Practice Problem Set 6

1. A parallel-plate capacitor has square plates 7.5 cm on a side, separated by 0.29 mm. The capacitor is charged to 12 V, then disconnected from the charging power supply.

   (a) Calculate the capacitance of this capacitor.

   (b) What is the total charge on each plate? What is the charge density on the plates?

   (c) What is the electric field between the plates?

   (d) A *dielectric* is an insulating material that modifies the external electric field in which it is placed. Suppose a sheet of glass is placed between the parallel-plate capacitor. Calculate the capacitance and total charge on each plate as you did above, this time including the glass sheet. Do the values increase or decrease? Now repeat these calculations for polyethylene and quartz. Which dielectric gives the greatest capacitance? Why might using a dielectric in a capacitor be useful in practice? (Hint: See Table 23.1)

Solution:

(a) The capacitance for a parallel-plate capacitor is given by Eq. (23.2):

$$C = \frac{\epsilon_0 A}{d}, \tag{1}$$

where A is plate area and d is separation between the plates. For a square plate of side length ℓ, the area is $A = \ell^2$. Plugging in the numbers, the capacitance is,

$$\begin{aligned} C &= \frac{\epsilon_0 \ell^2}{d} \\ &= \frac{(8.85 \times 10^{-12}\,\mathrm{F\,m^{-1}})(0.075\,\mathrm{m})^2}{2.9 \times 10^{-4}\,\mathrm{m}} \\ &= 1.7 \times 10^{-10}\,\mathrm{F} \end{aligned} \tag{2}$$

(b) From the definition of capacitance, Eq. (23.1), the total charge on each plate is,

$$\begin{aligned} Q &= CV \\ &= \frac{\epsilon_0 \ell^2 V}{d} \\ &= \frac{(8.85 \times 10^{-12}\,\mathrm{F\,m^{-1}})(0.075\,\mathrm{m})^2(12\,\mathrm{V})}{2.9 \times 10^{-4}\,\mathrm{m}} \\ &= 2.1 \times 10^{-9}\,\mathrm{C} \end{aligned} \tag{3}$$

The charge density is the charge on the plate divided by the plate's area:

$$\begin{aligned} \sigma &= \frac{Q}{A} \\ &= \frac{CV}{\ell^2} \\ &= \frac{\epsilon_0 V}{d} \\ &= \frac{(8.85 \times 10^{-12}\,\mathrm{F\,m^{-1}})(12\,\mathrm{V})}{2.9 \times 10^{-4}\,\mathrm{m}} \\ &= 3.7 \times 10^{-7}\,\mathrm{C\,m^{-2}} \end{aligned} \tag{4}$$

(c) The electric field between two oppositely charged plates is,

$$\begin{aligned} E &= \frac{\sigma}{\epsilon_0} \\ &= \frac{V}{d} \\ &= \frac{12\,\mathrm{V}}{2.9 \times 10^{-4}\,\mathrm{m}} \\ &= 4.1 \times 10^{4}\,\mathrm{V\,m^{-1}} \end{aligned} \tag{5}$$

(d) The dielectric constant for glass from Table 23.1 is $\kappa = 5.6$. The capacitance and total charge with the dielectric are simply the capacitance and total charge without the dielectric (which we'll denote as C_0 and Q_0, respectively) multiplied by the dielectric constant:

$$\begin{aligned} C &= \kappa C_0 \\ &= 5.6(1.7 \times 10^{-10}\,\mathrm{F}) \\ &= 9.5 \times 10^{-10}\,\mathrm{F} \end{aligned} \tag{6}$$

and,

$$\begin{aligned} Q &= \kappa Q_0 \\ &= 5.6(2.1 \times 10^{-9}\,\mathrm{C}) \\ &= 1.2 \times 10^{-8}\,\mathrm{C} \end{aligned} \tag{7}$$

Both the capacitance and total charge on each plate increase when the glass plate is added.

Polyethylene: $\kappa = 2.3$. The capacitance and total charge are,

$$C = 2.3(1.7 \times 10^{-10}\,\mathrm{F}) = 3.9 \times 10^{-10}\,\mathrm{F} \tag{8}$$

and,

$$Q = 2.3(2.1 \times 10^{-9}\,\mathrm{C}) = 4.8 \times 10^{-9}\,\mathrm{C} \tag{9}$$

Quartz: $\kappa = 3.8$. The capacitance and total charge are,

$$C = 3.8(1.7 \times 10^{-10}\,\mathrm{F}) = 6.5 \times 10^{-10}\,\mathrm{F} \tag{10}$$

and,

$$Q = 3.8(2.1 \times 10^{-9}\,\mathrm{C}) = 8.0 \times 10^{-9}\,\mathrm{C} \tag{11}$$

The dielectric which gives the greatest capacitance is the one with the greatest dielectric constant, which in this case is glass.

The two main advantages of using a dielectric in a capacitor are (1) it helps prevent the conducting plates from coming into direct electrical contact and (2) a high capacitance allows a greater stored charge for a given voltage.

