Closed universes with black holes but no event horizons as a solution to the black hole information problem

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ABSTRACT

We show that it is possible for the information paradox in black hole evaporation to be resolved classically. Using standard junction conditions, we attach the general closed spherically symmetric dust metric to a space–time satisfying all standard energy conditions but with a single point future c-boundary. The resulting Omega Point space–time, which has NO event horizons, nevertheless has black hole type trapped surfaces and hence black holes. However, since there are no event horizons, information eventually escapes from the black holes. We show that a scalar quintessence field with an appropriate exponential potential near the final singularity would give rise to an Omega Point final singularity.

Key words: black hole physics – gravitation – large-scale structure of Universe.

1 INTRODUCTION

One of the outstanding questions of black hole physics is to determine what happens to the information that falls into a black hole. Hawking (1976, 1982, 1996) has shown that a black hole radiates away its mass, and he pointed out that if a black hole were to completely evaporate – which it inevitably will in a universe that exists forever in cosmological proper time – a Planck size remnant would probably be inconsistent with the Bekenstein Bound (Bekenstein 1981, 1999;[1] Bekenstein & Schiller 1989; Wald 1994) – then any information exclusively inside the black hole would disappear from the universe, violating unitarity. Many solutions have been proposed to resolve this paradox. Hawking himself until recently believed that unitarity was indeed violated, but it has been argued that such a resolution would be inconsistent with locality and/or conservation of energy (Banks, Susskind & Peskin 1984). Hawking (2005) now believes that the unitarity-violating universes would be of measure zero in the multiverse, and so they can be neglected in the sum over histories. If Hawking's new position is correct, it would mean that unitarity is effectively valid. Susskind (1992, 1995) and t'Hooft (1990) propose, on the other hand, that all information inside a black hole is also completely encoded on its surface, so there is no net information inside the black hole. However, at the semiclassical level, this 'holographic principle' would not resolve the paradox, because

the generators of an event horizon – the black hole surface – cannot end in a space–time, but at a singularity which itself would annihilate any information on the horizon. To avoid unitarity violation, the information must get outside the black hole event horizon.

Bekenstein (1993) has pointed out that the Hawking radiation is not quite a blackbody spectrum, and thus it carries some information about the initial state of the black hole. He has shown that the permitted information outflow rate can be as large as the rate of decrease in black hole's entropy, and hence it is possible for information to gradually leak out of a black hole during evaporation. However, Bekenstein emphasized that he had not demonstrated that *all* the information got out, just that it was possible that it did, and if only one bit of information fails to escape, unitarity will be violated. Bekenstein also did not address the semiclassical event horizon problem. Ashtekar & Bojowald (2005) have proposed that it will be necessary to consider trapped surfaces in the context of full quantum gravity to resolve the information problem.

We will show in this paper that a purely classical gravity solution to the black hole information problem is possible and consistent with all observations: the universe may have no event horizons at all. In such a universe, there would be no black hole event horizons to prevent the exchange of information between one part of the cosmos and another. A space–time with no event horizons has a future c-boundary (Hawking & Ellis 1973, pp. 217–221) which is topologically a single point, and hence has been called (Tipler 1994) an *Omega Point space–time*. It can be shown (Tipler 1994) that if a space–time's future c-boundary is a single point, then the space–time necessarily admits compact Cauchy surfaces, and the global space–time topology is $S \times R^1$, where S is the topology of any Cauchy surface. Even in a universe with compact Cauchy

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[1] Bekenstein (1999) shows that the claim in Wald (1994), that the Bekenstein Bound is not an essential assumption in any derivation of the generalized Second Law and hence need not be true, is incorrect.

surfaces, we would expect black holes to evaporate to completion if the universe were to expand forever. Hence, a space–time which avoids the black hole information paradox because of the absence of event horizons would have to end in a final singularity before any black hole would have time to evaporate. Since the expected black hole lifetime is $10^{64}(M/M_{\odot})^3$ yr, (Wald 1994), our universe would have to expand to a maximum size and recontract before 10^{64} yr have passed. It can be shown (Barrow, Galloway & Tipler 1986; Barrow & Tipler 1986) that the only two simple topologies possible for universe with a maximal Cauchy hypersurface and satisfying the weak energy condition are S^3 and $S^2 \times S^1$.

We will construct in this paper a spherically symmetric S^3 universe with a black hole but with no event horizons. The space–time will be shown to satisfy all the standard energy conditions. Indeed, the stress energy tensor for the space–time will be just pressure-free dust in its expanding phase. We will discuss various definitions of 'black hole' in closed universes, and show that the space–time we construct has a black hole by any of these definitions. The parameters of the constructed space–time can be chosen so that the black hole is identical to a black hole in any dust spherically symmetric (Tolman–Bondi) S^3 universe – and hence it would be in appearance a black hole with event horizons according to any observations that could be carried out in the expanding phase of a closed universe. The null generators of what is apparently the event horizons stay close to the trapped surfaces during the expanding phase of the closed universe, and only expand out into the universe at large very close to the final singularity. This means that the standard astrophysical analysis of black holes and their collisions (e.g. Matzner et al. 1995,[2] Matzner et al. 1996; Price et al. 1999) can be trusted, since they are in the short run the same as in asymptotically flat space–time. Thus, our proposal is quite different from many proposals which eliminate event horizons by eliminating black hole type trapped surfaces. In our proposal, black hole trapped surfaces exist as usual, but they do not give rise to event horizons.

This paper will be organized as follows. In Section 2, we will construct a Friedmann–Robertson–Walker (FRW) universe which is a standard dust FRW closed universe until arbitrarily near the final singularity when we join it to a metric which satisfies all the energy conditions, which has a final singularity, but which has no event horizons. In Section 3, we show that the no event horizon metric constructed in Section 2 satisfies the Einstein equations for a scalar field with a suitably chosen exponential potential. In Section 4, we will generalize the FRW $w = -1/3$ universe to the spherically symmetric case, obtaining an inhomogeneous (but spherically symmetric) space–time which satisfies all the standard energy conditions, yet has a future c-boundary which is a single point. In Section 5, we join this modified version of the FRW event horizonless metric to a general Tolman–Bondi closed universe. In Section 6, we discuss various definitions for a black hole in a closed universe, and show that the Tolman–Bondi universe parameters of the metric in Section 5 can be chosen so that by any of these definitions the expanding phase of the universe has a black hole. In Section 7, we show that the recent supernova observations which strongly suggest that the universe is currently accelerating are consistent with a universe which recollapses to a final singularity before any black

hole has time to completely evaporate, provided the acceleration is due to quintessence with certain specified properties. Finally in our concluding Section 8, we will point out how our 'no event horizon' solution to the black hole information paradox naturally complements the 'holographic principle' resolution, which assumes that all information in a black hole interior is coded also on its surface.

