2 Permutations, Combinations, and the Binomial Theorem

2.1 Introduction

A permutation is an ordering, or arrangement, of the elements in a finite set. Of greater interest are the r-permutations and r-combinations, which are ordered and unordered selections, respectively, of r elements from a given finite set. The Binomial Theorem gives us a formula for $(x+y)^n$, where $n \in \mathbb{N}$. If you would like extra reading, please refer to Sections 5.3 and 5.4 in Rosen.

Upon completion of this chapter, you will be able to do the following:

- Compute the number of r-permutations and r-combinations of an n-set.
- Use the Binomial Theorem to find the expansion of $(a+b)^n$ for specified a, b and n.
- Use the Binomial Theorem directly to prove certain types of identities.
- Provide a combinatorial proof to a well-chosen combinatorial identity.

2.2 Overview and Definitions

A *permutation* π of $A = \{a_1, a_2, \ldots, a_n\}$ is an ordering $a_{\pi_1}, a_{\pi_2}, \ldots, a_{\pi_n}$ of the elements of A. Note that $i \neq j \rightarrow \pi_i \neq \pi_j$. For example some permutations of the set $A = \{a, b, c, d\}$ are a, b, c, d or d, b, c, a or d, a, c, b. There are $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ such permutation. Generally, there are $n!$ permutations of an n-element set. An r-permutation of an n-element set (or n**-set**) A is an ordering $a_{\pi_1}, a_{\pi_2}, \ldots, a_{\pi_r}$ of some r-subset of A. There are $P(n, r)$ of these. As an example, for the set $A = \{a, b, c, d\}$ some examples of 2-set permutations of elements of A are a, b or a, c or b, c, and so on. There are $P(4, 2)$ of those. Some examples of 3-set permutations of elements of A are a, b, c or a, c, d or b, c, a, and so on. There are $P(4, 3)$ of those.

An r-combination of an n-set A is simply an r-subset $\{a_{i_1}, a_{i_2}, \ldots, a_{i_r}\}$ of A. There are $C(n, r)$ of these. The number $C(n, r)$ is also commonly written $\binom{n}{r}$, which is called a *binomial coefficient*. These are associated with a mnemonic called *Pascal's Triangle* and a powerful result called the Binomial Theorem, which makes it simple to compute powers of binomials. The inductive proof of the binomial theorem is a bit messy, and that makes this a good time to introduce the idea of *combinatorial proof*. The sort of combinatorial proof that we work with here consists of arguing that both sides of an equation of two integer expressions are equal to the cardinality of the same set. It is a powerful proof technique, and is the last one that you will learn in MA1025.

2.3 Permutations and Combinations

For integers $n \geq 0$, the factorial $f(n) = n!$ is defined by

$$n! = \begin{cases} 1, & \text{if } n = 0; \\ n(n-1)!, & \text{if } n > 0. \end{cases}$$

A *permutation* of an n-set is an arrangement of its elements. In such an arrangement, there are n choices for the first element, $(n-1)$ choices for the second element, etc., so the number of possible permutations of an n-set is $n(n-1)(n-2)\cdots(2)(1) = n!$. There is little more to say about it. Of greater interest is the notion of an *r-permutation* of an n-set ($r \leq n$). This is an ordered selection of r elements from the set. The usual notation for the number of these, if repetition is forbidden, is $P(n,r)$, which is computed by taking the product of the first r numbers counting down from n. In other words,

$$\begin{aligned} P(n,r) &= n(n-1)(n-2)\cdots(n-r+1) = \\ &= \frac{n(n-1)\cdots(n-r+1)(n-r)(n-r-1)\cdots(2)(1)}{(n-r)(n-r-1)(n-r-2)\cdots(2)(1)} = \frac{n!}{(n-r)!}. \end{aligned}$$

This is sometimes called a *falling factorial*.

Example 1:

(a) The number of 3-digit decimal numbers with no repeated digit (leading zeros allowed) is $P(10,3) = 720$. Note that this can be obtained by the multiplication principle as well as $10 \cdot 9 \cdot 8 = 720$. If we are taking an r-permutation of an n-set with repetition *allowed*, the number of such arrangements is n^r. The number of 3-digit decimal numbers with repetition (and possible leading zeros) allowed is simply $10^3 = 1000$. This can also be obtained by the multiplication principle as $10 \cdot 10 \cdot 10$.

