The BINOMIAL Expansion.

A BINOMIAL EXPRESSION is one which has two terms, added or subtracted, which are raised to a given POWER.

$$(a+b)^n$$

At this stage the POWER n WILL <u>ALWAYS</u> BE A POSITIVE WHOLE NUMBER.
We are required to EXPAND this expression, ie, to expand the brackets. We will find that there is a formula that can be used for such an EXPANSION which will be provided in the formula booklet in the examination. There are alternative methods that can often be easier than using a formula.

Firstly Consider the pattern emerging from the following:

$(a+b)^1$ which is clearly equal to $a+b$

$$(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2$$

$$(a+b)^3 = (a+b)(a+b)(a+b)$$
$$= (a^2 + 2ab + b^2)(a+b)$$
$$= a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a+b)^4 = (a+b)(a+b)(a+b)(a+b)$$
$$= (a^3 + 3a^2b + 3ab^2 + b^3)(a+b)$$
$$= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

You can check this yourself if you are interested!!!! But I can assure you it IS what you should get!

Expansions of any higher powers than this are obviously getting time consuming and messy, so we turn to the famous mathematicians of the past and look at what they came up with!

Firstly we see that there is a clear pattern in the <u>**TERMS.**</u>

The expansion of $(a+b)^3$ had <u>terms</u> of the type, a^3, a^2b, ab^2 and b^3.
We notice that the powers of each term always add up to 3.

The expansion of $(a+b)^4$ had <u>terms</u> of the type, a^4, a^3b, a^2b^2, ab^3 and b^4.
We notice that the powers of each term always add up to 4.

We can assume that the expansion of $(a+b)^5$ will have terms of the type $a^5, a^4b, a^3b^2, a^2b^3, ab^4$ and b^5 because the powers of each term will always add up to 5.

What do you think will be the sorts of TERMS in the expansion of $(a+b)^6$?

If we are expanding $(a+b)^n$ then a GENERAL TERM look like $a^r b^{n-r}$ or $a^{n-r} b^r$ in fact these terms would have the same coefficient due to the SYMMETRY of the binomial expansion of this type.

BUT THE SUM OF THE POWERS WILL ALWAYS ADD TO n (Whatever n may be as long as it is a positive whole number)

The question is WHAT ARE THE <u>COEFFICIENTS?</u>
This means how many of each term there will be.

It is quite obvious that the first and last term will have a coefficient of 1 because
there will be only one possible <u>combination</u> of the first or last "letter".

For example in $(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2$

There is only one possible way of multiplying an a with an a to produce a^2.
Yet there are two ways that we can multiply an a with a b to produce ab
(first a with second b or first b with second a)
In the expansion of

$$(a+b)^3 = (a+b)(a+b)(a+b)$$
$$= a^3 + 3a^2b + 3ab^2 + b^3$$

we see there is only one way to obtain a^3
we see there is only one way to obtain b^3
and yet there are THREE ways that we can produce an a^2b

CAN YOU SEE THE THREE WAYS?

So the COEFFICIENT is all about THE NUMBER OF WAYS OF SELECTING the a's
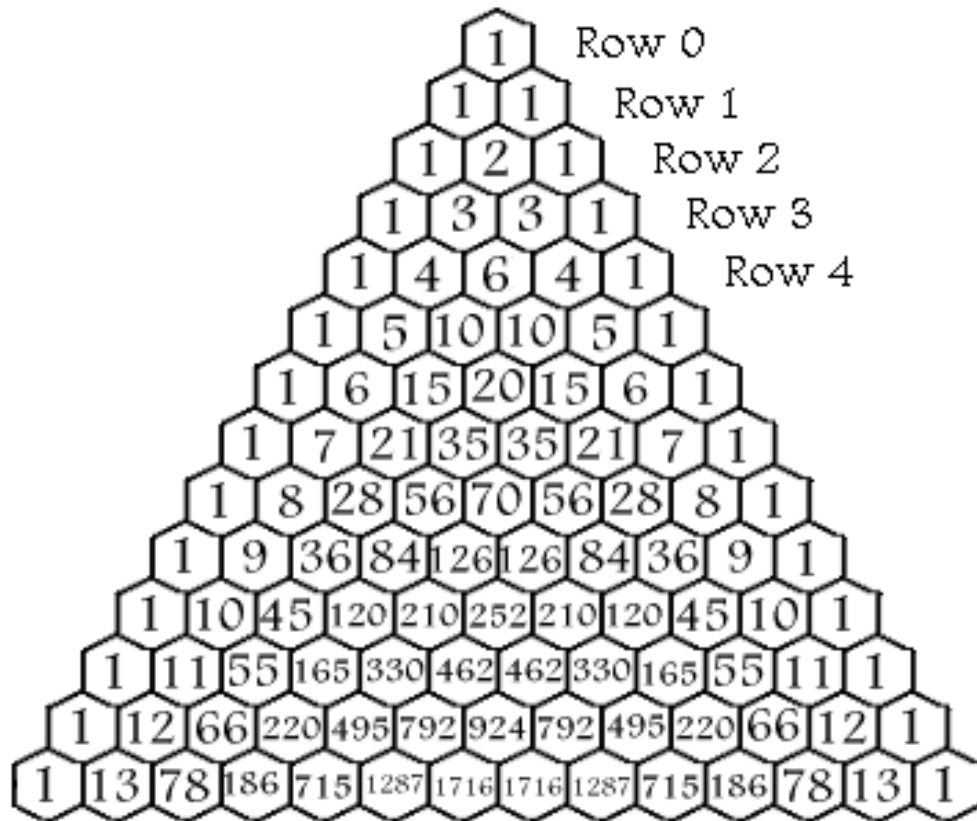with the b's.