2 A THREE-SPHERE FRIEDMANN–ROBERTSON–WALKER UNIVERSE WITH FINAL SINGULARITY BUT NO EVENT HORIZONS

The Friedmann equation for an S^3 closed universe is

$$\left(\frac{1}{R}\frac{dR}{dt}\right)^2 = \frac{8\pi G M}{3 R^{3(1+w)}} - \frac{1}{R^2} \tag{2.1}$$

where the pressure $p = w\rho$, with $w = \gamma - 1$, where γ is the adiabatic index and ρ the mass density. If $w = -1/3$, then

$$R(t) = \sqrt{\left(\frac{8\pi G M}{3} - 1\right)(t_{\rm f} - t)} \tag{2.2}$$

is a solution to equation (2.1) for $t < t_{\rm f}$ with a final singularity at $t = t_{\rm f}$, provided $(8\pi G M/3) > 1$. The second-order equation for the Friedmann universe,

$$\frac{1}{R}\frac{d^2 R}{dt^2} = -\frac{4\pi G}{3}(\rho + 3p), \tag{2.3}$$

is automatically satisfied for $p = -(1/3)\rho$ and $d^2R/dt^2 = 0$.

The closed FRW universe with the scalefactor (2.2), namely

$$ds^2 = -dt^2 + R_0^2(t_{\rm f} - t)^2[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\varphi^2)], \tag{2.4}$$

has no event horizons; that is, its future c-boundary consists of a single point – the Omega Point. Indeed, the equation for future-directed null geodesics, $ds^2 = 0$, can be integrated for radial null geodesics to give

$$\Delta\chi = \int^{t_{\rm f}}\frac{dt}{R(t)} = +\infty \tag{2.5}$$

which shows that radial null geodesics circumnavigate the universe an infinite number of times as the future c-boundary at $t = t_{\rm f}$ is approached. By homogeneity and isotropy, we can transpose the coordinate system so that any spatial location (χ, θ, ϕ) can reach any other location (χ', θ', ϕ') via a *radial* null geodesic segment, and equation (2.5) shows such radial geodesics can be exchanged an infinite number of times. Hence, all future endless time-like curves define the same c-boundary point: the future c-boundary is a single point.

A perfect fluid with $w = -1/3$ satisfies the weak, the strong, and the dominant energy conditions, since Hawking and Ellis have shown (1973, pp. 89–95) that for diagonalizable stress energy tensors (Type I matter), the weak energy condition will hold if $\rho \geqslant 0$ and $\rho + p \geqslant 0$, the strong energy condition will hold if $\rho + p \geqslant 0$ and $\rho + 3p \geqslant 0$, and the dominant energy condition will hold if $\rho \geqslant 0$ and $-\rho \leqslant p \leqslant \rho$.

It is possible to join the metric with scalefactor (2.2) to any closed FRW universe at *any* time in the collapsing phase. Consider, for example, the dust ($w = 0$) scalefactor

$$R(\tau) = \frac{R_{\rm max}}{2}(1 - \cos\tau), \tag{2.6}$$

$$t(\tau) = \frac{R_{\rm max}}{2}(\tau - \sin\tau), \tag{2.7}$$

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[2] See especially the 'horizon generators' of a pair of colliding black holes, which appeared on the cover of the issue of Science containing this paper (1995 November 10). The null geodesics so pictured will still exist, and will be astrophysically indistinguishable from the null geodesics pictured here, but they will not be the generators of a event horzion.

where $0 < \tau < 2\pi$ is the conformal time, and $R_{\max}$ is the radius of the universe at maximum expansion, which occurs when $\tau = \pi$. We will make the join at a conformal time $\pi \leqslant t_{\text{join}} < 2\pi$, which by equation (2.7) gives a proper time of

$$t_{\text{join}} \equiv t_{\text{join}}(\tau_{\text{join}}) = \frac{R_{\max}}{2}(\tau_{\text{join}} - \sin \tau_{\text{join}}).$$

The standard junction conditions (see e.g. Misner, Thorne & Wheeler 1973, section 21.13, pp. 551–556) require continuity of the metric and its first derivatives at the join:

$$R(\tau_{\text{join}}) = \frac{R_{\max}}{2}(1 - \cos \tau_{\text{join}}) \tag{2.8}$$

$$= R(t_{\text{join}}) = -Q t_{\text{join}} + A, \tag{2.9}$$

$$\left.\frac{\mathrm{d}R}{\mathrm{d}t}\right|_{\tau_{\text{join}}} = \left.\left(\frac{\mathrm{d}R/\mathrm{d}\tau}{\mathrm{d}t/\mathrm{d}\tau}\right)\right|_{\tau_{\text{join}}} = \tag{2.10}$$

$$\frac{\sin \tau_{\text{join}}}{1 - \cos \tau_{\text{join}}} = \left.\frac{\mathrm{d}R}{\mathrm{d}t}\right|_{t_{\text{join}}} = -Q, \tag{2.11}$$

where $Q \equiv \sqrt{8\pi G M/3 - 1}$ and $A \equiv t_{\text{f}} Q$. Solving for t_{f} and M yields

$$\frac{8\pi G M}{3} = \frac{2}{1 - \cos \tau_{\text{join}}}, \tag{2.12}$$

$$t_{\text{f}} = \frac{R_{\max}}{2}\left[\tau_{\text{join}} + \frac{2(\cos \tau_{\text{join}} - 1)}{\sin \tau_{\text{join}}}\right]. \tag{2.13}$$

Note that as $\tau_{\text{join}} \to \pi^{+}$, we have $t_{\text{f}} \to +\infty$, which means joining at the time of maximum expansion would yield the Einstein static universe thereafter. As $\tau_{\text{join}} \to 2\pi$, we have $t_{\text{f}} \to \pi R_{\max}$ and $M \to +\infty$, which means that an arbitrarily large total mass of the $w = -1/3$ matter is required to eliminate event horizons if the join is made arbitrarily close to the usual proper time end of a dust FRW universe, $t_{\text{f}} = \pi R_{\max}$.

The standard junction conditions yield a global metric which is C^{∞}, except at the join, where it is C^{1}. One can smoothen this metric to one which is C^{∞} everywhere and which satisfies the energy conditions everywhere by allowing w to vary smoothly from 0 to $-1/3$ in a neighbourhood $[t_{\text{join}}, t_{\text{join}} - \Delta t)$. By the constraint FRW equation $(R^{-1}\,\mathrm{d}R/\mathrm{d}t)^2 = -R^{-2} + 8\pi G\rho/3$ and the dynamical FRW equation $2R^{-1}\,\mathrm{d}^2R/\mathrm{d}t^2 = -(R^{-1}\,\mathrm{d}R/\mathrm{d}t)^2 - R^{-2} - 8\pi G p$, continuity in p and ρ would ensure that $R(t)$ is C^2, and repeatedly differentiating the dynamical equation would yield that $R(t)$ is C^{∞} (the constraint and dynamical FRW equations imply the conservation equation $T^{\mu\nu}{}_{;\nu} = 0$).

