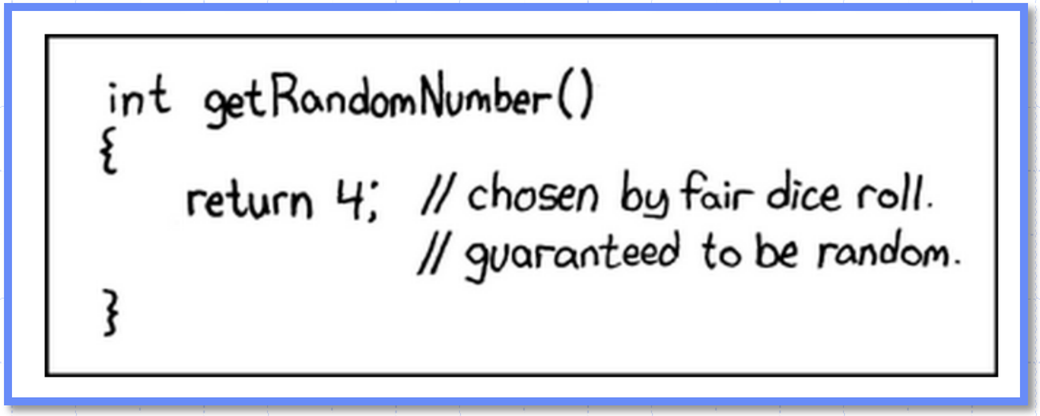
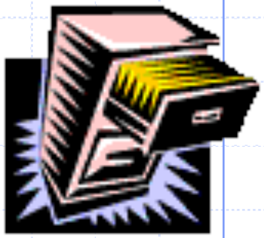
Presentation for use with the textbook Algorithm Design and Applications, by M. T. Goodrich and R. Tamassia, Wiley, 2015

Hash Tables



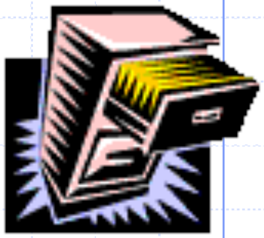
xkcd. http://xkcd.com/221/. “Random Number.” Used with permission under Creative Commons 2.5 License.

Recall the Map Operations

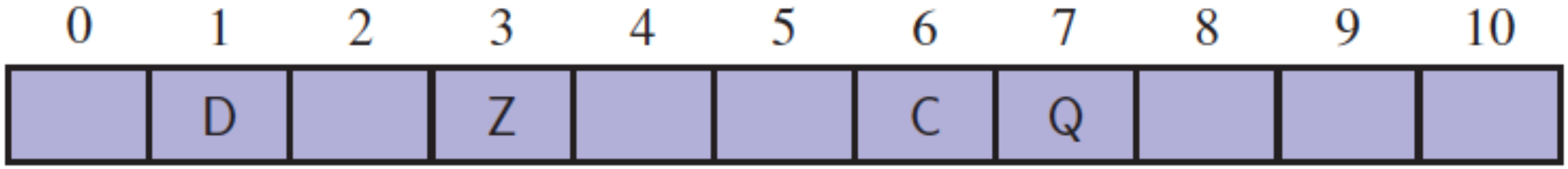


- get(k): if the map M has an entry with key k, return its associated value; else, return null
- put(k, v): insert entry (k, v) into the map M; if key k is not already in M, then return null; else, return old value associated with k
- remove(k): if the map M has an entry with key k, remove it from M and return its associated value; else, return null
- size(), isEmpty()

Intuitive Notion of a Map

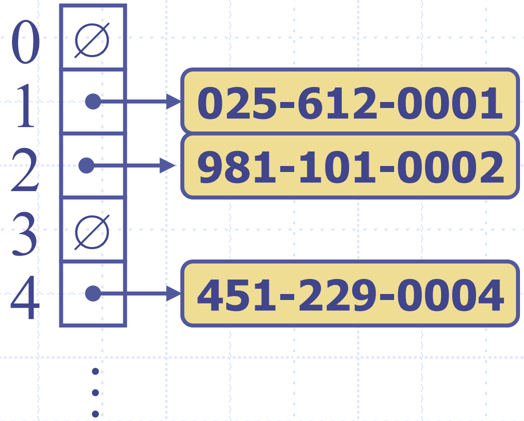


- Intuitively, a map M supports the abstraction of using keys as indices with a syntax such as M[k].
- As a mental warm-up, consider a restricted setting in which a map with n items uses keys that are known to be integers in a range from 0 to $N - 1$, for some $N \geq n$.

