$; if $x < 0$, then abs $x = -x$. Compute each quantity:

$$\text{abs}(3) \qquad \text{abs}\ 3 \qquad \text{abs}(-2) \qquad \text{abs}\ 4-5 \qquad \text{abs}(4-5) \qquad \text{abs}\ 4 - \text{abs}\ 5$$

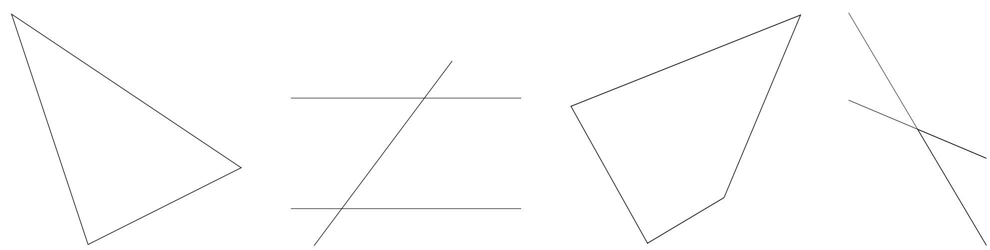
$$\text{abs}\ 7\ \text{abs}\ 1 \qquad \text{abs}\ 7\cdot 1 \qquad \text{abs}\ (7\cdot 1) \qquad 7\ \text{abs}\ 2$$

Practice problems (calculator allowed):

7. Solve for x if $\frac{x}{3.8} = \frac{2.63}{3.21}$.
8. Solve for x if $x^2 + 17.8 = 33.9$.
9. Solve for $\cos x$ if $13^2 = 9^2 + 10^2 - 18(10)\cos x$.
10. Find a negative angle which is coterminal with $251°$.
11. Find three angles coterminal with $1000°$.
12. Find an angle between $0°$ and $360°$ which is coterminal with $2016°$.
13. Find the distance between the points $(2, 6)$ and $(-3, 4)$.
14. Convert $153°$ to radians.
15. Convert 8.52 radians to degrees.
16. Find the length of an arc whose central angle is $125°$, if the radius of the circle is 15 units.
17. The area of a sector measuring $43°$ is 20 square units. What is the radius of the circle from which the sector is taken?
18. A pie with radius 6 inches is cut into 12 equal-sized slices. What is the area of each slice?
19. An angle measures $40°$ less than its supplement. What is the measure of the angle?

20. An angle has eleven times as much measure as its complement. What is the measure of the supplement of the angle?

21. In each picture, find x. (On an exam, you might be asked to find the measure of any angle in the picture.) In the second picture, the two lines which look parallel should be assumed parallel.



22. A mixing blade on a food processor extends out 3 inches from its center. If the blade is turning at 600 revolutions per minute, what is the linear velocity of the tip of the blade?

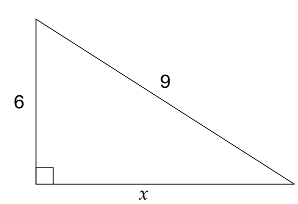
23. A ski lift operates by driving a wire rope, from which chairs are suspended, around two wheels (at the ends of the lift). If the wheel at the bottom of the lift is 15 feet in radius and turns at a rate of 8 revolutions per minute,

    a) how fast is someone sitting in a chair moving up the mountain?

    b) what is the angular velocity of the wheel at the top of the lift, if its radius is 12 feet?

24. If both legs of a right triangle have length 16.3, what is the length of the hypotenuse?

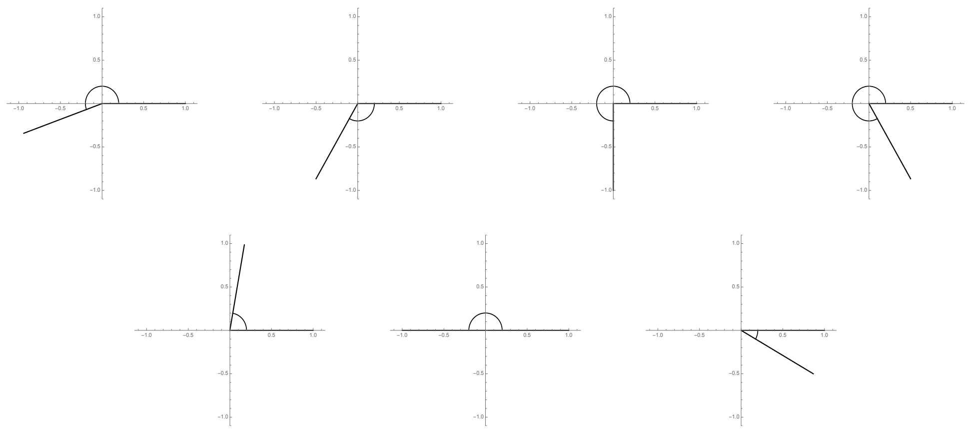
25. Find x:



Solutions to these practice problems

Note: I did these by hand; there may be errors.

1. Any picture of an angle which is more than $90°$ but less than $180°$ is fine:
2. $180°$ is a straight angle.
3.



4. $\frac{3\pi}{4} = 3 \cdot \frac{\pi}{4} = 3 \cdot 45° = 135°$.
   $\frac{7\pi}{6} = 7 \cdot \frac{\pi}{6} = 7 \cdot 30° = 210°$.
   $\frac{-\pi}{4} = -45°$.
   $\frac{-5\pi}{6} = -5 \cdot 30° = -150°$.
   $\frac{-\pi}{3} = -60°$.
   $\frac{\pi}{2} = 90°$.
5. $g(2+4) = g(6) = 6^2 = 36$.
   $g(2) + g(4) = 2^2 + 4^2 = 20$.
   $g(2) + 4 = 2^2 + 4 = 4 + 4 = 8$.
   $2 + g(4) = 2 + 4^2 = 2 + 16 = 18$.
   $3g(2) = 3(2^2) = 12$.
   $g(3 \cdot 2) = g(6) = 6^2 = 36$
   $2g(3 \cdot 2) = 2g(6) = 2(6^2) = 72$.
6. abs$(3) = 3(3) = 9$
   abs $3 = 3(3) = 9$ (abs 3 is the same as abs(3))
   abs$(-2) = -(-2) = 2$

abs $4-5=3(4)-5=7$
abs $(4-5)=$ abs$(-1)=-(-1)=1$
abs $4-$ abs $5=3(4)-3(5)=12-15=-3$
abs 7 abs $1=3(7)\cdot 3(1)=21\cdot 3=63$
abs $7\cdot 1=$ abs $7=3(7)=21$
abs$(7\cdot 1)=$abs $7=3(7)=21$ (this is the same as the previous problem)
7 abs $2=7(3\cdot 2)=42$

7. Cross-multiply, then divide by 3.21 to get $x=\frac{(2.63)(3.8)}{3.21}=3.1134$.
8. Subtract 17.8 from both sides, then take square roots to get $x=\pm\sqrt{4.01248}$.
9. Combine the like terms to get $169=181-180\cos x$. Subtract 181 from both sides, then divide by -180 to get $\cos x=\frac{1}{15}=.06666$.
10. Subtract $360°$ to get $-109°$.
11. Add or subtract $360°$ repeatedly to get $1360°$, $1720°$, $640°$ (other answers possible).
12. First, $\frac{2016°}{360°}=5.6$. Therefore we want $2016°-5\cdot 360°=216°$.
13. $D=\sqrt{(4-6)^2+(-3-2)^2}=\sqrt{4+25}=\sqrt{29}\approx 5.38$.
14. $153°\cdot\frac{\pi}{180°}=2.67$.
15. $8.52\cdot\frac{180°}{\pi}=488.16°$.
16. Make sure to convert the angle to radians: $s=r\theta=15\left(125\cdot\frac{\pi}{180°}\right)=32.72$ units.
17. First convert the angle to radians: $\theta=43°\cdot\frac{\pi}{180°}=.75$. Then use the area of a sector formula $A=\frac{1}{2}r^2\theta$; plug in $\theta=.75$ and $A=20$ to get $20=\frac{1}{2}r^2(.75)$. Solve for r to get $r=\sqrt{\frac{40}{.75}}=7.3$ units.
18. Each slice measures $\frac{2\pi}{12}=.524$ radians. Therefore $A=\frac{1}{2}r^2\theta=\frac{1}{2}(6^2)(.524)=9.43$ square inches.
19. Let the angle be x; we are told $x=(180°-x)-40°$. Solve for x to get $x=70°$.
20. Let the angle be x; we are told $x=11(90°-x)$. Solve for x to get $x=82.5°$. Thus the supplement of x is $180°-x=180°-82.5°=97.5°$.
21. a) The angles in a triangle must add to $180°$: $x+2x+(x+14°)=180°$; solve for x to get $x=41.5°$.

b) The angles must have the same measure: $3x - 20° = x + 42°$; solve for x to get $x = 31°$.

c) The angles in a quadrilateral must add to $360°$: $x+80°+125°+90° = 360°$; solve for x to get $x = 65°$.

d) The angles are supplementary: $(2x + 15°) + (x - 20°) = 180°$; solve for x to get $61.66°$.

22. The angular velocity is $\omega = 600 \cdot 2\pi = 3769.91$ radians per second. The linear velocity is therefore $v = r\omega = 3(3769.91) = 11309.7$ inches per second.

23. The angular velocity of the bottom wheel is $\omega = 8 \cdot 2\pi = 50.26$ radians per minute. The linear velocity of the bottom wheel is therefore $v = r\omega = 15(50.26) = 753.9$ feet per minute. The linear velocity at all points on the lift is the same, so the answer to (a) is 753.9 feet per minute. To answer (b), convert the linear velocity back to angular velocity: $\omega = \frac{v}{r} = \frac{753.9}{12} = 62.82$ radians per minute.

24. Call the length of the hypotenuse c; then use the Pythagorean Theorem:

$$
\begin{aligned}
c^2 &= a^2 + b^2 \\
c^2 &= 16.3^2 + 16.3^2 \\
c^2 &= 265.69 + 265.69 \\
c^2 &= 531.38 \\
c &= \sqrt{531.38} \approx 23.052.
\end{aligned}
$$

25. By the Pythagorean Theorem, we have

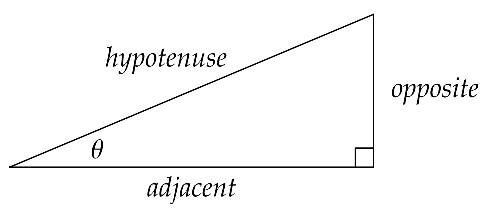
$$
\begin{aligned}
9^2 &= 6^2 + x^2 \\
81 &= 36 + x^2 \\
45 &= x^2 \\
\sqrt{45} &= x \\
6.708 &\approx x.
\end{aligned}
$$

Chapter 3

Sine and cosine

3.1 Sines and cosines of acute angles

Take an angle θ between $0°$ and $90°$. Put θ in a **right** triangle:



Definition 3.1 *Let $0 \leq \theta \leq 90°$. Then define the* **sine**, **cosine** *and* **tangent** *of θ to be:*

$$\sin\theta = \frac{\textit{opposite}}{\textit{hypotenuse}} \qquad \cos\theta = \frac{\textit{adjacent}}{\textit{hypotenuse}} \qquad \tan\theta = \frac{\textit{opposite}}{\textit{adjacent}}$$

These are **functions** of θ and depend only on θ and not on the particular right triangle with θ in it.

To help remember which is which, use the pneumonic device **SOHCAHTOA**:

SOH CAH TOA

It turns out that you can determine $\sin\theta$, $\cos\theta$ and $\tan\theta$ algebraically (see the next section). Typically, you compute sines and cosines with calculators (make sure the mode of the calculator is set correctly between degrees and radians).

EXAMPLE 1

Use a calculator to compute these quantities:

1. $\sin 28°$

2. $\cos 66.35°$

3. $\sin 71°$

   Answer: $.945519$

4. $\cos 25.8°$

   Answer: $.900319$

5. $\sin(15° + 35°)$

6. $\sin 15° + \sin 35°$

Remark: In # 3 above, it is totally correct to write "$\sin 71° = .945519$".

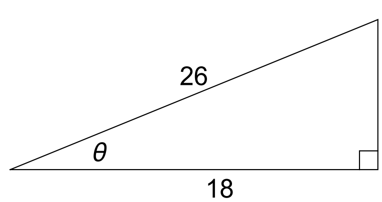
But is is totally **<u>NOT</u>** OK to write "$\sin = .945519$", because $\sin$ is a function, and functions without inputs don't do anything. In this nonsense, the "$\sin$" is what I call "naked". **NAKED TRIG FUNCTIONS LEAD TO DEDUCTIONS.**

EXAMPLE 2

Suppose ABC is a right triangle with sides of lengths a, b and c (where angle C is the right angle). If $a = 6$ and $b = 8$, find the sine and cosine of angle A, and find the sine and cosine of angle B.

EXAMPLE 3

Find the <u>exact</u> values of $\sin\theta$, $\cos\theta$ and $\tan\theta$:



$\sin\theta =$

$\cos\theta =$

$\tan\theta =$

Follow-up question: In the triangle above, find $\sin(90° - \theta)$ and $\cos(90° - \theta)$.

$\sin(90° - \theta) =$

$\cos(90° - \theta) =$

The example on the previous page generalizes:

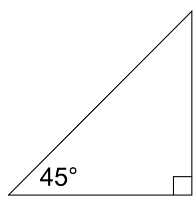
Theorem 3.2 (Cofunction identities) *For any angle θ,*

$$\sin(90^\circ - \theta) = \cos\theta \quad \textit{and} \quad \cos(90^\circ - \theta) = \sin\theta$$

Sines and cosines that you should memorize

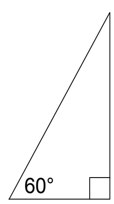
By a wide margin, the most common angles used in the world are 0°, 30°, 45°, 60° and 90°. I'm going to derive the sine, cosine and tangent of these angles below; **you should memorize (or internalize) these answers**:

$45^\circ = \frac{\pi}{4}$:



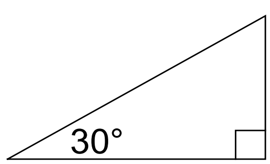
$\Rightarrow \sin 45^\circ =$ $\qquad$ $, \cos 45^\circ =$

$60^\circ = \frac{\pi}{3}$:



$\Rightarrow \sin 60^\circ =$ $\qquad$ $, \cos 60^\circ =$

$30^\circ = \frac{\pi}{6}$:



$\Rightarrow \sin 30^\circ =$ $\qquad$ $, \cos 30^\circ =$

$0^\circ = 0$:

$\Rightarrow \sin 0^\circ =$ $\qquad$ $, \cos 0^\circ =$

$90^\circ = \frac{\pi}{2}$:

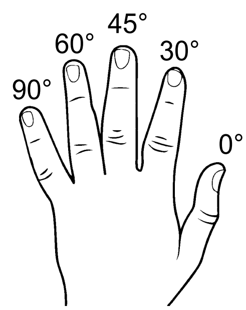
$\Rightarrow \sin 90^\circ =$ $\qquad$ $, \cos 90^\circ =$

Theorem 3.3 (Sines, cosines and tangents of special angles)

θ *in degrees*	θ *in radians*	$\sin\theta$	$\cos\theta$	$\tan\theta$
$0°$	0	0	1	0
$30°$	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$45°$	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$60°$	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$90°$	$\frac{\pi}{2}$	1	0	DNE

How do you remember these?

Either remember the triangles on the previous page, or use this **finger counting trick**:



To compute the sine or cosine of a special angle in Quadrant I:

1. Label the fingers on your left hand as above.
2. Use your right hand to grab the finger corresponding to the angle you want.
3. Count fingers:
   a) For cosine, count the fingers to the **left** of the finger you grabbed (towards $90°$).
   b) For sine, count the fingers to the **right** of the finger you grabbed (towards $0°$).
   c) Never count the finger that you grabbed.
4. The trig function you want is $\dfrac{\sqrt{\text{\# fingers counted in Step 2}}}{2}$.

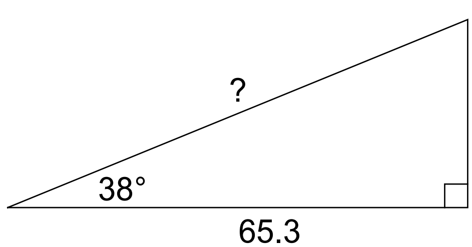
Quick example: $\cos 30° =$

EXAMPLE 4

Try to do these without looking back at the previous page, and without writing any scratch work or tables:

1. $\cos 45°$
2. $\cos \frac{\pi}{2}$
3. $\sin 30°$
4. $\sin 60°$
5. $\sin \frac{\pi}{3}$
6. $\cos \frac{\pi}{4}$
7. $\cos 0$
8. $\sin 90°$

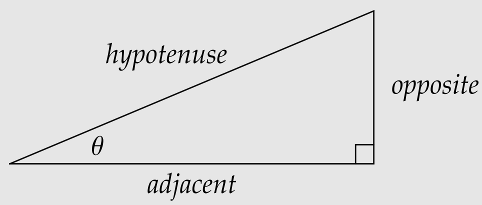
What do we do with sine and cosine? A preview:



3.2 More on sine and cosine

Four different ways to define sine and cosine

Definition 3.4 (Right triangle definition of sine and cosine) *(from the last section) Take an* ***acute*** *angle θ. Put θ in a* ***right*** *triangle:*

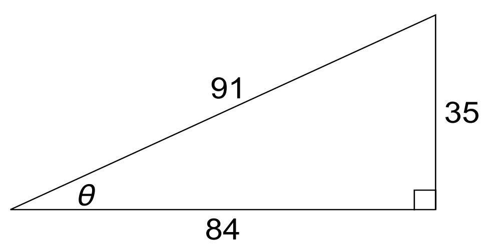


Let $0 \leq \theta \leq 90°$. Then define the **sine** *and* **cosine** *of θ to be:*

$$\sin\theta = \frac{opposite}{hypotenuse} \qquad \cos\theta = \frac{adjacent}{hypotenuse}$$

EXAMPLE 5

Find the exact values of $\sin\theta$ and $\cos\theta$:



Answer:

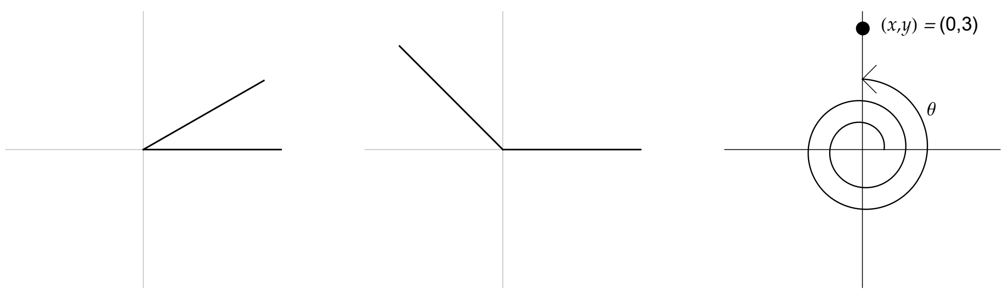
$$\sin\theta = \frac{opp}{hyp} = \frac{35}{91} \quad \text{and} \quad \cos\theta = \frac{adj}{hyp} = \frac{84}{91}.$$

DILEMMA: Definition 3.4 only works for acute angles (because those are the only angles you can put in a right triangle). To define the sine and cosine of angles that aren't between $0°$ and $90°$, we need other approaches:

Definition 3.5 (Angle definition of sine and cosine) *Take an angle θ (in degrees or radians) and draw it in standard position. Choose any point (x, y) on the terminal side of the angle (x and/or y could be positive, negative or zero). Let r be the distance from (x, y) to the origin, so that $r = \sqrt{x^2 + y^2}$ (r is always positive). Then define the* **sine** *and* **cosine** *of θ to be, respectively,*

$$\sin\theta = \sin(\theta) = \frac{y}{r} \quad \textit{and} \quad \cos\theta = \cos(\theta) = \frac{x}{r}.$$

Interestingly, this definition means that sine and cosine sometimes work out to be negative numbers!



Theorem 3.6 *If two angles are coterminal, then they have the same sine and cosine, i.e.*

$$\sin(\theta \pm 360°) = \sin\theta \quad \textit{and} \quad \cos(\theta \pm 360°) = \cos\theta.$$

In radians, this means

$$\sin(\theta \pm 2\pi) = \sin\theta \quad \textit{and} \quad \cos(\theta \pm 2\pi) = \cos\theta.$$

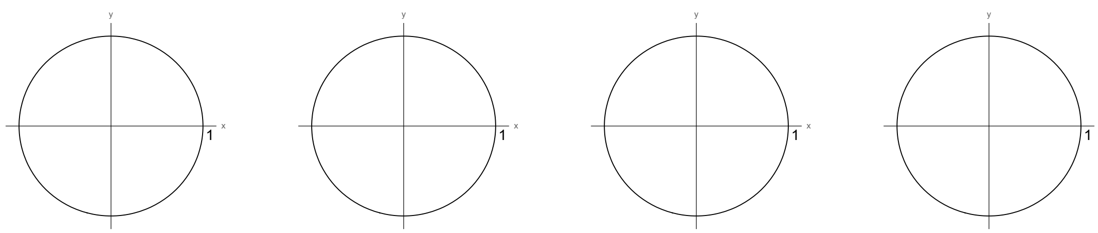
EXAMPLE 6

Suppose the point $(2, -3)$ is on the terminal side of angle θ when drawn in standard position. Find the exact values of $\cos\theta$ and $\sin\theta$ ("exact values" means no decimals):

DILEMMA: the previous definition only defines the sine and cosine of an angle. We also want to be able to take the sine and cosine of a number:

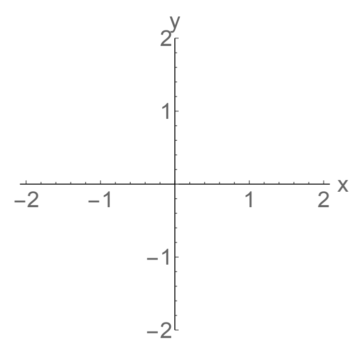
Definition 3.7 (Unit circle definition of sine and cosine) *Take a **number** θ. Starting at the point $(1,0)$, mark off an arc of length θ on the unit circle (go counterclockwise if $\theta > 0$ and clockwise if $\theta < 0$). (In other words, mark off an angle of θ radians with $r = 1$.) Call the point where the arc ends (x, y). Then define the **sine** and **cosine** of θ to be, respectively,*

$$\sin\theta = \sin(\theta) = y \quad \textit{and} \quad \cos\theta = \cos(\theta) = x.$$

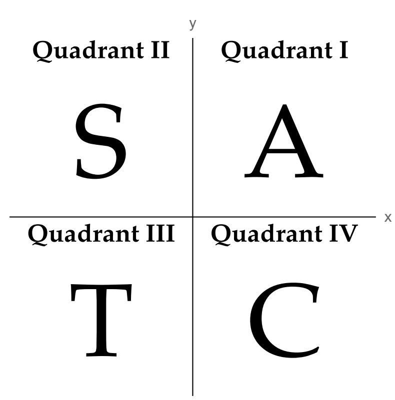


Signs of cosine and sine

Question: For what kinds of angles θ is $\sin\theta$ positive? (What about $\cos\theta$?)



There is a pneumonic device to help you remember this:



EXAMPLE 6

Compute the following quantities without a calculator:

1. $\cos \frac{\pi}{2}$
2. $\sin 270°$
3. $\sin 180°$
4. $\cos 0$
5. $\sin \pi$
6. $\cos 270°$
7. $\sin \frac{3\pi}{2}$
8. $\cos \frac{-3\pi}{2}$

EXAMPLE 7

Determine whether or not the following quantities are positive or negative:

1. $\cos 241°$
2. $\sin 815°$
3. $\cos(-23°)$
4. $\sin^2 1432815°$
5. $-3 \cos 110° \sin 205°$

A calculator will compute the sine and/or cosine of an angle or number (make sure the mode is set correctly in degrees or radians) using the [SIN] and/or [COS] buttons. Your calculator uses the following formula (which you don't need to remember):

Definition 3.8 (Algebra/calculus definition of sine and cosine) *If θ is either a real number or an angle expressed in radians, then*

$$\sin\theta = \theta - \frac{\theta^3}{3\cdot 2\cdot 1} + \frac{\theta^5}{5\cdot 4\cdot 3\cdot 2\cdot 1} - \frac{\theta^7}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} + ...$$

and

$$\cos\theta = 1 - \frac{\theta^2}{2\cdot 1} + \frac{\theta^4}{4\cdot 3\cdot 2\cdot 1} - \frac{\theta^6}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} + ...$$

The four definitions described above all describe the same functions. To solve problems, you move back and forth between the various definitions as necessary.

Order of operations with sine and cosine

EXAMPLE 8

Insert the "imaginary parenthesis" in each of the following expressions:

$\sin 1 + 1$

$\sin 2\theta$

$2\sin\theta$

$\sin -\theta$

$\sin 3^2$

NOTE: As with any function, there are imaginary parenthesis around everything following "$\sin$" or "$\cos$" up to the first addition or subtraction sign, or up until the next function.

Notation for exponents with sine/cosine:

$\sin x^2$ means $\sin(x^2)$, i.e. first square x then take the sine (same with cosine).

$\sin^3 x$ means $(\sin x)^3$, i.e. first take the sine, then cube the answer.

EXAMPLE 9

Compute the following quantities without a calculator:

1. $2\sin 90°$

2. $\sin 2 \cdot 90°$

3. $\cos^2 \frac{\pi}{4}$

4. $3\cos 0$

5. $\cos \pi + \pi$

6. $\cos(\pi + \pi)$

7. $4\sin^4 \frac{\pi}{3}$

The Pythagorean trig identity

We know that if (x, y) lies on the terminal side of angle θ, then

$$r = \sqrt{x^2 + y^2} \qquad \cos\theta = \frac{x}{r} \qquad \sin\theta = \frac{y}{r}$$

Suppose we compute $\cos^2\theta + \sin^2\theta$:

$$\begin{aligned} \cos^2\theta + \sin^2\theta &= \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 \\ &= \frac{x^2}{r^2} + \frac{y^2}{r^2} \\ &= \frac{x^2 + y^2}{r^2} \\ &= \frac{r^2}{r^2} = 1. \end{aligned}$$

We have proven the following important theorem:

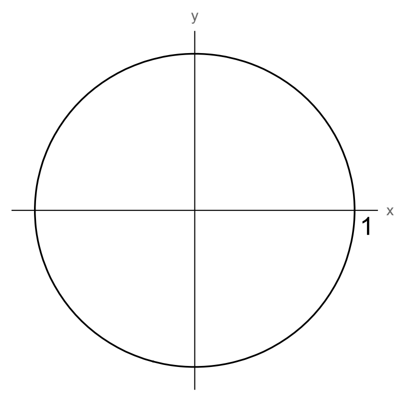
Theorem 3.9 (Pythagorean identity) *For any angle θ,*

$$\cos^2\theta + \sin^2\theta = 1.$$

This identity is useful because it (up to the sign of the answer) tells you how to find $\sin\theta$ from $\cos\theta$ and vice-versa.

EXAMPLE 10

Suppose $\sin\theta = \frac{4}{9}$. Find all possible values of $\cos\theta$. (Let's find exact values hee without using a calculator.)



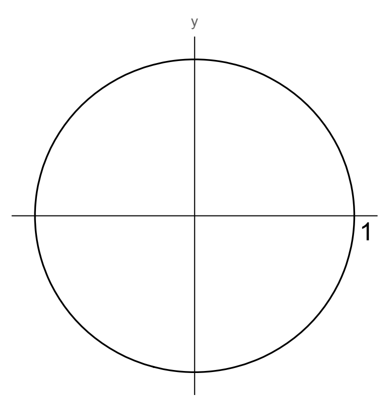
EXAMPLE 11

Suppose $\cos\theta = .35$ and $\sin\theta > 0$. Find $\sin\theta$.

Solution: First, we know $\cos^2\theta + \sin^2\theta = 1$. That means

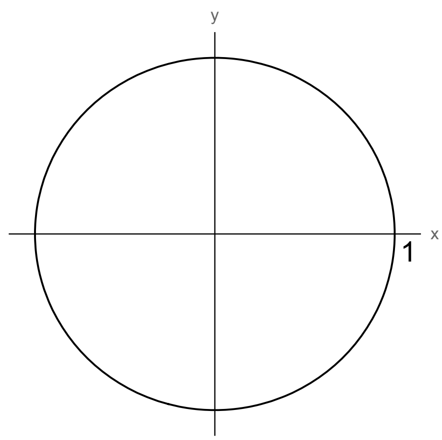
$$\begin{aligned}
(.35)^2 + \sin^2\theta &= 1\\
.1225 + \sin^2\theta &= 1\\
\sin^2\theta &= 1 - .1225\\
\sin^2\theta &= .8775\\
(\sin\theta)^2 &= .8775\\
\sin\theta &= \pm\sqrt{.8775}\\
\sin\theta &= .93675
\end{aligned}$$

(choose the positive square root since we are given that $\sin\theta > 0$).



Odd-even identities

Consider angles θ and $-\theta$. Suppose you computed the sine and cosine of these angles using the unit circle definition:



Theorem 3.10 (Odd-even identities) *For any angle or number θ,*

$$\cos(-\theta) = \cos\theta \qquad \textit{and} \qquad \sin(-\theta) = -\sin\theta.$$

EXAMPLE 12

1. Suppose θ is some angle such that $\cos\theta = \frac{1}{5}$. Find $\cos(-\theta)$.
2. Suppose θ is some angle such that $\sin\theta = \frac{-2}{3}$. Find $\sin(-\theta)$.

Computing inverse sine and cosine with a calculator

Motivating question: Suppose you know what $\sin\theta$ is or what $\cos\theta$ is. Can you find θ? If so, how?

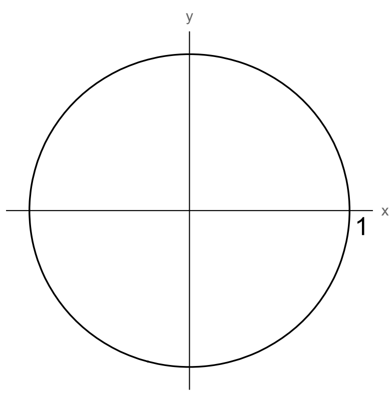
Answers:

EXAMPLE 13

Find an angle θ between $0°$ and $90°$ such that $\sin\theta = .6684$.

EXAMPLE 13 (VERSION 2)

Find all angles between $0°$ and $360°$ such that $\sin\theta = .6684$.

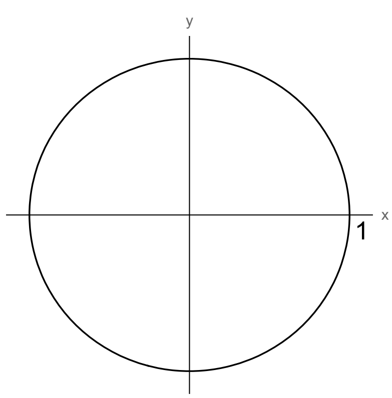


EXAMPLE 14

Find an angle θ between $0°$ and $180°$ such that $\cos\theta = -.3802$.

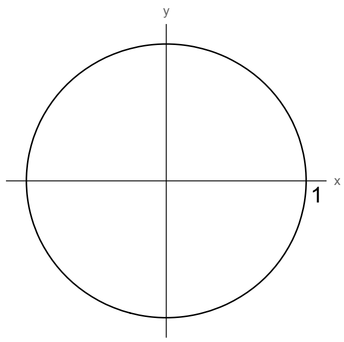
EXAMPLE 14 (VERSION 2)

Find all angles between $0°$ and $360°$ such that $\cos\theta = -.3802$.

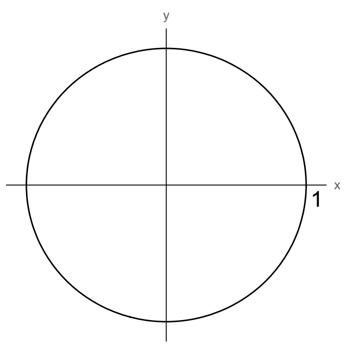


More on this

Suppose you are trying to solve $\sin\theta = q$. This means you are looking for a point on the unit circle whose ________________ equals q.



Now suppose you are trying to solve $\cos\theta = q$. This means you are looking for a point on the unit circle whose ________________ equals q.



To summarize:

Theorem 3.11 (The equation $\sin\theta = q$) *Let q be a number.*

- *If $q < -1$ or $q > 1$, then the equation $\sin\theta = q$ has no solution.*
- *If $q = -1$, the equation $\sin\theta = q$ has one solution between $0°$ and $360°$: $\theta = 270°$.*
- *If $q = 1$, the equation $\sin\theta = q$ has one solution between $0°$ and $360°$: $\theta = 90°$.*
- *If $-1 < q < 1$, the equation $\sin\theta = q$ has two solutions between $0°$ and $360°$. The first is obtained by typing* [SIN^{-1}] *q on the calculator; the other is $180°-$(the first answer).*

Theorem 3.12 (The equation $\cos\theta = q$) *Let q be a number.*

- *If $q < -1$ or $q > 1$, then the equation $\cos\theta = q$ has no solution.*
- *If $q = -1$, the equation $\cos\theta = q$ has one solution between $0°$ and $360°$: $\theta = 0°$.*
- *If $q = 1$, the equation $\cos\theta = q$ has one solution between $0°$ and $360°$: $\theta = 180°$.*
- *If $-1 < q < 1$, the equation $\cos\theta = q$ has two solutions between $0°$ and $360°$. The first is obtained by typing* [COS^{-1}] *q on the calculator; the other is $360°-$(the first answer).*

3.3 The meaning of sine and cosine

The essence of the cosine and sine functions is that they convert between angles and lengths in the context of the unit circle.

To understand how to solve problems with cosine and sine like those we have been doing, and to understand **why** those problems work the way that they do, we're going to practice drawing a picture that explain what a basic trig problem is asking.

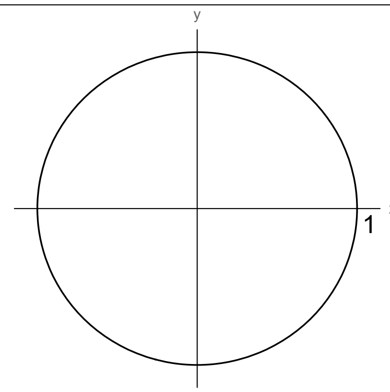
What I hope you find is that by drawing pictures, you can solve problems which are substantially more complicated than the simple ones we have already done.

Directions in the following examples:

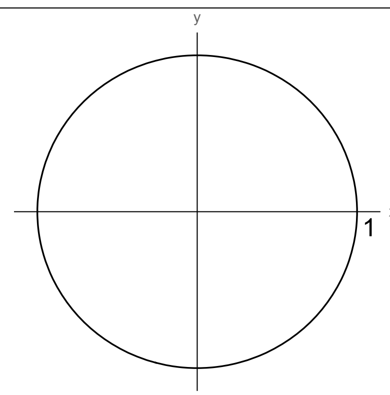
1. In the first column, evaluate the given quantity or solve the problem.
2. Indicate on the given unit circle what the problem is asking you to compute. In particular:
    - label any given quantities of the problem in your picture;
    - indicate with a "?" the quantity you are asked to find in the problem;
    - if θ is neither given nor the quantity you are asked to find, label where θ is in the picture.

EXAMPLE 15

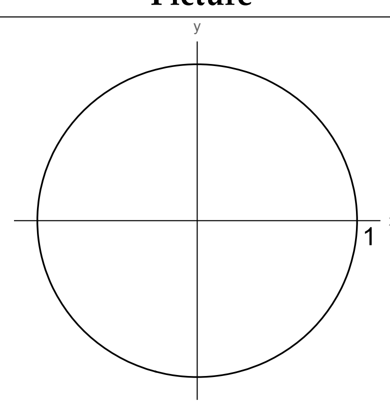
1. Compute $\sin 100°$.