Since $R^{\mu}{}_{\mu} = -8\pi G T^{\mu}{}_{\mu} = 16\pi G\rho = 16\pi G M/R^2$ for the $w = -1/3$ equation of state, where $R^{\mu}{}_{\mu}$ is the Ricci scalar, the Omega Point singularity at $t = t_{\text{f}}$ (at $R = 0$) is a p.p. (parallel propagated) curvature singularity (Hawking & Ellis 1973, p. 260). Budic & Sachs (1976) were the first to investigate Omega Point space–times, and he pointed out that they are most easily formed by suitably identifying Minkowski space, which would have locally extensible singularities.

3 A $w = -1/3$ PERFECT FLUID CAN BE GENERATED BY A QUINTESSENCE SCALAR FIELD WITH AN EXPONENTIAL POTENTIAL

We will now show that a scalar field with exponential potential will generate, at least in an FRW universe, a $w = -1/3$ perfect fluid

behaviour near the final singularity, that is, the $w = -1/3$ perfect fluid behaviour will be seen if the potential for the scalar field ϕ is of the form $V(\phi) = V_0 e^{B\phi}$, where V_0 and B are constants. Such a potential is often discussed as a particularly plausible potential for the inflation field which is thought to be responsible for inflation in the early universe, and as a model of the quintessence field which is responsible for the cosmological acceleration in the present epoch. This will show that a $w = -1/3$ equation of state is physically plausible near the final singularity of a closed universe, and thus that the absence of event horizons is physically possible.

The stress energy tensor for a scalar field ϕ with potential $V(\phi)$ is (Turner 1983):

$$T_{\alpha\beta} = \left[\phi_{;\alpha}\phi_{;\beta} - \frac{1}{2}g_{\alpha\beta}\left(\phi_{;\mu}\phi_{;\nu}g^{\mu\nu} + 2V(\phi)\right)\right]. \tag{3.1}$$

In the FRW universe, we have $\phi = \phi(t)$, so $\phi_{,i} = 0$ and $\phi_{,0} = \phi_{,0}$, where the i denotes a spatial coordinate, 0 the time coordinate t, and the semicolon and comma denote the covariant and partial derivatives, respectively. In a local orthonormal frame, we obtain

$$T_{\hat{0}\hat{0}} = \frac{1}{2}(\phi_{,\hat{0}})^2 + V(\phi) \tag{3.2}$$

and

$$T_{\hat{0}\hat{0}} + 3T_{\hat{i}\hat{i}} = 2\left[(\phi_{,\hat{0}})^2 - V(\phi)\right] \tag{3.3}$$

If $w = -1/3$, $T_{\hat{0}\hat{0}} + 3T_{\hat{i}\hat{i}} = 0$, which means

$$V(\phi) = (\phi_{,\hat{0}})^2 = (\phi_{,0})^2, \tag{3.4}$$

where we have used $\phi_{,\hat{0}} = \phi_{,0} = \mathrm{d}\phi/\mathrm{d}t$. Thus,

$$8\pi G T_{\hat{0}\hat{0}} = 12\pi G(\phi_{,0})^2 = G_{\hat{0}\hat{0}}$$

$$= \frac{3((R_{,0})^2 + 1)}{R^2} = \frac{3(R_0^2 + 1)}{R_0^2(t_{\text{f}} - t)^2}. \tag{3.5}$$

Taking the square root gives

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\sqrt{(R_0^2 + 1)/4\pi G}}{R_0(t_{\text{f}} - t)} \tag{3.6}$$

which can be immediately integrated to yield

$$\phi_0 - \phi = \sqrt{\left(\frac{1}{4\pi G}\right)\left(1 + \frac{1}{R_0^2}\right)} \ln(t_{\text{f}} - t), \tag{3.7}$$

where ϕ_0 is a constant. Equation (3.7) can be written in the form

$$(t_{\text{f}} - t)^{-1} = \exp\left[\frac{\phi - \phi_0}{\sqrt{(1/4\pi G)\left(1 + 1/R_0^2\right)}}\right]. \tag{3.8}$$

We thus obtain for the potential

$$V(\phi) = (\phi_{,0})^2 = \frac{(R_0^2 + 1)}{4\pi G R_0^2}\left[\frac{1}{(t_{\text{f}} - t)^2}\right] = V_0 e^{B\phi}, \tag{3.9}$$

where

$$B = \sqrt{\frac{16\pi G R_0^2}{R_0^2 + 1}} \tag{3.10}$$

and

$$V_0 = \frac{(R_0^2 + 1)}{4\pi G R_0^2} e^{-B\phi_0}. \tag{3.11}$$

It was pointed out in the previous section that the join between the dust- (or radiation-) dominated FRW part of the universe and

the $w = -1/3$ portion can be made at any time and at any radius. If the constants V_0 and B are fixed by the laws of physics, then as the above relation between these constants and the constants R_0 and ϕ_0 indicate, the physical laws would also restrict the radius of the join and the value of the scalar field at the join.

It is interesting to confirm that the potential (3.9) satisfies the second-order equation of motion for a scalar field in the FRW universe with $R(t) = R_0(t_f - t)$. The equation of motion with arbitrary scalar potential $V(\phi)$ is (Tipler 1994, p. 466; Barrow & Tipler 1986, p. 431):

$$\phi_{;\alpha}{}^{;\alpha} = \frac{\partial V(\phi)}{\partial \phi}. \tag{3.12}$$

In the FRW universe we have $\phi_{;\alpha}{}^{;\alpha} = (\phi_{,0})^{;0} = (\phi^{,0})_{;0} = (-\phi_{,0})_{;0}$, and in a coordinate basis, the identity $A^\alpha{}_{;\alpha} = (1/\sqrt{-g})(\sqrt{-g} A^\alpha)_{,\alpha}$, for any vector field A^α, applies. Thus, in an FRW coordinate basis, the scalar field equation of motion can be written in the form

$$\frac{1}{R^3}\left(R^3(-\phi_{,0})\right)_{,0} = \frac{\partial V(\phi)}{\partial \phi}, \tag{3.13}$$

which can be reduced to the standard expression $\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0$ as follows. In both a coordinate basis and an orthonormal basis, we have $A_{,0} = dA/dt$, for any function A. Thus, for an exponential potential, we have $\partial V(\phi)/\partial \phi = BV(\phi) = B(\phi_{,0})^2$. Using $R = R_0(t_f - t)$ and the expression (3.6) for $\phi_{,0}$, it is confirmed that expression (3.13) is indeed an identity.

An alternative derivation of the fact that the $w = -1/3$ equation of state can be generated by a scalar field with exponential potential would be to make use of Barrow's work (Barrow 1987; Burd & Barrow 1988) on scalar fields with exponential potentials in a flat space (FRW $k = 0$). Barrow (1987) in fact noted that in the far future, an exponential potential could give rise to the $w = -1/3$ equation of state in the $k = +1$ case, but he did not attempt to derive the constants (B and V_0 above) that would allow the $w = -1/3$ equation of state to be joined to a dust equation of state for earlier times, which is why we did the calculation above. In addition, Vilenkin (1984) (see also Turner 1999) has pointed out that a $w = -1/3$ equation of state can be generated by a tangled network of very light cosmic strings.