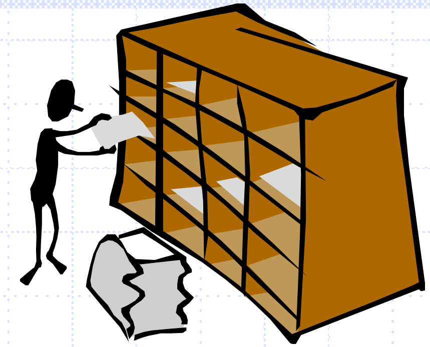


More General Kinds of Keys

- But what should we do if our keys are not integers in the range from 0 to N – 1?
  - Use a **hash function** to map general keys to corresponding indices in a table.
  - For instance, the last four digits of a Social Security number.



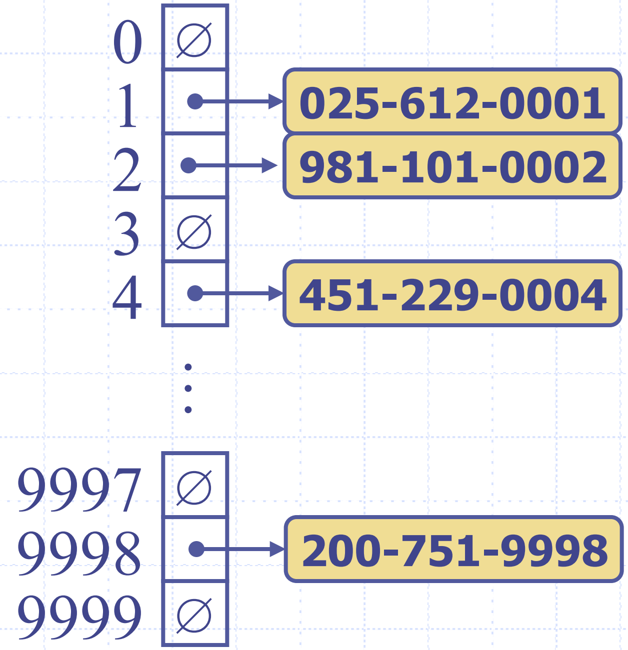
Hash Functions and Hash Tables



- A hash function h maps keys of a given type to integers in a fixed interval $[0, N-1]$
- Example:
  $$h(x) = x \bmod N$$
  is a hash function for integer keys
- The integer $h(x)$ is called the hash value of key x
- A hash table for a given key type consists of
  - Hash function h
  - Array (called table) of size N
- When implementing a map with a hash table, the goal is to store item (k, o) at index $i = h(k)$

Example

- We design a hash table for a map storing entries as (SSN, Name), where SSN (social security number) is a nine-digit positive integer
- Our hash table uses an array of size $N = 10{,}000$ and the hash function
  $h(x)$ = last four digits of x



Hash Functions

- A hash function is usually specified as the composition of two functions:

  Hash code:

  h_1: keys $\rightarrow$ integers

  Compression function:

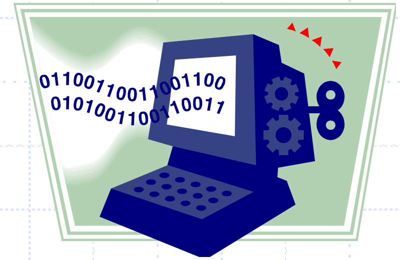
  h_2: integers $\rightarrow [0, N-1]$

- The hash code is applied first, and the compression function is applied next on the result, i.e.,

  $$h(x) = h_2(h_1(x))$$

- The goal of the hash function is to "disperse" the keys in an apparently random way

Hash Codes



- Memory address:
  - We reinterpret the memory address of the key object as an integer. Good in general, except for numeric and string keys
- Integer cast:
  - We reinterpret the bits of the key as an integer
  - Suitable for keys of length less than or equal to the number of bits of the integer type (e.g., byte, short, int and float)
- Component sum:
  - We partition the bits of the key into components of fixed length (e.g., 16 or 32 bits) and we sum the components (ignoring overflows)
  - Suitable for numeric keys of fixed length greater than or equal to the number of bits of the integer type.