Solution	Picture
	

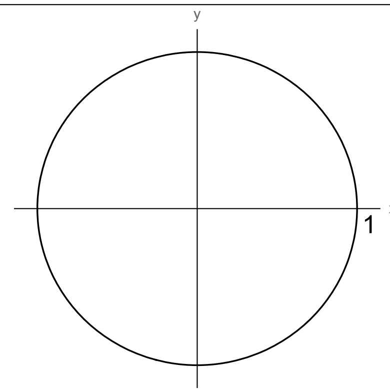
2. Compute $\cos 4372°$.

Solution	Picture
	

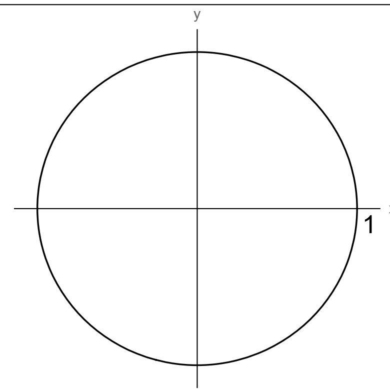
3. Compute $\cos(-35°)$.

Solution	Picture
	

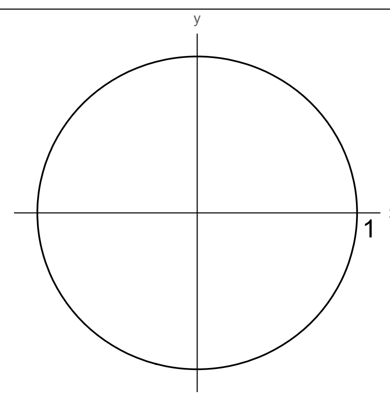
4. Compute $-\cos 35°$.

Solution	Picture
	

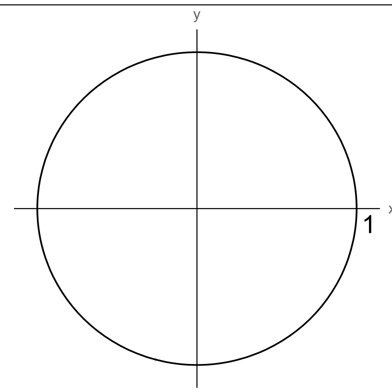
5. Compute $2\sin 100°$.

Solution	Picture
	

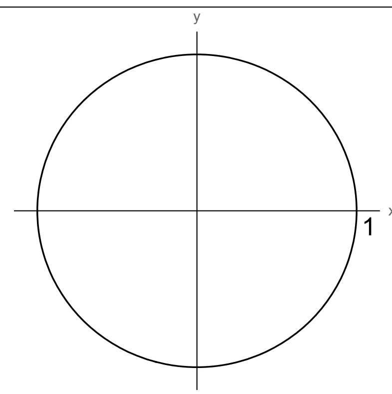
6. Compute $\sin 2 \cdot 100°$.

Solution	Picture
	

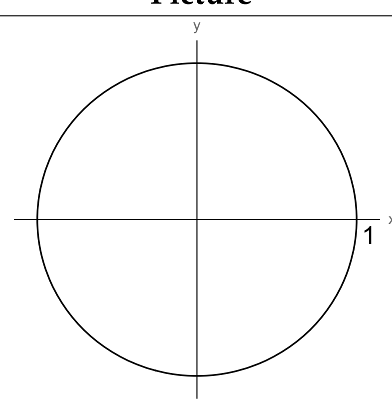
7. Find all values of θ between $0°$ and $360°$, if $\sin\theta = -.3$.

Solution	**Picture**
	

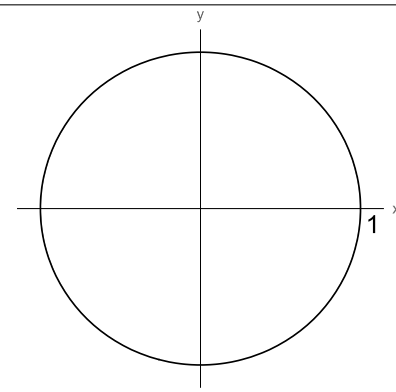
8. Find all values of θ between $0°$ and $360°$, if $\cos\theta = 1$.

Solution	**Picture**
	

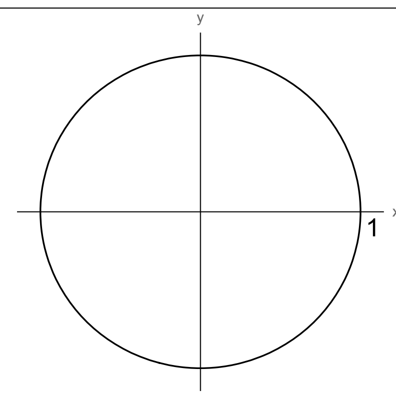
9. Find all values of θ between $0°$ and $360°$, if $\cos\theta = -1.35$.

Solution	**Picture**
	

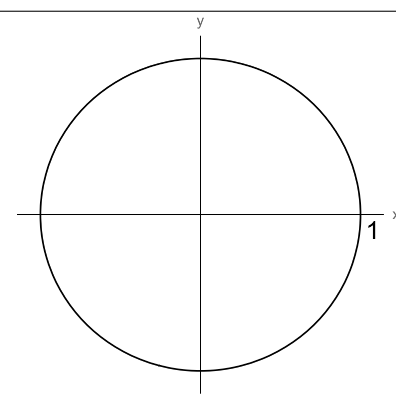
10. Find all values of $\cos\theta$, if $\sin\theta = -.7$.

Solution	Picture
	

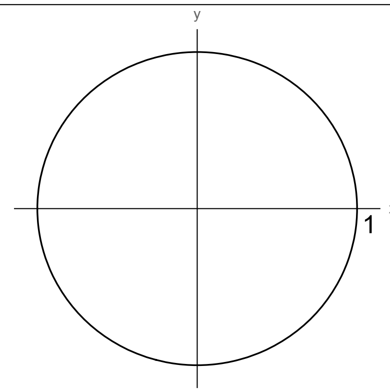
11. Find $\sin\theta$, if $\cos\theta = .95$ and $\sin\theta < 0$.

Solution	Picture
	

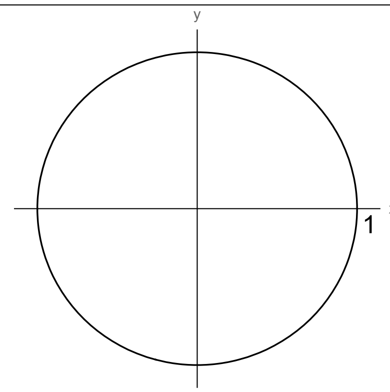
12. Find $\cos(-\theta)$, if $\cos\theta = \frac{3}{4}$.

Solution	Picture
	

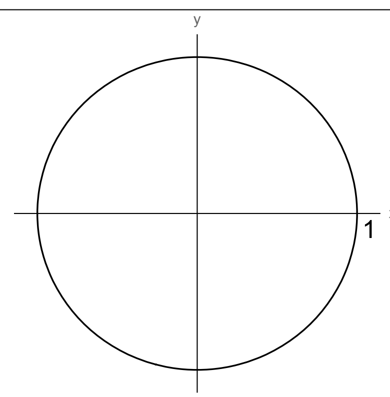
13. Find $\sin(-\theta)$, if $\sin\theta = \frac{2}{5}$.

Solution	Picture
	

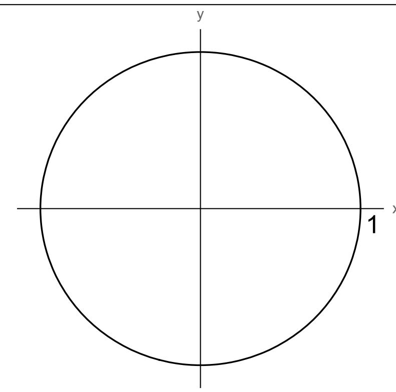
14. Find $\sin(90^\circ - \theta)$, if $\cos\theta = .15$ and $\sin\theta > 0$.

Solution	Picture
	

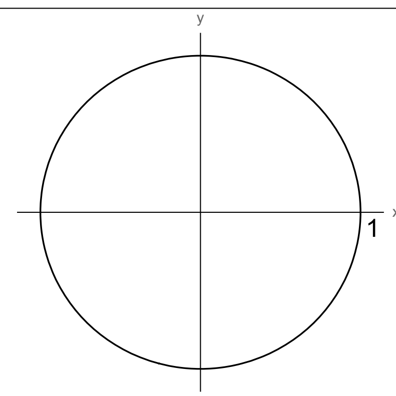
15. Find $\cos(360^\circ + \theta)$, if $\cos\theta = -.425$.

Solution	Picture
	

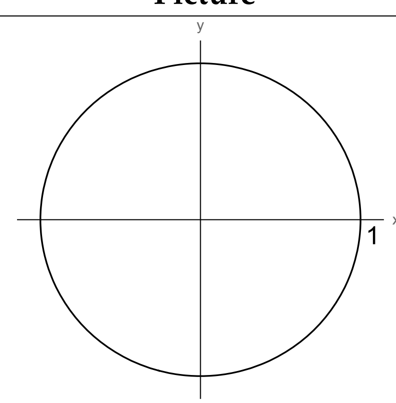
16. Find $\sin(\theta - 720°)$, if $\sin\theta = \frac{1}{3}$.

Solution	Picture
	

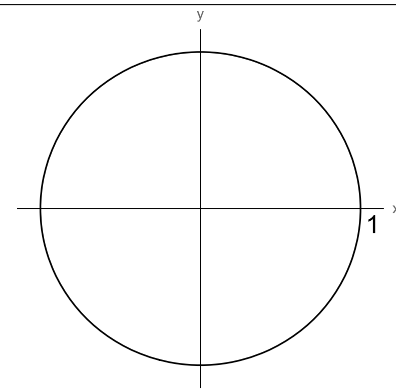
17. Find $\sin(\theta + 180°)$, if $\sin\theta = \frac{1}{5}$.

Solution	Picture
	

18. Find $\cos(180° - \theta)$, if $\cos\theta = \frac{2}{3}$ and $\sin\theta > 0$.

Solution	Picture
	

19. Find $\sin(90° - \theta)$, if $\cos\theta = .15$ and $\sin\theta > 0$.

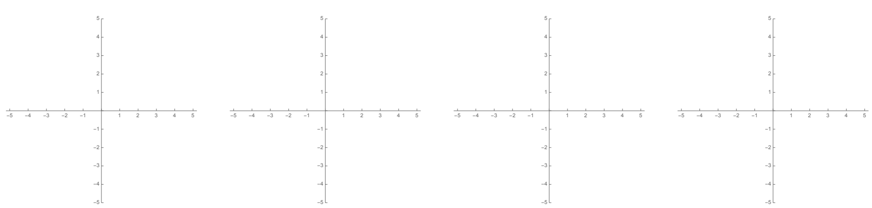
Solution	Picture
	

Symmetry and reference angles

The last three examples above illustrate principles involving trigonometry and symmetry.

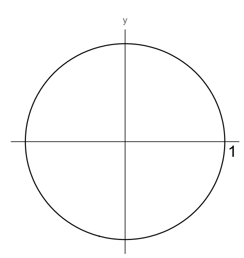
> **Definition 3.13** *Two angles are called* **symmetric** *if, when you draw them both in standard position, you can get from the terminal side of one to the terminal side of the other by reflecting across the x-axis and/or y-axis.*

Put another way, two symmetric angles either have the same slope, or their slopes are opposites.



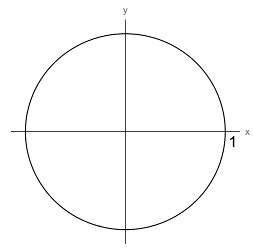
EXAMPLE 16

Find an angle in the second quadrant which is symmetric with $24°$.



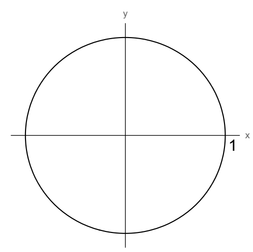
EXAMPLE 17

Find four angles between $0°$ and $360°$ which are symmetric with $160°$.

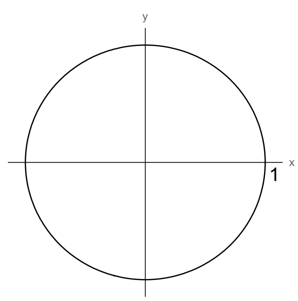


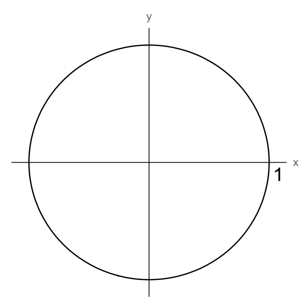
EXAMPLE 18

Suppose $0° < \theta < 90°$ and that (x, y) is the point on the unit circle at θ. What are the coordinates of the point on the unit circle in Quadrant IV, at an angle symmetric with θ?



Question: In general, what angles are symmetric with θ?





Theorem 3.14 *Let θ be an angle, and let $(x, y) = (\cos\theta, \sin\theta)$ be the coordinates of the point on the unit circle at angle θ. Then, the four different angles symmetric with θ, and the coordinates of the point on the unit circle at those angles are:*

$$\begin{array}{rcll}
\theta & \longleftrightarrow & (x, y) & \\
180° - \theta & \longleftrightarrow & (-x, y) & \textit{(reflect } \theta \textit{ across the } y\textit{-axis)} \\
\theta \pm 180° & \longleftrightarrow & (-x, -y) & \textit{(reflect } \theta \textit{ across both axes)} \\
360° - \theta & \longleftrightarrow & (x, -y) & \textit{(reflect } \theta \textit{ across the } x\textit{-axis).}
\end{array}$$

Theorem 3.15 *If angles θ_1 and θ_2 are symmetric, we know*

$$\sin\theta_1 = \pm\sin\theta_2 \quad \textit{and} \quad \cos\theta_1 = \pm\cos\theta_2.$$

Reference angles

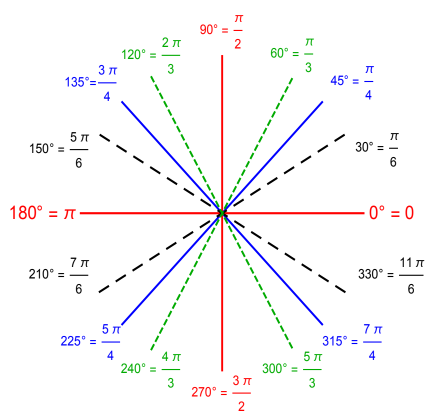
Every angle is symmetric with exactly one angle in the first quadrant.

Definition 3.16 *Given angle θ, the* **reference angle** *of θ is the angle θ' in the first quadrant (i.e. $0^\circ \le \theta' \le 90^\circ$) which is symmetric with θ.*

EXAMPLE 17

Find the reference angle of 193°.

Reference angles of special angles are easy to find, if the angles are in radians:



Theorem 3.17 *If θ is in radians, expressed as a fraction $\frac{a\pi}{b}$ where $\frac{a}{b}$* ***is in lowest terms*** *and $b = 2, 3, 4,$ or 6, then the reference angle of θ is $\frac{\pi}{b}$.*

Here is why reference angles are useful:

Theorem 3.18 *If θ' is the reference angle of θ, are symmetric, we know*

$$\sin\theta = \pm\sin\theta' \quad \textit{and} \quad \cos\theta = \pm\cos\theta'.$$

3.4 Computing sines and cosines of special angles

A "special angle" is any multiple of $30°$ or any multiple of $45°$, i.e. angles like

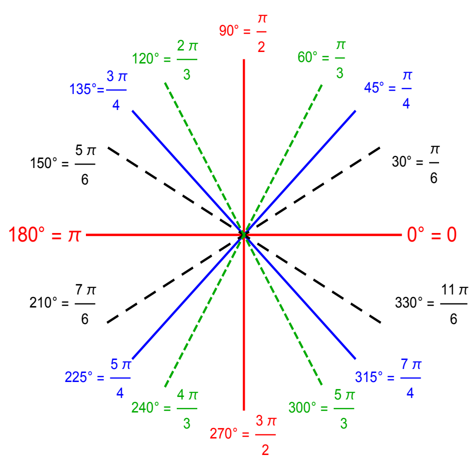
$$0°, 30°, 60°, 90°, 120°, 150°, -30°, -60°, -90°, -120°, -150°, \ldots$$

$$45°, 90°, 135°, 180°, 225°, -45°, -90°, -135°, -180°, -225°, \ldots$$

In radians, these special angles are any multiple of $\frac{\pi}{6}$, any multiple of $\frac{\pi}{4}$, any multiple of $\frac{\pi}{3}$, any multiple of $\frac{\pi}{2}$ or any multiple of π:

$$\frac{\pi}{6}, \frac{5\pi}{6}, \frac{-\pi}{6}, \frac{-5\pi}{6}, \frac{7\pi}{6}, \frac{-7\pi}{6}, \frac{\pi}{3}, \frac{-\pi}{3}, \frac{3\pi}{4}, \frac{-5\pi}{4}, \frac{3\pi}{2}, \frac{-5\pi}{2}, 0, \pi, -2\pi, 3\pi, \ldots$$

These are the angles you get when you divide a right angle into halves or thirds.



The goal of this section is to learn how to compute the exact values of the sine and/or cosine of any special angle **quickly** and **without using a calculator**.

Here's how we do this:

- If the angle θ is quadrantal (i.e. a multiple of $90°$), we find the point on the unit circle at θ. The x-coordinate of this point is $\cos\theta$, and the y-coordinate of this point is $\sin\theta$.
- If the angle θ is not quadrantal, we
  1. use the finger counting trick to determine $\sin\theta$ and/or $\cos\theta$ for angles in the first quadrant;
  2. and use symmetry / reference angles (together with the finger counting trick) for other angles.

STEPS	EXAMPLE A $\sin 240°$	EXAMPLE B $\cos \frac{-7\pi}{4}$	EXAMPLE C $\sin 900°$
1. Is θ quadrantal? (If necessary, convert (θ to degrees, and add or subtract $360°$ to figure this out.)			
2. If θ is quadrantal, find point on unit circle at angle θ. (This point must be either $(0, \pm 1)$ or $(\pm 1, 0)$.) Then read off answer: $\cos\theta = x$; $\sin\theta = y$.			
3. If θ is not quadrantal, determine two items: **First**, the quadrant θ lies in. (If necessary, convert θ to degrees, and add or subtract $360°$ to figure this out.) Use the ASTC rules to determine the sign of your final answer. **Second**, the reference angle of θ. It is: $30°$, if $\theta = \frac{N\pi}{6}$ or if θ is "shallow"; $45°$, if $\theta = \frac{N\pi}{4}$ or if θ is "diagonal"; $60°$, if $\theta = \frac{N\pi}{3}$ or if θ is "steep"; Take $\sin$ or $\cos$ of the reference angle using the finger-counting trick, and use the sign from earlier to get the final answer.			

Problem	Is θ quadrantal?	**If θ is quadrantal:** Find point on unit circle at θ	Write answer	**If θ is not quadrantal:** Find quadrant	Sign of answer	Reference angle	$\sin$ or $\cos$ of ref. angle	Write answer
$\sin \frac{-11\pi}{6}$								
$\cos \frac{3\pi}{2}$								
$\sin 300^\circ$								
$\cos \frac{5\pi}{4}$								
$\cos 720^\circ$								

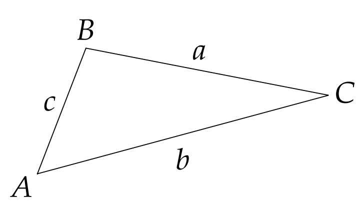
Chapter 4

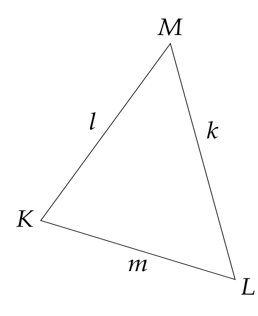
Solving triangles

To **solve** a triangle means to find the lengths of all three of its sides, and the measures of all three of its angles.

Notation

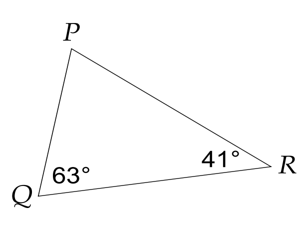
We will label the vertices of a triangle with capital letters and the side lengths with lowercase letters. As always, the side labelled with a certain lowercase letter will always be opposite the angle with the same capital letter.





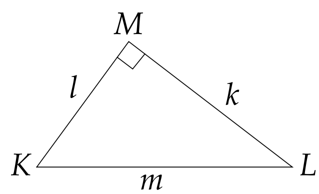
Always keep in mind

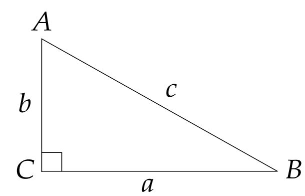
Whenever you know two angles of a triangle, you can figure the third, because the sum of the three angles in a triangle is $180°$.



4.1 Solving right triangles

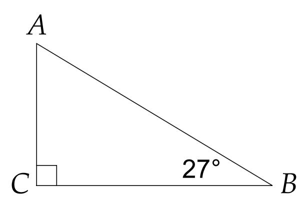
In this section, we discuss solving right triangles. In a right triangle, if you use letters ABC, C should be the right angle (in general, try to make the last letter alphabetically the right angle).



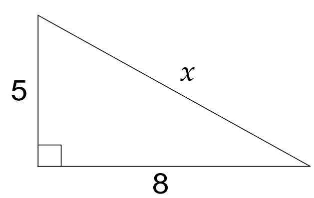


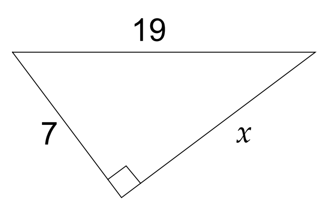
Keep these two things in mind when solving right triangles

(1) In a **right** triangle, if one of the acute angles measures θ, then the other one measures $90° - \theta$.



(2) In a **right** triangle, whenever you know the lengths of two sides of the (right) triangle, you can figure the length of the third, using the Pythagorean Theorem.





Situation # 1: You are given the lengths of two sides of the right triangle

In this situation, find the length of the third side using the Pythagorean Theorem, and then use inverse sine or inverse cosine to find the measure of an angle.

EXAMPLE 1

Solve right triangle ABC, if $a = 28.5$ and $b = 22.3$.

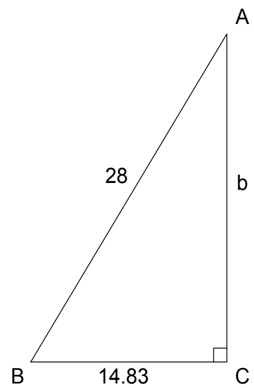
$\angle A =$	$\angle B =$	$\angle C =$
$a = 28.5$	$b = 22.3$	$c =$

Solution:

EXAMPLE 2

Solve right triangle ABC, if $a = 14.83$ and $c = 28$.

Solution: First, draw a picture:



Second, solve for b:

$14.83^2 + b^2 = 28^2$ (Pythagorean Thm)

$219.929 + b^2 = 784$

$b^2 = 564.071$

$b = \sqrt{564.071} = 23.75.$

Third, find angle B:

$\cos B = \frac{adj}{hyp} = \frac{14.83}{28} = .529643.$

$B = \cos^{-1}(.529643) = 58.02°.$

Last, find angle A:

$A = 90° - B = 90° - 58.02° = 31.98°.$

$\angle A = 31.98°$	$\angle B = 58.02°$	$\angle C = 90°$
$a = 14.83$	$b = 23.75$	$c = 28$

Situation # 2: You are given the length of one side of the triangle, and the measure of one of the angles

In this situation, write an equation connecting the given side length, the length of some other unknown side and the sine or cosine of the given angle. Use this to find a second side length.

EXAMPLE 3

Solve right triangle ABC, if $a = 14$ and $\angle A = 27°$.

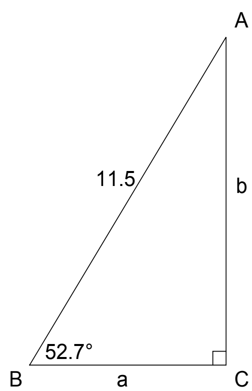
$\angle A = 27°$	$\angle B =$	$\angle C =$
$a = 14$	$b =$	$c =$

Solution:

EXAMPLE 4

Solve right triangle ABC, if $c = 11.5$ and $\angle B = 52.7°$.

Solution: First, draw a picture:



Second, find angle A:

$A = 90° - 52.7° = 37.3°$.

Next, solve for a (you could do b first):

$\cos 52.7° = \frac{adj}{hyp} = \frac{a}{11.5}$

$.605988 = \frac{a}{11.5}$

$11.5(.605988) = a$

$6.96887 = a$.

Last, find the remaining side (in our case, b):

$\sin 52.7° = \frac{opp}{hyp} = \frac{b}{11.5}$.

$.795473 = \frac{b}{11.5}$

$11.5(.795473) = b$

$9.14795 = b$.

$\angle A = 37.3°$	$\angle B = 52.7°$	$\angle C = 90°$
$a = 6.97$	$b = 9.15$	$c = 11.5$

Example 5

Solve right triangle ABC, if $b = 200$ and $\angle A = 40°$.

Solution: First, draw a picture.

Second, $\angle B = 90° - 40° = 50°$.

Third, find c using cosine:

$$\begin{aligned} \cos 40° &= \frac{200}{c} \\ .766044 &= \frac{200}{c} \\ .766044c &= 200 \\ c &= \frac{200}{.766044} = 261.081 \end{aligned}$$

Last, find a, using the Pythagorean Theorem:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ a^2 + 200^2 &= (261.081)^2 \\ a^2 + 40000 &= 68163.5 \\ a^2 &= 28163.5 \\ a &= \sqrt{21863.5} = 167.82. \end{aligned}$$

$\angle A = 40°$	$\angle B = 50°$	$\angle C = 90°$
$a = 167.82$	$b = 200$	$c = 261.08$

Story problems involving right triangles

In this situation, draw a picture and fill in the angles and/or side lengths that you know. Then proceed as on the previous pages.

Vocabulary: angle of elevation is an angle measured upward from the horizontal; angle of depression is an angle measured downward from the horizontal.

EXAMPLE 6

A plane is flying at an altitude of 12,500 feet. The pilot notices that her angle of depression to a radar tower is $25°$. How far is the plane from the radar tower?

EXAMPLE 7

A person walks 100 feet away from the base of the building, turns around and notices that the angle of elevation from their eyes to the top of the building is $72°$. If the person's eyes are 6 feet above the ground, how tall is the building?

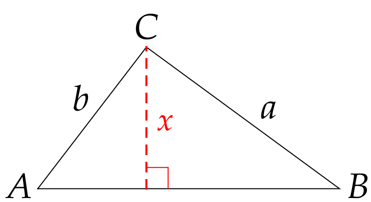
4.2 Solving arbitrary triangles: the Law of Sines

We now study rules which allow us to solve **any** triangle (whether or not it is a right triangle).

Theorem 4.1 (Law of Sines) *In any triangle with vertices labelled A, B and C and respective sides labelled a, b and c,*

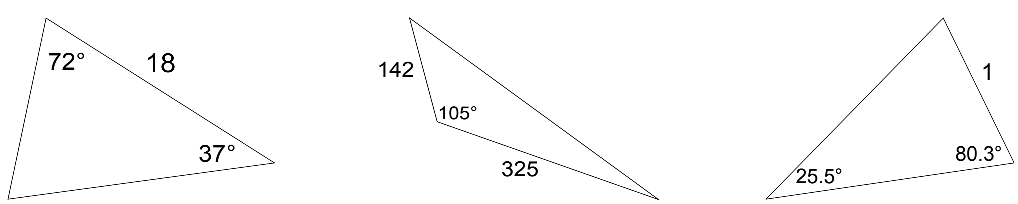
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

Why is this law true? In the diagram below, compute x two different ways:



Law of Sines Situation # 1: ASA

In this situation, you know two angles and the side length between the two known angles:



EXAMPLE 8

Solve triangle ABC, if $\angle A = 41°$, $\angle B = 66°$ and $c = 15.3$.

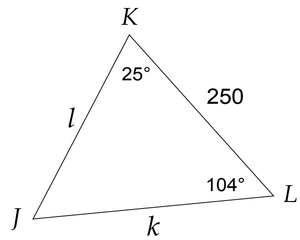
$\angle A \quad = 41°$	$\angle B \quad = 66°$	$\angle C \quad =$
$a \quad =$	$b \quad =$	$c \quad = 15.3$

Solution:

EXAMPLE 9

Solve triangle JKL, if $\angle K = 25°$, $\angle L = 104°$ and $j = 250$.

Solution: First, draw a picture (it is not important that the picture shows the angles or lengths accurately):



Second, find the third angle: $\angle J = 180° - 25° - 104° = 51°$.

Third, find side k using the Law of Sines (you could do l first):

$$\begin{aligned}
\frac{\sin K}{k} &= \frac{\sin J}{j} \\
\frac{\sin 25°}{k} &= \frac{\sin 51°}{250} \\
\frac{.422618}{k} &= \frac{.777146}{250} \quad \text{(cross-multiply to get the next line)} \\
.777146k &= 105.655 \\
k &= \frac{105.655}{.777146} = 135.95.
\end{aligned}$$

Last, find side l, again using the Law of Sines:

$$\begin{aligned}
\frac{\sin L}{l} &= \frac{\sin J}{j} \\
\frac{\sin 104°}{l} &= \frac{\sin 51°}{250} \\
\frac{.970296}{l} &= \frac{.777146}{250} \quad \text{(cross-multiply to get the next line)} \\
.777146l &= 242.574 \\
l &= \frac{242.574}{.777146} = 312.13.
\end{aligned}$$

WARNING: Do not use the Pythagorean Theorem. This is not a right triangle.

$\angle J = 51°$	$\angle K = 25°$	$\angle L = 104°$
$j = 250$	$k = 135.95$	$l = 312.13$

Law of Sines Situation # 2: AAS

In this situation, you know two angles and a side length which is not between the two known angles:

EXAMPLE 10

Solve triangle ABC if $\angle A = 71.3°$, $\angle B = 58.8°$ and $a = 2.75$.

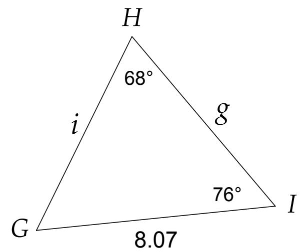
$\angle A = 71.3°$	$\angle B = 58.8°$	$\angle C =$
$a = 2.75$	$b =$	$c =$

Solution:

EXAMPLE 11

Solve triangle GHI, if $\angle H = 68°$, $\angle I = 76°$ and $h = 8.07$.

Solution: First, draw a picture (it is not important that the picture shows the angles or lengths accurately):



Second, find the third angle: $\angle G = 180° - 68° - 76° = 36°$.

Third, find side g using the Law of Sines (you could do i first):

$$\begin{aligned}
\frac{\sin G}{g} &= \frac{\sin H}{h} \\
\frac{\sin 36°}{g} &= \frac{\sin 68°}{8.07} \\
\frac{.587785}{g} &= \frac{.927184}{8.07} \\
.927184g &= 4.74343 \\
g &= \frac{4.74343}{.927184} = 5.11595.
\end{aligned}$$

Last, find side i, again using the Law of Sines:

$$\begin{aligned}
\frac{\sin I}{i} &= \frac{\sin H}{h} \\
\frac{\sin 76°}{i} &= \frac{\sin 68°}{8.07} \\
\frac{.970296}{i} &= \frac{.927184}{8.07} \\
.927184i &= 7.83029 \\
i &= \frac{7.83029}{.927184} = 8.44523.
\end{aligned}$$

$\angle G = 36°$	$\angle H = 68°$	$\angle I = 76°$
$g = 5.12$	$h = 8.07$	$i = 8.45$

Law of Sines Situation # 3: SSA

In this situation, you know two side lengths and an angle which is not between the two known side lengths. ***This situation is hard*** *and is called the ambiguous case of the Law of Sines.*

As a warmup, recall how you solve the equation $\sin\theta = q$:

Theorem 4.2 (Solving $\sin\theta = q$) *Let q be a constant and consider the equation $\sin\theta = q$.*

1. *If $q = 1$, then the equation $\sin\theta = q$ has one solution between $0°$ and $360°$: $q = 90°$.*
2. *If $q = -1$, then the equation $\sin\theta = q$ has one solution between $0°$ and $360°$: $q = 270°$.*
3. *If $-1 < q < 1$, then the equation $\sin\theta = q$ has two solutions between $0°$ and $360°$ ($\sin^{-1} q$ button and $180° - \sin^{-1} q$).*
4. *If $q < -1$ or $q > 1$, then the equation $\sin\theta = q$ has no solution.*

EXAMPLES

1. Find all θ between $0°$ and $360°$ such that $\sin\theta = .735$.

2. Find all θ between $0°$ and $360°$ such that $\sin\theta = -.45$.

   Solution: From a calculator, $\sin^{-1}(-.45) = -26.74°$.

   Adding $360°$ to make this between $0°$ and $360°$ gives $-26.74° + 360° = 333.26°$.

   The other solution is $180°$ minus the original answer,

   which is $180° - (-26.74°) = 206.74°$.

3. Find all θ between $0°$ and $360°$ such that $\sin\theta = 1$.

4. Find all θ between $0°$ and $360°$ such that $\sin\theta = 1.578$.

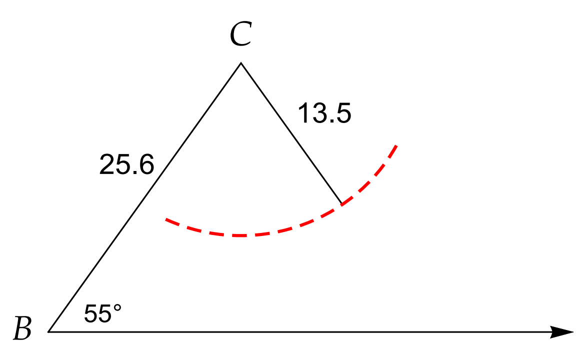
EXAMPLE 12

Solve triangle ABC, if $\angle B = 55°$, $b = 13.5$ and $a = 25.6$.

$\angle A$	$=$	$\angle B$	$= 55°$	$\angle C$	$=$
a	$= 25.6$	b	$= 13.5$	c	$=$

Solution:

What went wrong here (geometrically)?



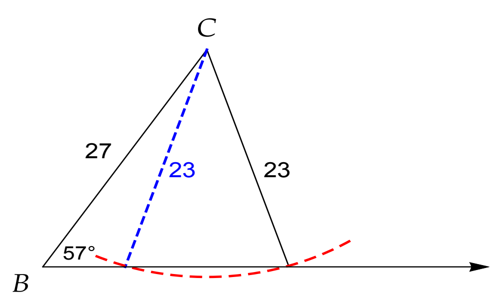
EXAMPLE 13

Solve triangle ABC, if $\angle B = 57°$, $a = 27$ and $b = 23$.

$\angle A =$	$\angle B = 57°$	$\angle C =$
$a = 27$	$b = 23$	$c =$

Solution:

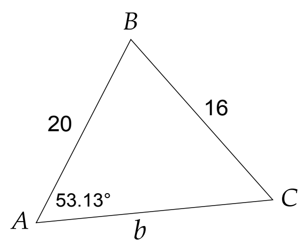
What happened here (geometrically)?



EXAMPLE 14

Solve triangle ABC, if $\angle A = 53.13°$, $c = 20$ and $a = 16$.