In joining two metrics with different equations of state, one effectively assumes that one form of matter disappears and is replaced by the other. More realistically, if a scalar field were to be present near a final singularity, we would expect it to be in addition to dust or radiation already present. In such a situation, a pure exponential potential uncoupled to the other forms of matter would *not* give rise to a single c-boundary point, if its stress–energy tensor increased as R^{-2}, since dust and radiation would increase as R^{-3} and R^{-4}, respectively; such a universe would inevitably become radiation dominated sufficiently near the singularity. However, in an FRW universe, we can always find, for any assumed mixture of dust and radiation, a suitable potential $V(\phi)$ which would have the effect of cancelling out the gravitational force of the dust and matter fields, leaving an effective pure exponential scalar field (this is in effect what happens after the join between the $w = -1/3$ equation-of-state and the $w = 0$ equation-of-state fluids).

However, the actual universe is not expected to be an FRW near the final singularity. Even if the universe were an FRW in the beginning, we would expect it to become curvature dominated near the final singularity, since the 'effective energy density' curvature perturbations around FRW grow as R^{-6}, much faster than the densities of dust or radiation (Misner, Thorne & Wheeler 1973, p. 807). Thus, in the actual universe, the elimination of event horizons would have to be carried out by the global collective interactions (of known

forces) which give rise to the Misner mixmaster horizon elimination mechanism, as described in Tipler (1994).

On the other hand, a pure scalar inflation (quintessence field) with exponential potential might be expected to be the entire matter content in the very early universe, and the initial singularity might be expected to be an FRW. In such a case the effect of such an inflation field would be to eliminate the particle horizons. In other words, with an exponential inflation (quintessence) field, the horizon problem of cosmology would be automatically resolved.

4 GENERALIZING THE FRW $w = -1/3$ OMEGA POINT SPACE–TIME TO THE SPHERICALLY SYMMETRIC CASE

The approach used in Section 2 for creating space–times with no event horizons can be generalized to yield a wider class of such space–times. Instead of using the metric (2.4), we introduce functions $N(\chi)$ and $Z(\chi)$, where N is positive on $[0, \pi]$ and Z is positive on $(0, \pi)$, vanishing at 0 and π. The metric we then use is

$$ds^2 = -dt^2 + (t_f - t)^2[N^2 d\chi^2 + Z^2(d\theta^2 + \sin^2\theta \, d\varphi^2)]. \tag{4.1}$$

Proposition 1. *A Tolman–Bondi space–time with metric (4.1) has a c-boundary which is a single point.*

Proof. To check that this space–time actually has no event horizons, we mimic the calculation of the same proposition for the $w = -1/3$ FRW universe in Section 2. Let N_{max} be the maximum value of N on $[0, \pi]$, and then

$$\Delta\chi = \int^{t_f} \frac{dt}{N(\chi)(t_f - t)} \geqslant \int^{t_f} \frac{dt}{N_{max}(t_f - t)} = +\infty. \tag{4.2}$$

Thus, in this class of space–times, radial null geodesics are capable of hitting every value of χ an infinite number of times. In order to conclude that every point in space can communicate with every other point, however, we must refine the argument given in Section 2 a bit, for we no longer have the symmetry of the three-sphere to exploit. We do, however, still have (two-)spherical symmetry. Therefore, we can say that a null geodesic may be sent from the origin to any (χ, θ, φ), and vice versa. Hence, given points $P_1 = (\chi_1, \theta_1, \varphi_1)$ and $P_2 = (\chi_2, \theta_2, \varphi_2)$ which desire to communicate with one another, there exists a piecewise C^∞ null curve from P_1 to P_2, consisting of a null curve from P_1 to the origin and then a null curve from the origin to P_2. Applying an elementary result of Penrose (1972, lemma 2.16), we conclude that there exists a time-like or null curve from P_1 to P_2, which is precisely what we wanted. QED.

We would like the space–time (4.1) to satisfy the weak, dominant, and strong energy conditions (Hawking & Ellis 1973). Let G be the Einstein tensor of this space–time. Using the equations for the non-zero components of the Einstein tensor in Kramer et al. (2003), we can compute G in the orthonormal basis $\omega^{\hat{\imath}}$, where

$$\omega^{\hat{0}} = dt, \quad \omega^{\hat{1}} = N(t_f - t) \, d\chi,$$

$$\omega^{\hat{2}} = Z(t_f - t) \, d\theta, \quad \omega^{\hat{3}} = Z(t_f - t)\sin\theta \, d\varphi.$$

In this basis, all off-diagonal terms of G are zero. Thus, all matter is of Type I (Hawking & Ellis 1973), and the energy conditions will hold if the following six conditions are satisfied:

$$G^{\hat{0}\hat{0}} \geqslant 0, \tag{4.3}$$

$$G^{\hat{0}\hat{0}} + G^{\hat{1}\hat{1}} + G^{\hat{2}\hat{2}} + G^{\hat{3}\hat{3}} \geqslant 0, \tag{4.4}$$

$$G^{\hat{0}\hat{0}} + G^{\hat{1}\hat{1}} \geqslant 0, \tag{4.5}$$

$$G^{\hat{0}\hat{0}} - G^{\hat{1}\hat{1}} \geqslant 0, \tag{4.6}$$

$$G^{\hat{0}\hat{0}} + G^{\hat{2}\hat{2}} = G^{\hat{0}\hat{0}} + G^{\hat{3}\hat{3}} \geqslant 0, \tag{4.7}$$

$$G^{\hat{0}\hat{0}} - G^{\hat{2}\hat{2}} = G^{\hat{0}\hat{0}} - G^{\hat{3}\hat{3}} \geqslant 0. \tag{4.8}$$

The strong energy conditions are equations (4.3) and (4.4), the weak are equations (4.3), (4.5) and (4.7), and the dominant are equations (4.3) and (4.5)–(4.8). Computing these expressions with our metric (4.1), we see that $G^{00} + G^{11} + G^{22} + G^{33} = 0$ identically, and equations (4.3), (4.5), (4.6), (4.7) and (4.8) are equivalent to, respectively,

$$3 + \frac{1}{Z^2} - \frac{1}{N^2}\left(\frac{2Z''}{Z} - \frac{2Z'N'}{ZN} + \frac{(Z')^2}{Z^2}\right) \geqslant 0, \tag{4.9}$$

$$2 - \frac{2}{N^2}\left(\frac{Z''}{Z} - \frac{Z'N'}{ZN}\right) \geqslant 0, \tag{4.10}$$

$$4 + \frac{2}{Z^2} - \frac{2}{N^2}\left(\frac{Z''}{Z} - \frac{Z'N'}{ZN} + \frac{(Z')^2}{Z^2}\right) \geqslant 0, \tag{4.11}$$

$$2 + \frac{1}{Z^2} - \frac{1}{N^2}\left(\frac{Z''}{Z} - \frac{Z'N'}{ZN} + \frac{(Z')^2}{Z^2}\right) \geqslant 0 \tag{4.12}$$

and

$$4 + \frac{1}{Z^2} - \frac{1}{N^2}\left(\frac{3Z''}{Z} - \frac{3Z'N'}{ZN} + \frac{(Z')^2}{Z^2}\right) \geqslant 0, \tag{4.13}$$

where prime ($'$) denotes differentiation with respect to χ (everything is a function of χ). In the FRW case, $N = R_0$, $Z = R_0 \sin\chi$, and the energy condition equations are all are equivalent to

$$1 + \frac{1}{R_0^2} \geqslant 0.$$

In other words, in the FRW case, the energy conditions are always satisfied.