(b) Here is a commonly-encountered sort of problem. The question: how many permutations of ABCDEFGH contain the string "DEF" as a substring? There are $8! = 40,320$ arrangements of ABCDEFGH, but most of them don't have DEF as a substring. The simplest approach is to treat DEF as a single "letter". You are now counting permutations of the 6-letter string ABC(DEF)GH; there are $6! = 720$ of these.

(c) Let $A = \{a, b, c\}$ be a 3-set. There are $P(3,2) = \frac{3!}{1!}$ ways of permuting two elements from A, namely: $(1)a, b$; $(2)b, a$; $(3)a, c$; $(4)c, a$; $(5)b, c$; and $(6)c, b$.

Now for r-combinations: In how many ways can we choose an r-subset (no repetition) of an n-set? Such a subset is called an r-*combination*. Let $C(n,r)$ denote the number of these. Note that we are partitioning the n-set into sets: one r-set and one $n-r$-set. So there are $C(n,r)$ ways to choose an r-set (producing one $n-r$-set each time time an r-set is chosen). For example there are $C(3,2) = \frac{3!}{2!1!} = 3$ possible ways of choosing a 2-set from a set $A = \{a,b,c\}$, namely $\{a,b\}$, {b, c} and $\{a,c\}$.

Now, we could construct an r-permutation of an set with n elements (an n-set) in two steps: first take an r-combination, then take a permutation of the r-combination. It follows by the Product Rule that $P(n,r) = r!C(n,r)$, but then

$$C(n,r) = \frac{1}{r!}P(n,r) = \frac{n!}{r!(n-r)!}.$$

With a bit of practice, you won't forget this.

The formula above is not a practical formula for hand computation, but we can find a better one:

$$C(n,r) = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\cdots(n-r+1)(n-r)!}{r!(n-r)!} = \frac{n(n-1)\cdots(n-r+1)}{r!}; \qquad (1)$$

note that there are exactly r factors in numerator and denominator of the last fraction.

Here is another useful result when it comes to hand computation. We can rearrange the formula for $C(n,r)$ as follows:

$$C(n,r) = \frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)!r!} = \frac{n!}{(n-r)!(n-(n-r))!} = C(n,n-r). \qquad (2)$$

This makes some potentially nasty computations pretty easy to carry out. For example, what is $C(100,98)$? By definition, $C(100,98) = \dfrac{100!}{98!2!}$, which is beyond the range of many calculators. And if we use formula (1) for hand computation of r-combinations, we'll have 98 factors in both numerator and denominator. But by (2) and (1) together, we have

$$C(100,98) = C(100,2) = \frac{(100)(99)}{2!} = 50 \cdot 99 = 4,950.$$

Example 2: The Department of Applied Mathematics has nineteen faculty members, of whom three are women. How many committees of four can be formed if

(a) the department chairman is not eligible?

(b) exactly one of the committee members must be a woman?

(c) exactly one must be a woman, and Professors Smith and Jones (both men) refuse to serve together.

Solution:

(a) If the chairman is ineligible, four committee members must be selected from the remaining eighteen faculty members. This can be done in $C(18, 4) = 3060$ ways. (Note that the order we pick the 4 committee members doesn't matter, so we use $C(18, 4)$)

(b) If exactly one member must be a woman, there are $C(3, 1)C(16, 3) = 3 \cdot 560 = 1680$ ways to form the committee.

(c) All of the committees formed in (b) qualify except those in which Smith and Jones are both members. There are $C(3, 1)C(14, 1) = 42$ committees that both Smith and Jones can serve on, so $C(3, 1)C(16, 3) - C(3, 1)C(14, 1) = 1680 - 42 = 1638$ committees can be formed containing exactly one woman and at most one of the grumps.

2.4 Combinatorial Proof

The algebraic proof of the identity $C(n, r) = C(n, n - r)$ has been presented before (see equation (2)). But there is another way, equally simple. This is called *combinatorial proof.* For our purposes, combinatorial proof is a technique by which we can prove an algebraic identity without using algebra, by finding a set whose cardinality is described by both sides of the equation. Here is a combinatorial proof that $C(n, r) = C(n, n - r)$.