Solution: First, draw a picture:



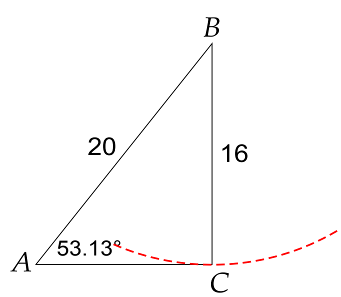
Next, find $\angle C$ using the Law of Sines:

$$\begin{aligned}
\frac{\sin C}{20} &= \frac{\sin 53.13°}{16} \\
\frac{\sin C}{20} &= \frac{.8}{16} \\
16 \sin C &= (.8)20 = 16 \\
\sin C &= 1 \\
C &=
\end{aligned}$$

Notice that this time, you only get one value of C! So there is only one triangle, which after some more work, ends up being

$\angle A = 53.13°$	$\angle B = 36.87°$	$\angle C = 90°$
$a = 16$	$b = 12$	$c = 20$

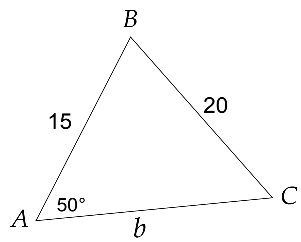
What happened here (geometrically)?



EXAMPLE 15

Solve triangle ABC, if $\angle A = 50°$, $c = 15$ and $a = 20$.

Solution: First, draw a picture:



Next, find $\angle C$ using the Law of Sines:

$$\begin{aligned} \frac{\sin C}{15} &= \frac{\sin 50°}{20} \\ \frac{\sin C}{15} &= \frac{.766}{20} \\ 20 \sin C &= (.766)15 = 11.325 \\ \sin C &= .5745 \\ C &= \sin^{-1}(.5745) \\ C &= 35.1° \text{ and } C = 180° - 35.1° = 144.9° \end{aligned}$$

The situation where $C = 35.1°$ works out to the following triangle:

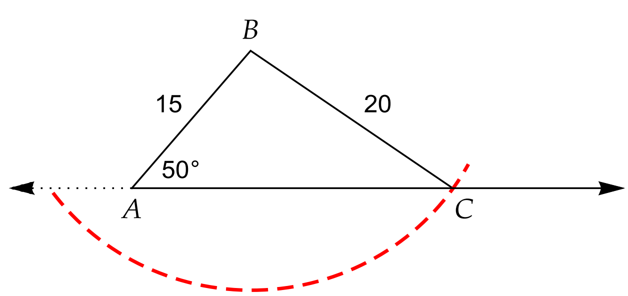
$\angle A = 50°$	$\angle B = 94.9°$	$\angle C = 35.1°$
$a = 20$	$b = 26.01$	$c = 15$

But the situation where $C = 144.9°$ is impossible, because in this case, angle B would be

$$B = 180° - A - C = 180° - 50° - 144.9° = -14.9°.$$

That means there is only one triangle.

The geometric picture:

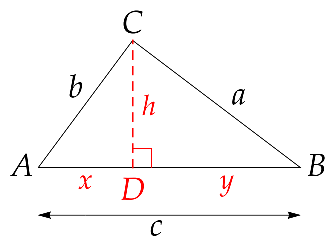


4.3 Solving arbitrary triangles: the Law of Cosines

Theorem 4.3 (Law of Cosines) *In any triangle with vertices labelled A, B and C and respective sides labelled a, b and c,*

$$
\begin{aligned}
c^2 &= a^2 + b^2 - 2ab\cos C \\
b^2 &= a^2 + c^2 - 2ac\cos B \\
a^2 &= b^2 + c^2 - 2bc\cos A
\end{aligned}
$$

Why is this law true? Consider the diagram below:



First, from the right triangle ACD, we see

$$\sin A = \frac{h}{b} \quad \Rightarrow h = b\sin A$$

and

$$\cos A = \frac{x}{b} \quad \Rightarrow \quad x = b\cos A$$

so $y = c - x = c - b\cos A$.

Second, we can apply the Pythagorean Theorem to right triangle BCD to get

$$y^2 + h^2 = a^2$$

Substitute the formulas we found above for y and h to get

$$(c - b\cos A)^2 + (b\sin A)^2 = a^2.$$

FOIL the left-hand side out to get

$$
\begin{aligned}
c^2 - 2bc\cos A + b^2\cos^2 A + b^2\sin A &= a^2 \\
c^2 - 2bc\cos A + b^2(\cos^2 A + \sin^2 A) &= a^2
\end{aligned}
$$

Last, use the Pythagorean Identity $\cos^2 A + \sin^2 A = 1$ to get

$$c^2 - 2bc\cos A + b^2(1) = a^2,$$

which can be rearranged as $a^2 = b^2 + c^2 - 2bc\cos A$, the Law of Cosines.

Law of Cosines Situation # 1: SSS

In this situation, you know all three side lengths of the triangle.

EXAMPLE 16

Solve triangle ABC, if $a = 14$, $b = 22$ and $c = 27$.

$\angle A =$	$\angle B =$	$\angle C =$
$a = 14$	$b = 22$	$c = 27$

Note: With the Law of Cosines, there is never a situation where you can get two triangles. But you might get no possible triangle:

EXAMPLE 17

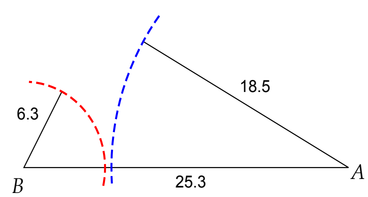
Solve triangle ABC, if $a = 6.3$, $b = 18.5$ and $c = 25.3$.

$\angle A =$	$\angle B =$	$\angle C =$
$a = 6.3$	$b = 18.5$	$c = 25.3$

Solution:

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A \\ 6.3^2 &= 18.5^2 + 25.3^2 - 2(18.5)(25.3)\cos A \\ 39.69 &= 342.25 + 640.09 - 936.1\cos A \\ 39.69 &= 982.34 - 936.1\cos A \\ -942.65 &= -936.1\cos A \\ 1.006 &= \cos A \end{aligned}$$

What went wrong here, geometrically?



Theorem 4.4 (Triangle Inequality) *If a, b and c are the three side lengths of a triangle, then*

$$c \leq a + b \qquad b \leq a + c \qquad a \leq b + c.$$

Put another way, any side length of a triangle is less than or equal to the sum of the other two side lengths.

Law of Cosines Situation # 2: SAS

In this situation, you know two side lengths and the measure of the angle between those two sides.

EXAMPLE 18

A surveyor wants to find the distance between two points A and B on the opposite side of a lake. She goes to a third point on the side of the lake, and figures that her distance to A is 340 meters, and her distance to B is 385 meters. She also notices that the angle between her line of sight to A and her line of sight to B is $163.5°$. What is the distance between A and B?

Summary of procedure to solve triangles

Note: if the triangle is a right triangle, then the triangle can be solved by the techniques of Section 4.1 (SOHCAHTOA and the Pythagorean Theorem, etc.).

Known information in triangle	Procedure for solving the triangle
SSS	Find one angle using the Law of Cosines. Then find a second angle using the Law of Cosines. Last, find the third angle (the angles add to $180°$).
SAS	Find the third side length using the Law of Cosines. Then find a second angle using the Law of Cosines. Last, find the third angle (the angles add to $180°$).
ASA	Find the third angle (the angles add to $180°$). Then use the Law of Sines to find the remaining sides.
AAS	Find the third angle (the angles add to $180°$). Then use the Law of Sines to find the remaining sides.
SSA	Find a second angle using the Law of Sines. **WARNING: ambiguous case** **(there may be zero, one or two triangles possible)** For each second angle you find, compute the third angle (the angles add to $180°$). *If the third angle is negative, throw out that triangle, leaving you with only one triangle.* Last, for each second angle, find the remaining side using either the Law of Sines or the Law of Cosines.
AAA	This is not a doable problem. The triangle could be any size.

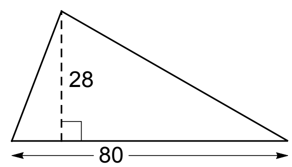
4.4 Area formulas for triangles

Theorem 4.5 (Traditional area formula) *If a triangle has base b and height h, then the area of the triangle is*

$$A = \frac{1}{2}bh.$$

EXAMPLE 19

Find the area of this triangle:



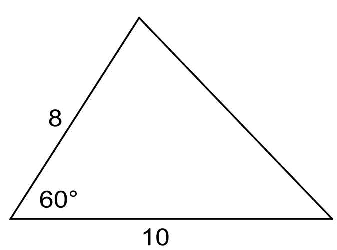
The drawback of the traditional formula is that the h is often unknown. Usually what you know about a triangle is its side lengths and angle measures, which don't directly tell you what h is. Here's a better version of the same formula for the area of a triangle, involving only side lengths and angle measures:

Theorem 4.6 (SAS area formula) *If a triangle has two sides a and b, with angle C between sides a and b, then the area of the triangle is*

$$A = \frac{1}{2}ab\sin C.$$

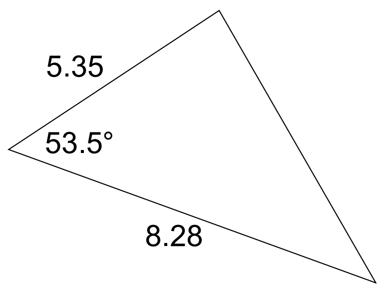
EXAMPLE 20

Find the exact area (no decimals) of this triangle:



Example 21

Find (a decimal approximation of) the area of this triangle:



Example 22

Find (a decimal approximation of) the area of triangle ABC, if $a = 25.7$, $b = 18.3$, $\angle A = 102^\circ$ and $\angle B = 55^\circ$.

The drawback of the SAS area formula is that you have to know an angle measure. Often all you know are the lengths of the sides of a triangle. In this case, you can use the formula on the next page:

Theorem 4.7 (Heron's formula) *Suppose that a triangle has side lengths a, b and c. Define the* **semiperimeter** *of the triangle to be half of its perimeter, i.e.*

$$s = \frac{a+b+c}{2}.$$

Then the area of the triangle is

$$A = \sqrt{s(s-a)(s-b)(s-c)}.$$

Why is this true? First, from rearranging the Law of Cosines, we get

$$\cos C = \frac{a^2+b^2-c^2}{2ab}.$$

Second, from the Pythagorean Identity $\cos^2 C + \sin^2 C = 1$, we get

$$\sin C = \sqrt{1-\cos^2 C}.$$

Take the area formula $Area = \frac{1}{2}ab\sin C$, plug in the formula above for $\sin C$ to get

$$Area = \frac{1}{2}ab\sqrt{1-\cos^2 C}.$$

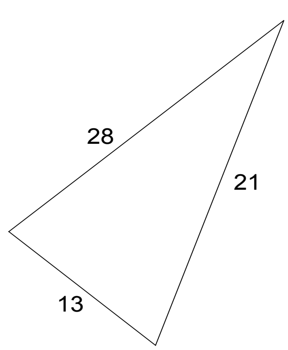
Then replace $\cos C$ with the top formula above to get

$$Area = \frac{1}{2}ab\sqrt{1-\left(\frac{a^2+b^2-c^2}{2ab}\right)^2}.$$

Do two pages of algebra on the right-hand side; eventually this simplifies to Heron's formula.

EXAMPLE 23

Find the area of the following triangle:



EXAMPLE 24

Find the area of triangle PQR if $p = 9$, $q = 10$ and $r = 17$.

Solution: First, find the semiperimeter s:

$$s = \frac{p+q+r}{2} = \frac{9+10+17}{2} = \frac{36}{2} = 18.$$

Next, use Heron's formula to find the area:

$$\begin{aligned} A &= \sqrt{s(s-p)(s-q)(s-r)} \\ &= \sqrt{18(18-9)(18-10)(18-17)} \\ &= \sqrt{18(9)(8)(1)} \\ &= \sqrt{1296} \\ &= 36. \end{aligned}$$

EXAMPLE 25

A wealthy rancher owns a triangular plot of land in Wyoming. He measures the lengths of the sides of his property as 19.5 mi, 22.0 mi and 25.7 mi. How many square miles of land does he own?

Solution: First, find the semiperimeter s:

$$s = \frac{p+q+r}{2} = \frac{19.5+22.0+25.7}{2} = \frac{67.2}{2} = 33.6.$$

Next, use Heron's formula to find the area:

$$\begin{aligned} A &= \sqrt{s(s-p)(s-q)(s-r)} \\ &= \sqrt{33.6(33.6-19.5)(33.6-22.0)(33.6-25.7)} \\ &= \sqrt{33.6(14.1)(11.6)(7.9)} \\ &= \sqrt{43415.37} \\ &= 208.36 \text{ sq mi}. \end{aligned}$$

4.5 Review material for Exam 2

Exam 2 content

What I will ask you to do without a calculator:

- **MOST IMPORTANT:** Compute the sine and/or cosine of any special angle (given in either degrees or radians)
- Given a picture of a right triangle with the side lengths labelled, find the sine and/or cosine of the angles in that triangle.
- Answer questions that involve the use of these identities:

  Pythagorean Identity: $\cos^2\theta + \sin^2\theta = 1$

  Cofunction identities: $\sin(90° - \theta) = \cos\theta$ and $\cos(90° - \theta) = \sin\theta$

  Odd-even identities: $\sin(-\theta) = -\sin\theta$ and $\cos(-\theta) = \cos\theta$

  Periodicity: $\sin(\theta \pm 360°) = \sin\theta$ and $\cos(\theta \pm 360°) = \cos\theta$
- Draw pictures to explain the meaning of basic trig problems involving cosine and/or sine
- Answer questions involving symmetry and reference angles
- Find the sine and/or cosine of an angle, given the coordinates of a point (x, y) on the terminal side of the angle
- Determine whether the sign of the sine and/or cosine of an angle is positive or negative based on its quadrant

What I will ask you to do with a calculator:

- **MOST IMPORTANT:** Solve triangles
  - solving right triangles using SOHCAHTOA and Pythagorean Thm
  - solving non-right triangles using the Laws of Sines and/or Cosines including the ambiguous case)
  - story problems which apply methods of solving triangles
- Anything you should be able to do without a calculator
- Find angle(s) whose sine or cosine is given (i.e. solve $\sin\theta = q$ and $\cos\theta = q$)
- Find the area of a triangle

Practice problems (no calculator)

On these questions, no decimal approximations are allowed–all answers must be exact.

1. Find the exact value of these quantities :

   a) $\sin -480°$ e) $\cos 450°$ i) $\cos 120°$ m) $\cos 180°$

   b) $\sin 750°$ f) $\cos -1440°$ j) $\cos -210°$ n) $\sin 330°$

   c) $\cos 150°$ g) $\sin 360°$ k) $\sin -60°$ o) $\cos 240°$

   d) $\sin 225°$ h) $\sin 420°$ l) $\cos 330°$ p) $\cos 270°$

   Note: Many more practice problems like these can be found later in this section.

2. Find the exact value of these quantities:

   a) $\sin 0$ e) $\sin \frac{\pi}{6}$ i) $\cos \frac{11\pi}{6}$ m) $\sin \frac{5\pi}{6}$

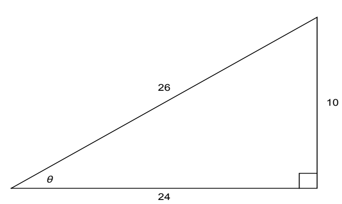
   b) $\cos \frac{5\pi}{3}$ f) $\sin \pi$ j) $\sin \frac{3\pi}{2}$ n) $\cos 2\pi$

   c) $\sin \frac{11\pi}{6}$ g) $\cos \frac{3\pi}{4}$ k) $\sin \frac{\pi}{3}$ o) $\sin \frac{5\pi}{4}$

   d) $\cos \frac{\pi}{3}$ h) $\sin \frac{2\pi}{3}$ l) $\cos \frac{2\pi}{3}$ p) $\cos 8\pi$

   Note: Many more practice problems like these can be found later in this section.

3. Find the exact values of $\sin\theta$ and $\cos\theta$, if θ is as in the following picture:



4. Suppose $\cos\theta = \frac{2}{5}$.

   a) What are the possible values of $\sin\theta$?
   b) What is $\cos(-\theta)$?
   c) What is $\sin(90° - \theta)$?
   d) What is $\cos(\theta + 720°)$?
   e) What are the two quadrants that θ might be in (when θ is drawn in standard position)?

5. Suppose $\sin\theta = \frac{1}{3}$ and $\cos\theta < 0$.
   a) What is $\sin(-\theta)$?
   b) What is $\cos\theta$?
   c) What is $\cos(-\theta)$?
   d) What is $\cos(360° + \theta)$?
   e) What quadrant is θ in (when θ is drawn in standard position)?
6. Suppose the point $(-3, 4)$ is on the terminal side of an angle θ, drawn in standard position. What are $\sin\theta$ and $\cos\theta$?
7. Determine whether the following quantities are positive or negative:

a) $\cos 324°$	c) $\sin 207°$	e) $\sin 113.75°$	g) $\cos 406°$
b) $\cos(-107.3°)$	d) $\cos 48°$	f) $\sin 280°$	h) $\sin(-580°)$

8. Suppose θ is in Quadrant II.
   a) Is $\sin\theta$ positive or negative?
   b) Is $\cos\theta$ positive or negative?
   c) What quadrant is $180° - \theta$ in?
   d) What quadrant is $-\theta$ in?
   e) What is the sign of $\sin(\theta + 90°)$?
   f) What is the sign of $\cos(\theta - 180°)$?
9. Draw a picture involving the unit circle to explain the quantity that is being asked for (you do not actually need to compute the quantity). In particular, your picture should indicate any given quantity, indicate where θ is, and indicate what you are asked to find with a "?".
   a) Find all angles between $0°$ and $360°$ such that $\sin\theta = -.725$.
   b) Find $\cos 250°$.
   c) If $\cos\theta = \frac{2}{3}$ and $\sin\theta < 0$, find $\sin\theta$.
   d) If $\sin\theta = -\frac{3}{4}$, find $\sin(-\theta)$.
   e) If $\cos\theta = \frac{2}{7}$ and $\sin\theta > 0$, find $\cos(\theta + 90°)$.
   f) If $\sin\theta = \frac{2}{5}$, find all possible values of $\cos\theta$.
10. Suppose $\left(\frac{3}{5}, \frac{-4}{5}\right)$ is the point on the unit circle at angle θ.
    a) What is $\cos\theta$?

b) What is the point on the unit circle at angle $\theta + 360°$?
c) What is the point on the unit circle at angle $180° - \theta$?
d) What is the point on the unit circle at angle $-\theta$?
e) What is the point on the unit circle at angle $\theta + 270°$?
f) What is $\sin(180° - \theta)$?
g) What is $\cos(90° - \theta)$?

11. Suppose (x, y) is the point on the unit circle at angle θ.

    a) At what angle (on the unit circle) does the point $(x, -y)$ sit?
    b) At what angle (on the unit circle) does the point $(-x, -y)$ sit?
    c) What is $\sin\theta$?
    d) What is $\sin(\theta + 180°)$?
    e) What is the point on the unit circle at angle $90° - \theta$?
    f) What is the point on the unit circle at angle $90° + \theta$?
    g) What is the point on the unit circle at angle $180° - \theta$?

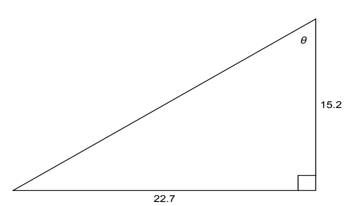
12. a) Find an angle in Quadrant 4 symmetric with $50°$.
    b) Find the reference angle of $230°$.
    c) Find four different angles between $0°$ and $360°$ which are symmetric with $110°$.

13. Find the area of triangle ABC if $a = 6$, $b = 7$ and $\angle C = 30°$.

Practice problems (calculator allowed)

In these problems, decimal approximations to answers are allowed.

14. Find $\sin\theta$ and $\cos\theta$, if θ is as in the following picture:



15. Suppose $\sin\theta = .63$ and $\cos\theta > 0$. Find $\cos\theta$.

16. Suppose $\cos\theta = -.42$ and $\sin\theta < 0$. Find $\sin\theta$.

17. Find all angles θ between $0°$ and $360°$ such that $\sin\theta = .46$.

18. Find all angles θ between $0°$ and $360°$ such that $\cos\theta = .825$.

19. Find all angles θ between $0°$ and $360°$ such that $\sin\theta = -1$.

20. Find all angles θ between $0°$ and $360°$ such that $\cos\theta = 1.24$.

21. Solve right triangle ABC if $\angle A = 35°$ and $c = 15.8$.

22. Solve right triangle ABC if $c = 25$ and $b = 20.6$.

23. Solve triangle ABC if $\angle A = 78°$, $\angle B = 43°$ and $c = 18$.

24. Solve triangle ABC if $\angle A = 25°$, $\angle B = 120°$ and $a = 5.3$.

25. Solve triangle ABC if $a = 19$, $b = 25$ and $c = 40$.

26. Solve triangle ABC if $a = 32.5$, $b = 15.8$ and $\angle C = 100°$.

27. Solve triangle ABC if $a = 16$, $\angle B = 73°$ and $b = 25$.

28. Solve triangle ABC if $a = 3.3$, $c = 1.8$ and $\angle C = 70°$.

29. Adam is standing on top of a hill, and Bruce is standing at the bottom of the hill. If the distance between Adam and Bruce is 120 feet, and Bruce's angle of elevation to Adam is $15°$, how far is Adam elevated above Bruce?

30. A ship is sailing due east. At noon, the captain notices that the angle between the ship's direction of motion and the line of sight to a lighthouse is $65°$. After the ship has travelled 3 miles, the captain notices that the angle between its direction of motion and the line of sight to the same lighthouse is $80°$. How far is the ship from the lighthouse when the captain makes this second observation?

31. A surveyor wants to know the distance between points P and Q, where Q is due south of P. He starts at point P and walks 25 feet due west until he gets to point R. While standing at point R, he notices that the angle between his line of sight to P and his line of sight to Q is $81.25°$. What is the distance between P and Q?

32. Find the area of triangle ABC if $a = 11$, $b = 8$ and $\angle C = 63°$.

33. Find the area of triangle ABC if $a = 13$, $b = 14$ and $c = 15$.

More practice with trig functions of special angles

34. $\sin -900°$
35. $\cos 930°$
36. $\sin 495°$
37. $\sin 240°$
38. $\sin 0$
39. $\sin 60°$
40. $\cos 135°$
41. $\sin 540°$
42. $\sin 300°$
43. $\cos 210°$
44. $\cos 225°$
45. $\sin 315°$
46. $\cos 360°$
47. $\sin 90°$
48. $\sin 180°$
49. $\sin 210°$
50. $\sin 765°$
51. $\sin -570°$
52. $\cos 315°$
53. $\sin 270°$
54. $\cos 0°$
55. $\cos 30°$
56. $\cos 45°$
57. $\sin 120°$
58. $\cos 300°$
59. $\sin 150°$
60. $\cos 480°$
61. $\cos -135°$
62. $\sin 30°$
63. $\sin 45°$
64. $\cos -570°$
65. $\cos 60°$
66. $\cos 90°$
67. $\sin 135°$
68. $\cos -405°$
69. $\cos -270°$
70. $\sin \frac{13\pi}{2}$
71. $\sin \frac{-7\pi}{2}$
72. $\cos \frac{-5\pi}{2}$
73. $\sin -6\pi$
74. $\cos -\pi$
75. $\cos \frac{-13\pi}{6}$
76. $\cos \frac{5\pi}{4}$
77. $\cos \pi$
78. $\sin \frac{3\pi}{4}$
79. $\cos 3\pi$
80. $\sin \frac{7\pi}{4}$
81. $\cos \frac{19\pi}{2}$
82. $\cos \frac{\pi}{6}$
83. $\sin \frac{13\pi}{4}$
84. $\cos \frac{-5\pi}{4}$
85. $\sin \frac{4\pi}{3}$
86. $\cos \frac{14\pi}{3}$
87. $\cos \frac{29\pi}{6}$
88. $\cos \frac{7\pi}{4}$
89. $\sin \frac{5\pi}{3}$
90. $\cos \frac{3\pi}{2}$
91. $\sin 7\pi$
92. $\sin \frac{-9\pi}{2}$
93. $\cos \frac{4\pi}{3}$
94. $\sin \frac{7\pi}{6}$
95. $\cos \frac{\pi}{4}$
96. $\sin 3\pi$
97. $\cos \frac{5\pi}{6}$
98. $\sin 2\pi$
99. $\sin \frac{-4\pi}{3}$
100. $\sin -3\pi$
101. $\cos -4\pi$
102. $\cos \frac{17\pi}{4}$
103. $\sin \frac{8\pi}{3}$
104. $\cos \frac{5\pi}{2}$
105. $\sin \frac{11\pi}{4}$
106. $\cos \frac{23\pi}{4}$
107. $\sin \frac{13\pi}{6}$
108. $\sin \frac{17\pi}{3}$
109. $\cos \frac{20\pi}{3}$
110. $\cos \frac{7\pi}{6}$
111. $\cos 0$
112. $\sin \frac{\pi}{4}$
113. $\sin \frac{19\pi}{6}$
114. $\cos \frac{23\pi}{6}$
115. $\sin \frac{-\pi}{4}$
116. $\sin \frac{7\pi}{2}$
117. $\cos \frac{-\pi}{3}$
118. $\sin \frac{-5\pi}{3}$
119. $\cos \frac{-15\pi}{4}$

Solutions

Note: I did these by hand; there may be errors.

1. a) $\sin -480° = -\frac{\sqrt{3}}{2}$
   b) $\sin 750° = \frac{1}{2}$
   c) $\cos 150° = -\frac{\sqrt{3}}{2}$
   d) $\sin 225° = \frac{-\sqrt{2}}{2}$
   e) $\cos 450° = 0$
   f) $\cos -1440° = 1$
   g) $\sin 360° = 0$
   h) $\sin 420° = \frac{\sqrt{3}}{2}$
   i) $\cos 120° = -\frac{1}{2}$
   j) $\cos -210° = -\frac{\sqrt{3}}{2}$
   k) $\sin -60° = -\frac{\sqrt{3}}{2}$
   l) $\cos 330° = \frac{\sqrt{3}}{2}$
   m) $\cos 180° = -1$
   n) $\sin 330° = -\frac{1}{2}$
   o) $\cos 240° = -\frac{1}{2}$
   p) $\cos 270° = 0$

2. a) $\sin 0 = 0$
   b) $\cos \frac{5\pi}{3} = \frac{1}{2}$
   c) $\sin \frac{11\pi}{6} = -\frac{1}{2}$
   d) $\cos \frac{\pi}{3} = \frac{1}{2}$
   e) $\sin \frac{\pi}{6} = \frac{1}{2}$
   f) $\sin \pi = 0$
   g) $\cos \frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$
   h) $\sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2}$
   i) $\cos \frac{11\pi}{6} = \frac{\sqrt{3}}{2}$
   j) $\sin \frac{3\pi}{2} = -1$
   k) $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$
   l) $\cos \frac{2\pi}{3} = -\frac{1}{2}$
   m) $\sin \frac{5\pi}{6} = \frac{1}{2}$
   n) $\cos 2\pi = 1$
   o) $\sin \frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$
   p) $\cos 8\pi = 1$

3. $\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{10}{26} = \frac{5}{13}$; $\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{24}{26} = \frac{12}{13}$.

4. a) Start with the Pythagorean identity $\cos^2\theta + \sin^2\theta = 1$. Plug in $\cos\theta = \frac{2}{5}$ to get $\left(\frac{2}{5}\right)^2 + \sin^2\theta = 1$, i.e. $\sin^2\theta + \frac{4}{25} = 1$, i.e. $\sin^2\theta = 1 - \frac{4}{25}$. Subtract the fractions on the right-hand side to get $\sin^2\theta = \frac{21}{25}$. Take the square root to get $\sin\theta = \pm\sqrt{\frac{21}{25}} = \pm\frac{\sqrt{21}}{5}$.
   b) $\cos(-\theta) = \cos\theta = \frac{2}{5}$ by the odd-even identity.
   c) $\sin(90° - \theta) = \cos\theta = \frac{2}{5}$ by the cofunction identity.
   d) $\cos(\theta + 720°) = \cos\theta = \frac{2}{5}$ by periodicity.
   e) Since $\cos\theta > 0$, θ is in Quadrant I or IV.

5. a) $\sin(-\theta) = -\sin\theta = \frac{-1}{3}$.
   b) Start with the Pythagorean identity $\cos^2\theta + \sin^2\theta = 1$. Plug in $\sin\theta = \frac{1}{3}$ to get $\cos^2\theta + \left(\frac{1}{3}\right)^2 = 1$, i.e. $\cos^2\theta = 1 - \frac{1}{9}$. Subtract the fractions on the right-hand side to get $\cos^2\theta = \frac{8}{9}$. Take the square root to get $\cos\theta = \pm\sqrt{\frac{8}{9}}$; since we are given that $\cos\theta < 0$, the answer must be $\cos\theta = -\sqrt{\frac{8}{9}} = -\frac{\sqrt{8}}{3}$.
   c) $\cos(-\theta) = \cos\theta = -\frac{\sqrt{8}}{3}$.
   d) By periodicity, $\cos(360° + \theta) = \cos\theta = -\frac{\sqrt{8}}{3}$.

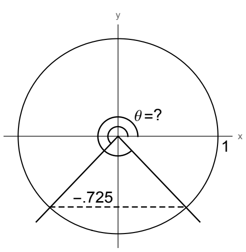
e) Since $\sin\theta > 0$ and $\cos\theta < 0$, θ must be in Quadrant II.

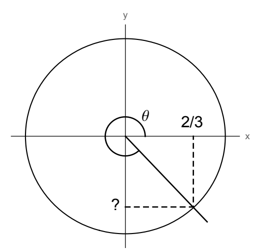
6. We have $x = -3$, $y = 4$ and $r = \sqrt{x^2+y^2} = \sqrt{(-3)^2+4^2} = \sqrt{9+16} = \sqrt{25} = 5$. Therefore $\sin\theta = \frac{y}{r} = \frac{4}{5}$ and $\cos\theta = \frac{x}{r} = \frac{-3}{5}$.

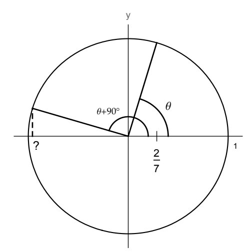
7. Determine the sign by finding the quadrant (answers on next page):

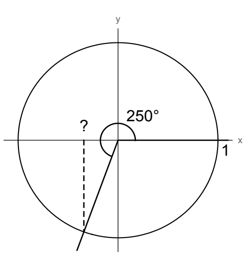
   a) $\cos 324° > 0$ (Quadrant IV)
   b) $\cos(-107.3°) < 0$ (Quadrant III)
   c) $\sin 207° < 0$ (Quadrant III)
   d) $\cos 48° > 0$ (Quadrant I)
   e) $\sin 113.75° > 0$ (Quadrant II)
   f) $\sin 280° < 0$ (Quadrant IV)
   g) $\cos 406° = \cos(406° - 360°) = \cos 46° > 0$ (Quadrant I)
   h) $\sin(-580°) = \sin(-580° + 720°) = \sin 140° > 0$ (Quadrant II)

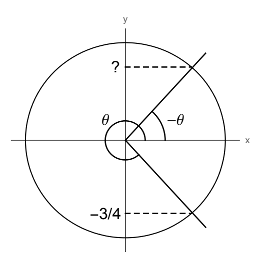
8. a) Since θ is in Quadrant II, $\sin\theta$ is positive.
   b) Since θ is in Quadrant II, $\cos\theta$ is negative.
   c) To get to $180° - \theta$, reflect across the y-axis. This puts you in Quadrant I.
   d) To get to $-\theta$, reflect across the x-axis. This puts you in Quadrant III.
   e) $\theta + 90°$ is in Quadrant III, so $\sin(\theta + 90°)$ is negative.
   f) From (c), $180° - \theta$ is in Quadrant I, so $\cos(180° - \theta)$ is positive.

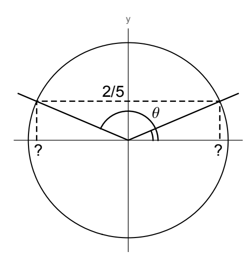
9. a) 

c) 

e) 

b) 

d) 

f) 

10. a) $\cos\theta = \frac{3}{5}$, the x-coordinate at angle θ.
    b) $\theta + 360°$ is coterminal with θ, so this point is also $\left(\frac{3}{5}, -\frac{4}{5}\right)$.
    c) Reflect the given point across the y-axis to get $\left(-\frac{3}{5}, -\frac{4}{5}\right)$.
    d) Reflect the given point across the x-axis to get $\left(\frac{3}{5}, \frac{4}{5}\right)$.

e) Rotate the given point around $270°$ counter-clockwise to get $\left(-\frac{4}{5}, -\frac{3}{5}\right)$.

f) Reflect the given point across both axes to get $\left(-\frac{3}{5}, \frac{4}{5}\right)$; then, $\sin(180° - \theta)$ is the y-coordinate of this point, i.e. $\frac{4}{5}$.

g) By a cofunction identity, $\cos(90° - \theta) = \sin\theta = -\frac{4}{5}$.

11. a) $-\theta$ (or $360° - \theta$).
    b) $\theta + 180°$ (or $\theta - 180°$).
    c) $\sin\theta = y$.
    d) $\sin(\theta + 180°) = -y$, since the point at that angle is $(-x, -y)$.
    e) (y, x).
    f) $(-y, x)$.
    g) $(-x, y)$.

12. a) $360° - 50° = 310°$.
    b) $230° - 180° = 50°$.
    c) $110°$, $180° - 110° = 70°$; $110° + 180° = 290°$; $360° - 110° = 250°$.

13. $A = \frac{1}{2}ab\sin C = \frac{1}{2}(6)(7)\sin 30° = \frac{1}{2}(6)(7)(\frac{1}{2}) = \frac{42}{4} = \frac{21}{2}$.

14. First, find the hypotenuse c by the Pythagorean Theorem: $c^2 = 22.7^2 + 15.2^2$, so $c^2 = 746.33$ and therefore $c = \sqrt{746.33} = 27.32$. Therefore $\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{22.7}{27.32} = .831$ and $\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{15.2}{27.32} = .556$.

15. Start with the Pythagorean identity: $\cos^2\theta + \sin^2\theta = 1$. Plug in $.63$ for $\sin\theta$ to get $\cos^2\theta + (.63)^2 = 1$, i.e. $\cos^2\theta + .397 = 1$ so $\cos^2\theta = 1 - .397 = .603$. Take the square root of both sides to get $\cos\theta = \pm\sqrt{.603} = \pm.776$. Since we are given that $\cos\theta > 0$, $\cos\theta = .776$.

16. Start with the Pythagorean identity: $\cos^2\theta + \sin^2\theta = 1$. Plug in $-.42$ for $\cos\theta$ to get $(-.42)^2 + \sin^2\theta = 1$, i.e. $.1764 + \sin^2\theta = 1$ so $\sin^2\theta = 1 - .1764 = .8236$. Take the square root of both sides to get $\sin\theta = \pm\sqrt{.8236} = \pm.908$. Since we are given that $\sin\theta < 0$, $\sin\theta = -.908$.

17. From a calculator, $\theta = \sin^{-1}(.46) = 27.4°$. A second angle is $180° - \theta = 152.6°$.

18. From a calculator, $\theta = \cos^{-1}(.825) = 34.4°$. A second angle is $360° - \theta = 325.6°$.

19. The only angle is at the bottom of the unit circle, i.e. $\theta = 270°$.

20. This has no solution because $1.24 > 1$.

21. $a = 9.06$, $b = 12.94$, $c = 15.8$, $\angle A = 35°$, $\angle B = 55°$, $\angle C = 90°$.

22. $a = 14.16$, $b = 20.6$, $c = 25$, $\angle A = 34.5°$, $\angle B = 55.5°$, $\angle C = 90°$.

23. This is ASA, so find the third angle and then use the Law of Sines: $a = 20.54$, $b = 14.32$, $c = 18$, $\angle A = 78°$, $\angle B = 43°$, $\angle C = 59°$.

24. This is AAS, so find the third angle and then use the Law of Sines: $a = 5.3$, $b = 10.86$, $c = 7.19$, $\angle A = 25°$, $\angle B = 120°$, $\angle C = 35°$.