We will now show that if the metric (4.1) defines a universe that is 'sufficiently large', it will automatically satisfy the energy conditions. Suppose we are given functions $N_0(\chi)$ and $Z_0(\chi)$ such that there exist constants R_1, R_2, ϵ_1, $\epsilon_2 > 0$ with $N_0(\chi) = R_1$ and $Z_0(\chi) = R_1\sin\chi$ for $0 \leqslant \chi < \epsilon_1$ and $N_0(\chi) = R_2$ and $Z_0(\chi) = R_2\sin\chi$ for $\pi - \epsilon_2 < \chi \leqslant \pi$. In other words, N_0 and Z_0 look like the N and Z from FRW universes near $\chi = 0$ and $\chi = \pi$. We then know that near $\chi = 0$ and π, the energy conditions are satisfied for $N = RN_0$ and $Z = RZ_0$, where R is an arbitrary positive constant. Since the expressions multiplied by $\frac{1}{N^2}$ in the energy conditions are bounded for $\chi \in [\epsilon_1, \pi - \epsilon_2]$, we may then find a constant multiplier R such that the metric (4.1) with $N = RN_0$ and $Z = RZ_0$ satisfies all the energy conditions everywhere. The current observational evidence indicates that the universe is very close to being spatially flat, so the actual universe satisfies the 'sufficiently large' criterion.

5 A THREE-SPHERE UNIVERSE CONTAINING A BLACK HOLE BUT HAVING NO EVENT HORIZONS

We will produce a large class of $S^3 \times R^1$ space–times which are in their expanding phase, special cases of the general spherically symmetric dust solution (Kramer et al. 2003) and which are eventually joined to a space–time of the type described in Section 4, so that they end with a c-boundary of a point (and hence have no event horizons), and satisfy the energy conditions everywhere.

5.1 General dust solution

The general spherically symmetric pressureless dust solution (Kramer et al. 2003) is

$$ds^2 = -dt^2 + (1 - f^2)^{-1}\left(\frac{\partial Y}{\partial \chi}\right)_t^2 d\chi^2 + Y^2(d\theta^2 + \sin^2\theta\, d\varphi^2), \tag{5.1}$$

where the notation $(\frac{\partial Y}{\partial \chi})_t$ denotes differentiation of Y with respect to χ where the independent variables are t, χ, θ and φ (subscripts to differentials in general will specify independent variables, with the assumption that θ and φ are always independent), f is an arbitrary function of χ alone taking values in $[0, 1]$ and Y and t are given, respectively, by

$$Y = (1 - \cos\eta)\frac{m(\chi)}{f(\chi)^2} \tag{5.2}$$

and

$$t = t_0(\chi) + (\eta - \sin\eta)\frac{m(\chi)}{f(\chi)^3}. \tag{5.3}$$

In the above expressions, t_0 is an arbitrary function of χ alone, m is another arbitrary function of χ positive on $(0, \pi)$, and η is defined by equation (5.3). The only restrictions on these free functions are that to maintain the non-degeneracy of the metric in a closed universe, f should be equal to 1 at one χ-value in the interior of $[0, \pi]$, at which point m', f' and t_0' should all be zero. The general dust metric becomes degenerate whenever $Y' = 0$ and $f \neq 1$ or $Y' \neq 0$ and $f = 1$. Such two-spheres of degeneracy correspond to shell-crossing singularities, and if these degeneracy spheres occur before the final singularity at $\eta = 2\pi$, they will give rise to a breakdown in global hyperbolicity, as is well known. We will assume that the free functions m, f and t_0 are so chosen that this does not occur.

The dust case of the FRW metric (the case where $w = 0$) is a special case of this general metric. Letting

$$f = \sin\chi, \qquad m = \frac{R_{\max}}{2}\sin^3\chi \quad \text{and} \quad t_0 = 0,$$

one obtains

$$t = \frac{R_{\max}}{2}(\eta - \sin\eta), \qquad Y = \frac{R_{\max}}{2}(1 - \cos\eta)\sin\chi,$$

and

$$\left(\frac{\partial Y}{\partial \chi}\right)_t = \frac{R_{\max}}{2}(1 - \cos\eta).$$

The resulting metric is

$$ds^2 = -dt^2 + \left[\frac{R_{\max}}{2}(1 - \cos\eta)\right]^2 [d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\, d\varphi^2)],$$

precisely the Friedmann collapsing dust S^3 solution.

5.2 The join

We have shown above that equation (2.4) can be joined in a C^1 manner to any collapsing dust FRW S^3 universe at *any* time in the collapsing phase by a suitable choice of the constants R_0 and t_f. We will now generalize this construction substantially, joining a certain class of Tolman–Bondi pressureless dust solutions (including the FRW S^3 collapsing dust solution) to universes of the sort of equation (4.1), so that we produce a large class of universes which start with pressureless dust and in the end have no event horizons.

In making this join, we will allow the hypersurface $\mathcal{J}$ along which the two metrics are joined to vary as a free function. For convenience, we will take $\mathcal{J}$ to be spherically symmetric, parametrized

as $(t_{\mathcal{J}}, \chi, \theta, \varphi)$ in the dust universe, where $\eta_{\mathcal{J}} = \eta_{\mathcal{J}}(\chi)$ is a free function of χ, and

$$t_{\mathcal{J}} \equiv t(\eta_{\mathcal{J}}(\chi), \chi) = t_0(\chi) + \frac{(\eta_{\mathcal{J}} - \sin\eta_{\mathcal{J}})m(\chi)}{f^3(\chi)}. \tag{5.4}$$

We will also assume that the t, χ, θ, and φ coordinates agree across $\mathcal{J}$. Therefore, we will have six degrees of freedom altogether: t_0, m, f, $\eta_{\mathcal{J}}$, N and Z, all free functions of χ.