Proof: We can partition an n-set into two subsets, with respective cardinalities r and $n - r$, in two ways: we can first select an r-combination, leaving behind its complement, which has cardinality $n - r$ and this can be done in $C(n, r)$ ways (the left hand side of the equation). Or we can first take an $(n - r)$-combination, then leaving behind *its* complement, which has cardinality r and this can be done in $C(n, n-r)$ ways (the right hand side of the equation). The number of possible outcomes is the same either way. It follows that $C(n, r) = C(n, n - r)$. $\square$

It's a remarkable method. It doesn't apply in every instance, but it does add an arrow to your quiver. There are times when it is far easier to devise a combinatorial proof than an algebraic proof, as we'll see shortly. Look for more examples of combinatorial proof in the next section.

2.5 The Binomial Theorem

It's time to begin using the alternate notation for $C(n, r)$, which is $\binom{n}{r}$. This is called a *binomial coefficient*, and is pronounced "n choose r". Perhaps you recall from the beginning of the module that if x and y are variables and $n \in \mathbb{N}$, then

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

This is the Binomial Theorem. Here is a combinatorial proof.

Proof: Expanding $(x+y)^n$, we get $(x+y)^n = (x+y)(x+y)\cdots(x+y)$, a product of n factors. What is the coefficient on $x^{n-k}y^k$? Every term in the expansion is the result of choosing either the x or the y from each factor. Since the power of y is k, we need to choose the y from k factors (there are $\binom{n}{k}$ ways to so), and to choose x from the remaining $n-k$ factors, so it follows that the coefficient on $x^{n-k}y^k$ is $\binom{n}{k}$. $\square$

Note that the coefficients on $x^{n-k}y^k$ and $x^k y^{n-k}$ in $(x+y)^n$ are the same, since $\binom{n}{k} = \binom{n}{n-k}$. It follows that an equivalent formulation is

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

The alternative to a combinatorial proof of the theorem is a proof by mathematical induction, which can be found following the examples illustrating uses of the theorem.

Example 3: We start with some straightforward applications of the theorem.

(a) What is the expansion of $(x+y)^5$?

(b) What is the expansion of $(2x-y)^4$?

(c) What is the coefficient on x^3y^3 in $(2x-3y)^6$?

Solution:

(a) By the binomial theorem,

$$\begin{aligned}(x+y)^5 &= \binom{5}{0}x^5y^0 + \binom{5}{1}x^4y^1 + \binom{5}{2}x^3y^2 + \binom{5}{3}x^2y^3 + \binom{5}{4}x^1y^4 + \binom{5}{5}x^0y^5 \\ &= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5.\end{aligned}$$

(b)

$$\begin{aligned}(2x-y)^5 &= ((2x)+(-y))^5 \\ &= \binom{5}{0}(2x)^5(-y)^0 + \binom{5}{1}(2x)^4(-y)^1 + \binom{5}{2}(2x)^3(-y)^2 + \binom{5}{3}(2x)^2(-y)^3 + \binom{5}{4}(2x)^1(- \\ &= 32x^5 - 30x^4y + 80x^3y^2 - 40x^2y^3 + 10xy^4 - y^5.\end{aligned}$$

(c) The term in $(2x-3y)^6$ containing x^3y^3 is

$$\binom{6}{3}(2x)^3(-3y)^3 = (20)(8x^3)(-27y^3) = -4320x^3y^3,$$

so the coefficient in question is -4320. The entire expansion of $(2x-3y)^6$ can be computed in similar fashion.

There are also some surprising ways to use the theorem. For example, let $n \in \mathbf{Z}^+$, and let $0 \le m \le n$. For any positive integer k, n^k can be expressed as a sum of powers of m and $n - m$. To see this, simply note that, by the Binomial Theorem,

$$n^k = \sum_{j=0}^{k} \binom{k}{j} m^k (n-m)^{k-j}.$$

For an example, $5^n = \displaystyle\sum_{k=0}^{n} \binom{n}{k} 3^k 2^{n-k}$.

Here are some additional examples of combinatorial proof.

Example 4: A nameless algebraic identity states that

$$\binom{2n}{2} = 2\binom{n}{2} + n^2.$$

Here is a combinatorial solution. Its use of color is just one of several ways to differentiate the elements of the two subsets introduced to drive the proof.