25. This is SSS, so use the Law of Cosines to find an angle (then use either law to find a second angle): $a = 19$, $b = 25$, $c = 40$, $\angle A = 21.26°$, $\angle B = 28.48°$, $\angle C = 130.26°$.

26. This is SAS, so use the Law of Cosines to find the third side (then use either law to find a second angle): $a = 32.5$, $b = 15.8$, $c = 38.53$, $\angle A = 56.17°$, $\angle B = 23.83°$, $\angle C = 100°$.

27. This is SSA, the ambiguous case of the Law of Sines; you will find that there is only one possible triangle:

$$a = 16, b = 25, c = 24.45, \angle A = 37.74°, \angle B = 73°, \angle C = 69.26°.$$

Note that $\angle A$ might be $180° - 37.74° = 142.26°$, but in this case $\angle C$ would be negative, so you can throw this situation out.

28. This is SSA, the ambiguous case of the Law of Sines. When you try this, you will get $\sin A = 1.722$ which has no solution, so no such triangle exists.

29. Draw right triangle ABC with Adam at position A and Bruce at position B (A should be above the right angle, and B should be to the left or right of the right angle). We are given $c = 120$ and $\angle B = 15°$ and want to know what b is. We know $\sin B = \frac{b}{c} = \frac{b}{120}$, so we have $b = 120 \sin B = 120 \sin 15° = 31.05$ feet.

30. Draw triangle ABC where C is the lighthouse, A is the position of the ship at noon, and B is the position of the ship at the second observation. We have $c = 3$, $\angle A = 65°$ and $\angle B = 180° - 80° = 100°$; our goal is to find a. This is an ASA situation; first find the third angle which is

$$\angle C = 180° - \angle A - \angle B = 180° - 65° - 100° = 15°.$$

Now use the Law of Sines; starting with $\frac{\sin A}{a} = \frac{\sin C}{c}$ we get $\frac{\sin 65°}{a} = \frac{\sin 15°}{3}$ so $a = \frac{3 \sin 65°}{\sin 15°} = 10.50$ miles.

31. Draw right triangle PQR (the right angle is at point P; R is to the left of P and Q is below P). We know $q = 25$ and $\angle R = 81.25°$. We want to know r; the quickest way to do this is to use tangent:

$$\tan 81.25° = \frac{\text{opp}}{\text{adj}} = \frac{r}{q} = \frac{r}{25}$$

so $r = 25 \tan 81.25° = 162.42$ feet.

(You can also do this by using sine or cosine to find p, then using the Pythagorean Theorem to find r.)

32. $A = \frac{1}{2}ab \sin C = \frac{1}{2}(11)(8) \sin 63° = 44(.891) = 39.2$ square units.

33. Use Heron's formula. First, find the semiperimeter:

$$s = \frac{a+b+c}{2} = \frac{13+14+15}{2} = 21$$

Now, the area is

$$\begin{aligned} A = \sqrt{s(s-a)(s-b)(s-c)} &= \sqrt{21(21-13)(21-14)(21-15)} \\ &= \sqrt{21(8)(7)(6)} \\ &= \sqrt{7056} \\ &= 84 \text{ square units.} \end{aligned}$$

34. $\sin -900° = 0$
35. $\cos 930° = -\frac{\sqrt{3}}{2}$
36. $\sin 495° = \frac{\sqrt{2}}{2}$
37. $\sin 240° = -\frac{\sqrt{3}}{2}$
38. $\sin 0 = 0$
39. $\sin 60° = \frac{\sqrt{3}}{2}$
40. $\cos 135° = -\frac{\sqrt{2}}{2}$
41. $\sin 540° = 0$
42. $\sin 300° = -\frac{\sqrt{3}}{2}$
43. $\cos 210° = -\frac{\sqrt{3}}{2}$
44. $\cos 225° = -\frac{\sqrt{2}}{2}$
45. $\sin 315° = -\frac{\sqrt{2}}{2}$
46. $\cos 360° = 1$
47. $\sin 90° = 1$
48. $\sin 180° = 0$
49. $\sin 210° = -\frac{1}{2}$
50. $\sin 765° = \frac{\sqrt{2}}{2}$
51. $\sin -570° = \frac{1}{2}$
52. $\cos 315° = \frac{\sqrt{2}}{2}$
53. $\sin 270° = -1$
54. $\cos 0° = 1$
55. $\cos 30° = \frac{\sqrt{3}}{2}$
56. $\cos 45° = \frac{\sqrt{2}}{2}$
57. $\sin 120° = \frac{\sqrt{3}}{2}$
58. $\cos 300° = \frac{1}{2}$
59. $\sin 150° = \frac{1}{2}$
60. $\cos 480° = -\frac{1}{2}$
61. $\cos -135° = -\frac{\sqrt{2}}{2}$
62. $\sin 30° = \frac{1}{2}$
63. $\sin 45° = \frac{\sqrt{2}}{2}$
64. $\cos -570° = -\frac{\sqrt{3}}{2}$
65. $\cos 60° = \frac{1}{2}$
66. $\cos 90° = 0$
67. $\sin 135° = \frac{\sqrt{2}}{2}$
68. $\cos -405° = \frac{\sqrt{2}}{2}$
69. $\cos -270° = 0$
70. $\sin \frac{13\pi}{2} = 1$
71. $\sin \frac{-7\pi}{2} = 1$
72. $\cos \frac{-5\pi}{2} = 0$
73. $\sin -6\pi = 0$
74. $\cos -\pi = -1$
75. $\cos \frac{-13\pi}{6} = \frac{\sqrt{3}}{2}$

76. $\cos\frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$
77. $\cos\pi = -1$
78. $\sin\frac{3\pi}{4} = \frac{\sqrt{2}}{2}$
79. $\cos 3\pi = -1$
80. $\sin\frac{7\pi}{4} = -\frac{\sqrt{2}}{2}$
81. $\cos\frac{19\pi}{2} = 0$
82. $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$
83. $\sin\frac{13\pi}{4} = -\frac{\sqrt{2}}{2}$
84. $\cos\frac{-5\pi}{4} = -\frac{\sqrt{2}}{2}$
85. $\sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$
86. $\cos\frac{14\pi}{3} = -\frac{1}{2}$
87. $\cos\frac{29\pi}{6} = -\frac{\sqrt{3}}{2}$
88. $\cos\frac{7\pi}{4} = \frac{\sqrt{2}}{2}$
89. $\sin\frac{5\pi}{3} = -\frac{\sqrt{3}}{2}$
90. $\cos\frac{3\pi}{2} = 0$
91. $\sin 7\pi = 0$
92. $\sin\frac{-9\pi}{2} = -1$
93. $\cos\frac{4\pi}{3} = -\frac{1}{2}$
94. $\sin\frac{7\pi}{6} = -\frac{1}{2}$
95. $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$
96. $\sin 3\pi = 0$
97. $\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$
98. $\sin 2\pi = 0$
99. $\sin\frac{-4\pi}{3} = \frac{\sqrt{3}}{2}$
100. $\sin -3\pi = 0$
101. $\cos -4\pi = 1$
102. $\cos\frac{17\pi}{4} = \frac{\sqrt{2}}{2}$
103. $\sin\frac{8\pi}{3} = \frac{\sqrt{3}}{2}$
104. $\cos\frac{5\pi}{2} = 0$
105. $\sin\frac{11\pi}{4} = \frac{\sqrt{2}}{2}$
106. $\cos\frac{23\pi}{4} = \frac{\sqrt{2}}{2}$
107. $\sin\frac{13\pi}{6} = \frac{1}{2}$
108. $\sin\frac{17\pi}{3} = -\frac{\sqrt{3}}{2}$
109. $\cos\frac{20\pi}{3} = -\frac{1}{2}$
110. $\cos\frac{7\pi}{6} = -\frac{\sqrt{3}}{2}$
111. $\cos 0 = 1$
112. $\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$
113. $\sin\frac{19\pi}{6} = -\frac{1}{2}$
114. $\cos\frac{23\pi}{6} = \frac{\sqrt{3}}{2}$
115. $\sin\frac{-\pi}{4} = -\frac{\sqrt{2}}{2}$
116. $\sin\frac{7\pi}{2} = -1$
117. $\cos\frac{-\pi}{3} = \frac{1}{2}$
118. $\sin\frac{-5\pi}{3} = \frac{\sqrt{3}}{2}$
119. $\cos\frac{-15\pi}{4} = \frac{\sqrt{2}}{2}$

Chapter 5

Trigonometric functions

5.1 The six trigonometric functions

So far in this course, we have extensively studied two trigonometric functions: sine and cosine.

Ratios of sine and cosine can be used to create four other trigonometric functions (giving a total of six trig functions). Here they are:

Definition 5.1 *Let θ be a number or an angle (either in a right triangle, or drawn in standard position, or wherever). Define the* **tangent**, **cotangent**, **secant** *and* **cosecant** *of θ to be, respectively,*

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

Remark: You can solve pretty much any trig problem with only cosine and sine. (Examples: solving triangles needs only the Law of Sines and the Law of Cosines; the SAS area formula uses only sine, etc.) These new trig functions are here mainly to streamline computations in future settings (especially calculus applications).

The six trig functions come in three pairs of reciprocals:

Theorem 5.2 (Reciprocal identities)

$$\sec\theta = \frac{1}{\cos\theta} \qquad \cos\theta = \frac{1}{\sec\theta}$$
$$\csc\theta = \frac{1}{\sin\theta} \qquad \sin\theta = \frac{1}{\csc\theta}$$
$$\cot\theta = \frac{1}{\tan\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

EXAMPLE 1

1. Suppose $\tan\theta = 3$. Find $\cot\theta$.

2. Suppose $\sin\theta = \frac{1}{4}$. Find $\csc\theta$.

3. Suppose $\sec\theta = 2.587$. Find $\cos\theta$.

   Solution: $\cos\theta = \frac{1}{\sec\theta} = \frac{1}{2.587} = .3865$.

4. Suppose $\cot\theta = \frac{4}{7}$. Find $\tan\theta$.

   Solution: $\tan\theta = \frac{1}{\cot\theta} = \frac{1}{\frac{4}{7}} = \frac{7}{4}$.

5. Suppose $\csc\theta = 5.35$ and $\cos\theta < 0$. Find $\cos\theta$.

Evaluating trig functions on a calculator

Your calculator (probably) has [SIN], [COS] and [TAN] buttons but probably does not have [SEC], [CSC] or [COT] buttons. To evaluate secants, cosecants and/or cotangents, you need to use the reciprocal identities on the previous page:

EXAMPLE 2

1. Find $\csc 117°$ (using a calculator).

2. Find $\tan 219°$ (using a calculator).

3. Find $\cot 76.5°$ (using a calculator).

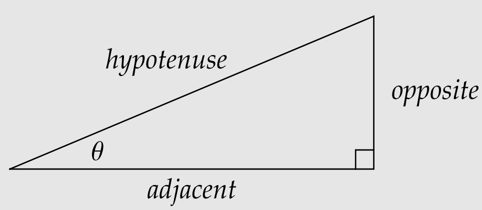
4. Find $\sec 283°$ (using a calculator).

   Solution: From a calculator, $\cos 283° = .224951$.

   Therefore $\sec 283° = \frac{1}{\cos 283°} = \frac{1}{.224951} = 4.44541$.

Right triangle definitions

Theorem 5.3 (Right triangle definition of the trig functions) *Take an* ***acute*** *angle θ. Put θ in a* ***right*** *triangle:*



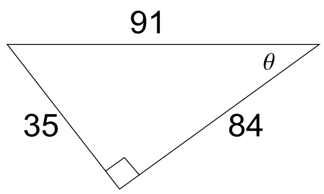
Let $0 \leq \theta \leq 90°$. Then:

SOH CAH TOA: $\sin\theta = \dfrac{opp}{hyp}$ $\quad\cos\theta = \dfrac{adj}{hyp}$ $\quad\tan\theta = \dfrac{opp}{adj}$

Reciprocals: $\csc\theta = \dfrac{hyp}{opp}$ $\quad\sec\theta = \dfrac{hyp}{adj}$ $\quad\cot\theta = \dfrac{adj}{opp}$

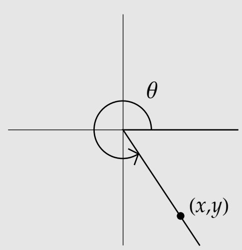
EXAMPLE 3

Find the values of all six trig functions:



Angle definitions of the trig functions

Theorem 5.4 (Angle definition of the trig funtions) *Take an angle θ (in degrees or radians) and draw it in standard position. Choose any point (x, y) on the terminal side of the angle (x and/or y could be positive, negative or zero). Let r be the distance from (x, y) to the origin, so that $r = \sqrt{x^2 + y^2}$ (r is always positive). Then:*



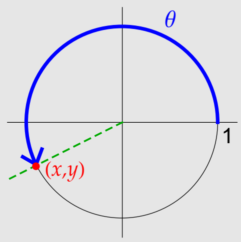
$$\begin{aligned} \sin\theta &= \sin(\theta) = \tfrac{y}{r} & \csc\theta &= \csc(\theta) = \\ \cos\theta &= \cos(\theta) = \tfrac{x}{r} & \sec\theta &= \sec(\theta) = \\ \tan\theta &= \tan(\theta) = & \cot\theta &= \cot(\theta) = \end{aligned}$$

EXAMPLE 4

Suppose that the point $(-2, 7)$ is on the terminal side of an angle θ when drawn in standard position. Find the values of all six trig functions of θ.

Unit circle definitions

Theorem 5.5 (Unit circle definition of the trig functions) *Take a **number** θ. Starting at the point $(1, 0)$, mark off an arc of length θ on the unit circle (go counterclockwise if $\theta > 0$ and clockwise if $\theta < 0$). (In other words, mark off an angle of θ radians with $r = 1$.) Call the point where the arc ends (x, y). Then:*

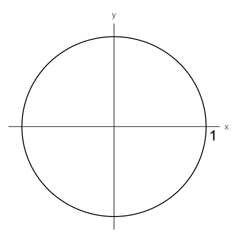


$$\begin{aligned} \sin\theta &= \sin(\theta) = y & \csc\theta &= \csc(\theta) = \tfrac{1}{y} \\ \cos\theta &= \cos(\theta) = x & \sec\theta &= \sec(\theta) = \tfrac{1}{x} \\ \tan\theta &= \tan(\theta) = & \cot\theta &= \cot(\theta) = \end{aligned}$$

EXAMPLE 5

For each given angle θ, find the values of all six trig functions of θ.

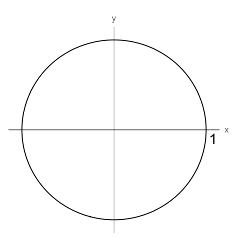
1. $\theta = \frac{\pi}{2}$



$\sin\dfrac{\pi}{2} =$ $\qquad$ $\cos\dfrac{\pi}{2} =$ $\qquad$ $\tan\dfrac{\pi}{2} =$

$\csc\dfrac{\pi}{2} =$ $\qquad$ $\sec\dfrac{\pi}{2} =$ $\qquad$ $\cot\dfrac{\pi}{2} =$

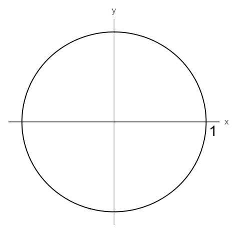
2. $\theta = 540°$



$\sin 540° =$ $\qquad$ $\cos 540° =$ $\qquad$ $\tan 540° =$

$\csc 540° =$ $\qquad$ $\sec 540° =$ $\qquad$ $\cot 540° =$

3. $\theta = \frac{3\pi}{2}$



$$\sin\frac{3\pi}{2} = \qquad \cos\frac{3\pi}{2} = \qquad \tan\frac{3\pi}{2} =$$

$$\csc\frac{3\pi}{2} = \qquad \sec\frac{3\pi}{2} = \qquad \cot\frac{3\pi}{2} =$$

Algebraic definitions

Recall that we saw algebraic definitions of sine and cosine (using powers of θ) earlier in this course. There are also algebraic definitions of the other four trig functions, but I won't write them down here.

Solving for angles if given the value of a trig function

Earlier in the course, we saw that to solve an equation like

$$\sin\theta = .386$$

you'd execute [SIN^{-1}] .386 on a calculator and obtain

$$\theta = \sin^{-1}(.386) = 22.7°.$$

This wasn't the only answer between $0°$ *and* $360°$*!* Another answer was $180° - \theta = 157.3°$.

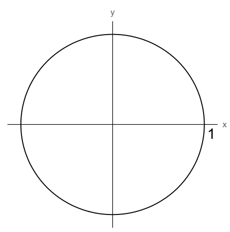
Your calculator has [SIN^{-1}], [COS^{-1}] and [TAN^{-1}] buttons, but it probably doesn't have [CSC^{-1}], [SEC^{-1}] or [COT^{-1}] buttons. How then, do you solve equations like

$$\sec\theta = 2.3 \qquad \cot\theta = .678 \qquad \csc\theta = 5 \qquad \text{etc.?}$$

EXAMPLE 6

For each equation, find an angle between $0°$ and $90°$ (equivalently, an angle in radians in the interval $[0, \frac{\pi}{2}]$) which solves the given equation. Then find all angles between $0°$ an $360°$ which solve the equation.

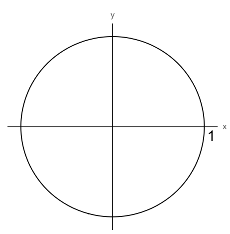
1. $\tan\theta = 1.36$



Angle between $0°$ and $90°$:

All angles between $0°$ and $360°$:

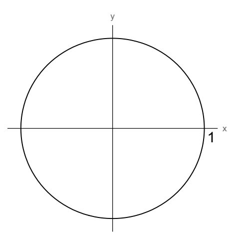
2. $\sec\theta = 2.3$



Angle between $0°$ and $90°$:

All angles between $0°$ and $360°$:

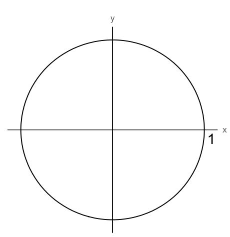
3. $\cot\theta = .678$



Angle between $0°$ and $90°$:

All angles between $0°$ and $360°$:

4. $\csc\theta = 5$



Angle between $0°$ and $90°$:

All angles between $0°$ and $360°$:

The general tricks to finding the second angle which solves a trig equation can be found on the next page:

Theorem 5.6 (Inverse sine, cosine and tangent tricks) *Let q be a number.*

- *To solve the equation $\sin\theta = q$, type* [SIN^{-1}] q *on the calculator. This gives the answer θ between $-90°$ and $90°$. Another answer to the same equation is always given by $180° - \theta$.*
- *To solve the equation $\cos\theta = q$, type* [COS^{-1}] q *on the calculator. This gives the answer θ between $0°$ and $180°$. Another answer to the same equation is always given by $360° - \theta$.*
- *To solve the equation $\tan\theta = q$, type* [TAN^{-1}] q *on the calculator. This gives the answer θ between $-90°$ and $90°$. Another answer to the same equation is always given by $\theta + 180°$.*

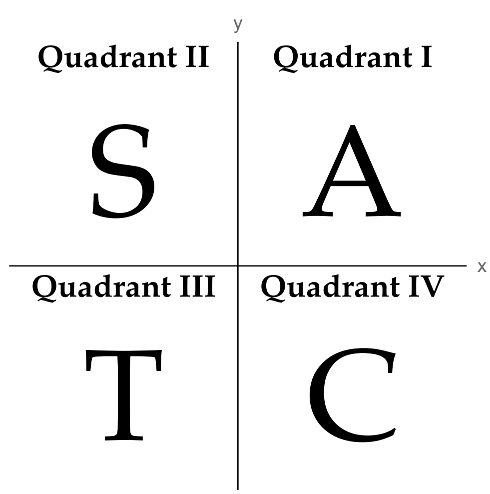
5.2 More on the six trigonometric functions

Signs of the six trig functions

Earlier in the course, we learned the pneumonic device

“**A**ll **S**cholars **T**ake **C**alculus”

which governs the signs of the trig functions. Since a quantity always has the same sign as its reciprocal, we can extend this rule to all six trig functions:



EXAMPLE 7

Determine (without a calculator) whether the following quantities are positive or negative:

1. $\csc 322^\circ$
2. $\cot 115^\circ$
3. $\tan 204^\circ$
4. $\tan 515^\circ$

EXAMPLE 8

Determine which quadrant θ lies in, from the given information:

1. $\sin\theta < 0$ and $\cot\theta > 0$
2. $\cos\theta > 0$ and $\csc\theta < 0$
3. $\cot\theta > 0$ and $\cos\theta > 0$
4. $\cos\theta = -.35$ and $\tan\theta < 0$
5. $\csc\theta = -4$ and $\cot\theta < 0$
6. $\theta = 258704335^\circ$ (using a calculator)
7. $\theta = 243$ radians (using a calculator)

Ranges of the trig functions

The **domain** of a function is the set of "allowable inputs" to the function, i.e. the set of inputs which do not cause an error in the function. The **range** of a function is the set of outputs of the function. Today, I want to discuss the ranges of the trig functions.

Example: $f(x) = x^2$.

Application: Consider the equations

$$x^2 = 16 \qquad x^2 = 5 \qquad x^2 = 0 \qquad x^2 = -3$$

Theorem 5.7 (Ranges of the six trig functions) *The ranges of the six trig functions are as follows:*

$$\begin{aligned} Range(\sin\theta) = Range(\cos\theta) &= [-1,1] = \{y : -1 \leq y \leq 1\} \\ Range(\tan\theta) = Range(\cot\theta) &= \mathbb{R} = \{\textit{all real numbers}\} \\ Range(\sec\theta) = Range(\csc\theta) &= (-\infty,-1] \cup [1,\infty) = \{y : y \leq -1 \textit{ or } y \geq 1\} \end{aligned}$$

EXAMPLE 9

Determine if the following statements are possible, or impossible.

1. $\sin\theta = .35$
2. $\sin\theta = 2.3$
3. $\sin\theta = -1.45$
4. $\cos\theta = -2.20$
5. $\cos\theta = 1$
6. $\sec\theta = \frac{2}{3}$
7. $\sec\theta = 4$
8. $\sec\theta = -3$
9. $\tan\theta = 0$
10. $\tan\theta = 3.5$
11. $\tan\theta = -1.2$
12. $\cot\theta = \frac{11}{3}$
13. $\cot\theta = -1.7$
14. $\cot\theta = .5$
15. $\csc\theta = \frac{-1}{4}$
16. $\csc\theta = 1$
17. $\csc\theta = 0$
18. $\csc\theta = -2.4$

Elementary trig identities

The three boxes below comprise, collectively, what are called the *elementary trig identities*. They should be memorized.

Theorem 5.8 (Reciprocal identities)

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sin\theta = \frac{1}{\csc\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

Theorem 5.9 (Quotient identities)

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Theorem 5.10 (Pythagorean identities)

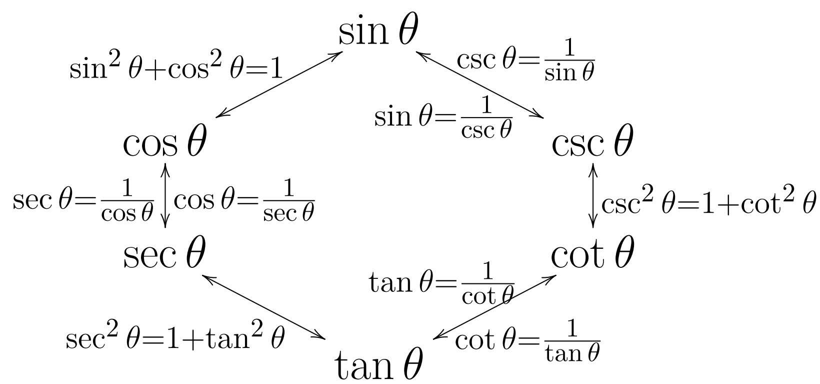
$$\sin^2\theta + \cos^2\theta = 1 \qquad \sec^2\theta = 1 + \tan^2\theta \qquad \csc^2\theta = 1 + \cot^2\theta$$

PROOFS (of the Pythagorean identities) We already learned $\sin^2\theta + \cos^2\theta = 1$.

The purpose of these identities is that **if you know the value of any one of the six trig functions, and you know the sign of one of the other trig functions, you can compute the values of all six trig functions** by following the arrows in the diagram on the next page (and using the quotient identities as shortcuts).

EXAMPLE 10

Find all possible values of $\tan\theta$, if $\sec\theta = -3$.



EXAMPLE 11

Find the values of all six trig functions of θ, from the given information:

1. $\sin\theta = \frac{2}{3}$ and $\cot\theta < 0$ (no calculator)

$\sin\theta =$	$\cos\theta =$	$\tan\theta =$
$\csc\theta =$	$\sec\theta =$	$\cot\theta =$

2. $\cot\theta = 4$ and $\csc\theta > 0$ (no calculator)

   First, since $\cot\theta > 0$ and $\csc\theta > 0$, θ is in Quadrant I, so all six trig functions are positive.

   Second, $\tan\theta = \frac{1}{\cot\theta} = \frac{1}{4}$. From here, there are two ways to proceed:

   Solution # 1: Use identity $\cot^2\theta + 1 = \csc^2\theta$:

$$\begin{aligned} 4^2 + 1 &= \csc^2\theta \\ 17 &= \csc^2\theta \\ \pm\sqrt{17} &= \csc\theta \\ \sqrt{17} &= \csc\theta \text{ (since we are in Quadrant I).} \end{aligned}$$

   Thus $\sin\theta = \frac{1}{\csc\theta} = \frac{1}{\sqrt{17}}$. Now, use a quotient identity:

$$\begin{aligned} \tan\theta &= \frac{\sin\theta}{\cos\theta} \\ \frac{1}{4} &= \frac{1/\sqrt{17}}{\cos\theta} \\ \cos\theta &= \frac{4}{\sqrt{17}} \end{aligned}$$

   Last, $\sec\theta = \frac{1}{\cos\theta} = \frac{\sqrt{17}}{4}$. Therefore:

$\sin\theta = \frac{1}{\sqrt{17}}$	$\cos\theta = \frac{4}{\sqrt{17}}$	$\tan\theta = \frac{1}{4}$
$\csc\theta = \sqrt{17}$	$\sec\theta = \frac{\sqrt{17}}{4}$	$\cot\theta = 4$

   Solution # 2:

3. $\tan\theta = -.346$ and $\sec\theta > 0$ (calculator OK)

$\sin\theta =$	$\cos\theta =$	$\tan\theta =$
$\csc\theta =$	$\sec\theta =$	$\cot\theta =$

4. $\cos\theta = .825$ and $\cot\theta < 0$ (calculator OK)

$\sin\theta =$	$\cos\theta =$	$\tan\theta =$
$\csc\theta =$	$\sec\theta =$	$\cot\theta =$

The other set of identities that it is good to memorize are these odd-even identities, which tell you how to deal with minus signs on the inside of a trig function:

Theorem 5.11 (Odd-even identities)

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta \qquad \tan(-\theta) = -\tan\theta$$

$$\csc(-\theta) = -\csc\theta \qquad \sec(-\theta) = \sec\theta \qquad \cot(-\theta) = -\cot\theta$$

PROOF We already learned $\sin(-\theta) = -\sin\theta$ and $\cos(-\theta) = \cos\theta$.

EXAMPLE 12

1. Suppose $\tan\theta = 3.25$. Find $\tan(-\theta)$.

2. Suppose $\sec\theta = -1.3$. Find $\sec(-\theta)$.

3. Suppose $\cot\theta = -\frac{3}{8}$. Find $\cot(-\theta)$.

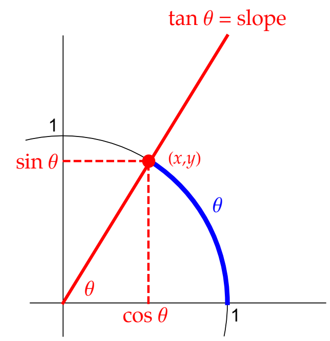
   Solution: Since $\cot(-\theta) = -\cot\theta$, we see that $\cot(-\theta) = -\left(-\frac{3}{8}\right) = \frac{3}{8}$.

4. Suppose $\sin\theta = \frac{2}{3}$. Find $\csc(-\theta)$.

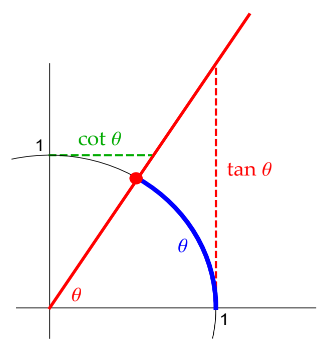
5.3 The meaning of the six trig functions

Earlier this semester we discussed how to interpret basic problems involving cosine and sine with pictures. Now, we do the same thing with all six trig functions. To do this, we first need some conceptual understanding of what each of the six trig functions mean on the unit circle.

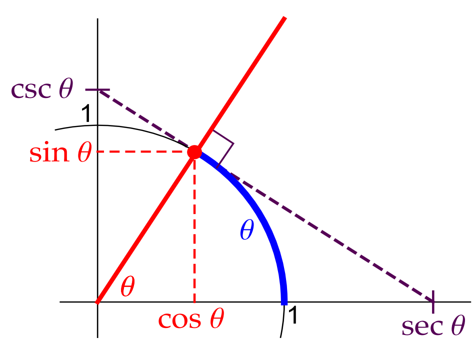
Picture # 1: cosine, sine and tangent



Picture # 2: tangent and cotangent



Picture # 1: secant and cosecant



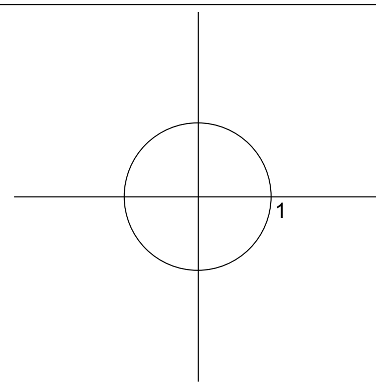
EXAMPLE 13

(a) In the first column, evaluate the given quantity or solve the problem.

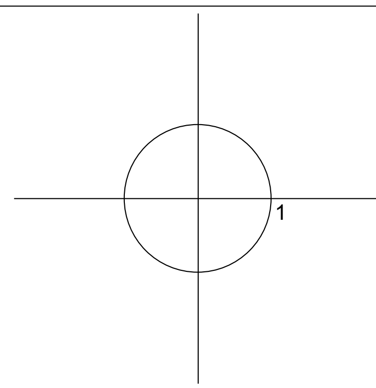
(b) Indicate on the given unit circle what the problem is asking you to compute. In particular:

- label the given quantities of the problem in your picture,
- indicate with a "?" the quantity you are asked to find in the problem, and
- if θ is neither what is given nor what is asked for, indicate where θ is.

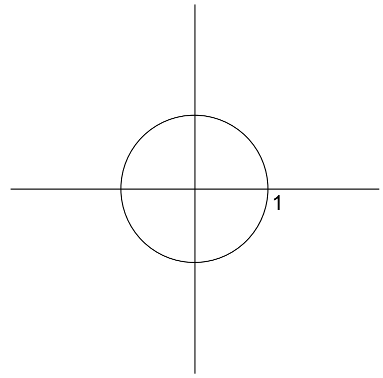
1. Compute $\cot 113°$.

Solution	**Picture**
	

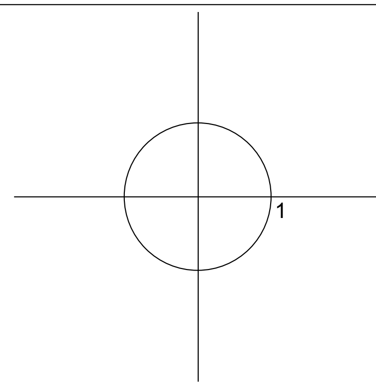
2. Compute $\sec 320°$.

Solution	**Picture**
	

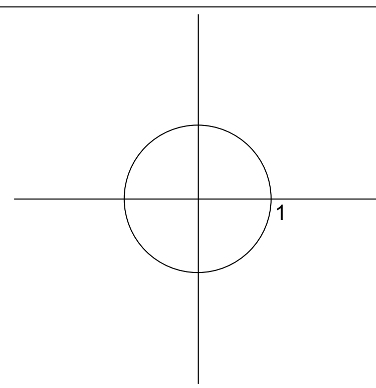
3. Compute $\csc 180°$.

Solution	Picture
	

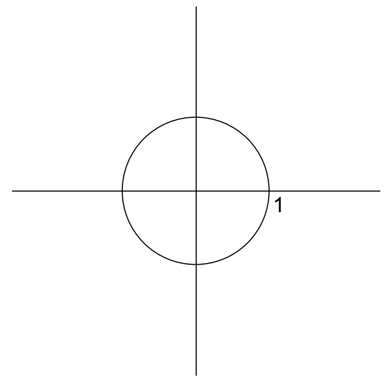
4. Compute $\tan 222°$.

Solution	Picture
	

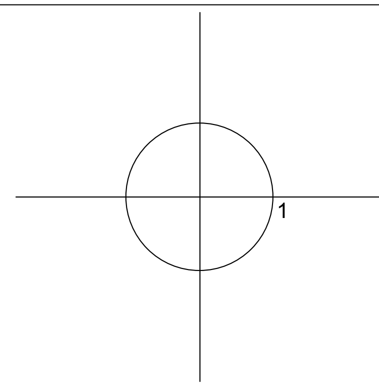
5. Find all values of θ between $0°$ and $360°$, if $\csc \theta = 1.5$.

Solution	Picture
	

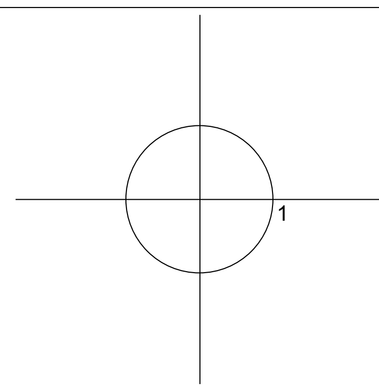
6. Find all values of θ between $0°$ and $360°$, if $\cot\theta = -\frac{2}{3}$.

Solution	**Picture**
	

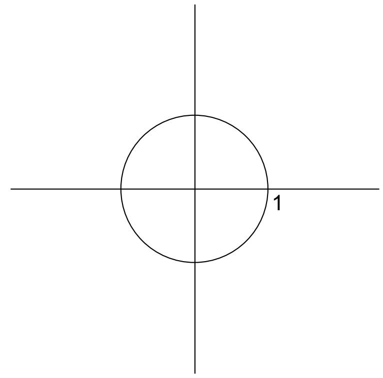
7. Find all values of θ between $0°$ and $360°$, if $\sec\theta = \frac{1}{2}$.

Solution	**Picture**
	

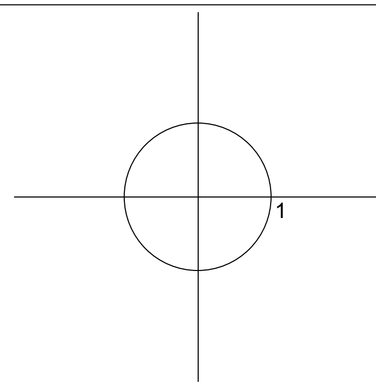
8. Find $\cos\theta$, if $\sec\theta = 1.75$.

Solution	**Picture**
	

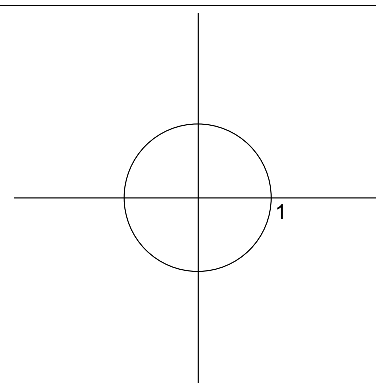
9. Find $\cot\theta$, if $\tan\theta = \frac{1}{5}$.