Notice that adding a dielectric is equivalent to replacing the permittivity of free space, ϵ_0, by a new parameter, simply called the *permittivity*, ϵ:

$$\epsilon = \kappa \epsilon_0. \tag{12}$$

2. A spherical conductor of radius R carries a total charge Q.

   (a) Determine the energy density, u_E, at each point over all space as a function of the distance r from the sphere's centre. Plot u_E as a function of r.

   (b) Use this energy density to compute the system's total energy, U, by integrating over all space.

Solution:

(a) Recall that the electric field for a conducting sphere of radius R with charge Q is,

$$E = \frac{Q}{4\pi\epsilon_0 r^2}, \quad r \geq R. \tag{13}$$

Inside the conducting sphere ($r < R$), the field is zero. We want to find the energy density, which is the amount of energy in an electric field per unit volume. We can calculate the energy density due to this charged sphere using Eq. (23.7):

$$\begin{aligned} u_E &= \frac{1}{2}\epsilon_0 E^2 \\ &= \frac{1}{2}\epsilon_0 \left(\frac{Q}{4\pi\epsilon_0 r^2}\right)^2 \\ &= \frac{Q^2}{32\pi^2\epsilon_0 r^4} \end{aligned} \tag{14}$$

outside the sphere ($r \geq R$). Because the electric field is zero inside the sphere, the energy density is also zero in this region. The energy density is plotted in Fig. 1.

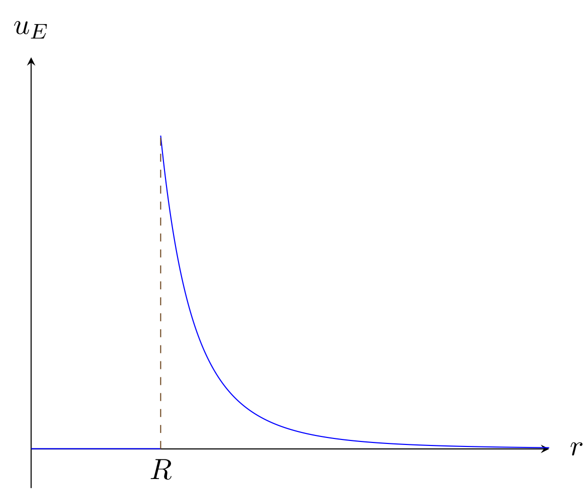

Figure 1: Plot of energy density as a function of r.

(b) The total energy in the electric field is the integral of the energy density we found above over all space:

$$U = \int_{\text{volume}} u_E \, dV, \tag{15}$$

where dV is the volume element of a thin spherical shell of thickness dr. This volume is equal to the product of the thickness dr and the surface area of the shell, $4\pi r^2$. So, $dV = 4\pi r^2 dr$. The system's total energy is then,

$$
\begin{aligned}
U &= \int_{\text{volume}} \frac{Q^2}{32\pi^2\epsilon_0 r^4}\,dV \\
&= \frac{Q^2}{32\pi^2\epsilon_0}\int_R^\infty \frac{4\pi r^2}{r^4}\,dr \\
&= \frac{Q^2}{8\pi\epsilon_0}\int_R^\infty \frac{dr}{r^2} \\
&= -\frac{Q^2}{8\pi\epsilon_0}\frac{1}{r}\bigg|_R^\infty \\
&= \frac{Q^2}{8\pi\epsilon_0 R}
\end{aligned} \tag{16}
$$

Notice that we integrated over r from R to ∞. We could have instead integrated from 0 to ∞, but since the energy density for $0 < r < R$ is zero, it does not contribute to the total energy.

3. Consider a parallel-plate capacitor where the separation between the plates can be varied. The maximum capacitance it can withstand is 120 pF. You charge the capacitor to a potential difference of 50 mV at maximum capacitance and then isolate it. With the capacitor still isolated, what plate separation is required so that it now has a potential difference of 30 V? The plate area is $3.1\,\text{cm}^2$.

   Solution:

   Since the capacitor is kept in isolation, the charge on the plates stays the same after the plate separation is changed. From the definition of capacitance, the charge on each plate is $Q = CV$, so the capacitance required to have a potential difference of 30 V can be found by equating the initial and final charge:

$$
\begin{aligned}
Q_i &= Q_f \\
C_i V_i &= C_f V_f \\
C_f &= \frac{C_i V_f}{V_i}
\end{aligned} \tag{17}
$$

   Using the now familiar equation for the capacitance of a parallel-plate capacitor, the plate separation d_f is,

$$
\begin{aligned}
d_f &= \frac{\epsilon_0 A}{C_f} \\
&= \epsilon_0 A \frac{V_i}{C_i V_f} \\
&= (8.85\times10^{-12}\,\text{F}\,\text{m}^{-1})(3.1\times10^{-4}\,\text{m}^2)\frac{30\,\text{V}}{(1.2\times10^{-10}\,\text{F})(0.05\,\text{V})} \\
&= 0.013\,72\,\text{m} \\
&\doteq 1.4\,\text{cm}
\end{aligned} \tag{18}
$$