Proof: The expression on the left-hand side is the number of 2-subsets of a $2n$-set. Let A be a $2n$-set, and suppose that A contains n red elements and n blue elements. We now choose all the possible 2-subsets, by counting all the choices: all the 2-subsets that have exactly 2 red elements, all the 2-subsets that have exactly 2 blue elements, and all the 2-subsets that have exactly one red element and one blue element. There are $\binom{n}{2}$ red 2-subsets, $\binom{n}{2}$ blue 2-subsets, and $\binom{n}{1}\binom{n}{1} = n^2$ subsets containing one red and one blue elements. By the Sum Rule, the number of 2-subsets of A is $2\binom{n}{2} + n^2$. □

Example 5: Here is a variation on the theme. Suppose we want to prove the identity,

$$\binom{2n}{3} = 2\binom{n}{3} + 2n\binom{n}{2}.$$

The same technique used in the preceding problem leads to the following argument.

Proof: The expression on the left counts the number of 3-subsets of a $2n$-set. Let A be a $2n$-set containing n red and n blue elements. There are $\binom{n}{3}$ red 3-subsets, $\binom{n}{3}$ blue 3-subsets, $\binom{n}{2}\binom{n}{1}$ 3-subsets with two red elements and one blue, and $\binom{n}{2}\binom{n}{1}$ 3-subsets with two blue elements and one red. Simplifying, we see that the number of 3-subsets of A is given by $\binom{n}{3} + \binom{n}{3} + \binom{n}{2}\binom{n}{1} + \binom{n}{2}\binom{n}{1} = 2\binom{n}{3} + 2n\binom{n}{2}$. The result follows. □

Example 6: Here's another, asking for a proof of the identity

$$\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}.$$

Proof: The left hand side has two factors: the first binomial coefficient is the number of ways to choose an r-subset of an n-set; the second is the number of ways to choose a k-subset from the r-set just chosen (which leaves the remaining $r-k$ elements). The result? It's a partition of the original n-set into subsets of cardinalities $n-r$, $r-k$, and k. We could just as well construct such a partition by first choosing the k-subset, then choosing the $(r-k)$-subset from the $(n-k)$-subset left behind. $\square$

Example 7: You might have seen a triangular array of binomial coefficients called Pascal's Triangle. The practical value of the triangle is questionable, but the value of the identity that generates the coefficients therein, called Pascal's Identity, is very useful. Pascal's Identity states that, for all $n, k \in \mathbf{Z}^+$,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

The identity can be easily proved using a combinatorial proof:

Proof: The left side of the identity is the number of k-subsets of an n-set. So suppose A is an n-set, and let $a \in A$. A given k-subset of A either contains a, or not. There are $\binom{n-1}{k-1}$ k-subsets of A that contain a, and $\binom{n-1}{k}$ k-subsets of A that do not. Together, they give all the k-subsets of an n-set, regardless if the k-subset contains a or not. $\square$

Before we look at the induction proof, here is one more thing we need to learn:

$$\sum_{j=0}^{k-1} \binom{k}{j} x^j = \sum_{j=1}^{k} \binom{k}{j-1} x^{j-1}.$$

To see this, try substituting $j-1$ in for j in the expression $\sum_{j=0}^{k-1} \binom{k}{j} x^j$.

Here is the inductive proof of the Binomial Theorem. The last step in the sequence uses Pascal's identity. The other steps involve simple manipulations of the summation indices, laws of exponents, the distributive law, etc.

Recall the statement of the theorem: **for all** $n \geq 0$**,** $(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$.

Proof: First note that $(x+y)^0 = 1 = \frac{0!}{0!0!} x^0 y^0 = \displaystyle\sum_{j=0}^{0} \binom{0}{j} x^{0-j} y^j$.