Hash Codes (cont.)

- Polynomial accumulation:
  - We partition the bits of the key into a sequence of components of fixed length (e.g., 8, 16 or 32 bits)
    $$a_0\, a_1\, \ldots\, a_{n-1}$$
  - We evaluate the polynomial
    $$p(z) = a_0 + a_1 z + a_2 z^2 + \ldots \ldots + a_{n-1} z^{n-1}$$
    at a fixed value z, ignoring overflows
  - Especially suitable for strings (e.g., the choice $z = 33$ gives at most 6 collisions on a set of 50,000 English words)
- Polynomial $p(z)$ can be evaluated in $O(n)$ time using Horner's rule:
  - The following polynomials are successively computed, each from the previous one in $O(1)$ time
    $$p_0(z) = a_{n-1}$$
    $$p_i(z) = a_{n-i-1} + z p_{i-1}(z) \quad (i = 1, 2, \ldots, n-1)$$
- We have $p(z) = p_{n-1}(z)$

Tabulation-Based Hashing

- Suppose each key can be viewed as a tuple, $k = (x_1, x_2, \ldots, x_d)$, for a fixed d, where each x_i is in the range $[0,M - 1]$.
- There is a class of hash functions we can use, which involve simple table lookups, known as **tabulation-based hashing**.
- We can initialize d tables, $T_1, T_2, \ldots, T_d$, of size M each, so that each $T_i[j]$ is a uniformly chosen independent random number in the range $[0,N - 1]$.
- We then can compute the hash function, h(k), as

$$h(k) = T_1[x_1] \oplus T_2[x_2] \oplus \ldots \oplus T_d[x_d],$$

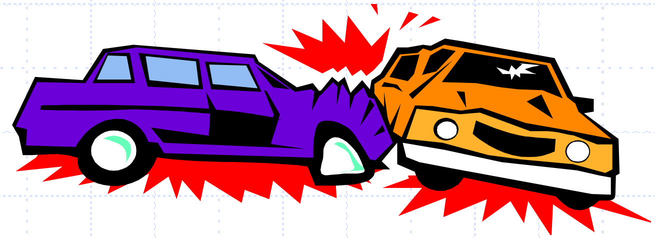
where "$\oplus$" denotes the bitwise exclusive-or function.

- Because the values in the tables are themselves chosen at random, such a function is itself fairly random. For instance, it can be shown that such a function will cause two distinct keys to collide at the same hash value with probability 1/N, which is what we would get from a perfectly random function.

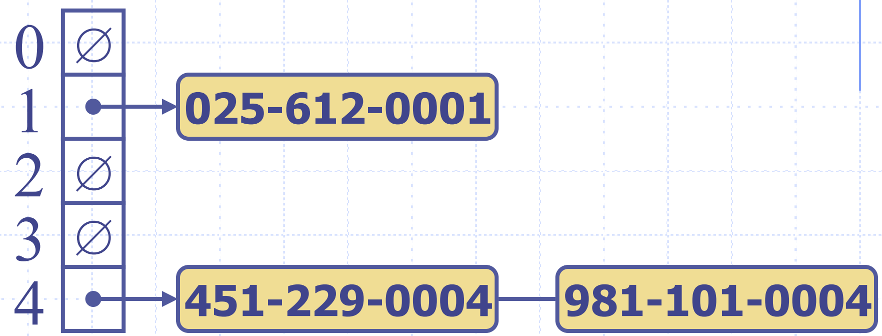
Compression Functions

- Division:
  - $h_2(y) = y \bmod N$
  - The size N of the hash table is usually chosen to be a prime
  - The reason has to do with number theory and is beyond the scope of this course
- Random linear hash function:
  - $h_2(y) = (ay + b) \bmod N$
  - a and b are random nonnegative integers such that
    $a \bmod N \neq 0$
  - Otherwise, every integer would map to the same value b