Solution	**Picture**
	

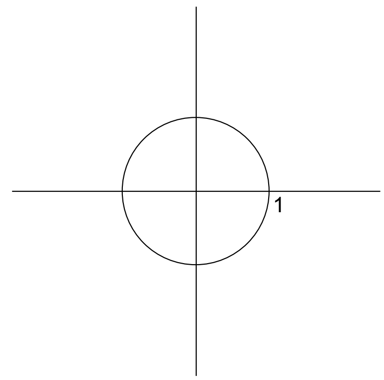
10. Find $\csc\theta$, if $\sin\theta = \frac{3}{5}$.

Solution	**Picture**
	

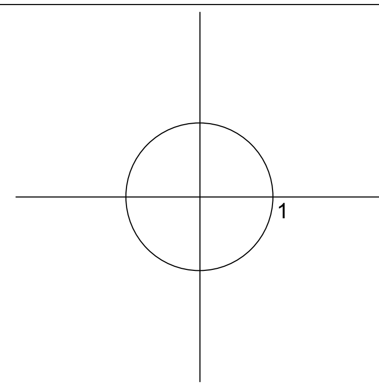
11. Find $\sec(-\theta)$, if $\sec\theta = \frac{5}{3}$.

Solution	**Picture**
	

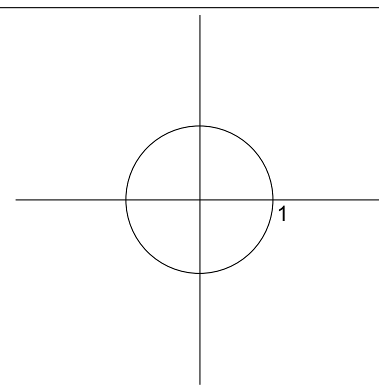
12. Find $\tan\theta$, if $\sec\theta = 2$ and $\tan\theta < 0$.

Solution	Picture
	

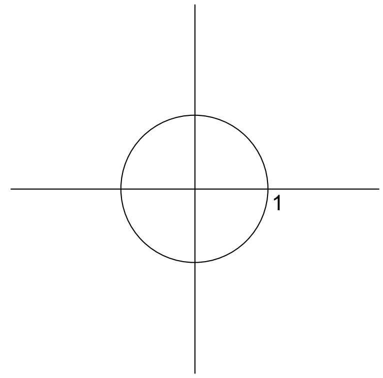
13. Find $\csc\theta$, if $\cot\theta = \frac{1}{2}$ and $\sin\theta < 0$.

Solution	Picture
	

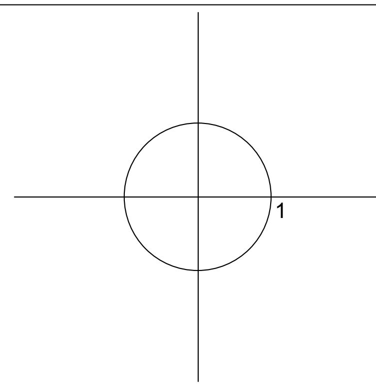
14. Find all possible values of $\sec\theta$, if $\tan\theta = \frac{4}{3}$.

Solution	Picture
	

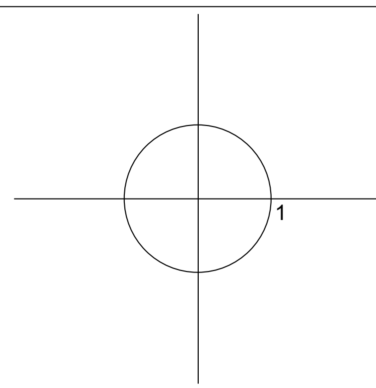
15. Find all possible values of $\tan\theta$, if $\csc\theta = -\frac{4}{3}$ and $\cos\theta > 0$.

Solution	**Picture**
	

16. Find all possible values of $\tan(\theta + \pi)$, if $\sin\theta = \frac{2}{3}$ and $\cos\theta < 0$.

Solution	**Picture**
	

17. Find all possible values of $\csc(180° - \theta)$, if $\cos\theta = .4$ and $\tan\theta < 0$.

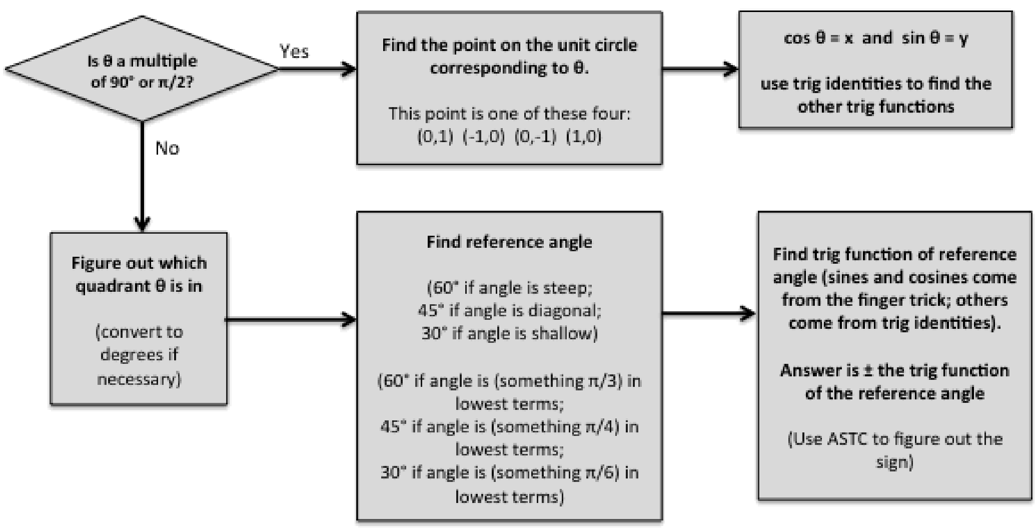
Solution	**Picture**
	

5.4 Evaluating trig functions at special angles

In Chapter 3, we discussed how to evaluate the sine and/or cosine of special angles. We now extend the ideas developed to all six trig functions. Essentially, this is the same procedure as before, except that you also need these identities:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \text{slope} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{1}{\text{slope}} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

Procedure to evaluate trig functions at special angles



EXAMPLE 14

1. Find $\sec\frac{7\pi}{6}$.

2. Find $\cot\frac{15\pi}{4}$.

3. Find $\tan\frac{-3\pi}{2}$.

Examples from the first quadrant

EXAMPLE 15

Compute the following:

1. $\sec 60^\circ$
2. $\csc \frac{\pi}{4}$
3. $\tan 0$
4. $\cot \frac{\pi}{3}$
5. $\sin 30^\circ$
6. $\tan 90^\circ$
7. $\cot 45^\circ$
8. $\csc 30^\circ$
9. $\csc 90^\circ$
10. $\sec 0^\circ$
11. $\sec \frac{\pi}{6}$

Theorem 5.12 (Values of the trig functions in Quadrant I) *This table contains the values of the trig functions for special angles between* $0°$ *and* $90°$*:*

θ *(degrees)*	θ *(radians)*	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
$0°$	0	0	1	0	DNE	1	DNE
$30°$	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}}$	2
$45°$	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
$60°$	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$	2	$\frac{2}{\sqrt{3}}$
$90°$	$\frac{\pi}{2}$	1	0	DNE	0	DNE	1

EXAMPLE 15

Compute the following:

1. $\sec(-120°)$

2. $\csc\left(\frac{-9\pi}{4}\right)$

3. $\cot\left(-\frac{2\pi}{3}\right)$

4. $\cot\frac{19\pi}{6}$

5. $\sec 180°$

6. $\tan 495°$

EXAMPLE 16

Compute the following:

1. $\cos\frac{11\pi}{6}$
2. $\sin 120°$
3. $\sec 225°$
4. $\cot 270°$
5. $\tan\frac{\pi}{3}$
6. $\csc(-150°)$
7. $\tan\frac{7\pi}{2}$
8. $\cot\frac{5\pi}{6}$
9. $\csc 0$
10. $\sin\frac{11\pi}{4}$
11. $\sec(-855°)$
12. $\tan 540°$

Examples with more complicated expressions

EXAMPLE 17

Compute the following:

1. $\cos 2 \cdot 90°$
2. $2\cos 90°$
3. $2\sec 2 \cdot \frac{\pi}{3}$
4. $\sin \frac{3\pi}{4} + \cos \frac{3\pi}{2}$
5. $\cot \frac{-2\pi}{3} \csc \frac{7\pi}{6}$
6. $\tan \frac{3\pi}{4} - 2\tan \frac{5\pi}{4}$
7. $\cot \left(\frac{7\pi}{6} + \frac{\pi}{6}\right)$
8. $\cot \frac{7\pi}{6} + \frac{\pi}{6}$
9. $\sin^2 135°$
10. $\csc^2 \frac{3\pi}{2}$
11. $\sec^4 240°$
12. $3\cot^4(-315°)$
13. $\sin^2 70° + \cos^2 70°$
14. $\sin 40° + \sin(-40°)$
15. $\sin 80° - \cos 10°$

5.5 Solving diagrams with trig expressions

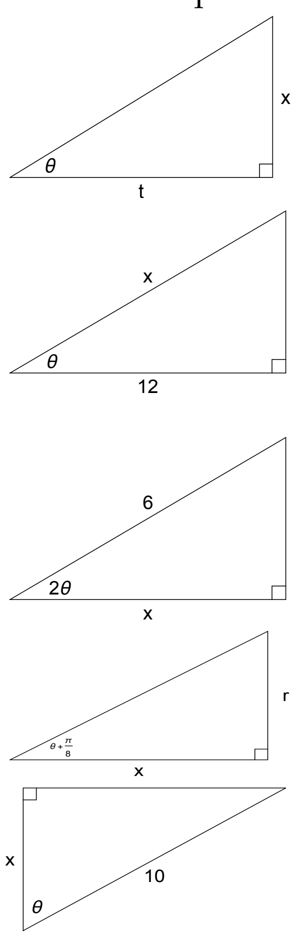
<u>Goal</u>: given a picture, write an equation which gives some quantity (like a length, an angle, area, volume, or something else) in terms of other given information in the picture.

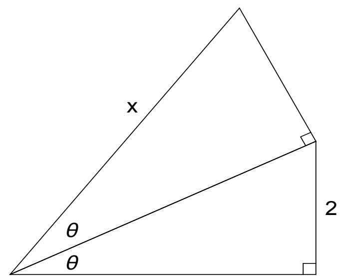
Often, the given information in the picture is either changing as time passes, or consists of unknown quantities you have to somehow solve for. That means you have to write equations/formulas for variables *in terms of other variables*.

In calculus, writing formulas with division in them is often bad (it makes the calculus harder), so we want to write formulas that do not contain division, whenever possible.

EXAMPLE 18

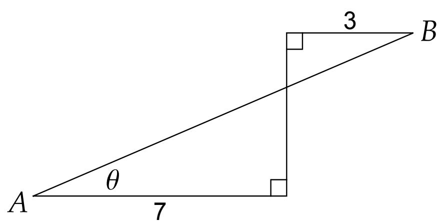
Write an equation for x, in terms of the other numbers and/or variables in the problem. Your equation should not contain division in it.





EXAMPLE 19

Let x be the distance from A to B. Write an equation for x, in terms of θ. Your equation should not contain division in it.



EXAMPLE 20

A 12 foot ladder leans up against a house. Find the distance from the bottom of the ladder to the house, in terms of the angle the ladder makes with the ground.

Chapter 6

Vectors

6.1 What is a vector?

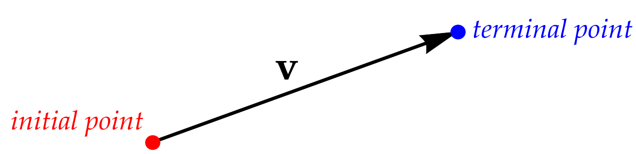
Some quantities in nature are described only by a single number, which says how great or small it is:

But to describe other quantities, you need to give both a number which represents how "big" the quantity is, and a "direction" of the quantity:

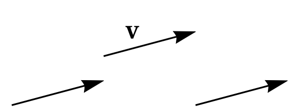
Definition 6.1 *A* **scalar** *is a number. A* **vector** *is a quantity which has two attributes: a size (called its* **magnitude** *or* **norm** *or* **length***), and a direction.*

Two vectors are **equal** if they have the same magnitude and same direction.

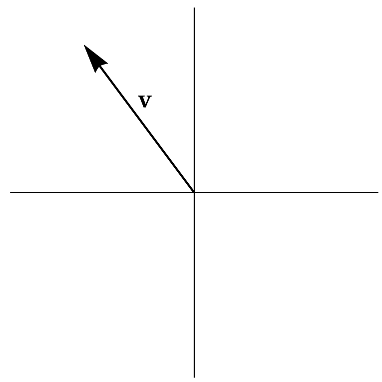
Vectors are often represented pictorially by arrows. In this context, the place the vector starts is called the **initial point** of the vector, and the place the vector ends is called the **terminal point** of the vector.



That said, vectors are really more like "floating arrows" than arrows with fixed position.



There is a standard way to draw vectors similar to the way we draw angles. A vector is in **standard position** if it is drawn on an $x, y-$plane with its initial point at the origin. When a vector is drawn in standard position, we name it by its terminal point (which has two coordinates). The coordinates of a vector are called the **horizontal** and **vertical components** of the vector. The positive angle from the x-axis to the vector is called the **direction angle** of the vector.



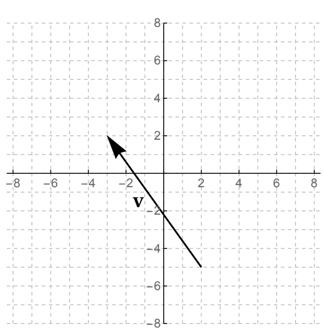
Two vectors drawn in this way are equal if and only if they have the same horizontal component and same vertical component:

$$\langle a, b\rangle = \langle c, d\rangle \qquad \text{means } a = c \text{ and } b = d.$$

More generally, the components of any vector drawn anywhere on an $x, y-$plane can be found by subtracting the initial point from the terminal point:

EXAMPLE 1

Find the components of a vector which has initial point $(2, -5)$ and terminal point $(-3, 2)$.



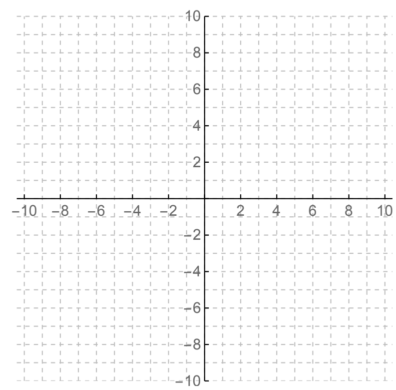
6.2 Vector operations

Theorem 6.2 (Magnitude formula) *The magnitude of vector* $\mathbf{u} = \langle a, b\rangle$ *is*

$$|\mathbf{u}| = \sqrt{a^2 + b^2}.$$

EXAMPLE 2

Find the magnitude of $\langle 2, -7\rangle$. Explain (using a picture) what this computation means.



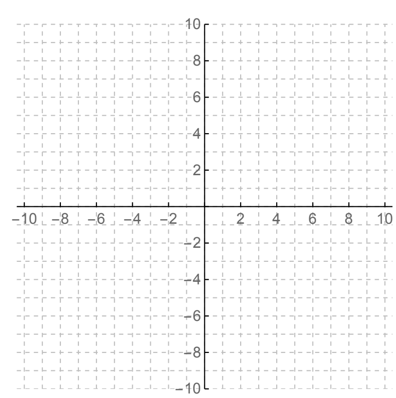
Theorem 6.3 (Direction angle formula) *The direction angle of vector* $\mathbf{u} = \langle a, b\rangle$ *is*

$$\theta = \tan^{-1}\frac{b}{a},$$

but you may have to add $180°$ *so that the angle is in the correct quadrant.*

EXAMPLE 3

Find the direction angle of $\langle 8, 5\rangle$. Explain (using a picture) what this computation means.



EXAMPLE 4

Find the magnitude and direction angle of $\langle -3, 5 \rangle$.

Theorem 6.4 (Horizontal and vertical component formulas) *Suppose* $\mathbf{v}$ *has magnitude* $|\mathbf{v}|$ *and direction angle* θ*. Then*

$$\mathbf{v} = \langle |\mathbf{v}| \cos\theta, |\mathbf{v}| \sin\theta \rangle .$$

EXAMPLE 5

Suppose that vector $\mathbf{w}$ has magnitude 6 and direction angle $140°$. Write $\mathbf{w}$ in component form (i.e. as $\langle a, b \rangle$).

EXAMPLE 6

If $|\mathbf{v}| = 25.3$ and $\mathbf{v}$ has direction angle $\theta = 292°$, find the horizontal component of $\mathbf{v}$.

Addition of vectors

Definition 6.5 (Addition of vectors) *Given two vectors* $\mathbf{v} = \langle a, b\rangle$ *and* $\mathbf{w} = \langle c, d\rangle$, *the* **sum** *of* $\mathbf{v}$ *and* $\mathbf{w}$ *is the vector*

$$\mathbf{v} + \mathbf{w} = \langle a + c, b + d\rangle \,.$$

The sum of two vectors is also called the **resultant** *of the vectors.*

This type of addition is called "coordinate-wise" or "component-wise" addition, because you add component-by-component. Subtraction of vectors is similar: given $\mathbf{v}$ and $\mathbf{w}$, to find $\mathbf{v} - \mathbf{w}$ you subtract coordinate-wise.

EXAMPLE 7

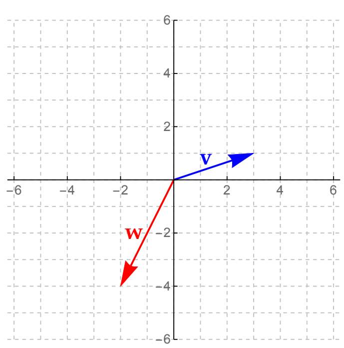
Let $\mathbf{v} = \langle 3, -2\rangle$ and let $\mathbf{w} = \langle 7, 5\rangle$. Compute the following quantities:

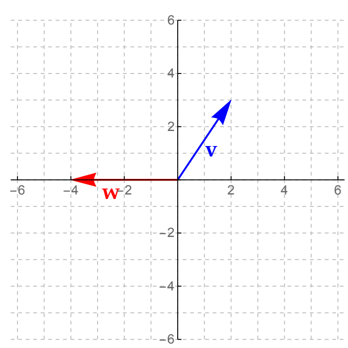
$\mathbf{v} + \mathbf{w}$

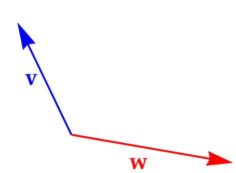
$\mathbf{w} - \mathbf{v}$

$\mathbf{w} + \mathbf{w}$

We can also think of the addition of two vectors geometrically using what is called "head-to-tail" or "parallelogram" addition:







Question: Is $|\mathbf{u} + \mathbf{v}| = |\mathbf{u}| + |\mathbf{v}|$?

Definition 6.6 (Scalar multiplication of vectors) *Given a vector $\mathbf{v} = \langle a, b\rangle$ and a scalar c, the* **(scalar) product** *of $c\mathbf{v}$ is the vector*

$$c\mathbf{v} = \langle ca, cb\rangle\,.$$

The vector $-\mathbf{v} = \langle -a, -b\rangle$ is called the **opposite** *or* **additive inverse** *of $\mathbf{v}$; it has the same magnitude as $\mathbf{v}$ but points in the opposite direction. The vector $\mathbf{0} = \overrightarrow{0} = \langle 0, 0\rangle$ is called the* **zero vector**. *For any vector $\mathbf{v}$, $0\mathbf{v} = \mathbf{0}$ and $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.*

The addition and scalar multiplication of vectors is commutative, associative and distributive. That means that the usual rules of arithmetic hold:

$$\begin{aligned}
\mathbf{v} + \mathbf{w} &= \mathbf{w} + \mathbf{v} \\
\mathbf{u} + (\mathbf{v} + \mathbf{w}) &= (\mathbf{u} + \mathbf{v}) + \mathbf{w} \\
\mathbf{v} + \mathbf{0} &= \mathbf{v} \\
\mathbf{v} + (-\mathbf{v}) &= \mathbf{0} \\
0\mathbf{v} &= \mathbf{0} \\
1\mathbf{v} &= \mathbf{v} \\
(cd)\mathbf{v} &= c(d\mathbf{v}) \\
c(\mathbf{v} + \mathbf{w}) &= c\mathbf{v} + c\mathbf{w} \\
(c + d)\mathbf{v} &= c\mathbf{v} + d\mathbf{v}
\end{aligned}$$

EXAMPLE 8

Let $\mathbf{u} = \langle 0, -3\rangle$, let $\mathbf{v} = \langle 3, -1\rangle$ and let $\mathbf{w} = \langle 7, 2\rangle$. Compute the following quantities:

$4\mathbf{w}$

$2\mathbf{u} + \mathbf{w}$

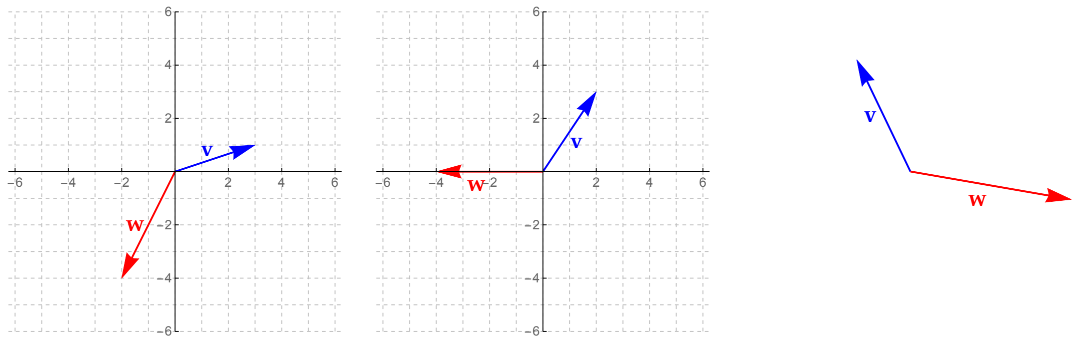
$5(\mathbf{u} + \mathbf{w}) - \mathbf{v}$

$-3\mathbf{u} - 4\mathbf{v}$

$0\mathbf{v} - 2\mathbf{u}$

$\mathbf{0} + 3\mathbf{v}$

We think of scalar multiplication as "shrinking" or "stretching" the vector:



Applications of vector addition and scalar multiplication

EXAMPLE 9

Suppose a plane takes off on a bearing $35°$ degrees west of north and flies for 100 miles. The plane then turns on a bearing of $70°$ west of north and flies for another 200 miles. How far from the airport is the plane?

Solution # 1: (uses component formulas and vector addition)

Solution # 2: (uses the Law of Cosines)

Note: In the navigation problems in your book, the direction angle is often given as a **bearing**. Unless told otherwise, a bearing is an angle measured east of north, i.e. a bearing of $40°$ is $40°$ east of north.

EXAMPLE 10

Two people push on a large box. One person applies a force of 70 newtons and another person applies a force of 85 newtons. If the angle between these two forces is $25°$, what is the magnitude of the equilibriant force? (An equilibriant force is a third force which cancels out the first two).

Dot products

Definition 6.7 *Given two vectors* $\mathbf{v} = \langle a, b \rangle$ *and* $\mathbf{w} = \langle c, d \rangle$*, the* **dot product** *of* $\mathbf{v}$ *and* $\mathbf{w}$ *is*

$$\mathbf{v} \cdot \mathbf{w} = ac + bd.$$

Note that the dot product of two vectors is a <u>*number*</u>*, not a vector.*

Example: Let $\mathbf{u} = \langle 0, -3 \rangle$, let $\mathbf{v} = \langle 3, -1 \rangle$ and let $\mathbf{w} = \langle 7, 2 \rangle$. Compute the following quantities:

$\mathbf{u} \cdot \mathbf{v}$

$2\mathbf{u} \cdot \mathbf{w}$

$\mathbf{v} \cdot (\mathbf{v} - \mathbf{w})$

Usual rules of arithmetic hold for dot product:

$$\begin{aligned} \mathbf{v} \cdot \mathbf{w} &= \mathbf{w} \cdot \mathbf{v} \\ \mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) &= \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} \\ (\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} &= \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w} \\ \mathbf{v} \cdot \mathbf{0} &= 0 \\ (c\mathbf{v}) \cdot \mathbf{w} &= \mathbf{v} \cdot c\mathbf{w} = c(\mathbf{v} \cdot \mathbf{w}) \end{aligned}$$

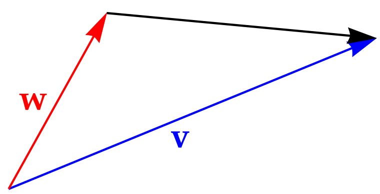
WARNING: You can't do $\mathbf{u} \cdot \mathbf{v} \cdot \mathbf{w}$:

Why does anyone care about dot products?

Theorem 6.8 (Magnitude formula with dot products) *Let* $\mathbf{v}$ *be a vector. Then*

$$|\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}} \quad \textit{and} \quad \mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2.$$

Now, let's bring addition into the picture:



Let's compute $|\mathbf{v} - \mathbf{w}|$ two different ways.

(1) From the preceding theorem applied to $\mathbf{v} - \mathbf{w}$,

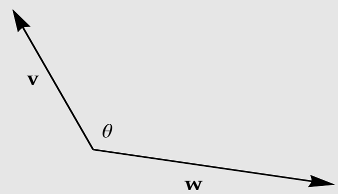
$$\begin{aligned} |\mathbf{v} - \mathbf{w}|^2 &= (\mathbf{v} - \mathbf{w}) \cdot (\mathbf{v} - \mathbf{w}) \\ &= \mathbf{v} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{w} - \mathbf{w} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{w} \\ &= |\mathbf{v}|^2 - 2\mathbf{v} \cdot \mathbf{w} + |\mathbf{w}|^2 \end{aligned}$$

(2) Use the Law of Cosines:

On the previous page, we showed:

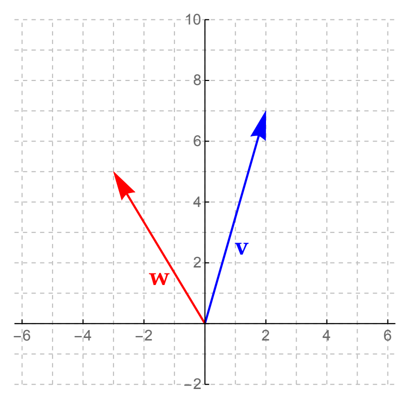
Theorem 6.9 (Angle formula with dot products) *Let* $\mathbf{v}$ *and* $\mathbf{w}$ *be two vectors and let* θ *be the angle between them. Then*

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\,\cos\theta.$$



EXAMPLE 11

Find the angle between the vectors $\langle 2,7\rangle$ and $\langle -3,5\rangle$. Then draw a picture to illustrate what you have computed.



Definition 6.10 *Two vectors* $\mathbf{v}$ *and* $\mathbf{w}$ *are called* **orthogonal** *(a.k.a.* **perpendicular***) if the angle between them is a right angle, equivalently that* $\mathbf{v} \cdot \mathbf{w} = 0$*. If* $\mathbf{v}$ *and* $\mathbf{w}$ *are orthogonal, we write* $\mathbf{v} \perp \mathbf{w}$*.*

EXAMPLE 12

Let $\mathbf{u} = \langle -4, 14\rangle$, let $\mathbf{v} = \langle 5,-3\rangle$ and let $\mathbf{w} = \langle 7,2\rangle$. Determine which two of $\mathbf{u}$, $\mathbf{v}$ and/or $\mathbf{w}$ are orthogonal.

EXAMPLE 13

Let $\mathbf{v} = \langle -3, 8 \rangle$, and let $\mathbf{w} = \langle 16, y \rangle$. If $\mathbf{v} \perp \mathbf{w}$, then find y.

6.3 Review material for Exam 3

Exam 3 content

What I will ask you to do without a calculator:

- Given a picture of a right triangle with the side lengths labelled, find any trig function of the angles in that triangle.
- Answer questions that involve the use of the reciprocal, quotient, Pythagorean and/or odd-even identities.
- Find the six trig functions of an angle, given the coordinates of a point (x, y) on the terminal side of the angle.
- Find the six trig functions of an angle, given the value of one trig function and the sign of a second trig function.
- Determine whether a trig equation is "possible" or "impossible" to solve (based on the ranges of the trig functions).
- Determine whether the sign of the trig function of an angle is positive or negative.
- Determine the quadrant an angle sits in, given the value of two of its trig functions.
- Compute any trig function of any special angle, and evaluate formulas involving trig functions of special angles and/or trig identities.
- Write formulas for some quantity in a picture in terms of other lengths (including story problems).
- Plot vectors in standard position; perform vector addition and scalar multiplication using pictures alone.

What I will ask you to do with a calculator:

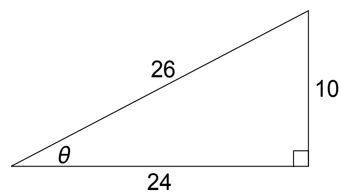
- Anything you should be able to do without a calculator.
- Find decimal approximations to values of trig functions.
- Find angle(s) for whom a trig function is given (i.e. solve $\tan\theta = q$, $\sec\theta = q$, etc.)
- Add vectors; multiply vectors by scalars; compute dot products, compute lengths of vectors; compute angles between vectors; find the coordinates of a vector given its length and angle (and vice-versa).
- Solve story problems with vectors

Practice problems (no calculator)

On these questions, no decimal approximations are allowed–all answers must be exact.

1. State the two identities that are called "Quotient Identities".

2. Find the exact values of all six trig functions of θ, if θ is as in the following picture:



3. Suppose $\tan\theta = 5$.
   a) What is $\tan(-\theta)$?
   b) What is $\cot\theta$?
   c) What is $\tan(\theta + 360°)$?

4. a) If $\cot\theta > 0$ and $\sin\theta < 0$, what quadrant is θ in?
   b) If $\sec\theta > 0$ and $\tan\theta > 0$, what quadrant is θ in?
   c) If $\csc\theta > 0$ and $\cos\theta < 0$, what quadrant is θ in?
   d) Suppose θ is in Quadrant II. Which trig functions of θ are positive?
   e) Suppose θ is in Quadrant III. Which trig functions of θ are positive?

5. Suppose $\cot\theta = \frac{2}{3}$ and $\csc\theta < 0$. Find the values of all six trig functions of θ.

6. Suppose $\sin\theta = \frac{-1}{4}$ and $\tan\theta < 0$. Find the values of all six trig functions of θ.

7. Suppose the point $(4, -3)$ is on the terminal side of an angle θ, drawn in standard position. What are the values of the six trig functions of θ?

8. Determine (without using a calculator) whether the following quantities are positive or negative:

   a) $\csc 314°$
   b) $\cot(-107.3°)$
   c) $\tan 207°$
   d) $\cos 48°$
   e) $\tan 113.75°$
   f) $\sin 280°$
   g) $\sec 406°$
   h) $\csc(-580°)$

9. Determine whether the following equations are "possible" to solve, or "impossible" to solve:

   a) $\tan\theta = -.46$
   b) $\cot\theta = -.45$
   c) $\sin\theta = .25$
   d) $\sec\theta = -4.24$
   e) $\tan\theta = 0$
   f) $\sec\theta = .8$
   g) $\csc\theta = 3$
   h) $\sin\theta = -1.25$
   i) $\cos\theta = \frac{7}{11}$
   j) $\sec\theta = -1$
   k) $\cot\theta = 2.8$
   l) $\csc\theta = \frac{2}{3}$

10. Suppose θ is in Quadrant IV. Determine whether each of the following quantities is positive or negative:

    a) $\sec\theta$
    b) $\csc\theta$
    c) $\cot\theta$
    d) $\sec(180° - \theta)$
    e) $\tan(-\theta)$
    f) $\sin(\theta + 90°)$
    g) $\cos(\theta - 180°)$
    h) $\sec(\theta + 180°)$
    i) $\cot(\theta + 720°)$
    j) $\cot(-\theta)$
    k) $\csc(180° - \theta)$
    l) $\tan(\theta + 90°)$

11. Draw a picture involving the unit circle to explain the quantity that is being asked for (you do not actually need to compute the quantity). In particular, your picture should indicate any given quantity, indicate where θ is, and indicate what you are asked to find with a "?".

    a) Find all angles between $0°$ and $360°$ such that $\tan\theta = .6$.
    b) Find all angles between $0°$ and $360°$ such that $\csc\theta = 2$.
    c) Find $\sec 130°$.
    d) If $\cos\theta = \frac{1}{3}$, find $\sec\theta$.
    e) If $\csc\theta = 1.35$ and $\cos\theta > 0$, find $\tan\theta$.
    f) Find $\cot 238°$.
    g) If $\sec\theta = \frac{5}{2}$ and $\sin\theta < 0$, find $\sin\theta$.

12. Find the exact value of these quantities:

    a) $\tan 0$
    b) $\csc 60°$
    c) $\cot 135°$
    d) $\sec 540°$
    e) $\sin 300°$
    f) $\cot 210°$
    g) $\sec 225°$
    h) $\cos 315°$
    i) $\cot 360°$
    j) $\tan 90°$
    k) $\sec 180°$
    l) $\csc 210°$
    m) $\sin 765°$
    n) $\tan -570°$
    o) $\cos 315°$
    p) $\sin 270°$
    q) $\csc 0°$
    r) $\sec 30°$
    s) $\csc 45°$
    t) $\cot 120°$

13. Find the exact value of these quantities:

a) $\csc -480°$
b) $\sin -900°$
c) $\sec 750°$
d) $\cos 150°$
e) $\tan 225°$
f) $\cot 450°$
g) $\csc -1440°$
h) $\sec 360°$
i) $\tan 420°$
j) $\cot 120°$
k) $\cos 930°$
l) $\tan 495°$
m) $\cot -210°$
n) $\csc -60°$
o) $\sin 330°$
p) $\cos 240°$

14. Find the exact value of these quantities:

a) $\sec 0$
b) $\cot \frac{5\pi}{3}$
c) $\cot \frac{11\pi}{6}$
d) $\tan \frac{\pi}{3}$
e) $\sin \frac{5\pi}{4}$
f) $\cot 8\pi$
g) $\csc \frac{13\pi}{2}$
h) $\sec \frac{-7\pi}{2}$
i) $\cot \frac{-5\pi}{2}$
j) $\tan -6\pi$
k) $\csc \frac{\pi}{6}$
l) $\sin \pi$
m) $\cos \frac{3\pi}{4}$
n) $\tan \frac{7\pi}{6}$
o) $\cot \frac{11\pi}{6}$
p) $\sin \frac{3\pi}{2}$
q) $\cos \frac{\pi}{3}$
r) $\tan \frac{2\pi}{3}$
s) $\cot \frac{5\pi}{6}$
t) $\csc 2\pi$

15. Find the exact value of these quantities:

a) $\cos -\pi$
b) $\cot \frac{-13\pi}{6}$
c) $\cos \frac{5\pi}{4}$
d) $\tan \pi$
e) $\cot \frac{7\pi}{4}$
f) $\csc \frac{5\pi}{3}$
g) $\cot \frac{3\pi}{2}$
h) $\csc 7\pi$
i) $\sin \frac{-9\pi}{2}$
j) $\cot \frac{4\pi}{3}$
k) $\sec \frac{3\pi}{4}$
l) $\cos 3\pi$
m) $\sin \frac{7\pi}{4}$
n) $\cos \frac{19\pi}{2}$
o) $\sec \frac{\pi}{6}$
p) $\csc \frac{13\pi}{4}$
q) $\cos \frac{-5\pi}{4}$
r) $\sin \frac{4\pi}{3}$
s) $\cos \frac{14\pi}{3}$
t) $\tan \frac{29\pi}{6}$

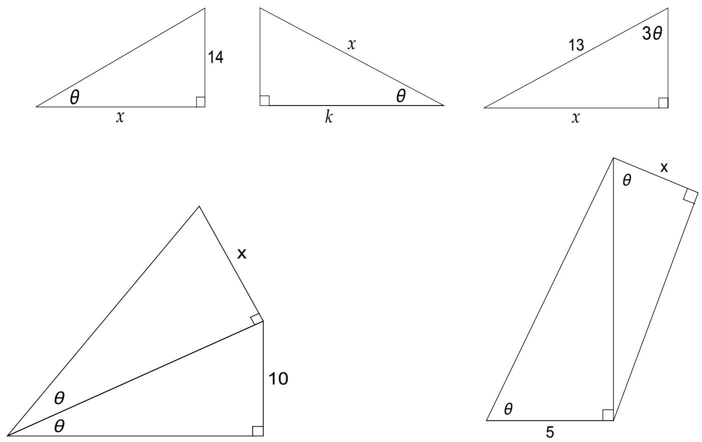
16. Find the exact value of these quantities:

a) $\sin^2 \frac{\pi}{6}$
b) $\cos^2 \frac{\pi}{4}$
c) $4 \sec 3\pi$
d) $\cot 2 \cdot 150°$
e) $3 \sin 2 \cdot 45°$
f) $\sin \left(\frac{-4\pi}{3} + \frac{2\pi}{3}\right)$
g) $\tan (90° - 45°)$
h) $\tan 90° - \tan 45°$
i) $\csc (120° + 210°)$
j) $\sin \left(\frac{8\pi}{3} + \frac{4\pi}{3}\right)$

17. Find the exact value of these quantities:

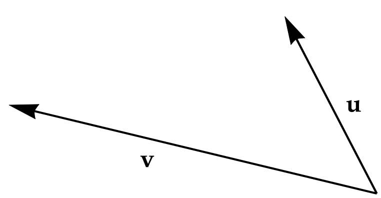
a) $\cos^2 \frac{\pi}{5} + \sin^2 \frac{\pi}{5}$
b) $\sin 27° + \sin(-27°)$
c) $\sec^2 77° - \tan^2 77°$
d) $\cot^2 \frac{3\pi}{8} - \csc^2 \frac{3\pi}{8}$
e) $\sin 50° - \cos 40°$
f) $\sin^2 220° + \cos^2 220°$
g) $\tan^2 \frac{\pi}{5} - \sec^2 \frac{\pi}{5}$
h) $\cos \frac{\pi}{7} - \cos \left(\frac{-\pi}{7}\right)$

18. Write a formula which for x in terms of the other quantities in each picture:



19. A ladder which is 15 feet long leans up against a wall. Find a formula for the distance from the bottom of the ladder to the wall, in terms of the angle the ladder makes with the wall.

