

# Domain and Range Examples

The *domain* of a function  $f$  is the set of all values  $x$  for which  $f(x)$  is defined.

The *range* of a function  $f$  is the set of all values that  $f(x)$  takes on as  $x$  runs through the domain of  $f$ . That is, it is the set of all  $y$  values for which there is an  $x$  value such that

$$y = f(x).$$

For many functions, the domain is easy to determine. Often, all we have to do is look for which  $x$  cause a problem when evaluating  $f(x)$ ; those  $x$  are *not* in the domain of  $f$ , and the domain is everything else.

For example, suppose  $f(x) = \frac{5}{x(x-1)}$ . To compute  $f(x)$  we have to multiply  $x$  times  $x - 1$ . This part causes no trouble: we can multiply any two number together. Then, we divide 5 by the product just calculated. This goes well, unless that product is zero. That is, the only  $x$  values that cause trouble are those for which

$$x(x - 1) = 0.$$

Since the left hand side of this equation is factored, we see that the only  $x$  for which  $x(x - 1) = 0$  are  $x = 0$  and  $x = 1$ . Thus, those are the only two values not in the domain of  $f$ , and so the domain of  $f$  is everything else. We might say: the domain of  $f$  is all  $x \neq 0, x \neq 1$ .

Square roots can cause us problems as well. The square root of  $x$  is only defined for  $x \geq 0$ . In other words, the domain of  $\sqrt{x}$  is all  $x \geq 0$ .

Functions made up from the square root function inherit this restriction. For example, let

$$g(x) = \sqrt{x - 1} + \sqrt{x + 5}.$$

In order for  $g(x)$  to be defined,  $x - 1$  and  $x + 5$  must both be greater than 0. So, if  $x$  is in the domain,

$$x - 1 \geq 0 \text{ and } x + 5 \geq 0$$

or, simplified,

$$x \geq 1 \text{ and } x \geq -5.$$

Now, if  $x \geq 1$  then  $x$  is automatically greater than or equal to  $-5$ , so the first condition is all we need: the domain of  $g$  is all  $x \geq 1$ .

It is worth knowing how to evaluate the domain of a function made from another function whose domain we know. For instance, suppose we know  $f$  has the domain  $2 \leq x \leq 11$ . Then, let  $g(x) = f(5x - 8)$ . For  $g(x)$  to be defined,  $f(5x - 8)$  must be defined. In order for  $f(5x - 8)$  to be defined,  $5x - 8$  must be in the domain of  $f$ . That is,

$$2 \leq 5x - 8 \leq 11.$$

Solving this, we find the domain of  $g$  is

$$2 \leq x \leq \frac{19}{5}.$$

Notice that if we let  $h(x) = 7f(x)$ , then  $h$  has the same domain as  $f$ : if  $f(x)$  makes sense, then so does  $7f(x)$ .

Range is often a more difficult matter. For many functions, even relatively simple ones, determining the range can be quite challenging, and ranges are often impossible to determine exactly. We won't say too much specifically about ranges in Math 120.

We can say something about the range of a function built up from another function whose range we know.

For instance, suppose the range of a function  $f$  is  $-5 \leq y \leq 8$  (note I use the variable  $y$  here, since we often think of the range of a function as all possible  $y$  values in its graph).

Then let  $k(x) = 3f(x) + 2$ . What is the range of  $k$ ?

Since we know the range of  $f$ , we can say that for all  $y$  values such that

$$-5 \leq y \leq 8$$

there is an  $x$  such that  $f(x) = y$ . Multiplying this inequality by 3, we can say that for all  $y$  such that

$$-15 \leq 3y \leq 24$$

there is an  $x$  such that  $y = 3f(x)$ . Adding 2, we can say that for all  $y$  such that

$$-13 \leq 3y + 2 \leq 26$$

there is an  $x$  such that  $y = 3f(x) + 2$ . In other words, the range of  $k(x)$  is the interval

$$-13 \leq y \leq 26.$$

Note that if we defined a function like  $m(x) = f(8x - 4)$ , where  $x$  has been replaced by a linear function of  $x$ , then this function  $m$  will have the same range as  $f$ . To see this, suppose  $z$  is in the range of  $f$ . That means that there exists an  $x$  such that

$$f(x) = z.$$

Now, let  $x = 8a - 4$  and solve for  $a$ . We get  $a = \frac{x+4}{8}$ , and  $m(a) = f(x) = z$ . Thus, for any  $z$  in the range of  $f$ ,  $z$  is in the range of  $m$ , so the ranges are identical.