The mathematical formula for calculating THE NUMBER OF WAYS OF SELECTING r
ITEMS FROM n ITEMS was part of the work of a French Mathematician called
BLAISE PASCAL.

PASCAL DEFINED THE NUMBER OF WAYS OF SELECTING r
OBJECTS FROM n OBJECTS AS:

$$^nC_r = \frac{n!}{(n-r)!r!} \text{ Sometimes written as } \binom{n}{r} = \frac{n!}{(n-r)!r!}$$

He also came up with a famous number sequence that helps to find the coefficients.
This is called **PASCALS TRIANGLE.**

Let us first look at PASCALS TRIANGLE and how it can help find coefficients for low values of n.

Can you spot how the number sequence is generated?

CAN YOU SEE ANY SIMILARITIES WITH THE COEFFICIENTS OF THE BINOMIAL EXPANSIONS THAT WE HAVE ALREADY LOOKED AT?

REMEMBER

$$(a+b)^2 = a^2 + 2ab + b^2$$

Can you see the numbers 1, 2, 1 in ROW 2 of PASCAL'S TRIANGLE?.

$$(a+b)^3 = (a+b)(a+b)(a+b)$$
$$= a^3 + 3a^2b + 3ab^2 + b^3$$

Can you see the numbers 1, 3, 3, 1 in ROW 3 of PASCAL'S TRIANGLE?

$$(a+b)^4 = (a+b)(a+b)(a+b)(a+b)$$
$$= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Can you see the numbers 1, 4, 6, 4, 1 in ROW 4 of PASCAL'S TRIANGLE?

SO WE CAN USE PASCAL'S TRIANGLE TO GENERATE THE COEFFICIENTS OF A BINOMIAL EXPANSION. BUT…….. It is time consuming for larger values of n.

Pascal also came up with a general formula for the coefficients.

FACTORIALS!

The meaning of n!   is
 n factorial
(NOTHING TO SHOUT ABOUT)

<u>DEFINITION</u>
When we repeatedly multiply by each digit decreasing by one at a time we are finding
a FACTORIAL.

5 factorial is written with an exclamation mark 5!

$5! = 5 \times 4 \times 3 \times 2 \times 1$ =120 This can be found on most scientific calculators.

We can use factorial notations to define any multiplication of this type, even if the
stopping number is not 1.

$15 \times 14 \times 13 \times 12 = \dfrac{15!}{11!}$ because 11! Will Cancel out the unwanted part of the multiplication.

WE ALSO NOTE THAT TO AVOID PROBLEMS LATER WE MUST
DEFINE ZERO FACTORIAL AS 1.

$$0! = 1$$

 This is to avoid compications and to ensure that there are no exceptions

A combination of r objects from n objects is denoted by

$$^{n}C_{r}$$

or some texts prefer the notation

$$\binom{n}{r}$$

This is defined as

$$^{n}C_{r} = \dfrac{n!}{(n-r)!\,r!}$$

Find the button on your scientific calculator but REMEMBER that it can
not be used in the C1 exam.
This is best seen by doing lots of examples.

EXAMPLE 1

Write out the BINOMIAL EXPANSION of $(a+b)^3$.

Hence expand $(3x+2)^3$, simplifying every term of the expansion.

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

Now replace a with (3x) and b with (2)

NOTE THE IMPORTANCE OF THE BRACKETS AROUND (3x) IN PARTICULAR!!!!!
THIS IS SO AS NOT TO FORGET TO SQUARE OR CUBE THE 3

$$(3x+2)^3 = (3x)^3 + 3(3x)^2(2) + 3(3x)(2)^2 + (2)^3$$

$$(3x+2)^3 = 27x^3 + 3(9x^2)(2) + 3(3x)4 + 8$$

$$= 27x^3 + 54x^2 + 36x + 8$$

Make sure that you understand where each of the numbers have come from!

EXAMPLE 2

Write out the BINOMIAL EXPANSION of $(a+b)^4$

Hence Expand $\left(3x-\dfrac{1}{3x}\right)^4$, simplifying each term of the expansion.