20. a) Given the picture below, sketch $\mathbf{u} + \mathbf{v}$.
    b) Given the picture below, sketch $2\mathbf{u}$.
    c) Given the picture below, sketch $-\mathbf{u}$.
    d) Given the picture below, sketch $\mathbf{v} - 2\mathbf{u}$.



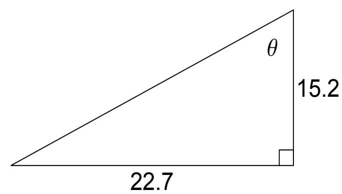
Practice problems (calculator allowed)

In these problems, decimal approximations to answers are allowed.

20. Find decimal approximations of the following:

a) $\cot 76°$
b) $\csc 119.5°$
c) $\sec 3.5$
d) $\cot 125° - \cot 40°$
e) $\tan^2 27°$
f) $\sec(41° + 23°)$
g) $\tan 4 \cdot 20°$
h) $3 \sec 17°$

21. Find the values of all six trig functions of θ, if θ is as in the following picture:



22. Suppose $\cot\theta = 3.63$ and $\cos\theta > 0$. Find $\sin\theta$ and $\sec\theta$.

23. Suppose $\cos\theta = -.42$ and $\tan\theta > 0$. Find $\sin\theta$ and $\csc\theta$.

24. Suppose $\sec\theta = 2.785$ and $\csc\theta < 0$. Find $\cot\theta$ and $\csc\theta$.

25. Find all angles θ between $0°$ and $360°$ satisfying the following equations. If there is no solution, write “DNE” or “impossible”.

a) $\tan\theta = -.46$.
b) $\cot\theta = 2.45$.
c) $\sin\theta = .25$.
d) $\sec\theta = -4.24$.
e) $\tan\theta = 0$
f) $\sec\theta = .8$.
g) $\csc\theta = 3$.
h) $\sec\theta = -1$.

26. Throughout this question, let $\mathbf{u} = \langle 2, -5\rangle$, $\mathbf{v} = \langle -1, -3\rangle$ and $\mathbf{w} = 4\mathbf{i} + \mathbf{j}$.

a) Sketch $\mathbf{u}$ in standard position.
b) Compute $3\mathbf{u}$.
c) Compute $2\mathbf{v} - \mathbf{w}$.
d) Compute $4\mathbf{u} + 2\mathbf{w}$.
e) Find the magnitude of $\mathbf{u}$.
f) Find the direction angle of $\mathbf{v}$.
g) Compute $\mathbf{u} \cdot \mathbf{w}$.
h) Compute $\mathbf{v} \cdot (\mathbf{v} - \mathbf{w})$.
i) Find the angle between $\mathbf{u}$ and $\mathbf{v}$.
j) Find the resultant of $\mathbf{u}$ and $\mathbf{v}$
k) Write down a nonzero vector which is orthogonal to $\mathbf{u}$.

27. Let $\mathbf{q}$ be the vector whose direction angle is 120° and whose magnitude is 7. Find the horizontal and vertical components of $\mathbf{q}$.

28. Two forces act on a point, with respective magnitudes 30 and 45 newtons. If the angle between the forces is 70°, find the magnitude of the equilibriant.

29. A ship leaves port on a bearing (east of north) of 28° and travels 82 miles. Then the ship turns due east and travels another 43 miles. How far is the ship from port?

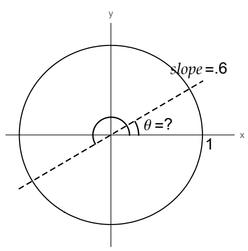
Solutions to these practice problems

As always, I did these by hand; there may be errors.

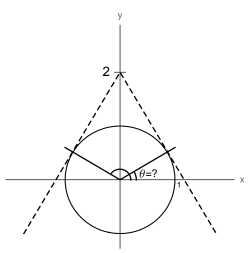
1. $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and $\cot\theta = \frac{\cos\theta}{\sin\theta}$.
2. $\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{10}{26} = \frac{5}{13}$; $\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{24}{26} = \frac{12}{13}$; $\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{10}{24} = \frac{5}{12}$; $\cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{24}{10} = \frac{12}{5}$; $\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{26}{24} = \frac{13}{12}$; $\csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{26}{10} = \frac{13}{5}$.
3. a) $\tan(-\theta) = -\tan\theta = -5$.
   b) $\cot\theta = \frac{1}{\tan t} = \frac{1}{5}$.
   c) $\tan(\theta + 360°) = \tan\theta = 5$.
4. a) III
   b) I
   c) II
   d) sine and cosecant
   e) tangent and cotangent
5. We are in Quadrant III, so the only two trig functions with positive values are tangent and cotangent. Draw a triangle with adjacent side 2 and opposite side 3; the hypotenuse is therefore $\sqrt{2^2+3^2} = \sqrt{13}$. Then $\tan\theta = \frac{3}{2}$, $\cot\theta = \frac{2}{3}$, $\csc\theta = \frac{-\sqrt{13}}{3}$, $\sin\theta = \frac{-3}{\sqrt{13}}$, $\cos\theta = \frac{-2}{\sqrt{13}}$, $\sec\theta = \frac{-\sqrt{13}}{2}$.
6. We are in Quadrant IV, so the only two trig functions with positive values are cosine and secant. Draw a triangle with opposite side 1 and hypotenuse 4; then solve for the adjacent side to get $\sqrt{4^2-1^2} = \sqrt{15}$. Then, $\sin\theta = \frac{-1}{4}$, $\cos\theta = \frac{\sqrt{15}}{4}$; $\tan\theta = \frac{-1}{\sqrt{15}}$, $\cot\theta = -\sqrt{15}$, $\sec\theta = \frac{4}{\sqrt{15}}$, and $\csc\theta = -4$.
7. We are given $x = 4$ and $y = -3$; first compute $r = \sqrt{x^2+y^2} = \sqrt{4^2+(-3)^2} = \sqrt{25} = 5$. Then $\sin\theta = \frac{y}{r} = \frac{-3}{5}$; $\cos\theta = \frac{4}{5}$; $\tan\theta = \frac{y}{x} = \frac{-3}{4}$; $\cot\theta = \frac{x}{y} = \frac{4}{-3}$; $\csc\theta = \frac{r}{y} = \frac{5}{-3}$; $\sec\theta = \frac{r}{x} = \frac{5}{4}$.
8. a) negative (Quadrant IV)
   b) positive (Quadrant III)
   c) positive (Quadrant III)
   d) positive (Quadrant I)
   e) negative (Quadrant II)
   f) negative (Quadrant IV)
   g) positive (Quadrant I)
   h) positive (Quadrant II)
9. a) possible
   b) possible
   c) possible
   d) possible
   e) possible
   f) impossible
   g) possible
   h) impossible
   i) possible
   j) possible
   k) possible
   l) impossible

10. a) $\sec\theta$ is positive in Quadrant IV.
  b) $\csc\theta$ is negative in Quadrant IV.
  c) $\cot\theta$ is negative in Quadrant IV.
  d) $180^\circ - \theta$ is in Quadrant III, so $\sec(180^\circ - \theta)$ is negative.
  e) $-\theta$ is in Quadrant I, so $\tan(-\theta)$ is positive.
  f) $\theta + 90^\circ$ is in Quadrant I, so $\sin(\theta + 90^\circ)$ is positive.
  g) $\theta - 180^\circ$ is in Quadrant II, so $\cos(\theta - 180^\circ)$ is negative.
  h) $\theta + 180^\circ$ is in Quadrant II, so $\sec(\theta + 180^\circ)$ is negative.
  i) $\theta + 720^\circ$ is the same as θ, so $\cot(\theta + 720^\circ)$ is negative.
  j) $-\theta$ is in Quadrant I, so $\cot(-\theta)$ is positive.
  k) $180^\circ - \theta$ is in Quadrant III, so $\csc(180^\circ - \theta)$ is negative.
  l) $90^\circ + \theta$ is in Quadrant I, so $\tan(\theta + 90^\circ)$ is positive.

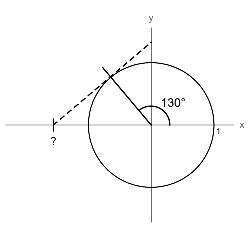
11. a)



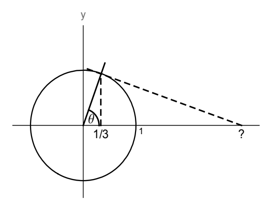
b)



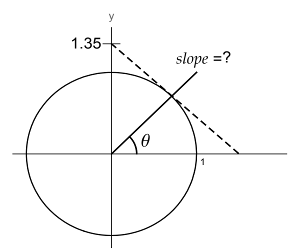
c)



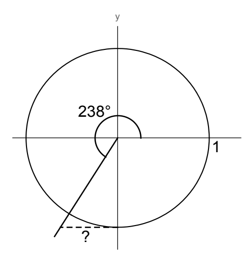
d)



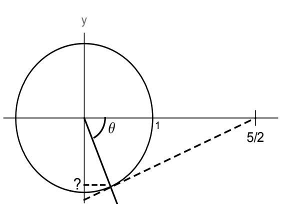
e)



f)



g)



12.
a) $\tan 0 = 0$
b) $\csc 60° = \frac{2}{\sqrt{3}}$
c) $\cot 135° = -1$
d) $\sec 540° = -1$
e) $\sin 300° = \frac{-\sqrt{3}}{2}$
f) $\cot 210° = \sqrt{3}$
g) $\sec 225° = -\sqrt{2}$
h) $\cos 315° = \frac{\sqrt{2}}{2}$
i) $\cot 360°$ DNE
j) $\tan 90°$ DNE
k) $\sec 180° = -1$
l) $\csc 210° = -2$
m) $\sin 765° = \frac{\sqrt{2}}{2}$
n) $\tan -570° = -\sqrt{3}$
o) $\cos 315° = \frac{\sqrt{2}}{2}$
p) $\sin 270° = -1$
q) $\csc 0°$ DNE
r) $\sec 30° = \frac{2}{\sqrt{3}}$
s) $\csc 45° = \sqrt{2}$
t) $\cot 120° = -\sqrt{3}$

13.
a) $\csc -480° = \frac{2}{-\sqrt{3}}$
b) $\sin -900° = 0$
c) $\sec 750° = \frac{2}{\sqrt{3}}$
d) $\cos 150° = \frac{-\sqrt{3}}{2}$
e) $\tan 225° = 1$
f) $\cot 450° = 0$
g) $\csc -1440°$ DNE
h) $\sec 360° = 1$
i) $\tan 420° = \sqrt{3}$
j) $\cot 120° = \frac{-1}{\sqrt{3}}$
k) $\cos 930° = \frac{-\sqrt{3}}{2}$
l) $\tan 495° = -1$
m) $\cot -210° = -\sqrt{3}$
n) $\csc -60° = \frac{-2}{\sqrt{3}}$
o) $\sin 330° = \frac{-1}{2}$
p) $\cos 240° = \frac{-1}{2}$

14.
a) $\sec 0 = 1$
b) $\cot \frac{5\pi}{3} = \frac{-1}{\sqrt{3}}$
c) $\cot \frac{11\pi}{6} = -\sqrt{3}$
d) $\tan \frac{\pi}{3} = \sqrt{3}$
e) $\sin \frac{5\pi}{4} = \frac{-\sqrt{2}}{2}$
f) $\cot 8\pi$ DNE
g) $\csc \frac{13\pi}{2} = 1$
h) $\sec \frac{-7\pi}{2}$ DNE
i) $\cot \frac{-5\pi}{2} = 0$
j) $\tan -6\pi = 0$
k) $\csc \frac{\pi}{6} = 2$
l) $\sin \pi = 0$
m) $\cos \frac{3\pi}{4} = \frac{-\sqrt{2}}{2}$
n) $\tan \frac{7\pi}{6} = \frac{1}{\sqrt{3}}$
o) $\cot \frac{11\pi}{6} = -\sqrt{3}$
p) $\sin \frac{3\pi}{2} = -1$
q) $\cos \frac{\pi}{3} = \frac{1}{2}$
r) $\tan \frac{2\pi}{3} = -\sqrt{3}$
s) $\cot \frac{5\pi}{6} = -\sqrt{3}$
t) $\csc 2\pi$ DNE

15.
a) $\cos -\pi = -1$
b) $\cot \frac{-13\pi}{6} = -\sqrt{3}$
c) $\cos \frac{5\pi}{4} = \frac{-\sqrt{2}}{2}$
d) $\tan \pi = 0$
e) $\cot \frac{7\pi}{4} = -1$
f) $\csc \frac{5\pi}{3} = \frac{-2}{\sqrt{3}}$
g) $\cot \frac{3\pi}{2} = 0$
h) $\csc 7\pi$ DNE
i) $\sin \frac{-9\pi}{2} = -1$
j) $\cot \frac{4\pi}{3} = \frac{1}{\sqrt{3}}$
k) $\sec \frac{3\pi}{4} = -\sqrt{2}$
l) $\cos 3\pi = -1$
m) $\sin \frac{7\pi}{4} = \frac{-\sqrt{2}}{2}$
n) $\cos \frac{19\pi}{2} = 0$
o) $\sec \frac{\pi}{6} = \frac{2}{\sqrt{3}}$
p) $\csc \frac{13\pi}{4} = -\sqrt{2}$
q) $\cos \frac{-5\pi}{4} = \frac{-\sqrt{2}}{2}$
r) $\sin \frac{4\pi}{3} = \frac{-\sqrt{3}}{2}$
s) $\cos \frac{14\pi}{3} = \frac{-1}{2}$
t) $\tan \frac{29\pi}{6} = \frac{-1}{\sqrt{3}}$

16.
a) $\sin^2 \frac{\pi}{6} = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$
b) $\cos^2 \frac{\pi}{4} = \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2}$
c) $4 \sec 3\pi = 4(-1) = -4$

d) $\cot 2 \cdot 150^\circ = \cot 300^\circ = -\sqrt{3}$

e) $3 \sin 2 \cdot 45^\circ = 3 \sin 90^\circ = 3(1) = 3$

f) $\sin\left(\frac{-4\pi}{3} + \frac{2\pi}{3}\right) = \sin\left(\frac{-2\pi}{3}\right) = \frac{-\sqrt{3}}{2}$

g) $\tan(90^\circ - 45^\circ) = \tan 45^\circ = 1$

h) $\tan 90^\circ - \tan 45^\circ$ DNE

i) $\csc(120^\circ + 210^\circ) = \csc 330^\circ = -2$

j) $\sin\left(\frac{8\pi}{3} + \frac{4\pi}{3}\right) = \sin 4\pi = 0$

17. a) $\cos^2 \frac{\pi}{5} + \sin^2 \frac{\pi}{5} = 1$

b) $\sin 27^\circ + \sin(-27^\circ) = 0$

c) $\sec^2 77^\circ - \tan^2 77^\circ = 1$

d) $\cot^2 \frac{3\pi}{8} - \csc^2 \frac{3\pi}{8} = -1$

e) $\sin 50^\circ - \cos 40^\circ = 0$

f) $\sin^2 220^\circ + \cos^2 220^\circ = 1$

g) $\tan^2 \frac{\pi}{5} - \sec^2 \frac{\pi}{5} = -1$

h) $\cos \frac{\pi}{7} - \cos\left(\frac{-\pi}{7}\right) = 0$

18. a) $x = 14 \cot \theta$

b) $x = k \sec \theta$

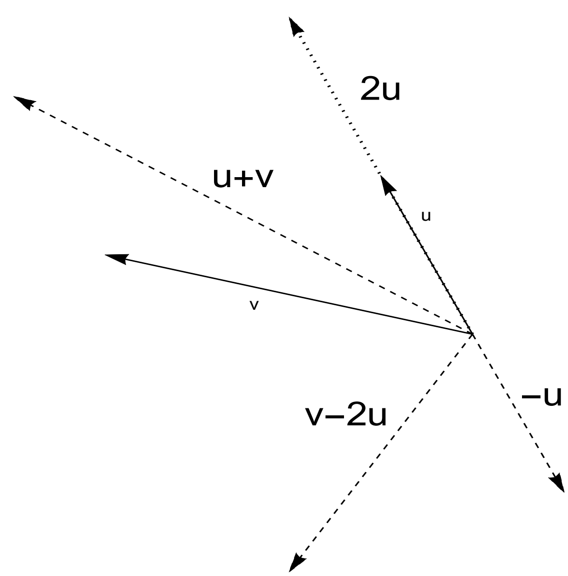
c) $x = 13 \sin 3\theta$

d) First, let w be the hypotenuse of the bottom triangle; then $w = 10 \csc \theta$. Next, $x = w \tan \theta$ so by substituting, $x = 10 \csc \theta \tan \theta$.

e) First, let w be the vertical line segment; then $w = 5 \tan \theta$. Next, $x = w \cos \theta$ so by substituting, $x = 5 \tan \theta \cos \theta$ (this simplifies to $x = 5 \sin \theta$, but you don't need to perform this simplification).

19. $x = 15 \sin \theta$.

20.



21. a) $\cot 76° = .249328$
    b) $\csc 119.5° = 1.14896$
    c) $\sec 3.5 = -1.06786$
    (note 3.5 is radians, not degrees)
    d) $\cot 125° - \cot 40° = -1.89196$
    e) $\tan^2 27° = .259616$
    f) $\sec(41° + 23°) = \sec 64° = 2.28117$
    g) $\tan 4 \cdot 20° = \tan 80° = 5.67128$
    h) $3 \sec 17° = 3(1.04569) = 3.13708$

22. First, find the hypotenuse c by the Pythagorean Theorem: $c^2 = 22.7^2 + 15.2^2$, so $c^2 = 746.33$ and therefore $c = \sqrt{746.33} = 27.32$. Therefore $\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{22.7}{27.32} = .831$; $\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{15.2}{27.32} = .556$; $\cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{15.2}{22.7} = .669$; $\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{22.7}{15.2} = 1.493$; $\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{27.32}{15.2} = 1.797$; $\csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{27.32}{22.7} = 1.203$.

23. We are in Quadrant I. Draw a right triangle with adjacent side 3.63 and opposite side 1; then compute the hypotenuse which is $\sqrt{3.63^2 + 1} = 3.765$. Therefore $\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{1}{3.765} = .265$ and $\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{3.765}{3.63} = 1.037$.

24. We are in Quadrant III. Draw a right triangle with adjacent side .42 and hypotenuse 1; then compute the opposite side which is $\sqrt{1^2 - (.42)^2} = .907$. Therefore $\sin\theta = -\frac{\text{opp}}{\text{hyp}} = -\frac{.907}{1} = -.907$ and $\csc\theta = -\frac{\text{hyp}}{\text{oppj}} = -\frac{1}{.907} = 1.102$.

25. We are in Quadrant IV. Draw a right triangle with hypotenuse 2.785 and adjacent side 1; then compute the opposite side which is $\sqrt{2.785^2 - 1^2} = 2.599$. Therefore $\cot\theta = -\frac{\text{adj}}{\text{opp}} = -\frac{1}{2.599} = -.384$ and $\csc\theta = -\frac{\text{hyp}}{\text{oppj}} = -\frac{2.785}{2.599} = -1.071$.

26. a) From a calculator, $\theta = \tan^{-1}(-.46) = -24.7°$. This is not between 0° and 360°; to get the solutions in the appropriate range, add 180° to get $180° + \theta = 155.3°$ and $360° + \theta = 335.3°$.
    b) We have $\tan\theta = \frac{1}{2.45} = .408$. So from a calculator, $\theta = \tan^{-1}(.408) = 22.2°$; another solution is $180° + \theta = 202.2°$.
    c) From a calculator, $\theta = \sin^{-1}(.25) = 14.5°$. Another solution is $180° - 14.5° = 165.5°$.
    d) We have $\cos\theta = \frac{1}{-4.24} = -.235$; then from a calculator $\theta = \cos^{-1}(-.235) = 103.6°$. Another solution is $360° - \theta = 256.4°$.
    e) From a calculator, $\theta = \tan^{-1} 0 = 0$; another solution is $\theta + 180° = 180°$.
    f) This has no solution because $\sec\theta$ cannot be between -1 and 1.

g) We have $\sin\theta = \frac{1}{3}$ so from a calculator, $\theta = \sin^{-1}\frac{1}{3} = 19.5°$; another solution is $180° - \theta = 160.5°$.

h) $\theta = 180°$ is the only solution between $0°$ and $360°$.

27. a) Draw an x, y-plane and draw an arrow which starts at $(0,0)$ and ends at $(2,-5)$ (the arrow should point southeast).

b) $3\mathbf{u} = \langle 6, -15\rangle$.

c) $2\mathbf{v} - \mathbf{w} = \langle -6, -7\rangle$.

d) $4\mathbf{u} + 2\mathbf{w} = \langle 16, -18\rangle$.

e) $|\mathbf{u}| = \sqrt{2^2 + (-5)^2} = \sqrt{29}$.

f) $\theta = \tan^{-1}\frac{-3}{-1} = \tan^{-1} 3 = 71.56°$, but this is the wrong quadrant (we need to be in Quadrant III). So add $180°$ to get $251.56°$.

g) $\mathbf{u}\cdot\mathbf{w} = 2(4) + (-5)1 = 3$.

h) $\mathbf{v}\cdot(\mathbf{v} - \mathbf{w}) = \langle -1, -3\rangle \cdot \langle -5, -4\rangle = 17$.

i) We know the angle θ satisfies $\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} = \frac{13}{\sqrt{29}\cdot\sqrt{10}} = .7633$. Therefore $\theta = \cos^{-1}(.7633) = 40.23°$.

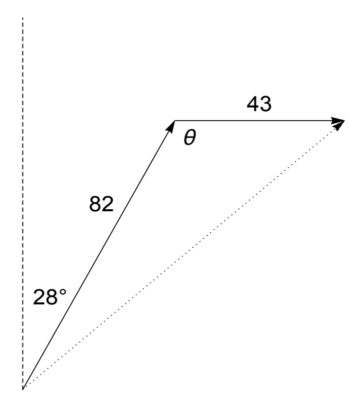
j) $\mathbf{u} + \mathbf{v} = \langle 1, -8\rangle$.

k) We need a vector whose dot product with $\mathbf{u}$ is zero; one of many such vectors is $\langle 5, 2\rangle$.

28. The horizontal component is $x = 7\cos 120° = 7 \cdot \frac{-1}{2} = -3.5$ and the vertical component is $y = 7\sin 120° = \frac{7\sqrt{3}}{2} = 6.06$. So the vector is $\langle -3.5, 6.06\rangle$.

29. Let the two forces be $\mathbf{v}$ and $\mathbf{w}$; then the magnitude of the equilibriant is $|\mathbf{v} + \mathbf{w}|$. Assume the first force is horizontal, so that its components are $\mathbf{v} = \langle 30, 0\rangle$. Then the second force has components $\mathbf{w} = \langle 45\cos 70°, 45\sin 70°\rangle = \langle 15.39, 42.29\rangle$. Adding these, we see that $\mathbf{v} + \mathbf{w} = \langle 30 + 15.39, 0 + 42.29\rangle = \langle 45.39, 42.29\rangle$. The magnitude of this is $|\mathbf{v} + \mathbf{w}| = \sqrt{45.39^2 + 42.29^2} = 62.036$.

30. The situation is described by the following picture:



Solution # 1: The angle θ indicated in the picture above is $90^\circ + 28^\circ = 118^\circ$. The distance from the ship to the port is the length of the dashed vector, which by the Law of Cosines is $\sqrt{82^2 + 43^2 - 2(82)(43)\cos 118^\circ} = \sqrt{11883.7} = 109.012$ miles.

Solution # 2: You can also solve this by finding the components of the two vectors, adding them, and then finding the magnitude. The first vector has components $\langle 82\cos 62^\circ, 82\sin 62^\circ\rangle = \langle 38.497, 72.402\rangle$ and the second vector has components $\langle 43, 0\rangle$, so the dashed vector has components

$$\langle 38.497 + 43, 72.402 + 0\rangle = \langle 81.497, 72.402\rangle .$$

The magnitude of this vector is $\sqrt{(81.497)^2 + (72.402)^2} = 109.012$ miles.

Chapter 7

Trigonometric identities

7.1 Elementary trig identities

Here is a list of trig identities that it is good to know:

Theorem 7.1 (Reciprocal identities)

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sin\theta = \frac{1}{\csc\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

Theorem 7.2 (Quotient identities)

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Theorem 7.3 (Pythagorean identities)

$$\sin^2\theta + \cos^2\theta = 1 \qquad \sec^2\theta = 1 + \tan^2\theta \qquad \csc^2\theta = 1 + \cot^2\theta$$

Theorem 7.4 (Odd-even identities)

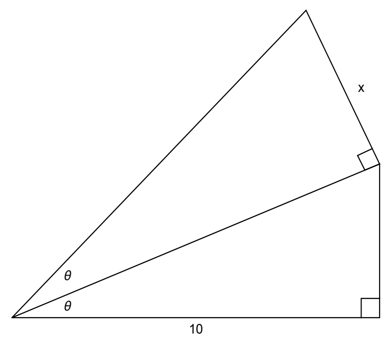
$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta \qquad \tan(-\theta) = -\tan\theta$$

$$\csc(-\theta) = -\csc\theta \qquad \sec(-\theta) = \sec\theta \qquad \cot(-\theta) = -\cot\theta$$

The reason it is good to know these identities is that they allow you to simplify more complicated trig expressions that may arise in a problem.

EXAMPLE (FROM CHAPTER 5)

Write an equation for x in terms of θ, and simplify your answer:



Alternate forms of the Pythagorean identities

Consider the statement "$3 + 5 = 8$". There are several ways to restate this fact, all of which are equivalent:

There are also some incorrect ways to restate this:

Each of the Pythagorean identities can be rewritten in similar ways:

$$\sin^2\theta + \cos^2\theta = 1 \qquad \sec^2\theta = 1 + \tan^2\theta \qquad \csc^2\theta = 1 + \cot^2\theta$$

EXAMPLE 1

If the following expressions can be simplified, simplify them; otherwise, do nothing.

$\tan^2\theta + 1$	$\csc^2\theta + \sin^2\theta$
$\tan^2\theta - 1$	$\cot^2\theta + 1$
$1 - \sec^2\theta$	$1 - \sin^2\theta$
$\sin^2\theta + \cos^2\theta$	$\csc^2\theta - 1$
$\sec^2\theta - \tan^2\theta$	$\csc^2\theta - \sec^2\theta$
$1 + \csc^2\theta$	$1 - \cos^2\theta$
$1 + \cos^2\theta$	$\sec^2\theta - 1$
$1 + \tan^2\theta$	$\cot^2\theta - 1$
$1 - \csc^2\theta$	$1 + \cot^2\theta$

Simplifying trig expressions

To simplify a trig expression:

1. Use the odd-even identities to remove any $-$ signs from inside the trig functions.
2. If possible, simplify the expression using a Pythagorean identity.
3. Write whatever is left in terms of sines and cosines, using the quotient and reciprocal identities.
4. Simplify using algebra.

EXAMPLE 2

Simplify each expression as much as possible, and write your answer so that no quotients appear in the final answer.

$\tan\theta\cos\theta$

$\csc\theta\cos\theta\tan\theta$

$\frac{\tan(-\theta)}{\sec\theta}$

$\frac{1-\sin^2(-\theta)}{1+\cot^2(-\theta)}$

Verifying identities

Suppose you are given a weird looking equation (or "identity") involving trig functions. To verify that the identity is true, work out both sides in terms of sines and cosines, using algebra to simplify when possible, and check that both sides work out to the same thing. Until you know the sides are equal, don't claim they are equal by writing a "=".

EXAMPLE 3

Verify that each equation is an identity:

$\frac{\tan\theta}{\sec\theta} = \sin\theta$

$\sin^2 x(1+\cot^2 x) = 1$

$\frac{\csc\theta\sec\theta}{\cot\theta} = \tan^2\theta + 1$

7.2 Sum and difference identities

We have seen several times in this course that if α and β are two angles, then in general,

$$\sin(\alpha+\beta) \neq \sin\alpha + \sin\beta$$

(and similarly for cosine, tangent, secant, etc.)

Question: Can you figure $\sin(\alpha+\beta)$ from $\sin\alpha$ and $\sin\beta$?

Answer: Yes, but it's complicated; you need the following identities which do not need to be memorized.

Theorem 7.5 (Sum Identities)

$$\begin{aligned}
\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\
\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\
\tan(\alpha+\beta) &= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}
\end{aligned}$$

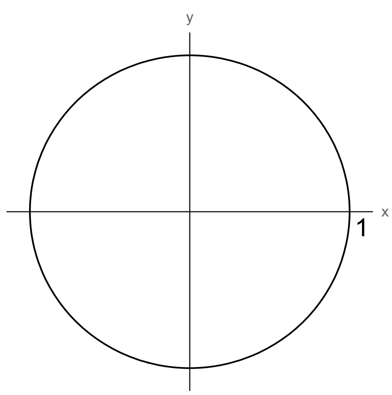
Theorem 7.6 (Difference Identities)

$$\begin{aligned}
\sin(\alpha-\beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta \\
\cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\
\tan(\alpha-\beta) &= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}
\end{aligned}$$

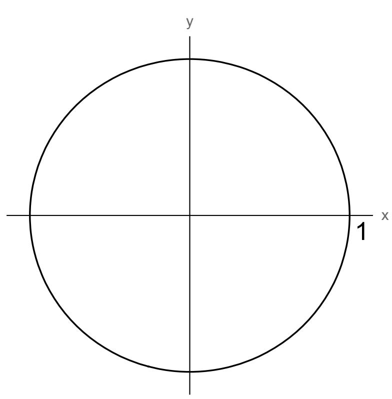
The next few pages give some examples of how you use these identities.

EXAMPLE 4

1. Find $\sin(\alpha+\beta)$ if α is in Quadrant II, β is in Quadrant I, $\sin\alpha = \frac{3}{5}$ and $\sin\beta = \frac{2}{3}$.



2. Find $\tan(\alpha-\beta)$ if $\tan\alpha = 2$, $\tan\beta = 3$, $\sin\alpha < 0$ and $\cos\beta > 0$.



3. Find the exact value of $\sin 165°$.

4. Find the exact value of $\cos 15°$.

   Solution: Think of $15°$ as $60° - 45°$. Then

$$\begin{aligned}
\cos 15° &= \cos(60° - \cos 45°) \\
&= \cos 60° \cos 45° + \sin 60° \sin 45° \quad \text{(by the difference identity)} \\
&= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} \\
&= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} \\
&= \frac{\sqrt{2} + \sqrt{6}}{4}.
\end{aligned}$$

7.3 More identities

Earlier we learned that trig functions do not respect multiplication and division, i.e.

$$\sin 2\theta \neq 2\sin\theta \qquad\qquad \cos\frac{\theta}{2} \neq \frac{\cos\theta}{2}$$

So how can you simplify or rewrite expressions like $\sin 2\theta$ or $\cos\frac{\theta}{2}$?

Theorem 7.7 (Double-angle Identities)

$$\begin{aligned}
\sin 2\theta &= 2\sin\theta\cos\theta \\
\cos 2\theta &= 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \cos^2\theta - \sin^2\theta \\
\tan 2\theta &= \frac{2\tan\theta}{1-\tan^2\theta}
\end{aligned}$$

Theorem 7.8 (Half-angle Identities)

$$\begin{aligned}
\sin\frac{\theta}{2} &= \pm\sqrt{\frac{1-\cos\theta}{2}} \\
\cos\frac{\theta}{2} &= \pm\sqrt{\frac{1+\cos\theta}{2}} \\
\tan\frac{\theta}{2} &= \frac{1-\cos\theta}{\sin\theta}
\end{aligned}$$

EXAMPLE 5

1. Find $\sin 2\theta$, if $\sin\theta = \frac{3}{4}$ and $\cos\theta < 0$.

2. Find the exact value of $\sin 22.5°$.

Solution:

$$\sin 22.5° = \sin \frac{45°}{2} = \sqrt{\frac{1-\cos 45°}{2}} = \sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}}.$$

Here are some more identities that allow you to convert between sums of trig expressions and products of trig expressions. They are occasionally used in acoustics and the study of waves (light waves, sound waves, radio waves, etc.), but if you ever needed one of these identities, you'd just look it up (in fact, your professor doesn't even know these off the top of his head... that said, he is aware of their existence and you should be too).

Theorem 7.9 (Sum-to-Product Identities)

$$\begin{aligned}
\sin\alpha + \sin\beta &= 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}\\
\sin\alpha - \sin\beta &= 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}\\
\cos\alpha + \cos\beta &= 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}\\
\cos\alpha - \cos\beta &= -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}
\end{aligned}$$

Theorem 7.10 (Product-to-Sum Identities)

$$\begin{aligned}
\sin\alpha\sin\beta &= \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]\\
\cos\alpha\cos\beta &= \frac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]\\
\sin\alpha\cos\beta &= \frac{1}{2}\left[\sin(\alpha-\beta) + \sin(\alpha+\beta)\right]
\end{aligned}$$

Chapter 8

Graphs of the trigonometric functions

8.1 Sinusodal graphs

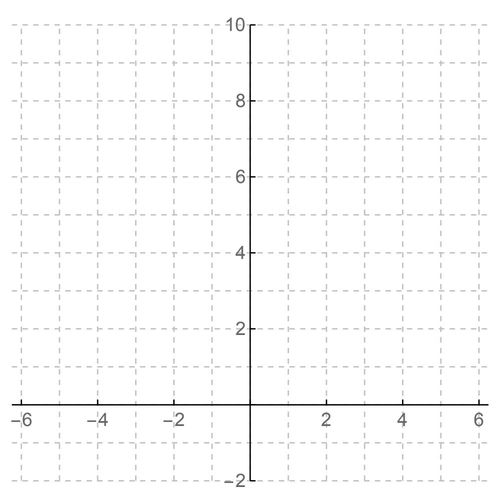
What is a "graph"?