To make the join C^1, we first must make it continuous across $\mathcal{J}$. This in particular means that the metric coefficients will agree along $\mathcal{J}$ itself. Therefore, the metric coefficients will agree in all derivatives along vectors tangent to $\mathcal{J}$, and thus in order to check that the join is C^1, one must only check that the first derivatives of the metric coefficients agree in a direction independent of the tangent spaces of $\mathcal{J}$. The direction we choose is $(\partial/\partial t)_{\chi}$. This direction is linearly independent of the tangent spaces of $\mathcal{J}$ because $(\partial/\partial\eta)_{\chi}$ is never tangent to $\mathcal{J}$ (since $\mathcal{J}$ is parametrized with η a function of χ), and

$$\left(\frac{\partial}{\partial t}\right)_{\chi} = \frac{f^3}{m(1 - \cos\eta)}\left(\frac{\partial}{\partial\eta}\right)_{\chi}.$$

Since we are assuming that the coordinates are the same on the dust universe as on the universe (equation 4.1), the off-diagonal coefficients already agree (they are 0 on both sides of $\mathcal{J}$), and g_{tt} is -1 on both sides of $\mathcal{J}$. Furthermore, since we have spherical symmetry on both sides of $\mathcal{J}$, we need to check only one of $g_{\theta\theta}$ and $g_{\varphi\varphi}$. We are left with *four* junction conditions:

$$\frac{(Y')^2}{1 - f^2} = N^2(t_{\mathrm{f}} - t_{\mathcal{J}})^2 \quad (g_{\chi\chi} \text{ continuous}), \tag{5.5}$$

$$Y^2 = Z^2(t_{\mathrm{f}} - t_{\mathcal{J}})^2 \quad (g_{\theta\theta} \text{ continuous}), \tag{5.6}$$

$$\frac{2Y'\dot{Y}'}{1 - f^2} = -2N^2(t_{\mathrm{f}} - t_{\mathcal{J}}) \qquad \left(g_{\chi\chi} \, C^1\right), \tag{5.7}$$

$$2Y\dot{Y} = -2Z^2(t_{\mathrm{f}} - t_{\mathcal{J}}) \qquad \left(g_{\theta\theta} \, C^1\right). \tag{5.8}$$

Here the dot (˙) denotes application of $(\partial/\partial t)_{\chi}$ and the prime (′) denotes application of $(\partial/\partial\chi)_t$. These tangent vectors arise from the coordinate system $(t, \chi, \theta, \varphi)$, and thus they commute with one another, so that equation (5.7) makes sense.

Therefore, we have six free functions – $t_0(\chi)$, $m(\chi)$ and $f(\chi)$ on the Tolman–Bondi side of the join, and $\eta_{\mathcal{J}}(\chi)$, $N(\chi)$ and $Z(\chi)$ on the final singularity side of the join – and four differential equations relating them. One would expect that these four equations would determine four of the functions in terms of the other two, and this is exactly what happens. We find it convenient to choose $m(\chi)$ and $f(\chi)$ as the arbitrary functions, and expressing all other functions in terms of these two. After some manipulation of the above equations, we obtain

$$\frac{(1 + \cos\eta_{\mathcal{J}})^2}{|\sin^3\eta_{\mathcal{J}}|} = C\frac{m}{f^3}, \tag{5.9}$$

$$t_0 = t_{\mathrm{f}} + \left[\frac{2(1 - \cos\eta_{\mathcal{J}})}{\sin\eta_{\mathcal{J}}} - \eta_{\mathcal{J}}\right]\frac{m}{f^3}, \tag{5.10}$$

$$N = \frac{|\dot{Y}'(\eta_{\mathcal{J}}, \chi)|}{\sqrt{1 - f^2}}, \tag{5.11}$$

$$Z = |\dot{Y}(\eta_{\mathcal{J}}, \chi)|. \tag{5.12}$$

where C is a constant of integration. Equation (5.9) can be inverted to give $\eta_{\mathcal{J}}(\chi)$. $\eta_{\mathcal{J}}$ is then inserted into equations (5.10), (5.11) and (5.12) to yield $t_0(\chi)$, $N(\chi)$ and $Z(\chi)$, respectively, in terms of the arbitrary functions $m(\chi)$ and $f(\chi)$, and the constants t_{f} and C. Note that the allowed Tolman–Bondi dust metrics are no longer completely general, since the function t_0 is now fixed rather than being completely arbitrary. The constants t_{f} and C allow the join to be made as far in the future in proper time (the constant t_{f}) and in η time (the constant C) as one wishes. To see the latter, note that equation (5.9) is of the form $F(\eta_{\mathcal{J}}) = Cm/f^3$, where F increases monotonically from 0 to $+\infty$ as $\eta_{\mathcal{J}}$ ranges from π to 2π. Thus, if we make C arbitrarily large, $\eta_{\mathcal{J}}$ can be made arbitrarily close to 2π, that is, as close as we wish to the final singularity. Furthermore, one observes that the boundary requirement of t_0' (that it is 0 whenever Y' and hence m' and f' are 0) is automatically satisfied, since by the junction conditions

$$t_0' = \frac{m}{f^3}\left(\frac{m'}{m} - 3\frac{f'}{f}\right)\left[\frac{(1 - \cos\eta_{\mathcal{J}})\sin\eta_{\mathcal{J}}}{2 + \cos\eta_{\mathcal{J}}} - \eta_{\mathcal{J}} + \sin\eta_{\mathcal{J}}\right].$$

The join in Section 2 is in fact a special case of this construction. Consider the FRW choices for m and f:

$$m(\chi) = \frac{R_{\max}}{2}\sin^3\chi \quad \text{and} \quad f(\chi) = \sin\chi,$$

then $m/f^3 = 1$ identically, so that by equation (5.9) $\eta_{\mathcal{J}}$ will be a constant, and then by equation (5.10) t_0 will be a constant. Choosing t_{f} appropriately, we may make t_0 identically zero, so that our Tolman–Bondi universe is in fact the FRW collapsing dust universe. As noted above, we may make the join at any time in the collapsing phase, just as in the FRW construction of Section 2, and a simple calculation reveals that the N and Z forced by the join are precisely those which give the $w = -1/3$ universe.

5.3 Join with a possible 'weak' shell-crossing singularity

In order to make the metric coefficients differentiable across $\mathcal{J}$, we had to impose four conditions on six functions. We would, however, like to join a completely arbitrary Tolman–Bondi metric to the metric (4.1), and this will require eliminating one of the equations. The junction condition which on physical grounds is the least important is equation (5.7), the requirement that $g_{\chi\chi}$ be C^1. If $g_{\chi\chi}$ is not C^1 at $\mathcal{J}$, then the curvature will be a δ function on $\mathcal{J}$, but this δ function will correspond to a shell-crossing singularity, a singularity that is generally agreed to be unphysical. (Note also that requiring $g_{\chi\chi}$ be C^1 across $\mathcal{J}$ actually requires that the radii Y of the constant χ spheres have one of its *second* derivatives, $\dot{Y}'$, be continuous across $\mathcal{J}$.) Thus, we drop the junction condition (5.7). A little manipulation yields

$$\eta_{\mathcal{J}} - 2\tan\frac{\eta_{\mathcal{J}}}{2} = \frac{f^3(t_{\mathrm{f}} - t_0)}{m}, \tag{5.13}$$

$$t_{\mathcal{J}} = t_0(\chi) + (\eta_{\mathcal{J}} - \sin\eta_{\mathcal{J}})\frac{m(\chi)}{f^3(\chi)}, \tag{5.14}$$

$$N(\chi) = \frac{|Y'(\eta_{\mathcal{J}}, \chi)|}{(t_{\mathrm{f}} - t_{\mathcal{J}})\sqrt{1 - f^2}}, \tag{5.15}$$