Collision Handling



- Collisions occur when different elements are mapped to the same cell



- Separate Chaining: let each cell in the table point to a linked list of entries that map there
- Separate chaining is simple, but requires additional memory outside the table

Map with Separate Chaining

- Delegate operations to a list-based map at each cell:

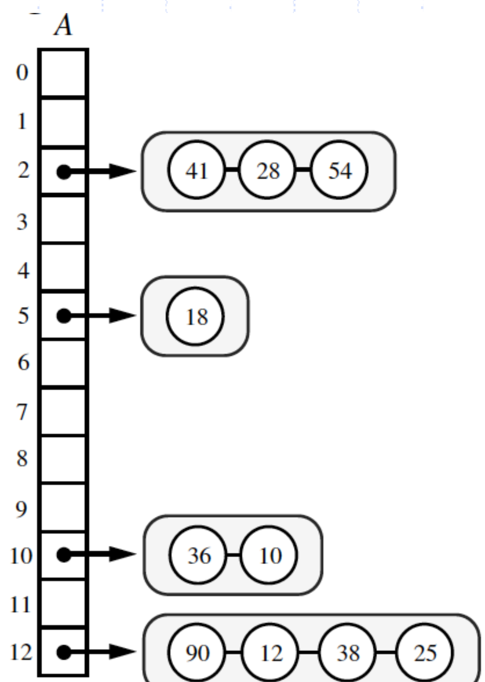
```
Algorithm get(k):
return A[h(k)].get(k)

Algorithm put(k,v):
t = A[h(k)].put(k,v)
if t = null then                          {k is a new key}
    n = n + 1
return t

Algorithm remove(k):
t = A[h(k)].remove(k)
if t ≠ null then                          {k was found}
    n = n - 1
return t
```

Performance of Separate Chaining

- Let us assume that our hash function, h, maps keys to independent uniform random values in the range [0,N−1].
- Thus, if we let X be a random variable representing the number of items that map to a bucket, i, in the array A, then the expected value of X, E(X) = n/N, where n is the number of items in the map, since each of the N locations in A is equally likely for each item to be placed.
- This parameter, n/N, which is the ratio of the number of items in a hash table, n, and the capacity of the table, N, is called the **load factor** of the hash table.
- If it is O(1), then the above analysis says that the expected time for hash table operations is O(1) when collisions are handled with separate chaining.



Linear Probing

- Open addressing: the colliding item is placed in a different cell of the table
- Linear probing: handles collisions by placing the colliding item in the next (circularly) available table cell
- Each table cell inspected is referred to as a "probe"
- Colliding items lump together, causing future collisions to cause a longer sequence of probes

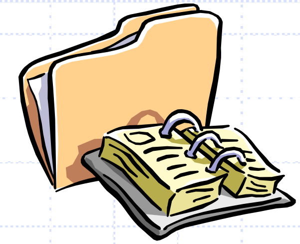
- Example:
  - $h(x) = x \bmod 13$
  - Insert keys 18, 41, 22, 44, 59, 32, 31, 73, in this order

0	1	2	3	4	5	6	7	8	9	10	11	12

⇩

		41			18	44	59	32	22	31	73	
0	1	2	3	4	5	6	7	8	9	10	11	12

Search with Linear Probing



- Consider a hash table A that uses linear probing
- get(k)
  - We start at cell $h(k)$
  - We probe consecutive locations until one of the following occurs
    - An item with key k is found, or
    - An empty cell is found, or
    - N cells have been unsuccessfully probed

```
Algorithm get(k)
   i ← h(k)
   p ← 0
   repeat
      c ← A[i]
      if c = ∅
         return null
      else if c.getKey () = k
         return c.getValue()
      else
         i ← (i + 1) mod N
         p ← p + 1
   until   p = N
   return null
```

Updates with Linear Probing

- To handle insertions and deletions, we introduce a special object, called ***DEFUNCT***, which replaces deleted elements
- remove(k)
  - We search for an entry with key k
  - If such an entry, (k, v), is found, we move elements to fill the "hole" created by its removal.
- put(k, v)
  - We throw an exception if the table is full
  - We start at cell $h(k)$
  - We probe consecutive cells until a A cell i is found that is empty.
  - We store (k, v) in cell i

Pseudo-code for get and put

- get(k):