Given a function f, the **graph** of f is a picture of the points (x, y) in the xy-plane for which $y = f(x)$.

MOTIVATING EXAMPLE

Sketch the graph of $y = x^2$ (also known as $f(x) = x^2$)

x	$y = f(x) = x^2$
-2	
-1	
0	
1	
2	
3	



A crude picture of the graph of $y = x^2$:

A crude picture like this one is usually enough in the context of application problems, because the crude picture contains all the "essential" properties of the graph (i.e. that it is a curve that is symmetric about the y-axis, turns at $(0, 0)$, is a parabola, etc.).

Our goal is to be able to quickly sketch crude graphs of (relatively simple) functions involving the six trig functions. We want to be able to do this without using a calculator, and in general, more quickly than we could if we used a calculator.

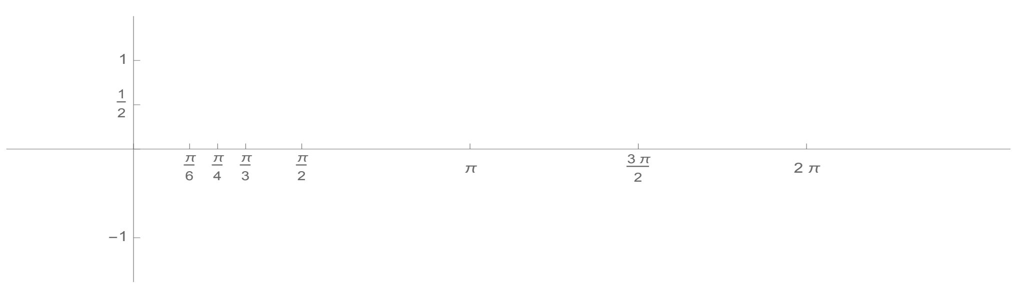
If the function isn't "relatively simple", then it isn't terribly useful to learn how to graph it, given that calculators and computers draw accurate graphs of complicated functions.

Graphs of $\sin x$ and $\cos x$

Let's start with the graph of $y = \sin x$.

Note: in the context of graphing, x is always assumed to be in radians.

x	$y = \sin x$
0	
$\frac{\pi}{6}$	
$\frac{\pi}{4}$	
$\frac{\pi}{3}$	
$\frac{\pi}{2}$	
π	
$\frac{3\pi}{2}$	
2π	



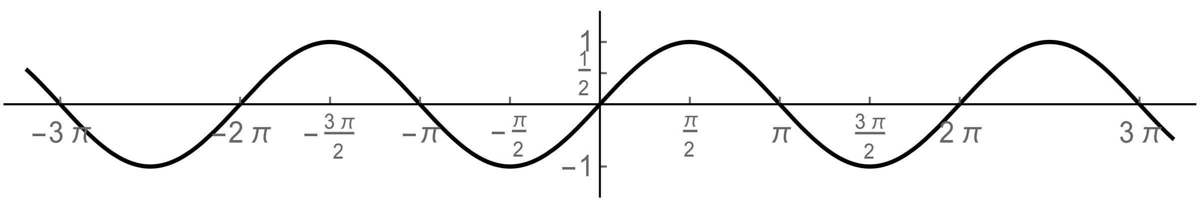
This is the graph of $y = \sin x$ for x between 0 and 2π. What happens for other x?

Definition 8.1 *The* **period** *of a function is the distance in the x-direction it takes for the graph of the function to start repeating itself. Algebraically, the period is the smallest positive number p such that $f(x+p) = f(x)$ for all x.*

EXAMPLES

- The period of $y = \sin x$ is 2π.
- The period of $y = \cos x$ is 2π (because $\cos(x + 2\pi) = \cos x$).
- There is no period of $y = x^2$ (because the graph doesn't repeat itself).

To summarize, here is a picture of the graph of $y = \sin x$:



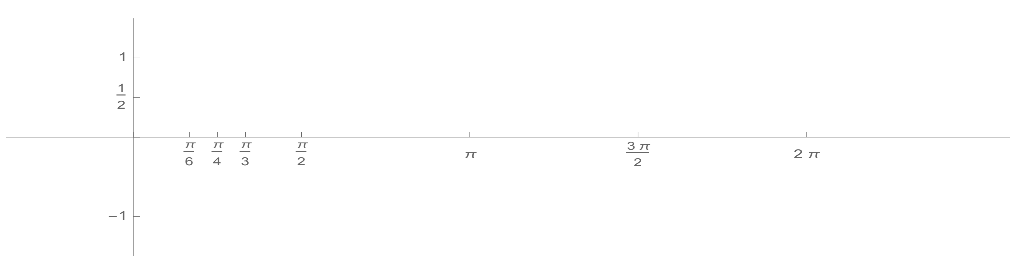
The crude picture you need to be able to draw is this:

This crude picture shows the following important information about $y = \sin x$:
- the graph goes through $(0, 0)$;
- one complete period of the graph is 2π;
- the graph goes as far up as 1 and as far down as -1;
- the general shape of the graph is a wave;
- from the point $(0, 0)$, the graph heads upward as you move to the right

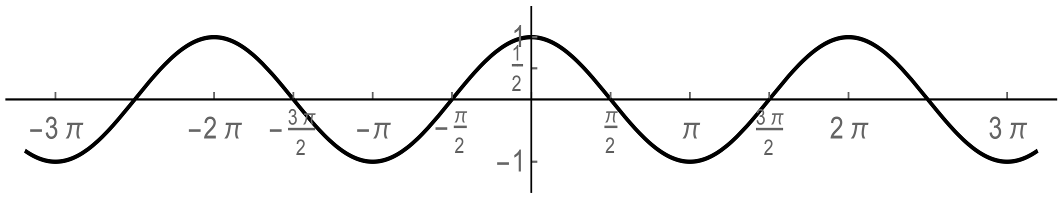
Now let's do $y = \cos x$. As with $\sin x$, the graph has period 2π so we only need to know what the shape looks like from $x = 0$ to $x = 2\pi$:

x	$y = \cos x$
0	
$\frac{\pi}{6}$	
$\frac{\pi}{4}$	
$\frac{\pi}{3}$	
$\frac{\pi}{2}$	
π	-1
$\frac{3\pi}{2}$	0
2π	1

Here is the graph of $y = \cos x$ for x between 0 and 2π:



Here is the graph of $y = \cos x$:



The crude picture you need to be able to draw is this:

This crude picture shows the following important information about $y = \cos x$:
- the graph goes through $(0, 1)$;
- one complete period of the graph is 2π;
- the graph goes as far up as 1 and as far down as -1;
- the general shape of the graph is a wave;
- from the point $(0, 1)$, the graph heads downward as you move to the right

Transformations on sine and cosine

Start with a function that has some formula. A **transformation** of a function refers to a function whose formula comes from the first function by including some extra constant(s) somewhere.

EXAMPLES

- Transformations of the function $y = x^2$ include

$$y = x^2 + 2 \qquad y = \frac{-1}{4}x^2 \qquad y = (x-2)^2 \qquad y = (3x)^2 - 1 \qquad \text{etc.}$$

- Transformations of the function $y = \sin x$ include

$$y = 2\sin x \qquad y = \sin(x+3) \qquad y = \sin 6x - 5 \qquad y = \frac{2}{3}\sin 2x - 1 \qquad \text{etc.}$$

- Transformations of the function $y = \cos x$ include

$$y = \frac{3}{4}\cos x \qquad y = -\cos(x-\pi) \qquad y = \cos 7x - 3 \qquad y = -8\cos 4x \qquad \text{etc.}$$

Any function which is a transformation of $y = \sin x$ or $y = \cos x$ is called a **sinusoidal** function; its graph is called a **sinusoidal** graph.

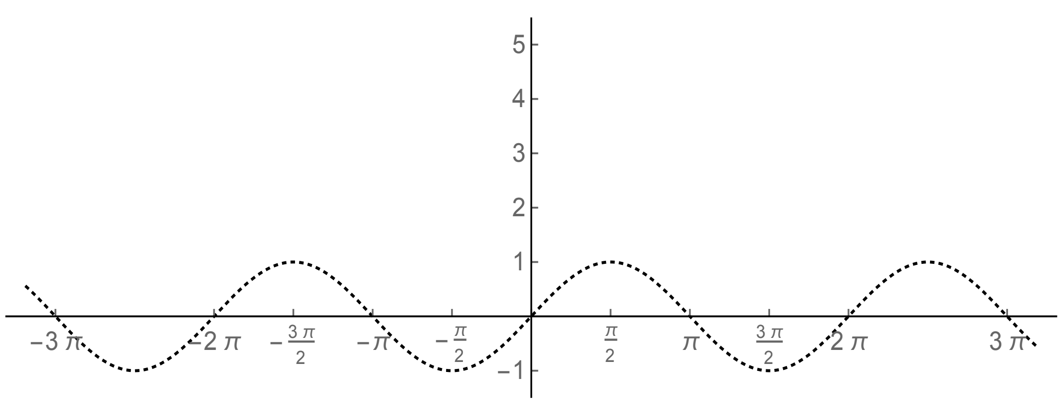
This section is about how to (quickly and crudely) graph transformations of trig functions.

How to graph $y = \sin x + D$ **and** $y = \cos x + D$

MOTIVATING EXAMPLE

Graph $y = \sin x + 3$.

x	$\sin x$	point on $y = \sin x$	$\sin x + 3$	point on $y = \sin x + 3$
0	0	$(0, 0)$		
$\frac{\pi}{6}$	$\frac{1}{2} = .5$	$(\frac{\pi}{6}, .5)$		
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2} \approx .707$	$(\frac{\pi}{4}, .707)$		
$\frac{\pi}{2}$	1	$(\frac{\pi}{2}, 1)$		
π	0	$(\pi, 0)$		
$\frac{3\pi}{2}$	-1	$(\frac{3\pi}{2}, -1)$		
2π	0	$(2\pi, 0)$		



Theorem 8.2 (Vertical shift) *The graph of $y = \sin x + D$ is the graph of $y = \sin x$, shifted upward/downward by D units (shifted up if $D > 0$ and down if $D < 0$).*

The graph of $y = \cos x + D$ is the graph of $y = \cos x$, shifted upward/downward by D units (shifted up if $D > 0$ and down if $D < 0$).

EXAMPLE 1

Graph $y = \sin x - 2$.

EXAMPLE 2

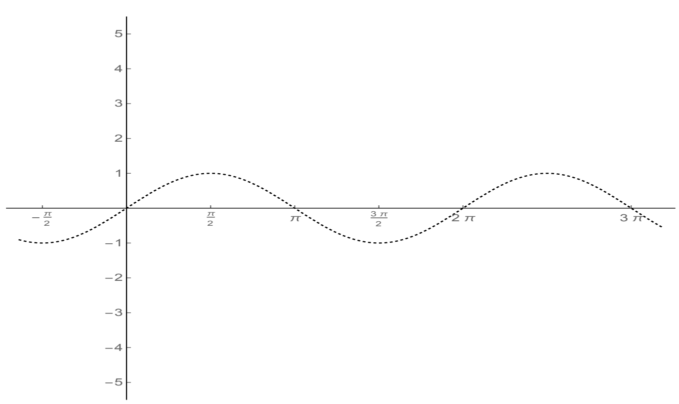
Graph $y = \cos x + 1$.

How to graph $y = A\sin x$ and $y = A\cos x$

MOTIVATING EXAMPLE

Graph $y = 3\sin x$.

x	$\sin x$	point on $y=\sin x$	$3\sin x$	point on $y = 3\sin x$
0	0	$(0,0)$		
$\frac{\pi}{6}$	$\frac{1}{2} = .5$	$(\frac{\pi}{6}, .5)$		
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2} \approx .707$	$(\frac{\pi}{4}, .707)$		
$\frac{\pi}{2}$	1	$(\frac{\pi}{2}, 1)$		
π	0	$(\pi, 0)$		
$\frac{3\pi}{2}$	-1	$(\frac{3\pi}{2}, -1)$		
2π	0	$(2\pi, 0)$		



Theorem 8.3 (Vertical stretch) *The graph of $y = A \sin x$ is the graph of $y = \sin x$, stretched upward/downward by a factor of A. If $A < 0$, the graph is flipped upside down. A is called the* **amplitude** *of $y = A \sin x$.*

The graph of $y = A \cos x$ is the graph of $y = \cos x$, stretched upward/downward by a factor of A. If $A < 0$, the graph is flipped upside down. A is called the **amplitude** *of $y = A \sin x$.*

EXAMPLE 3

Graph $y = -4 \sin x$.

EXAMPLE 4

Graph $y = 5 \cos x$.

EXAMPLE 5

Graph $y = \frac{1}{2} \cos x$.

How to graph $y = \sin(x - C)$ **and** $y = \cos(x - C)$

Theorem 8.4 (Horizontal shift) *The graph of* $y = \sin(x - C)$ *is the graph of* $y = \sin x$*, shifted leftward/rightward by* C *units (shifted rightward if* $C > 0$ *and leftward if* $C < 0$*).*

The graph of $y = \cos(x - C)$ *is the graph of* $y = \cos x$*, shifted leftward/rightward by* C *units (shifted rightward if* $C > 0$ *and leftward if* $C < 0$*).*

EXAMPLE 6

Graph $y = \sin(x - \frac{\pi}{2})$.

EXAMPLE 7

Graph $y = \cos(x + 2)$.

EXAMPLE 8

Graph $y = \sin(x + \frac{3\pi}{4})$.

Graphing $y = \sin Bx$ **and** $y = \cos Bx$

Theorem 8.5 (Horizontal stretch) *The graph of* $y = \sin Bx$ *is the graph of* $y = \sin x$*, stretched/compressed in the horizontal direction so that its period is* $\frac{2\pi}{B}$*.*

The graph of $y = \cos Bx$ *is the graph of* $y = \cos x$*, stretched/compressed in the horizontal direction so that its period is* $\frac{2\pi}{B}$*.*

EXAMPLE 9

Graph $y = \sin 4x$.

EXAMPLE 10

Graph $y = \cos \pi x$.

EXAMPLE 11

Graph $y = \sin \frac{x}{6}$.

Putting all this together

What if you have a function with more than one kind of shift?

Procedure to graph a shifted sinusoidal graph

1. Write the function in the standard form:

$$y = A\sin(B(x - C)) + D \qquad \text{or} \qquad y = A\cos(B(x - C)) + D$$

   Once in this form, A is the vertical stretch (a.k.a. amplitude), B is the horizontal stretch, C is the horizontal shift, and D is the vertical shift.

2. Draw dashed crosshairs at the point (C, D). This accounts for the horizontal and vertical shift.

3. On the vertical crosshair, go up and down A units and mark the y-coordinates of those heights. This tells you how far up and down the graph should go.

4. Starting in the appropriate place, draw one period of the graph. Then, mark where the period finished (this should be at $x = C + \frac{2\pi}{B}$, since the period is $\frac{2\pi}{B}$).

EXAMPLE 12

Graph $y = 4\cos 2x$.

EXAMPLE 13

Graph $y = \sin(x - \frac{\pi}{2}) - 3$.

Solution: This graph has a horizontal shift of $\frac{\pi}{2}$ units to the right, and a vertical shift of 3 units down.

EXAMPLE 13

Graph $y = -\cos\frac{x}{4} + 5$.

8.2 Graphs of tangent, cotangent, secant and cosecant

The graph of $y = \csc x$

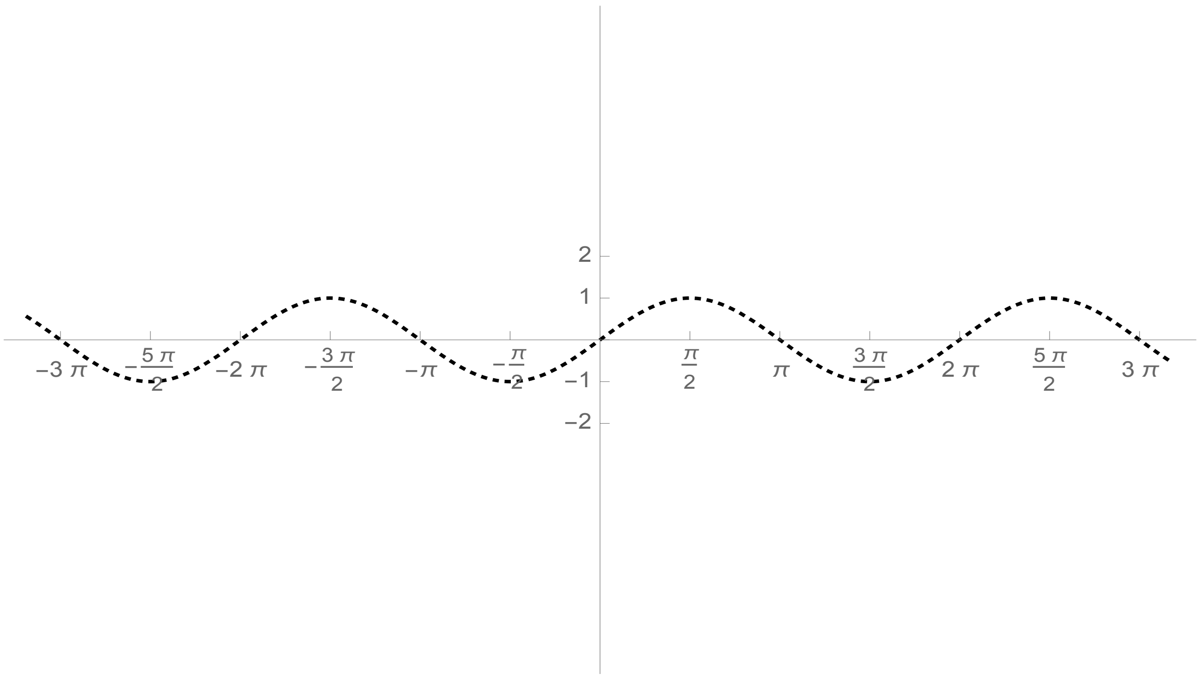
The function $y = \csc x$ can also be written as $y = \frac{1}{\sin x}$. Therefore the graph of $y = \csc x$ should be related to the graph of $y = \sin x$. But how, exactly?

Let's try a table of values for $y = \csc x$:

x	$\sin x$	point on $y = \sin x$	$\csc x = \frac{1}{\sin x}$	point on $y = \csc x$
0	0	$(0, 0)$		
$\frac{\pi}{6}$	$\frac{1}{2} = .5$	$(\frac{\pi}{6}, .5)$	$\frac{1}{\frac{1}{2}} = 2$	$(\frac{\pi}{6}, 2)$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2} \approx .707$	$(\frac{\pi}{4}, .707)$	$\frac{1}{\frac{\sqrt{2}}{2}} = \sqrt{2} \approx 1.414$	$(\frac{\pi}{4}, 1.414)$
$\frac{\pi}{2}$	1	$(\frac{\pi}{2}, 1)$	$\frac{1}{1} = 1$	$(\frac{\pi}{2}, 1)$
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2} \approx .707$	$(\frac{3\pi}{4}, .707)$	$\frac{1}{\frac{\sqrt{2}}{2}} = \sqrt{2} \approx 1.414$	$(\frac{3\pi}{4}, 1.414)$
$\frac{5\pi}{6}$	$\frac{1}{2} = .5$	$(\frac{5\pi}{6}, .5)$	$\frac{1}{\frac{1}{2}} = 2$	$(\frac{5\pi}{6}, 2)$
π	0	$(\pi, 0)$	DNE	DNE
$\frac{7\pi}{6}$	$\frac{-1}{2} = -.5$	$(\frac{5\pi}{6}, -.5)$	$\frac{1}{\frac{-1}{2}} = -2$	$(\frac{\pi}{6}, -2)$
$\frac{3\pi}{2}$	-1	$(\frac{3\pi}{2}, -1)$	$\frac{1}{-1} = -1$	$(\frac{3\pi}{2}, -1)$
$\frac{11\pi}{6}$	$\frac{-1}{2} = -.5$	$(\frac{11\pi}{6}, -.5)$	$\frac{1}{\frac{-1}{2}} = -2$	$(\frac{11\pi}{6}, -2)$
2π	0	$(2\pi, 0)$	DNE	DNE

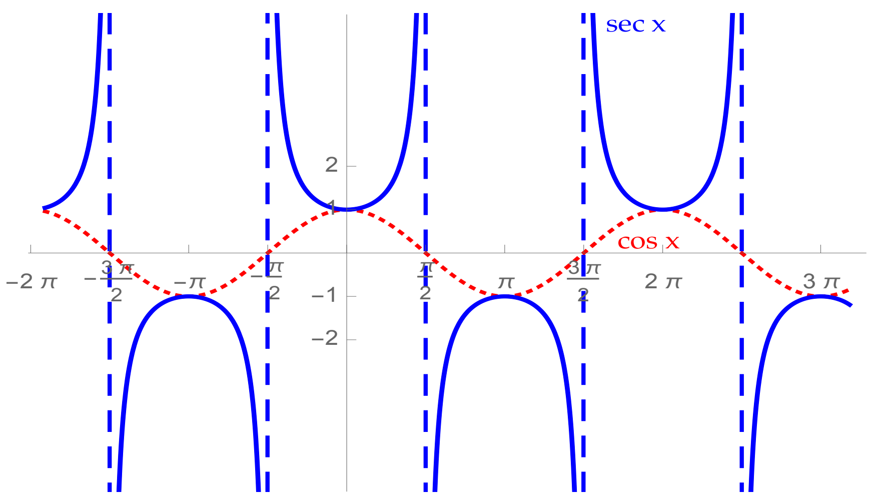
Observations:

Based on the observations on the previous page, we can sketch the graph of $y = \csc x$:



The graph of $y = \sec x$

The graph of $y = \sec x$ is related to the graph of $y = \cos x$ in the same way that the graph of $y = \csc x$ is related to $y = \sin x$:



The graph of $y = \tan x$

We can write $y = \tan x = \frac{\sin x}{\cos x}$. This means:

- when $\sin x = 0$, $\tan x$

  The values of x where this happens are

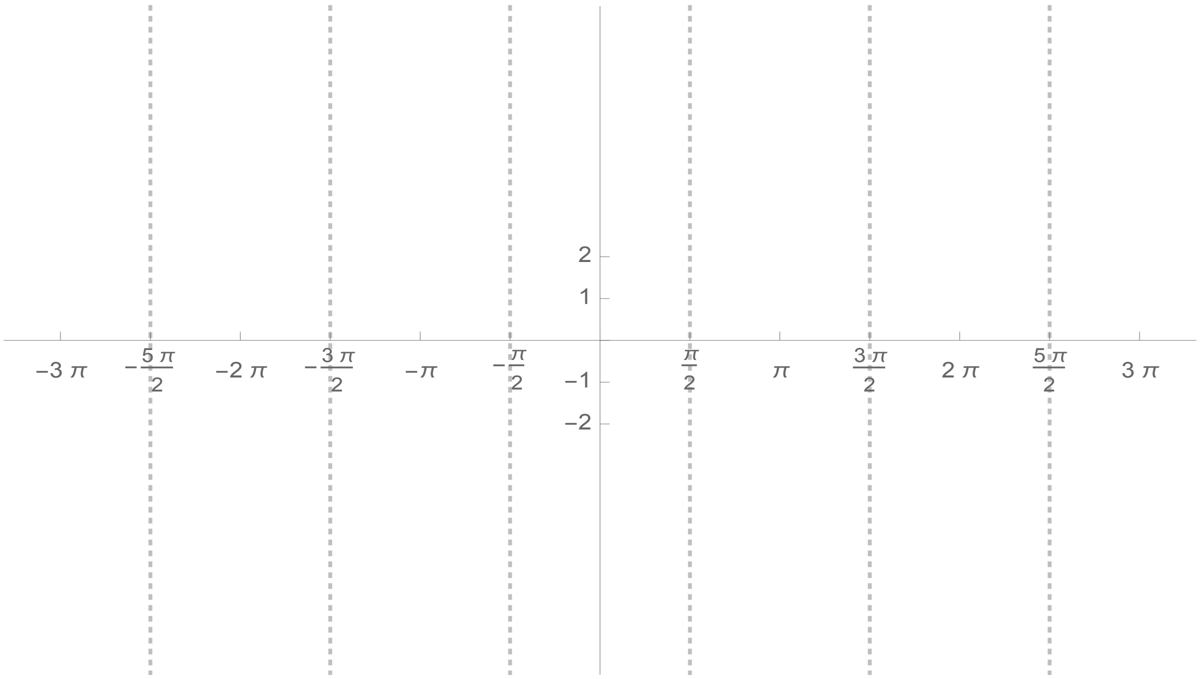
- when $\cos x = 0$, $\tan x$

  The values of x where this happens are

We can fill in the rest of the graph of $y = \tan x$ by making a table of values:

x	$y = \tan x$
0	0
$\frac{\pi}{6}$	$\frac{1}{\sqrt{3}} \approx .577$
$\frac{\pi}{4}$	1
$\frac{\pi}{3}$	$\sqrt{3} \approx 1.73$
$\frac{2\pi}{3}$	$-\sqrt{3} \approx -1.73$
$\frac{3\pi}{4}$	-1
$\frac{5\pi}{6}$	$\frac{-1}{\sqrt{3}} \approx -.577$

Putting this together, we get the graph on the next page:



The graph of $y = \cot x$

We can write $y = \cot x = \frac{\cos x}{\sin x}$. This means:

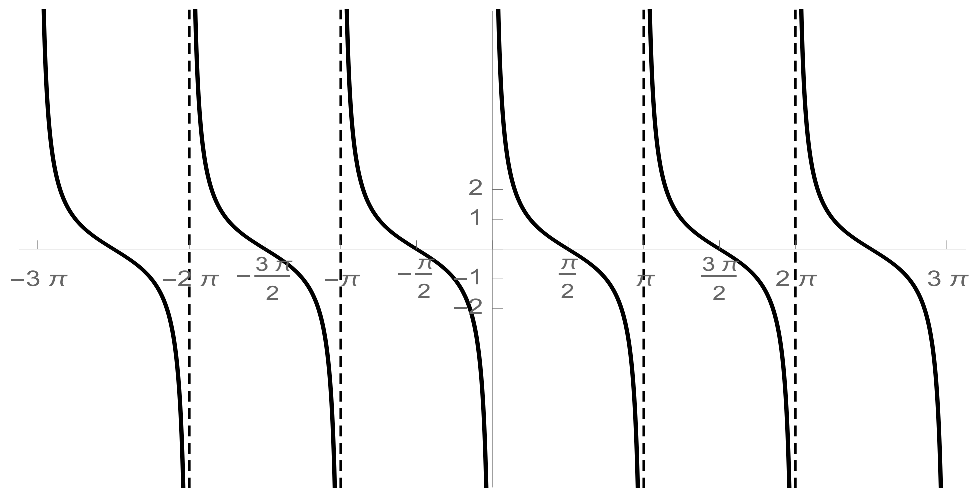
- when $\sin x = 0$, $\cot x$

  The values of x where this happens are $x = 0, \pi, 2\pi, 3\pi, ..., -\pi, -2\pi, -3\pi, ...$

- when $\cos x = 0$, $\cot x$

  The values of x where this happens are $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, ..., \frac{-\pi}{2}, \frac{-3\pi}{2}, ...$

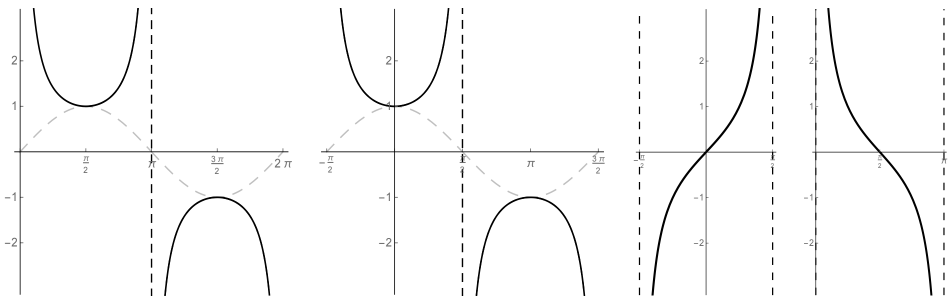
Putting this together with a table of values similar to the one we did for $y = \tan x$, we get the graph of $y = \cot x$ on the next page:



2
1
−1
−2
−3 π
−2 π
−3π/2
−π
−π/2
π/2
π
3π/2
2 π
3 π

Summary of Section 8.2

$y = \csc x$ $\qquad$ $y = \sec x$ $\qquad$ $y = \tan x$ $\qquad$ $y = \cot x$



For each of these four functions, there are dashed vertical lines called **asymptotes** which the graph does not cross (the graph appears to "merge" into the asymptote but never actually touches it). Here are the essential properties of these functions, which are reflected in the crude graphs drawn above:

FUNCTION	LOCATIONS OF ASYMPTOTES	PLACES WHERE GRAPH CROSSES x-AXIS	PERIOD	PIECES OF THE GRAPH ROUGHLY LOOK LIKE
$y = \csc x$	$x = 0, \pi, 2\pi, \ldots$ $x = -\pi, -2\pi, -3\pi, \ldots$ (where $\sin x = 0$)	None	2π	parabolas
$y = \sec x$	$x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$ $x = \frac{-\pi}{2}, \frac{-3\pi}{2}, \frac{-5\pi}{2}, \ldots$ (where $\cos x = 0$)	None	2π	parabolas
$y = \tan x$	$x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$ $x = \frac{-\pi}{2}, \frac{-3\pi}{2}, \frac{-5\pi}{2}, \ldots$ (where $\cos x = 0$)	$x = 0, \pi, 2\pi, \ldots$ $x = -\pi, -2\pi, -3\pi, \ldots$ (where $\sin x = 0$)	π	cubics going up from left to right
$y = \cot x$	$x = 0, \pi, 2\pi, \ldots$ $x = -\pi, -2\pi, -3\pi, \ldots$ (where $\sin x = 0$)	$x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$ $x = \frac{-\pi}{2}, \frac{-3\pi}{2}, \frac{-5\pi}{2}, \ldots$ (where $\cos x = 0$)	π	cubics going down from left to right

8.3 Review material for final exam

Most important skills:

1. Solve triangles
2. Given the value of one trig function and the sign of another, find the values of the other trig functions
3. Find trig functions of special angles
4. Use trig identities and basic facts about the trig functions

What I might ask you to do without a calculator:

- Convert angles from radians to degrees, if the radian measure is $\frac{A\pi}{B}$ with $B = 1, 2, 3, 4, 6$
- Given a picture of a right triangle with the side lengths labelled, find the six trig functions of either the two acute angles in that triangle.
- Find the six trig functions of an angle, given the coordinates of a point (x, y) on the terminal side of the angle.
- Find the six trig functions of an angle, given the value of one trig function and the sign of a second trig function.
- Determine whether a trig equation is "possible" or "impossible" to solve (based on the ranges of the trig functions).
- Determine whether the sign of the trig function of an angle is positive or negative.
- Draw pictures to explain the meaning of basic trig problems involving the trig functions.
- Answer questions involving symmetry and reference angles.
- Find the sine and/or cosine of an angle, given the coordinates of a point (x, y) on the terminal side of the angle.
- Determine the quadrant an angle sits in, given the value of two of its trig functions.
- Compute any trig function of any special angle, and evaluate formulas involving trig functions of special angles.
- Answer questions that involve the use of trig identities.
- Write formulas for some quantity in a picture in terms of other lengths (including story problems).
- Given a picture, write a formula for one side length in terms of the other quantities in the picture.
- Plot vectors in standard position; perform vector addition and scalar multiplication pictorially.
- Sketch crude graphs of trig functions, including shifts of sine and cosine

What I might ask you to do with a calculator:

- Anything you should be able to do without a calculator.

- Find angles which are coterminal with a given angle
- Find an angle between 0° and 360° which is coterminal with a given angle
- Convert any angle between degrees and radians
- Find arc lengths and sector areas
- Solve "angle pictures" that involve angle addition, right and straight angles, complementary and supplementary angles, parallel lines and transversals, angles in a polygon, etc.
- Solve problems involving linear and angular velocity (gears and pulleys, etc.)
- Find any trig function of any angle
- Solve a basic trig equation (i.e. solve $\tan\theta = q$, $\sec\theta = q$, $\cos\theta = q$, etc.)
- Solve triangles
- Find the area of a triangle
- Add vectors; multiply vectors by scalars; compute dot products, compute lengths of vectors; compute angles between vectors; find the coordinates of a vector given its length and angle (and vice-versa); determine if two vectors are orthogonal.
- Story problems which apply methods of solving triangles and/or vectors

Practice problems on the material covered after Exam 3 (all are "no calculator allowed")

1. Simplify the following expressions with a Pythagorean identity, if possible (if not, do nothing):

   a) $\tan^2\theta - 1$ c) $\sin^2\theta + 1$ e) $\sec^2\theta + 1$ g) $1 - \csc^2\theta$

   b) $1 - \cos^2\theta$ d) $\tan^2\theta + 1$ f) $\sec^2\theta - 1$ h) $\cot^2\theta + 1$

2. Simplify the following expressions as much a possible, writing them with no division and no minus signs inside the trig functions:

   a) $\frac{\sec\theta}{\tan\theta}$ b) $\cos(-\theta)\tan\theta\csc\theta$ c) $\frac{\csc\theta\sec\theta}{\cot\theta}$

3. Find the exact value of each quantity:

   a) $\sin 75^\circ$ d) $\tan 105^\circ$

   b) $\cos 165^\circ$ e) $\sin^2\frac{\pi}{8} + \cos^2\frac{\pi}{8}$

   c) $\sec 15^\circ$ f) $\sec^2 20^\circ - \tan^2 20^\circ$

4. Suppose $\sin\theta = \frac{2}{3}$ and $\cos\theta > 0$. Find $\sin 2\theta$ and $\cos 2\theta$.

5. Suppose $\tan\theta = -3$ and $\sin\theta < 0$. Find $\tan 2\theta$ and $\tan\frac{\theta}{2}$.

6. Sketch a graph of each function:

a) $y = \sin x$
b) $y = \cos x$
c) $y = \tan x$
d) $y = \cot x$
e) $y = \sec x$
f) $y = \csc x$

7. Sketch a graph of each function:

   a) $y = -2\sin x$
   b) $y = -\cos 3x$
   c) $y = \cos(-x)$
   d) $y = 4\sin x - 2$
   e) $y = \cos(x - \frac{\pi}{4})$
   f) $y = \sin \frac{x}{3}$
   g) $y = -\sin(x+\pi)+2$
   h) $y = \frac{1}{4}\cos 4x + 2$
   i) $y = -3\sin \frac{x}{2} - 3$

Practice problems on the material past students have had trouble with:

8. Convert 13.75 radians to degrees.

9. Convert -5.09 radians to degrees.

10. Convert $2456°$ to radians.

11. Convert $145.7°$ to radians.

12. Find all solutions (between $0°$ and $360°$) to the following equations; if there is no solution, say so:

    a) $\sin\theta = .43$
    b) $\cos\theta = 2.55$
    c) $\sec\theta = 3.08$
    d) $\csc\theta = -.35$
    e) $\tan\theta = 1.825$
    f) $\cot\theta = 2.37$
    g) $\cot\theta = -.3$
    h) $\tan\theta = -.25$
    i) $\sin\theta = 0$
    j) $\cos\theta = -.775$

13. Let $\mathbf{v} = \langle 3, -2\rangle$, $\mathbf{w} = \langle -4, -5\rangle$ and let $\mathbf{x} = \langle 0, 7\rangle$.

    a) Compute $\mathbf{v} + \mathbf{w}$.
    b) Compute $\mathbf{v} \cdot \mathbf{w}$.
    c) Compute $3\mathbf{v} \cdot \mathbf{x}$.
    d) Compute $(\mathbf{v} - \mathbf{x}) \cdot \mathbf{w}$.
    e) Compute $||\mathbf{x}||$.
    f) Compute the angle between $\mathbf{v}$ and $\mathbf{w}$.