$$Z(\chi) = |\dot{Y}(\eta_{\mathcal{J}}, \chi)|. \tag{5.16}$$

We proceed as in the previous section, solving first for $\eta_{\mathcal{J}}$ and then substituting this into the other equations. Note that in order to invert equation (5.13), solving for $\eta_{\mathcal{J}}$, the fact that t_0 must be less than t_f (the final t-value of the universe) implies that the left-hand side should be positive for all values of χ. However, since the positive values of $u - 2 \tan(u/2)$ for $u \in [0, 2\pi]$ are all at least 2π, we must therefore have the right-hand side being at least 2π for all values of χ. Since m, f and t_0 are all defined on the same compact interval, we may (and must) choose the constant t_f so large that the right-hand side is always at least 2π. Note that the differentiability of N in equation (5.15) will be guaranteed when $Y' = 0$ and $f = 1$ because the value of $(Y')/(1 - f^2)$ as χ approaches such a point is well defined and positive, and thus $(t_f - t_0)$ will be just the square root of this positive value. This means that the more standard type of shell-crossing singularity ($Y' = 0$ but $f \neq 1$, so that $g_{\chi\chi} = 0$) is assumed not to occur on $\mathcal{J}$. For this reason, we called the allowed singularity a 'weak' shell-crossing singularity.

Therefore, if we make the appropriate choice of t_f as described above, we may join an arbitrary Tolman–Bondi closed dust metric to the metric (4.1), provided we allow for a possible shell-crossing singularity. The only additional restriction we must impose on the Tolman–Bondi functions m, f and t_0 is that they must be chosen to make the universe 'sufficiently large' as discussed in Section 4.

6 BLACK HOLES

One interesting consequence of the above constructions is that they provide examples of space–times satisfying the energy conditions which can contain black holes, but do not contain event horizons. In order for this statement to make sense, however, we need a good definition of a black hole in a closed universe, for in a closed universe the black hole singularity is actually just a component of the final singularity (cf. Wheeler & Qadir 1985). We will discuss in detail three such definitions, the first due to Hayward (1994), the second due to Tipler (1977), and the third due to Wheeler & Qadir (1985).

In the standard definition of a black hole (cf. Wald 1984, p. 300, or Misner et al. 1973, p. 924), the black hole B is the space–time region $B \equiv M - J^-(\mathcal{I}^+)$, where M is the space–time manifold, $\mathcal{I}^+$ is 'scri plus' – future null infinity, and $J^-(S)$ is the *causal past* of a set S, which is to say that $J^-(S)$ is the set of all space–time points p which can be reached by a past-directed time-like or null curve from S to p. [Discussions of global general relativity and the definitions of concepts used in this discipline can be found in Wald (1984, chapter 8; Misner et al. 1973, chapter 34; or Hawking & Ellis 1973; Penrose 1972).] This definition cannot be applied in a closed universe, because $\mathcal{I}^+$ does not exist in a closed universe with a final singularity. However, this standard definition of a black hole is never used in practice. When astrophysicists search for black holes, they look for gravitational fields implying the presence of trapped or marginally trapped surfaces. In asymptotically flat space–times (i) all trapped surfaces can be proven to be inside of a black hole (in the standard definition), and (ii) black holes are expected to evolve rapidly to a Schwarzschild or Kerr black hole, in which there are trapped surfaces arbitrarily close to the boundary of the black hole $\partial J^-(\mathcal{I}^+)$ – the event horizon. Now trapped surfaces *can* be in closed universes.

6.1 Trapped and marginally trapped surfaces

Thus, the fundamental concept in Hayward's and Tipler's definitions of a black hole is that of a trapped surface (cf. Hayward 1994).
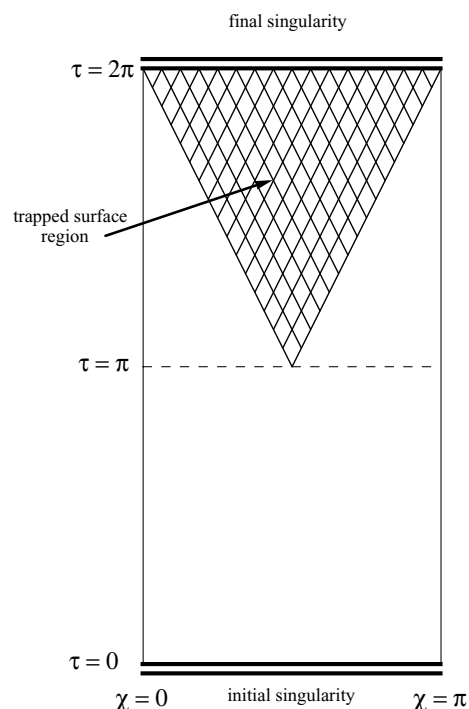
Figure 1. Trapped surfaces in the three-sphere FRW dust universe. The points in the cross-hatched region represent two-spheres which are trapped surfaces. The leftmost vertical line is the worldline of the origin of spatial coordinates, and the rightmost vertical line is the worldline of the spatial antipodal point. As pictured, the trapped surfaces begin with the collapse of the universe at a conformal time π. The boundary of the trapped surface region is formed of marginally trapped surfaces. Although it appears that there are no trapped surfaces through the origin of coordinates, this is an illusion caused by the fact that all two-spheres are concentric on the origin of coordinates. If the origin of coordinates were moved half way to the antipode (to $\chi = \pi/2$), in the collapsing phase one would see a trapped surface surface through the original origin of coordinates .

Let $\mathcal{S}$ be a compact space-like two-surface embedded in our space–time manifold. Let P be a point of $\mathcal{S}$. There are precisely two null directions normal to $\mathcal{S}$ at P. Suppose furthermore that these null directions can be expressed as two vector fields N_+ and N_- defined on all of $\mathcal{S}$. We can choose both of N_+ and N_- to be future-directed. Now, allowing $\mathcal{S}$ to evolve along N_+ and N_-, we can measure its area at every instant, and logarithmically differentiate the resulting function with respect to the evolution parameter. Call these quantities θ_+ and θ_-, respectively.

Definition 1. *$\mathcal{S}$ is called a* **(future) trapped surface** *if both $\theta_+ < 0$ and $\theta_- < 0$. If one of these quantities is zero and the other is negative, then $\mathcal{S}$ is called a* **marginally trapped surface**.