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Now simply replace a with (3x) and (not so simply!) replace b with $\left(-\dfrac{1}{3x}\right)$

DO NOT UNDERESTIMATE THE IMPORTANCE OF BRACKETS!!!!

$$(3x-\tfrac{1}{3x})^4 = (3x)^4 + 4(3x)^3\left(-\frac{1}{3x}\right) + 6(3x)^2\left(-\frac{1}{3x}\right)^2 + 4(3x)\left(-\frac{1}{3x}\right)^3 + \left(-\frac{1}{3x}\right)^4$$

NOTICE THAT ALL THAT HAS BEEN DONE SO FAR IS A DIRECT "REPLACE" a AND b WITH THE RELECENT EXPRESSIONS.

NOW SIMPLIFY

$$(3x-\tfrac{1}{3x})^4 = (3x)^4 + 4(3x)^3\left(-\frac{1}{3x}\right) + 6(3x)^2\left(-\frac{1}{3x}\right)^2 + 4(3x)\left(-\frac{1}{3x}\right)^3 + \left(-\frac{1}{3x}\right)^4$$

$$= 81x^4 + 4(27x^3)\left(-\frac{1}{3x}\right) + 6(9x^2)\left(-\frac{1}{3x}\right)^2 + 4(3x)\left(-\frac{1}{3x}\right)^3 + \left(-\frac{1}{3x}\right)^4$$

$$= 81x^4 - \frac{4(27x^3)}{3x} + \frac{6(9x^2)}{9x^2} - \frac{4(3x)}{27x^3} + \frac{1}{81x^4}$$

ARE YOU WITH IT SO FAR???

NOW IT IS TIME TO CANCEL THE FRACTIONS INCLUDING THE POWERS

$$(3x-\tfrac{1}{3x})^4 = 81x^4 - \frac{4(27x^3)}{3x} + \frac{6(9x^2)}{9x^2} - \frac{4(3x)}{27x^3} + \frac{1}{81x^4}$$

$$= 81x^4 - \frac{4(9x^2)}{1} + \frac{6}{1} - \frac{4}{9x^2} + \frac{1}{81x^4}$$

$$= 81x^4 - 36x^2 + 6 - \frac{4}{9x^2} + \frac{1}{81x^4}$$

MAKE SURE THAT YOU TAKE YOUR TIME AND FOLLOW EACH STEP OF THE SIMPLIFICATION!!! JOB DONE!!!

When we are not dealing with $(a+b)^n$ but have other expressions in place of a and b, we must be very careful to use brackets. We also note that the FINAL COEFFICIENTS are not simply those given by Pascal's Triangle.

EXAMPLE 3

The coefficient of x^2 in the expansion of $(1+2x)^n$ is 40.
Given that n is a positive integer, find the value of n.

The TERM of x^2 in $(1+2x)^n$ will be, BY DEFINITION $^nC_2(2x)^2$

Remember that

$$^nC_r = \frac{n!}{(n-r)!r!}$$

$$^nC_2 = \frac{n!}{(n-2)!2!}$$

So If we consider the TERM of x^2 in $(1+2x)^n$ we know it must be $40x^2$

$$^nC_2(2x)^2 = 40x^2$$

$$\frac{n!}{(n-2)!2!}(4x^2) = 40x^2$$

Ignoring the x^2 and considering only the COEFFICIENTS

$$\frac{4n!}{(n-2)!2!} = 40$$ but we can cancel part of the n! with the (n-2)! By definition.

$$\frac{4n(n-1)(n-2)(n-3)(n-4)\ldots\ldots\ldots3\times2\times1}{(n-2)(n-3)(n-4)\ldots\ldots3\times2\times1\times2!} = 40$$

$$\frac{4n(n-1)}{2!} = 40$$

Now we are in a position to SOLVE the equation

$$\frac{2n(n-1)}{1} = 40$$

$$n(n-1) = 20$$

$$n^2 - n - 20 = 0$$

$$(n-5)(n+4) = 0$$

$$n = 5 \text{ or } n = -4$$

The POSITIVE SOLUTION IS REQUIRED SO n=5

MORE
Write down the binomial expansion of $(1 + 3x)^n$ in ascending powers of x as far as the x^2 term.
If the coefficient of x^2 is six times the coefficient of x, find the value of n.

We do not require the whole expansion

$$(1 + 3x)^n = 1 + {}^nC_1(3x) + {}^nC_2(3x)^2$$

$$= 1 + n \times 3x + \frac{n(n-1)}{2!}9x^2 + \ldots\ldots$$

$$= 1 + 3nx + \frac{9n(n-1)}{2}x^2 + \ldots\ldots$$

FACT THAT WE ARE TOLD
Coefficient of x^2 = 6 × Coefficient of x.

$$\frac{9n(n-1)}{2} = 6 \times 3n$$

$$9n(n-1) = 36n$$

$$9n(n-1) - 36n = 0$$

$$9n(n-1-4) = 0$$

$$9n(n-5) = 0$$

$$n = 0 \quad \text{or} \quad n-5 = 0$$

$$n = 5 \text{ is the relevant solution}$$