  $i \leftarrow h(k)$
  while $A[i] \neq$ NULL **do**
      if $A[i]$.key $= k$ **then**
          return $A[i]$
      $i \leftarrow (i+1) \bmod N$
  return NULL

- put(k, v):

  $i \leftarrow h(k)$
  while $A[i] \neq$ NULL **do**
      if $A[i]$.key $= k$ **then**
          $A[i] \leftarrow (k, v)$ // replace the old (k, v')
      $i \leftarrow (i+1) \bmod N$
  $A[i] \leftarrow (k, v)$

Pseudo-code for remove

- remove(k):

```
i ← h(k)
while A[i] ≠ NULL do
    if A[i].key = k then
        temp ← A[i]
        A[i] ← NULL
        Call Shift(i) to restore A to a stable state without k
        return temp
    i ← (i + 1) mod N
return NULL
```

- Shift(i):

```
s ← 1        // the current shift amount
while A[(i + s) mod N] ≠ NULL do
    j ← h(A[(i + s) mod N].key)        // preferred index for this item
    if j ∉ (i, i + s] mod N then
        A[i] ← A[(i + s) mod N]        // fill in the "hole"
        A[(i + s) mod N] ← NULL        // move the "hole"
        i ← (i + s) mod N
        s ← 1
    else
        s ← s + 1
```

Performance of Linear Probing

- In the worst case, searches, insertions and removals on a hash table take $O(n)$ time
- The worst case occurs when all the keys inserted into the map collide
- The load factor $\alpha = n/N$ affects the performance of a hash table
- Assuming that the hash values are like random numbers, it can be shown that the expected number of probes for an insertion with open addressing is

$$1 / (1 - \alpha)$$

- The expected running time of all the dictionary ADT operations in a hash table is $O(1)$ with constant load < 1
- In practice, hashing is very fast provided the load factor is not close to 100%
- Applications of hash tables:
  - small databases
  - compilers
  - browser caches

A More Careful Analysis of Linear Probing

- Recall that, in the linear-probing scheme for handling collisions, whenever an insertion at a cell i would cause a collision, then we instead insert the new item in the first cell of i+1, i+2, and so on, until we find an empty cell.

Let $X_1, X_2, \ldots, X_n$ be a set of mutually independent indicator random variables, such that each X_i is 1 with some probability $p_i > 0$ and 0 otherwise. Let $X = \sum_{i=1}^{n} X_i$ be the sum of these random variables, and let μ denote the mean of X, that is, $\mu = E(X) = \sum_{i=1}^{n} p_i$. The following bound, which is due to Chernoff (and which we derive in Section 19.5), establishes that, for $\delta > 0$,

$$\Pr(X > (1+\delta)\mu) < \left[\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right]^{\mu}.$$

- For this analysis, let us assume that we are storing n items in a hash table of size N = 2n, that is, our hash table has a load factor of 1/2.

A More Careful Analysis of Linear Probing, 2

Let X denote a random variable equal to the number of probes that we would perform in doing a search or update operation in our hash table for some key, k. Furthermore, let X_i be a 0/1 indicator random variable that is 1 if and only if $i = h(k)$, and let Y_i be a random variable that is equal to the length of a run of contiguous nonempty cells that begins at position i, wrapping around the end of the table if necessary. By the way that linear probing works, and because we assume that our hash function $h(k)$ is random,

$$X = \sum_{i=0}^{N-1} X_i(Y_i + 1),$$

which implies that

$$\begin{aligned} E(X) &= \sum_{i=0}^{N-1} \frac{1}{2n} E(Y_i + 1) \\ &= 1 + (1/2n)E\left(\sum_{i=0}^{N-1} Y_i\right). \end{aligned}$$

- Thus, if we can bound the expected value of the sum of Y_i's, then we can bound the expected time for a search or update operation in a linear-probing hashing scheme.

A More Careful Analysis of Linear Probing, 2

Let X denote a random variable equal to the number of probes that we would perform in doing a search or update operation in our hash table for some key, k. Furthermore, let X_i be a 0/1 indicator random variable that is 1 if and only if $i = h(k)$, and let Y_i be a random variable that is equal to the length of a run of contiguous nonempty cells that begins at position i, wrapping around the end of the table if necessary. By the way that linear probing works, and because we assume that our hash function $h(k)$ is random,

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- Thus, if we can bound the expected value of the sum of Y_i's, then we can bound the expected time for a search or update operation in a linear-probing hashing scheme.

