CLARK

Trigonometric Substitutions

Math 121 Calculus II

D Joyce, Spring 2013

Now that we have trig functions and their inverses, we can use trig subs. They're special kinds of substitution that involves these functions. For these, you start out with an integral that doesn't have any trig functions in them, but you introduce trig functions to evaluate the integrals. These depend on knowing

(1) the Pythagorean identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad \sec^2\theta = 1 + \tan^2\theta$$

(2) the definitions

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

(3) the derivatives

$$(\sin\theta)' = \cos\theta \qquad (\cos\theta)' = -\sin\theta$$
$$(\tan\theta)' = \sec^2\theta \qquad (\sec\theta)' = \sec\theta\tan\theta$$

There are three kinds of trig subs. You use them when you see as part of the integrand one of the expressions $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, or $\sqrt{x^2 - a^2}$, where a is some constant. In each kind you substitute for x a certain trig function of a new variable θ. The substitution will simplify the integrand since it will eliminate the square root. Here's a table summarizing the substitution to make in each of the three kinds.

If use see	use the sub	so that	and
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$dx = a\cos\theta\,d\theta$	$\sqrt{a^2 - x^2} = a\cos\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$	$dx = a\sec^2\theta\,d\theta$	$\sqrt{a^2 + x^2} = a\sec\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$dx = a\sec\theta\tan\theta\,d\theta$	$\sqrt{x^2 - a^2} = a\tan\theta$

In each line, the last entry follows from the second entry by one of the Pythagorean identities.

There are also right triangles you can draw to make the connections between x, a, and θ. The three triangles below refer to the three trig subs, respectively.