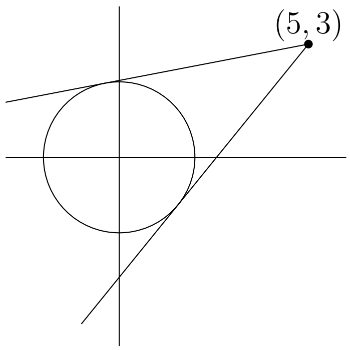
Tangent lines to a circle

This example will illustrate how to find the tangent lines to a given circle which pass through a given point.

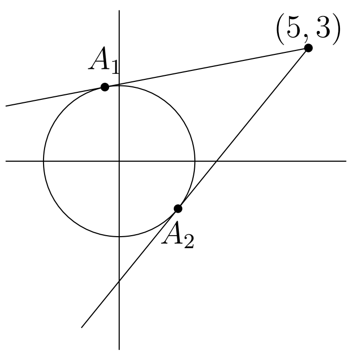
Suppose our circle has center $(0, 0)$ and radius 2, and we are interested in tangent lines to the circle that pass through $(5, 3)$.

The picture we might draw of this situation looks like this.



We are interested in finding the equations of these tangent lines (i.e., the lines which pass through exactly one point of the circle, and pass through $(5, 3)$).

The key is to find the *points of tangency*, labeled A_1 and A_2 in the next figure.



The trick to doing this is to introduce variables for the coordinates for one of these points. Let's say one of these points is (a, b).

(Note: I strongly recommend using variables other than x and y here, so that we can easily remember that (a, b) is a point of tangency; x and y are too generic, and we will want to use them later for other purposes (such as expressing our line equations).)

Then this is the key idea: **Since we have two variables (i.e., unknowns), we want to come up with two equations that these variables satisfy.** If we can do that, then we can use algebra to solve for our unknowns.

How do we come up with two equations?

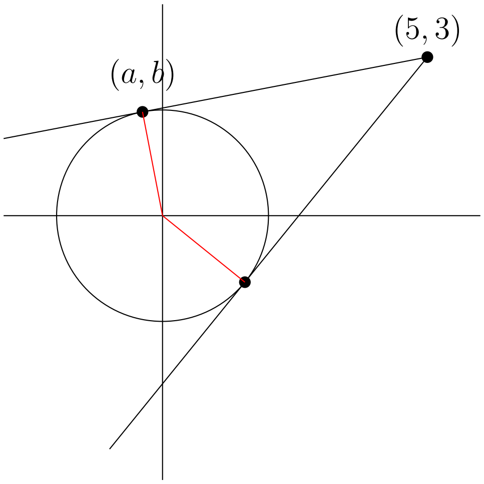
First, we know that (a, b) is a point on our circle, and so (a, b) satisfies the equation of the circle. Thus,

$$a^2 + b^2 = 2^2 = 4 \tag{1}$$

So we have one equation.

To get a second equation, we need to use the fact that the line through (a, b) and $(5, 3)$ is tangent to the circle. A geometric consequence of this is that **this line is perpendicular to the *radius***

drawn through the center of the circle and the point (a, b).



In the figure above, the radii are drawn in red. You can see that the red lines are perpendicular to the tangent lines.

Let the slope of the red line through (a, b) and the origin $(0, 0)$ be m_1. Then

$$m_1 = \frac{b-0}{a-0} = \frac{b}{a}.$$

Let the slope of the tangent line through (a, b) and $(5, 3)$ be m_2.

Then

$$m_2 = \frac{3-b}{5-a}.$$

Now, **since the red line and the tangent line are perpendicular, the relationship between their slopes gives us $m_2 = -\frac{1}{m_1}$.** This is the second equation we have been looking for.

Thus,

$$\frac{3-b}{5-a} = -\frac{1}{\frac{b}{a}} = -\frac{a}{b}. \tag{2}$$

Simplifying this equation, we find

$$3b - b^2 = -5a + a^2$$

so

$$3b + 5a = a^2 + b^2.$$

But we already know $a^2 + b^2 = 4$ by equation (1), so

$$3b + 5a = 4.$$

Solving for b, we have

$$b = \frac{4-5a}{3}. \tag{3}$$

Substituting this into equation (1), we have

$$\begin{aligned}
a^2 + \left(\frac{4-5a}{3}\right)^2 &= 4 \\
a^2 + \frac{(4-5a)^2}{9} &= 4 \\
9a^2 + (4-5a)^2 &= 36 \\
34a^2 - 40a - 20 &= 0
\end{aligned}$$

from which we find

$$a = \frac{40 \pm \sqrt{40^2 - 4(34)(-20)}}{68}$$

so $a = -0.3783339250$ or $a = 1.5548045132$.

Notice that, although mentally we were thinking that we were solving for *one* of the points of tangency, we have actually found both at the same time, since there is no algebraic distinction between the points (i.e., the equations are exactly the same for the two points).

Plugging into equation (3), we find the corresponding b values, and so our points of tangency A_1 and A_2 are

$$(-0.3783339250, 1.9638898750) \text{ and } (1.5548045132, -1.2580075220).$$

The tangents lines pass through these points and the point $(5, 3)$, so we find the equations of the tangent lines are

$$y = 3 + \frac{3 - 1.9638898750}{5 - (-0.3783339250)}(x-5) = 0.1926451833x + 2.0367740833$$

and

$$y = 3 + \frac{3 - (-1.2580075220)}{5 - 1.5548045132}(x-5) = 1.2359262452x - 3.1796312262.$$