A More Careful Analysis of Linear Probing, 3

Consider, then, a maximal contiguous sequence, S, of k nonempty table cells, that is, a contiguous group of occupied cells that has empty cells next to its opposite ends. Any search or update operation that lands in S will, in the worst case, march all the way to the end of S. That is, if a search lands in the first cell of S, it would make k wasted probes, if it lands in the second cell of S, it would make $k-1$ wasted probes, and so on. So the total cost of all the searches that land in S can be at most k^2. Thus, if we let $Z_{i,k}$ be a 0/1 indicator random variable for the existence of a maximal sequence of nonempty cells of length k, then

$$\sum_{i=0}^{N-1} Y_i \le \sum_{i=0}^{N-1} \sum_{k=1}^{2n} k^2 Z_{i,k}.$$

Put another way, it is as if we are "charging" each maximal sequence of nonempty cells for all the searches that land in that sequence.

A More Careful Analysis of Linear Probing, 4

So, to bound the expected value of the sum of the Y_i's, we need to bound the probability that $Z_{i,k}$ is 1, which is something we can do using the Chernoff bound given above. Let Z_k denote the number of items that are mapped to a given sequence of k cells in our table. Then,

$$\Pr(Z_{i,k} = 1) \le \Pr(Z_k \ge k).$$

Because the load factor of our table is $1/2$, $E(Z_k) = k/2$. Thus, by the above Chernoff bound,

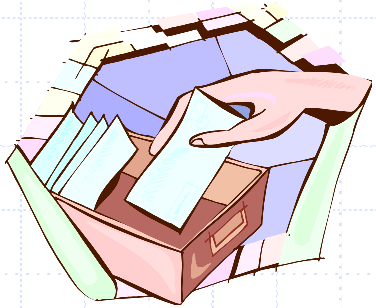
$$\begin{aligned} \Pr(Z_k \ge k) &= \Pr(Z_k \ge 2(k/2)) \\ &\le (e/4)^{k/2} \\ &< 2^{-k/4}. \end{aligned}$$

Therefore, putting all the above pieces together,

$$\begin{aligned} E(X) &= 1 + (1/2n)E\left(\sum_{i=0}^{N-1} Y_i\right) \\ &\le 1 + (1/2n)\sum_{i=0}^{N-1}\sum_{k=1}^{2n} k^2\, 2^{-k/4} \\ &\le 1 + \sum_{k=1}^{\infty} k^2\, 2^{-k/4} \\ &= O(1). \end{aligned}$$

That is, the expected running time for doing a search or update operation with linear probing is $O(1)$, so long as the load factor in our hash table is at most $1/2$.

Double Hashing



- Double hashing uses a secondary hash function $d(k)$ and handles collisions by placing an item in the first available cell of the series

  $$(i + jd(k)) \bmod N$$

  for $j = 0, 1, \ldots, N-1$
- The secondary hash function $d(k)$ cannot have zero values
- The table size N must be a prime to allow probing of all the cells
- Common choice of compression function for the secondary hash function:

  $$d_2(k) = q - k \bmod q$$

  where
  - $q < N$
  - q is a prime
- The possible values for $d_2(k)$ are

  $$1, 2, \ldots, q$$

Example of Double Hashing

- Consider a hash table storing integer keys that handles collision with double hashing
  - $N = 13$
  - $h(k) = k \bmod 13$
  - $d(k) = 7 - k \bmod 7$
- Insert keys 18, 41, 22, 44, 59, 32, 31, 73, in this order

k	$h(k)$	$d(k)$	Probes		
18	5	3	5		
41	2	1	2		
22	9	6	9		
44	5	5	5	10	
59	7	4	7		
32	6	3	6		
31	5	4	5	9	0
73	8	4	8		

0	1	2	3	4	5	6	7	8	9	10	11	12

⇩

31		41			18	32	59	73	22	44		
0	1	2	3	4	5	6	7	8	9	10	11	12