For the inductive step, let $k \geq 0$, and assume that $(x+y)^k = \displaystyle\sum_{j=1}^{k} \binom{k}{j} x^{n-j} y^j$. Then

$$\begin{aligned}
(x+y)^{k+1} &= (x+y)(x+y)^k \\
&= x(x+y)^k + y(x+y)^k \\
&= x \sum_{j=0}^{k} \binom{k}{j} x^{k-j} y^j + y \sum_{j=0}^{k} \binom{k}{j} x^{k-j} y^j
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{k} \binom{k}{j} x^{k+1-j} y^j + \sum_{j=0}^{k} \binom{k}{j} x^{k-j} y^{j+1} \\
&= x^{k+1} + \sum_{j=1}^{k} \binom{k}{j} x^{k+1-j} y^j + \sum_{j=0}^{k-1} \binom{k}{j} x^{k-j} y^{j+1} + y^{k+1} \\
&= x^{k+1} + \sum_{j=1}^{k} \binom{k}{j} x^{k+1-j} y^j + \sum_{j=1}^{k} \binom{k}{j-1} x^{k+1-j} y^j + y^{k+1} \\
&= x^{k+1} + \sum_{j=1}^{n} \left[\binom{k}{j} + \binom{k}{j-1} \right] x^{k+1-j} y^j + y^{k+1} \\
&= \sum_{j=1}^{k+1} \binom{k+1}{j} x^{(k+1)-j} y^j,
\end{aligned}
$$

and the result follows by induction. □

2.6 Exercises

For solutions, click here.

1. If $|A| = m$ and $|B| = n$, how many functions $f : A \to B$ are one-to-one?

2. How many strings of length three over $\Sigma = \{a, e, i, o, u\}$ have no repeated letter?

3. How many strings over $\Sigma = \{a, e, i, o, u\}$ have no repeated letter?

4. What is the number of 3-element subsets of $\{1, 2, \ldots, 10\}$?

5. Joe's Pizzeria offers two styles of crust and the following optional toppings: extra cheese, pepperoni, sausage, mushrooms, green peppers, artichokes, onions, and anchovies. Joe claims that he offers over 500 different pizzas. Is this true?

6. In how many ways can six hardcover and four paperback books be arranged on a shelf? In how many ways can they be arranged if no two paperbacks can be adjacent?

7. What is the coefficient on x^3y^3 in $(2x - y)^6$?

8. Use the Binomial Theorem to show that, for any $n \in \mathbb{Z}$, 3^n can be expressed as a linear combination of powers of 2, with the largest exponent being n.

9. Use the Binomial Theorem to prove that $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

10. Use a combinatorial proof to show that $\binom{3n}{2} = \binom{2n}{2} + \binom{n}{2} + 2n^2$.

Solutions to Exercises in Chapter 2: Permutations, Combinations, and the Binomial Theorem

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1. If $|A| = m$ and $|B| = n$, how many functions $f : A \to B$ are one-to-one?

   Solution: There are $P(n, m)$ one-to-one functions from A to B. This problem can be solved using the product rule alone, and so could have been included in the exercises for Chapter 1.

2. How many strings of length three over $\Sigma = \{a, e, i, o, u\}$ have no repeated letter?

   Solution: $P(5, 3) = 60$.

3. How many strings over $\Sigma = \{a, e, i, o, u\}$ have no repeated letter?

   Solution: A string over Σ with no repeated letter can contain 0, 1, 2, 3, 4, or 5 symbols, so the number of such strings is

$$\sum_{k=0}^{5} P(5, k) = 1 + 5 + 20 + 60 + 120 + 120 = 326.$$

4. What is the number of 3-element subsets of $A = \{1, 2, \ldots, 10\}$?

   Solution: There are $C(10, 3) = 120$ 3-element subsets of A.

5. Joe's Pizzeria offers two styles of crust and the following optional toppings: extra cheese, pepperoni, sausage, mushrooms, green peppers, artichokes, onions, and anchovies. Joe claims that he offers over 500 different pizzas. Is this true?

   Solution: Yes, it's true. There are two choices of crust, and $2^8 = 256$ choices of topping. By the Product Rule, there are 512 distinct pizzas available.

6. In how many ways can six hardcover and four paperback books be arranged on a shelf? In how many ways can they be arranged if no two paperbacks can be adjacent?

   Solution: There are $10! = 3,628,800$ arrangements of the eleven books. To find the number with no adjacent paperbacks, we count the permutations of the hardbacks, the ways to choose four nonadjacent locations for the paperbacks, and the permutations of the paperbacks separately. There are $6! = 720$ arrangements of the hardbacks, there are $\binom{7}{4} = 35$ ways to select four nonadjacent spaces for the four paperbacks, and there are $4! = 24$ ways to place the paperbacks in the selected spaces. By the Product Rule, the grand total is $720 \cdot 35 \cdot 24 = 604,800$ ways to get the job done.