14. An airplane flies 150 miles at an angle $35°$ north of west, then turns and flies 220 miles at an angle $20°$ west of north. How far is the airplane from where it started?

15. Two people push on a heavy box: one pushes due east with a force of 130 Newtons, and one pushes at an angle $40°$ south of west with a force of 50 Newtons. What angle (measured south of east) will the box move?

Solutions

1. a) $\tan^2\theta - 1$ cannot be simplified
   b) $1-\cos^2\theta = \sin^2\theta$
   c) $\sin^2\theta + 1$ cannot be simplified
   d) $\tan^2\theta + 1 = \sec^2\theta$
   e) $\sec^2\theta + 1$ cannot be simplified
   f) $\sec^2\theta - 1 = \tan^2\theta$
   g) $1-\csc^2\theta = -\cot^2\theta$
   h) $\cot^2\theta + 1 = \csc^2\theta$

2. a) $\frac{\sec\theta}{\tan\theta} = \frac{\frac{1}{\cos\theta}}{\frac{\sin\theta}{\cos\theta}} = \frac{1}{\cos\theta}\cdot\frac{\cos\theta}{\sin\theta} = \frac{1}{\sin\theta} = \csc\theta.$
   b) $\cos(-\theta)\tan\theta\csc\theta = \cos\theta\frac{\sin\theta}{\cos\theta}\frac{1}{\sin\theta} = 1.$
   c) $\frac{\csc\theta\sec\theta}{\cot\theta} = \frac{\frac{1}{\sin\theta}\frac{1}{\cos\theta}}{\frac{\cos\theta}{\sin\theta}} = \frac{1}{\sin\theta}\frac{1}{\cos\theta}\cdot\frac{\sin\theta}{\cos\theta} = \frac{1}{\cos^2\theta} = \sec^2\theta.$

3. a) $\sin 75^\circ = \sin(30^\circ+45^\circ) = \sin 30^\circ\cos 45^\circ + \cos 45^\circ \sin 30^\circ = \frac{1}{2}\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} = \frac{\sqrt{2}+\sqrt{6}}{4}.$
   b) $\cos 165^\circ = \cos(120^\circ+45^\circ) = \cos 120^\circ\cos 45^\circ - \sin 120^\circ\sin 45^\circ = \frac{-1}{2}\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} = \frac{-\sqrt{2}-\sqrt{6}}{4}.$
   c) $\sec 15^\circ = \frac{1}{\cos 15^\circ} = \frac{1}{\cos(45^\circ-30^\circ)} = \frac{1}{\cos 45^\circ\cos 30^\circ+\sin 45^\circ\sin 30^\circ} = \frac{1}{\frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2}+\frac{\sqrt{2}}{2}\frac{1}{2}} = \frac{1}{\frac{\sqrt{6}+\sqrt{2}}{4}} = \frac{4}{\sqrt{6}+\sqrt{2}}.$
   d) $\tan 105^\circ = \tan(60^\circ+45^\circ) = \frac{\tan 60^\circ+\tan 45^\circ}{1-\tan 60^\circ\tan 45^\circ} = \frac{\sqrt{3}+1}{1-\sqrt{3}1} = \frac{\sqrt{3}+1}{1-\sqrt{3}}.$
   e) $\sin^2\frac{\pi}{8} + \cos^2\frac{\pi}{8} = 1$
   f) $\sec^2 20^\circ - \tan^2 20^\circ = 1$

4. Draw a triangle with opposite side 2 and hypotenuse 3; solve for the adjacent side to get $\sqrt{5}$. Therefore, since $\cos\theta > 0$, $\cos\theta = \frac{\sqrt{5}}{3}$. That means

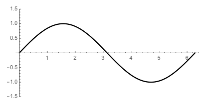
$$\sin 2\theta = 2\sin\theta\cos\theta = 2\frac{2}{3}\frac{\sqrt{5}}{3} = \frac{4\sqrt{5}}{9}$$

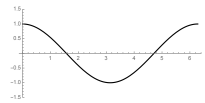
and

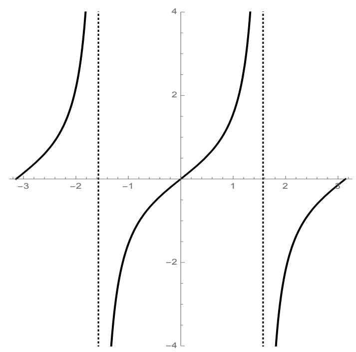
$$\cos 2\theta = 1 - 2\sin^2\theta = 1 - 2\left(\frac{2}{3}\right)^2 = 1 - 2\left(\frac{4}{9}\right) = \frac{1}{9}.$$

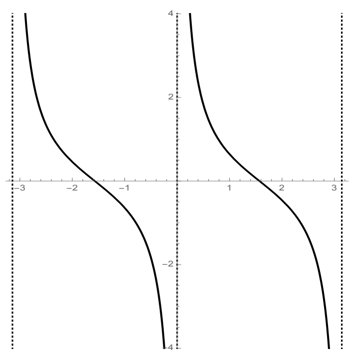
5. $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2(-3)}{1-(-3)^2} = \frac{-6}{-8} = \frac{3}{4}$. To find $\tan\frac{\theta}{2}$, we need to draw a triangle with opposite side 3 and adjacent side 1; the hypotenuse is therefore $\sqrt{3^2+1^2} = \sqrt{10}$. That makes $\sin\theta = \frac{-3}{\sqrt{10}}$ and $\cos\theta = \frac{1}{\sqrt{10}}$. Finally,

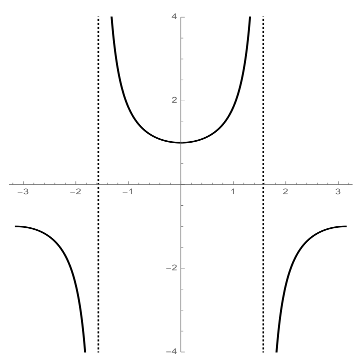
$$\tan\frac{\theta}{2} = \frac{1-\cos\theta}{\sin\theta} = \frac{1-\frac{1}{\sqrt{10}}}{\frac{-3}{\sqrt{10}}} = \frac{\sqrt{10}-1}{-3}.$$

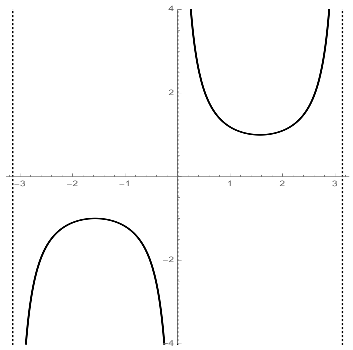
6. a) $y = \sin x$: 

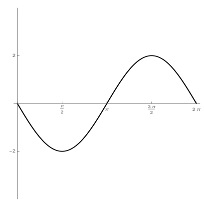
b) $y = \cos x$: 

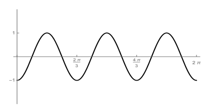
c) $y = \tan x$: 

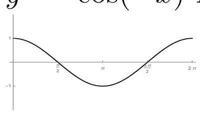
d) $y = \cot x$: 

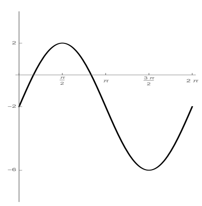
e) $y = \sec x$: 

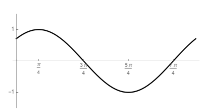
f) $y = \csc x$: 

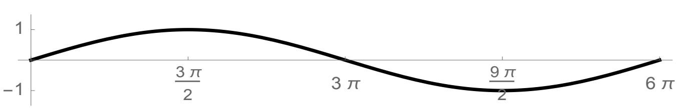
7. a) $y = -2\sin x$ (amplitude is 2; flipped through x-axis): 

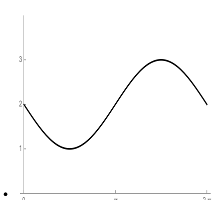
b) $y = -\cos 3x$ (period is $\frac{2\pi}{3}$; flipped through x-axis): 

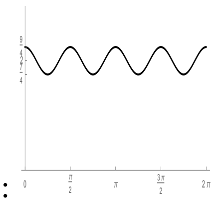
c) $y = \cos(-x)$ has the same graph as $y = \cos x$, since $\cos(-x) = \cos x$: 

d) $y = 4\sin x - 2$ (shifted down 2, amplitude 4): 

e) $y = \cos(x - \frac{\pi}{4})$ (shift right by $\frac{\pi}{4}$): 

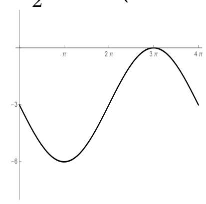
f) $y = \sin\frac{x}{3}$ (period is $\frac{2\pi}{1/3} = 6\pi$): 

g) $y = -\sin(x+\pi)+2$ (shift left by π and up 2, flipped over): 



h) $y = \frac{1}{4}\cos 4x + 2$ (shift up 2, amplitude $\frac{1}{4}$, period $\frac{2\pi}{4} = \frac{\pi}{2}$):

i) $y = -3\sin\frac{x}{2} - 3$ (shift down 3, amplitude 3 and flipped over, period



$\frac{2\pi}{1/2} = 4\pi$):

8. $13.75 \cdot \frac{180°}{\pi} = 787.8°$.

9. $-5.09 \cdot \frac{180°}{\pi} = -291.6°$.

10. $2456° \cdot \frac{\pi}{180°} = 42.86$ radians.

11. $145.7° \cdot \frac{\pi}{180°} = 2.54$ radians.

12. Find all solutions (between $0°$ and $360°$) to the following equations; if there is no solution, say so:

a) $\theta = \sin^{-1} .43 = 25.5°$; the other answer is $180° - \theta = 154.5°$.

b) No solution

c) Take reciprocals to get $\cos\theta = \frac{1}{3.08} = .3247$; then $\theta = \cos^{-1}(.3247) = 71.1°$; the other answer is $360° - 71.1° = 288.9°$.

d) Take reciprocals to get $\csc\theta = -.35$

e) $\theta = \tan^{-1} 1.825 = 61.3°$; the other answer is $180° + \theta = 241.3°$.

f) Take reciprocals to get $\tan\theta = \frac{1}{2.37} = .421$; then $\theta = \tan^{-1} .421 = 22.9°$; the other answer is $180° + \theta = 202.9°$.

g) Take reciprocals to get $\tan\theta = \frac{1}{-.3} = -3.333$; then $\theta = \tan^{-1}(-3.3333) = -73.3°$... add $360°$ to get an angle between $0°$ and $360°$ which is $286.7°$; the other answer is $180° + \theta = 106.7°$.

h) $\theta = \tan^{-1} -.25 = -14°$; add $360°$ to get $346°$; for the other angle, take $\theta + 180° = 166°$.

i) $\theta = \sin^{-1} 0 = 0°$; the other answer is $180° - \theta = 180°$.

j) $\theta = \cos^{-1} -.775 = 140.8°$; the other answer is $360° - \theta = 219.2°$.

13. a) $\mathbf{v} + \mathbf{w} = \langle 7, -7\rangle$.

b) $\mathbf{v} \cdot \mathbf{w} = 3(-4) + (-2)(-5) = -2.$

c) $3\mathbf{v} \cdot \mathbf{x} = \langle 9, -6 \rangle \cdot \langle 0, 7 \rangle = -42.$

d) $(\mathbf{v} - \mathbf{x}) \cdot \mathbf{w} = \langle 3, -9 \rangle \cdot \langle -4, -5 \rangle = -12 + 45 = 33.$

e) $||\mathbf{x}|| = \sqrt{0^2 + 7^2} = 7.$

f) Let θ be the angle, then

$$\begin{aligned}
\mathbf{v} \cdot \mathbf{w} &= ||\mathbf{v}|| \, ||\mathbf{w}|| \cos\theta \\
-2 &= \sqrt{3^2 + (-2)^2}\sqrt{(-4)^2 + (-5)^2} \cos\theta \\
-2 &= \sqrt{13}\sqrt{41} \cos\theta \\
\frac{-2}{\sqrt{13}\sqrt{41}} &= \cos\theta \\
-.0866 &= \cos\theta \\
\Rightarrow \theta &= \cos^{-1}(-.0866) = 94.97^\circ.
\end{aligned}$$

14. Let $\mathbf{v}$ represent the first part of the journey; $\mathbf{v}$ has length 150 and direction angle $180^\circ - 35^\circ = 145^\circ$, so the components of $\mathbf{v}$ are

$$\mathbf{v} = \langle 150 \cos 145^\circ, 150 \sin 145^\circ \rangle = \langle -122.9, 86 \rangle .$$

Now, let $\mathbf{w}$ represent the second part of the journey; $\mathbf{w}$ has length 220 and direction angle $90^\circ + 20^\circ = 110^\circ$, so the components of $\mathbf{w}$ are

$$\mathbf{w} = \langle 220 \cos 110^\circ, 220 \sin 110^\circ \rangle = \langle -75.2, 206.7 \rangle .$$

So the airplane is now at $\mathbf{v} + \mathbf{w} = \langle -198.1, 292.7 \rangle$. The distance from the starting point is the length of this vector, which is

$$||\mathbf{v} + \mathbf{w}|| = \sqrt{(-198.1)^2 + 292.7^2} = 353.44 \text{ miles.}$$

15. Let $\mathbf{v}$ represent the force from the first person. Then $\mathbf{v} = \langle 130, 0 \rangle$. Now let $\mathbf{w}$ represent the force from the second person; $\mathbf{w}$ has length 50 and direction angle $180^\circ + 40^\circ = 220^\circ$. So the components of $\mathbf{w}$ are

$$\mathbf{w} = \langle 50 \cos 220^\circ, 50 \sin 220^\circ \rangle = \langle -38.3, -32.1 \rangle .$$

The box moves in the direction of $\mathbf{v} + \mathbf{w} = \langle 91.7, -32.1 \rangle$; since this vector is in Quadrant IV, the angle of this vector is

$$\theta = \tan^{-1} \frac{-32.1}{91.7} = -19.3^\circ;$$

in other words, the box moves at an angle 19.3° south of east.

8.4 Additional conceptual questions for review

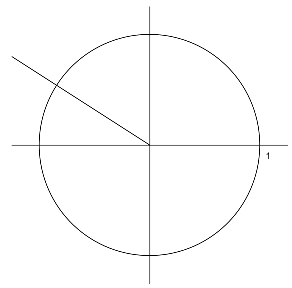
In response to some issues I have noticed on past students' exams, I wrote the following questions which I suggest you look at before the final exam (some of these may appear on your final). These are conceptual questions which you may/should respond to in your own words, but your answers should convey the correct meaning.

Suggested blocks of problems: 1-16; 17-30; 31-47, 48-55.

1. In the context of a right triangle, what does "$\sin\theta$" mean?
2. In the context of the unit circle, what does "$\sin\theta$" mean?
3. Explain why $\sin\theta$ is positive in Quadrants I and II.
4. What is the difference between the meanings of "$\sin\theta = .65$" and "$\sin .65$"?
5. Draw pictures to illustrate your answers to the previous question.
6. What is the difference between the meanings of "$\sin .65$" and "$\sin^{-1} .65$"?
7. What is the difference in meaning between the expressions "$\sin(\alpha+\beta)$", "$\sin\alpha + \sin\beta$" and "$\sin\alpha + \beta$"?
8. Of the three expressions in the previous problem, which can be rewritten in a meaningful way? How can it be rewritten?
9. Classify the following statement as true or false: $\sin 180° + \sin 180° = \sin 360°$.
10. Is your answer to the preceding question indicative of some more general fact, or is it a fluke based on the particular numbers in the problem?
11. What is meant by the expression "$\sin^2\theta$"?
12. What is the difference in meaning between the expressions "$\sin 2\theta$" and "$2\sin\theta$"?
13. Explain in your own words why $\sin(\theta + 360°) = \sin\theta$.
14. What is the relationship between $\sin(\theta + 180°)$ and $\sin\theta$? Why does that relationship hold? (Think about how you explained the previous problem.)
15. Consider the equation $\cos^2\theta + \sin^2\theta = 1$.
    a) Suppose you are given $\sin\theta = .75$. Substitute into the above equation. In this case, can you solve for $\cos\theta$? Why or why not?
    b) Suppose you are given $\sin\theta = .75$. Substitute into the above equation. If you know $\cos\theta < 0$, can you solve for $\cos\theta$? Why or why not?

c) Suppose you are given $\theta = .75$. Substitute into the above equation. In this case, can you solve for $\cos\theta$? Why or why not?

16. In the picture below, where is θ? Where is $\sin\theta$? Where is $\cos\theta$?



17. In the context of a right triangle, what does "$\tan\theta$" mean?

18. In the context of the unit circle, what does "$\cos\theta$" mean?

19. Explain why $\tan\theta$ is positive in Quadrants I and III.

20. What is the difference between the meanings of "$\cos\theta = .65$" and "$\cos .65$"?

21. Draw pictures to illustrate your answers to the previous question.

22. What is the difference between the meanings of "$\tan 2.35$" and "$\tan^{-1} 2.35$"?

23. What is the difference in meaning between the expressions "$\cos(\alpha+\beta)$", "$\cos\alpha + \cos\beta$" and "$\cos\alpha + \beta$"?

24. Of the three expressions in the previous problem, which can be rewritten in a meaningful way? How can it be rewritten?

25. What is meant by the expression "$\cos^2\theta$"?

26. What is the difference in meaning between the expressions "$\tan 3\theta$" and "$3\tan\theta$"?

27. Explain in your own words why $\tan(\theta + 360°) = \tan\theta$.

28. What is the relationship between $\tan(\theta + 180°)$ and $\tan\theta$? Why does that relationship hold? (Think about how you explained the previous problem.)

29. Without using a calculator, which is larger $\cos 42°$ or $\cos 50°$? Why?

30. Consider the equation $\cos^2\theta + \sin^2\theta = 1$.

    a) Suppose you are given $\cos\theta = .42$. Substitute into the above equation. In this case, can you solve for $\sin\theta$? Why or why not?

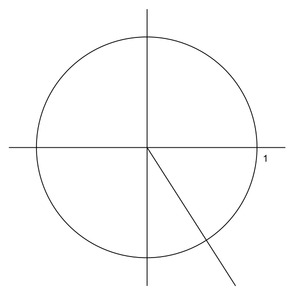
b) Suppose you are given $\cos\theta = .42$. Substitute into the above equation. If you know $\sin\theta > 0$, can you solve for $\sin\theta$? Why or why not?

c) Suppose you are given $\theta = .42$. Substitute into the above equation. In this case, can you solve for $\sin\theta$? Why or why not?

31. In the context of a right triangle, what does "$\sec\theta$" mean?

32. To evaluate $\sec\theta$, do you first compute $\cos\theta$, or first take a reciprocal of θ?

33. In the picture below, where is θ? Where is $\sin\theta$? Where is $\sec\theta$?



34. In the context of the unit circle, what does "$\tan\theta$" mean?

35. Explain in your own words what is meant by "$\cot\theta$".

36. Explain why $\cos\theta$ is positive in Quadrants I and IV.

37. What is the difference between the meanings of "$\tan\theta = 3.25$" and "$\tan 3.25$"?

38. Draw pictures to illustrate your answers to the previous question.

39. What is the difference between the meanings of "$\cos .8$" and "$\cos^{-1} .8$"?

40. If you are solving $\sin\theta = q$, one answer is given by $\sin^{-1} q$. Why is the other answer given by $180° - q$?

41. What is the difference in meaning between the expressions "$\tan(\alpha - \beta)$", "$\tan\alpha - \tan\beta$" and "$\tan\alpha - \beta$"?

42. Of the three expressions in the previous problem, which can be rewritten in a meaningful way? How can it be rewritten?

43. What is meant by the expression "$\tan^4\theta$"?

44. What is the difference in meaning between the expressions "$\cos\frac{\theta}{2}$" and "$\frac{\cos\theta}{2}$"?

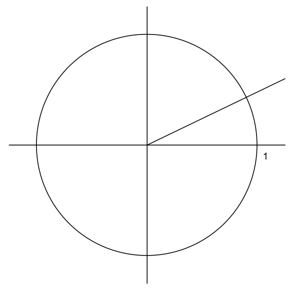
45. Explain in your own words why $\cos(-\theta) = \cos\theta$.

46. Explain why the equation $\cos\theta = 1.25$ has no solution.

47. Explain in your own words why $\sin(-\theta) = -\sin\theta$.

48. Explain in your own words what is meant by "$\sec\theta$".

49. In the picture below, where is θ? Where is $\cos\theta$? Where is $\tan\theta$?



50. Suppose you are given an equation like $\sec\theta = 6$. To solve for θ, should you start by taking a reciprocal of 6, or start by doing $\cos^{-1} 6$? What is the reason for your answer?

51. If you are solving $\tan\theta = q$, one answer is given by $\theta = \tan^{-1} q$. Why is the other answer given by $180° + \theta$?

52. If you are solving $\cos\theta = q$, one answer is given by $\theta = \tan^{-1} q$. Why is the other answer given by $360° - \theta$?

53. Suppose you are given an equation like $\csc\theta = 3.5$. To solve for θ, should you start by taking a reciprocal of 3.5, or start by doing $\sin^{-1} 3.5$? What is the reason for your answer?

54. Without using a calculator, which is larger, $\sin 105°$ or $\sin 110°$? Why?

55. Without using a calculator, which is larger $\tan 73°$ or $\tan 80°$? Why?

Answers

1. If θ is an angle in a right triangle, $\sin\theta$ is the length of the opposite side divided by the length of the hypotenuse.

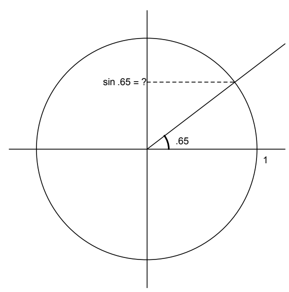
2. Given a point on the unit circle sitting at angle θ, $\sin\theta$ is the y-coordinate of that point.

3. $\sin\theta$ refers to the y-coordinate of a point on the unit circle; quadrants I and II are exactly the two quadrants where y-coordinates of points are positive.

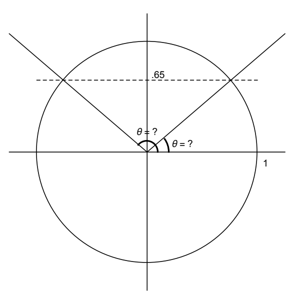
4. "$\sin .65$" is the y-coordinate on the unit circle associated to an angle measuring $.65$ radians.

   $\sin\theta = .65$ means $.65$ is the sine of some angle θ.

5. $\sin .65$: here, you have an angle of $.65$ radians, and you are looking for the y-coordinate:

   $\sin\theta = .65$: here, you are told the y-coordinate is $.65$, and you are looking for the value(s) of θ that produce that y-coordinate:

6. "$\sin .65$" means that $.65$ is an angle, and we need to take the sine of that angle.

   $\sin^{-1} .65$ is an angle θ such that $\sin\theta = .65$.

7. "$\sin(\alpha+\beta)$" means first add α and β, then take the sine of that angle (i.e. find the y-coordinate on the unit circle at angle $\alpha+\beta$).

"$\sin\alpha + \sin\beta$" means separately take the sine of each angle (i.e. find y-coordinates on the unit circle ate each angle α and β), then add those answers.

"$\sin\alpha + \beta$" means take the sine of α (i.e. find y-coordinate on unit circle at angle α), then add β. In other words, you don't even really treat β as an angle.

8. There is a trig identity telling us how to rewrite $\sin(\alpha + \beta)$, which we could look up if we needed it.

9. This is true, because $0 + 0 = 0$.

10. The fact that the previous statement was true was a fluke. It does not represent any kind of general fact; in general, you would not expect $\sin(\alpha + \beta)$ to be equal to $\sin\alpha$ + $\sin\beta$.

11. $\sin^2\theta = (\sin\theta)^2$, i.e. first take the sine of θ, then square that answer.

12. "$\sin 2\theta$" means take angle θ, first double it, then take the sine (i.e. find y-coordinate of point on unit circle sitting at angle 2θ).

    "$2\sin\theta$" means take the sine of θ, then double it (i.e. find y-coordinate of point on unit circle sitting at angle θ, then multiply that y-coordinate by 2).

13. On the unit circle, θ and $\theta + 360°$ both give you the same point (because they point in the same direction). Therefore these angles are associated to the same y-coordinates, meaning their sines are equal.

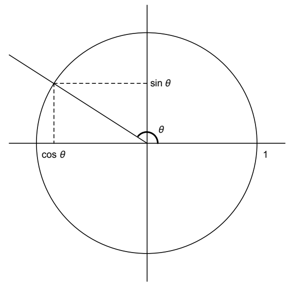
14. If you drew θ and $\theta + 180°$ on the same unit circle, those angles would point in opposite directions. That means their associated y-coordinates are opposites. That means $\sin(\theta + 180°) = -\sin\theta$.

15. a) Substituting, we get $\cos^2\theta + (.75)^2 = 1$. We can almost, but not quite, solve for $\cos\theta$ here, because at the end we would get $\cos\theta = \pm\sqrt{\text{something}}$, and we don't know whether or not to take the positive or negative square root to get $\cos\theta$ (i.e. $\cos\theta$ would have two possible values).

    b) Substituting the same way as in (a) to get $\cos^2\theta + (.75)^2 = 1$, this time we can solve for $\cos\theta$ because we know that $\cos\theta$ must be the negative square root.

    c) Substituting $.75$ in for θ instead of $\sin\theta$, we get $\cos^2(.75) + \sin^2(.75) = 1$. This is no longer an equation with a variable in it, so there is nothing you can or would want to solve for. (This is just a particular statement of the Pythagorean identity.)

16. As shown below, θ is the angle; $\sin\theta$ is the y-coordinate and $\cos\theta$ is the x-coordinate:



17. If θ is an angle in a right triangle, $\tan\theta$ is the length of the opposite side divided by the length of the adjacent side.

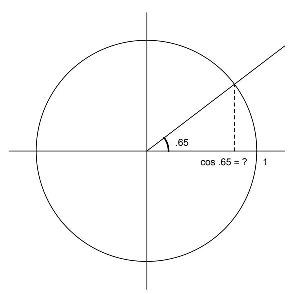
18. Given a point on the unit circle sitting at angle θ, $\cos\theta$ is the x-coordinate of that point.

19. $\tan\theta$ measures the slope of a line sitting at angle θ. If $\tan\theta > 0$, that means the line must have positive slope, i.e. go up from left to right. If θ is in Quadrant I or III, then it goes up from left to right, as wanted.

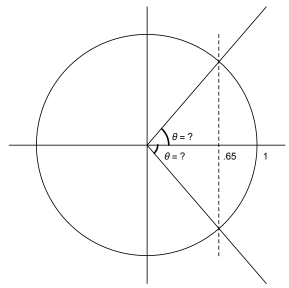
20. "$\cos .65$" means that $.65$ is an angle, and we need to take the cosine of that angle (i.e. find the x-coordinate of a point on the unit circle at an angle of $.65$ radians).

    $\cos^{-1} .65$ is an angle θ such that $\cos\theta = .65$.

21. $\cos .65$: here, you have an angle of $.65$ radians, and you are looking for the x-coordinate:



    $\cos\theta = .65$: here, you are told the x-coordinate is $.65$, and you are looking for the value(s) of θ that produce that x-coordinate:



22. "$\tan 2.35$" is the slope of a line which is at an angle of 2.35 radians.

    "$\tan^{-1} 2.35$" is an angle whose slope is 2.35.

23. "$\cos(\alpha + \beta)$" means first add α and β, then take the cosine of that angle (i.e. find the x-coordinate on the unit circle at angle $\alpha + \beta$).

    "$\cos\alpha + \cos\beta$" means separately take the cosine of each angle (i.e. find x-coordinates on the unit circle at each angle α and β), then add those answers.

    "$\cos\alpha + \beta$" means take the cosine of α (i.e. find x-coordinate on unit circle at angle α), then add β. In other words, you don't even really treat β as an angle.

24. $\cos(\alpha+\beta)$ can be rewritten as $\cos\alpha\cos\beta - \sin\alpha\sin\beta$ using a trig identity that you can look up if you need to use it.

25. "$\cos^2\theta$" means start with angle θ, take its cosine, then square that answer.

26. "$\tan 3\theta$" means start with angle θ, first multiply that angle by 3, then take the tangent (i.e. find the slope at 3θ).

    "$3\tan\theta$" means start with angle θ and first find its tangent (i.e. slope), then multiply that by 3.

27. On the unit circle, θ and $\theta + 360°$ both give you the same point (because they point in the same direction). Therefore these angles are have the same slopes, meaning their tangents are equal.

28. On the unit circle, θ and $\theta + 180°$ point in exactly the opposite directions. But this means they lie on the same line through the origin, so these angles are have the same slopes, meaning their tangents are equal. In other words, $\tan(\theta + 180°) = \tan\theta$.

29. $\cos 42° > \cos 50°$ because the x-coordinate on the unit circle at angle $42°$ is greater than the x-coordinate on the unit circle at angle $50°$.

30. a) Substituting, we get $\sin^2\theta + (.42)^2 = 1$. We can almost, but not quite, solve for $\sin\theta$ here, because at the end we would get $\sin\theta = \pm\sqrt{\text{something}}$, and we don't know whether or not to take the positive or negative square root to get $\sin\theta$ (i.e. $\sin\theta$ would have two possible values).

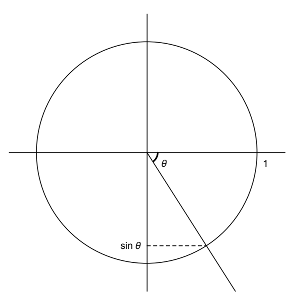
    b) Substituting the same way as in (a) to get $\sin^2\theta + (..42)^2 = 1$, this time we can solve for $\sin\theta$ because we know that $\sin\theta$ must be the positive square root.

    c) Substituting $.42$ in for θ instead of $\cos\theta$, we get $\cos^2(.42) + \sin^2(.42) = 1$. This is no longer an equation with a variable in it, so there is nothing you can or would want to solve for. (This is just a particular statement of the Pythagorean identity.)

31. If θ is an angle in a right triangle, $\sec\theta$ is the length of the hypotenuse divided by the length of the adjacent side.

32. Since $\sec\theta = \frac{1}{\cos\theta}$, to find $\sec\theta$ we first find $\cos\theta$, then take the reciprocal.

33. As shown below, θ is the angle and $\sin\theta$ is the y-coordinate of the point on the unit circle at that angle. $\sec\theta$ isn't really in this picture anywhere (it is the reciprocal of $\cos\theta$, which is the x-coordinate):



34. "$\tan\theta$" is the slope of a line drawn in standard position at angle θ.

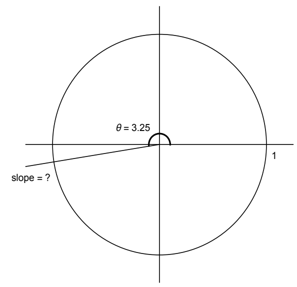
35. "$\cot\theta$" is the reciprocal of $\tan\theta$.

36. Since $\cos\theta$ is the x-coordinate of a point on the unit circle; quadrants I and IV are exactly the two quadrants where x-coordinates of points are positive.

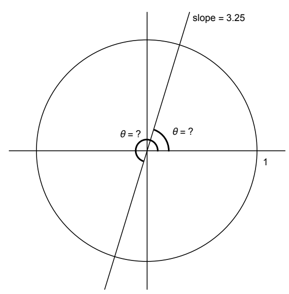
37. "$\tan 3.25$" is the slope of a line which is at an angle of 3.25 radians.

    "$\tan^{-1} 3.25$" is an angle whose slope is 3.25.

38. $\tan 3.25$: here, you have an angle of 3.25 radians, and you are looking for the slope



$\tan\theta = 3.25$: here, you are told the slope is 3.27, and you are looking for the value(s) of θ that produce that x-coordinate:



39. "$\cos .8$" means you are given an angle of $\theta = .8$ and you take the cosine, i.e. find the x-coordinate of the point on the unit circle at that angle.

    "$\cos^{-1} .8$" means you are looking for an angle θ whose cosine is $.8$.

40. You are being told that the y-coordinate of a point on the unit circle is q. There are two angles which have y-coordinate q. If you call the first one θ, then the second one is $180° - \theta$ because you get to it by reflecting θ across the y-axis. (Look at the second picture in # 5 and pretend that the $.65$ was q.)

41. "$\tan(\alpha - \beta)$" means subtract angles α and β to get angle $\alpha - \beta$; then take the tangent to find the slope at angle $\alpha - \beta$.

    "$\tan\alpha - \tan\beta$" means find the slopes at angles α and β independently, then subtract them.

    "$\tan\alpha - \beta$" means find the slope at angle α and then subtract β (i.e. you don't really think of β as an angle).

42. $\tan(\alpha - \beta)$ can be rewritten using a trig identity, which you could look up if you needed to use it.

43. $\tan^4\theta = (\tan\theta)^4$, i.e. start with θ, take its tangent (i.e. slope), then take that number to the fourth power.

44. $\cos\frac{\theta}{2}$ means take angle θ, first make the angle half as big by multiplying by $\frac{1}{2}$, then take the cosine (i.e. this is the x-coordinate on the unit circle at angle $\frac{\theta}{2}$).

    $\frac{\cos\theta}{2}$ means take the cosine of angle θ and divide that by 2 (i.e. find the x-coordinate on the unit circle at angle θ and take $\frac{1}{2}$ of that x-coordinate).

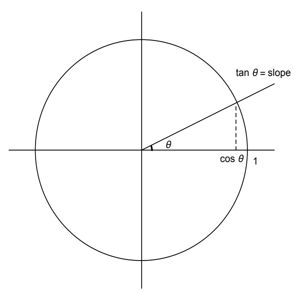
45. Given angle θ, you get to angle $-\theta$ by reflecting across the x-axis. That means the x-coordinate of a point on the unit circle at angle θ is the same as the x-coordinate of a point on the unit circle at angle $-\theta$, meaning these two angles have the same cosine.

46. $\cos\theta$ is the x-coordinate of a point on the unit circle at angle θ. But the unit circle only goes as far to the right as $x = 1$, so there is no point on the unit circle with x-coordinate 1.25, meaning no such θ exists.

47. Given angle θ, you get to angle $-\theta$ by reflecting across the x-axis. That means the y-coordinate of a point on the unit circle at angle θ is (-1) times the y-coordinate of a point on the unit circle at angle $-\theta$, meaning these two angles have sines which are (-1) times each other.

48. $\csc\theta$ is the reciprocal of $\sin\theta$.

49. As shown below, θ is the angle; $\cos\theta$ is the x-coordinate of the point on the unit circle at angle θ; $\tan\theta$ is the slope of the line at angle θ:



50. Start by taking a reciprocal of 6 to get $\cos\theta = \frac{1}{6}$, because $\cos\theta = 1\frac{\sec\theta}{=}\frac{1}{6}$.

51. $\tan\theta = q$ is asking you to find angle(s) whose slope is q. If one such angle is θ, the other angle is one that points in exactly the opposite direction as θ. This angle must be $180° + \theta$.

52. $\cos\theta = q$ is asking you to find angle(s) whose x-coordinate is q. If one such angle is θ, the other angle is one that is the reflection across the x-axis of θ. This angle must be $360° - \theta$.

53. Start by taking a reciprocal of 3.5 to get $\sin\theta = \frac{1}{3.5}$, because $\sin\theta = 1\frac{\csc\theta}{=}\frac{1}{3.5}$.

54. $\sin 105^\circ > \sin 110^\circ$, because the y-coordinate of the point on the unit circle at angle 105° is greater than the y-coordinate of the point on the unit circle at angle 110°.

55. $\tan 73^\circ < \tan 80^\circ$ because the slope of an 80° angle is greater than the slope of a 73° angle.