7. What is the coefficient on x^3y^3 in $(2x - y)^6$?

   Solution: The coefficient on x^3y^3 is $\binom{6}{3} \cdot 2^3 \cdot (-1)^3 = -160$.

8. Use the Binomial Theorem to show that, for any $n \in \mathbb{N}$, 3^n can be expressed as a linear combination of powers of 2, with the largest exponent being n.

   Proof: $3^n = (2+1)^n = \displaystyle\sum_{k=0}^{n} \binom{n}{k} 2^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k} 2^k.$ $\square$

9. Use the Binomial Theorem to prove that $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = 2^n.$

   Proof: $2^n = (1+1)^n = \displaystyle\sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k}.$ $\square$

10. Use a combinatorial proof to show that $\binom{3n}{2} = \binom{2n}{2} + \binom{n}{2} + 2n^2$.

    Proof: The number on the left is the number of 2-subsets of a $3n$-set. Let A be a $3n$-set containing, say, $2n$ red and n blue elements. There are $\binom{2n}{2}$ red 2-subsets, $\binom{n}{2}$ blue 2-subsets, and $\binom{2n}{1}\binom{n}{1} = 2n^2$ mixed subsets. The result follows. $\square$

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Exercises in Rosen in section 5.3 : $9, 11, 13, 17, 19, 21, 23$, and 37, and in section 5.4 exercises $5, 9, 21, 22a, 28$, and 29.

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Self-Quiz on Permutations, Combinations, and the Binomial Theorem

1. In how many ways can we construct a license number consisting of one decimal digit followed by three uppercase alphabetical characters followed by three decimal digits?

2. In how many ways can five men and five women be arranged in a line for a photograph so that men and women alternate?

3. What is the coefficient on x^2y^5 in $(x+y)^7$? In $(x-2y)^7$?

4. Give a combinatorial proof that $\displaystyle\binom{3n}{3} = 3\binom{n}{3} + 6n\binom{n}{2} + \binom{n}{1}^3.$

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Solution to Self-Quiz on Permutations, Combinations, and the Binomial Theorem

1. In how many ways can we construct a license number consisting of one decimal digit followed by three uppercase alphabetical characters followed by three decimal digits, if no characters or digits can be repeated?

   Solution: The number of ways to do this is $10P(26,3)P(9,3) = 10\frac{26!}{23!}\frac{9!}{6!} = 10(26 \cdot 25 \cdot 24)(9 \cdot 8 \cdot 7) = 78624000$.

2. In how many ways can five men and five women be arranged in a line for a photograph so that men and women alternate?

   Solution: There are $5! = 120$ ways to arrange the men, and $5! = 120$ ways to arrange the women. There are then two ways to interlace the two: mwmwmwmwmw and wmwmwmwmwm. The total, then, is $120^2 \cdot 2 = 28,800$.

3. What is the coefficient on x^2y^5 in $(x+y)^7$? In $(x-2y)^7$?

   Solution: The coefficient on x^2y^5 in $(x+y)^7$ is $\binom{7}{2} = 21$. The coefficient on x^2y^5 in $(x-2y)^7$ is $\binom{7}{2}(-2)^5 = -672$.

4. Give a combinatorial proof that $\displaystyle\binom{3n}{3} = 3n + 6n\binom{n}{2} + \binom{n}{3}^3$.

   Proof: Suppose that we have a $3n$-set, call it A. Then $\displaystyle\binom{3n}{3}$ is the number of $3-$subsets of A. There many ways to proceed. Suppose that A contains n elements each of three colors, say red, blue, and green. There are $3\binom{n}{3}$ ways to choose three elements of the same color. There are $3\binom{n}{2}2\binom{n}{1} = 6n\binom{n}{2}$ ways to choose two elements of one color and one element of a second color. Finally, there are $\binom{n}{1}^3$ ways to choose one element of each color. The result follows. $\square$

   Note: the assumption that A contains n elements each of three colors is only one way of forcing A to be the disjoint union of three n-sets, which in turn was dictated by the right-hand side of the identity to be proven. Had the identity been, say, $\binom{3n}{3} = \binom{2n}{3} + \binom{n}{3} + n\binom{2n}{2} + 2n\binom{n}{2}$, we would have partitioned A as the union of a $2n$-subset and an n-subset. (Try it!)