The intuition here is that light rays emitted from a trapped surface will converge, no matter whether they are sent 'outwards' or 'inwards'. This is certainly a necessary property of a black hole, and it would be sufficient if it were not for the fact that the cosmological singularity produces a wealth of trapped surfaces as the universe collapses. In an FRW closed universe, for example, there is a trapped surface passing through every spatial point in the collapsing phase, as shown in Fig. 1. As illustrated in that figure, the size of the trapped surfaces will approach zero even in the conformally related space–time, as the spatial size of the universe approaches zero. A universe which collapses into an Omega Point will also have trapped surfaces
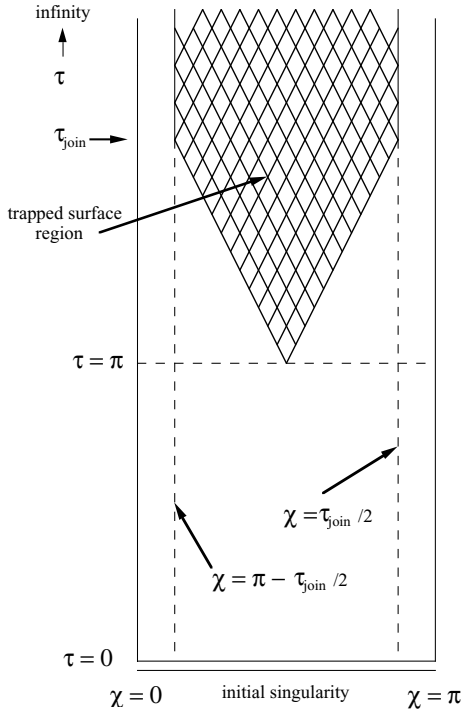
Figure 2. Trapped surfaces in a three-sphere FRW dust universe joined to an Omega Point metric. The size of the trapped surfaces does not approach zero in the conformally related metric, which is pictured.

passing through every spatial point in the collapsing phase, as illustrated in Fig. 2 though there is a lower bound to the size of the trapped surfaces in the conformally related space–time, a lower bound that depends on the conformal time of the join. In order to distinguish these cosmological trapped surfaces from non-cosmological ones – black hole type trapped surfaces – we need additional criteria.

6.2 Hayward's black hole definition

Marginally trapped surfaces are important because in some sense they are where the horizon of a prospective black hole should be. To distinguish trapped surfaces arising from black holes from those arising from the collapse of the universe, therefore, Hayward considers what should be happening in a neighbourhood of a marginally trapped surface which arises because of a black hole. Without loss of generality let $\theta_+ = 0$, $\theta_- < 0$ along the marginally trapped surface $\mathcal{S}$. N_+ can then sensibly be called 'outward', and N_- 'inward'. Outward-directed light rays run instantaneously parallel to the surface, and inward-directed light rays converge. In the case of black hole based marginally trapped surfaces, however, we would like to say that outward light rays just outside $\mathcal{S}$ diverge, while outward light rays just inside $\mathcal{S}$ converge. This can be accomplished mathematically by extending the embedding of $\mathcal{S}$ to a 'double-null foliation' (cf. Hayward 1994 and Ashtekar & Krishnan (2004)) in the direction of N_+ and N_-, extending θ_+ appropriately, and computing the sign of $\mathcal{L}_-\theta_+$, where $\mathcal{L}_-$ denotes the Lie derivative in the direction of N_-.

Definition 2. *A marginally trapped surface with $\theta_+ = 0$ (respectively $\theta_- = 0$) is called* **inner** *if $\mathcal{L}_-\theta_+$ (respectively $\mathcal{L}_+\theta_-$) is positive,* **outer** *if $\mathcal{L}_-\theta_+$ (respectively $\mathcal{L}_+\theta_-$) is negative, and degenerate otherwise.*

As it makes sense, inner marginally trapped surfaces correspond to cosmological collapse, and outer marginally trapped surfaces correspond to non-cosmological collapse, that is, to the marginally trapped surfaces we would expect to find inside black holes.

As mentioned above, in asymptotically flat space–times, all the trapped surfaces are inside the black hole, and furthermore, all future-directed causal (time-like or null) curves from any trapped surface $\mathcal{T}_i$ can also be shown to be inside the black hole. Thus, if B is the black hole region, we must have $J^+(\cup_i \mathcal{T}_i) \subset B$ in order to capture the astrophysically defining feature of a black hole in a black hole definition applicable to a closed universe. Also, in asymptotically flat space–times, any space–time point p whose causal future eventually enters the causal future of a trapped surface can be proven to be inside a black hole. This means that we should also include in the black hole B all points p such that null generators of $\partial J^+(p)$ eventually become null generators of $\partial J^+(\cup \mathcal{T}_i)$, or intersect $J^+(\cup_i \mathcal{T}_i)$ in its acausal initial boundary. This gives

Definition 3. (Hayward) *a black hole is the set $J^+(J^+(\cup \mathcal{T}_i)) \cup \{p|$ null generators of $\partial J^+(p)$ eventually become null generators of $\partial J^+(\cup \mathcal{T}_i)$ or intersect $\partial J^+(\cup \mathcal{T}_i)$ at points where this boundary is acausal, where $\cup \mathcal{T}_i$ is the union of all outer marginally trapped surfaces.*

Eardley (1998), however, has shown that the set of marginally trapped surfaces are ill behaved, and in fact marginally trapped surfaces do not have a smooth limit when they are deformed towards the horizon in asymptotically flat space–times. Nevertheless, Eardley proposed simply identifying the black hole region with the union of all marginally trapped surfaces, thereby avoiding the complication with precisely defining the points p in the black hole but not in $J^+(\cup_i \mathcal{T}_i)$. Eardley's proposal has been followed by the more recent results of Andersson et al. (2005) and Schnetter & Krishnan (2006). The non-smoothness of the set of marginally trapped surfaces, however, does suggest that there may be a problem with Hayward's definition.

6.3 Tipler's black hole definition

Tipler's (1977) criterion is related to Hayward's, but perhaps is a bit simpler (see Hayward 1994 for a short discussion of how these criteria relate), and refers only to the trapped surfaces, omitting the marginally trapped surfaces.

Instead of using a double-null foliation to test whether a given marginally trapped surface corresponds to the cosmological collapse or to a local black hole, Tipler instead supposes that the marginally trapped surface in question is contained in the boundary of a space-like hypersurface with boundary $\mathcal{T}$ whose interior $\mathcal{T} - \partial\mathcal{T}$ is foliated by trapped surfaces. He then (assuming without loss of generality that $\theta_+ = 0$, $\theta_- < 0$) makes the following.

Definition 4. *If the family of null vectors N_- (which are all on $\partial\mathcal{T}$) point in the direction of $\mathcal{T}$, then all trapped surfaces which can be obtained from trapped surfaces in $\mathcal{T}$ by an acausal homotopy foliated by trapped surfaces will be called* **non-cosmological**.

In particular, any trapped surface in $\mathcal{T}$ is non-cosmological in this case. Thus, black hole type trapped surfaces would be non-cosmological trapped surfaces, and we have

Definition 5. (Tipler) *a black hole is the set $J^+(J^+(\cup \mathcal{T}_i)) \cup \{p|$ null generators of $\partial J^+(p)$ eventually become null generators of $\partial J^+(\cup \mathcal{T}_i)$ or intersect $\partial J^+(\cup \mathcal{T}_i)$ at points where this boundary is*

acausal, where $\cup \mathcal{T}_i$ is the union of all non-cosmological trapped surfaces

6.4 Hayward–Tipler black holes in Tolman–Bondi closed universes

Specializing to the Tolman Dust case, we already have a convenient foliation by two-spheres, and we will use this foliation to assist in evaluating the two definitions outlined in the previous section.

First of all, note that the normal bundle to this foliation is spanned by the vector fields $\partial/\partial t$ and $\partial/\partial \chi$. Therefore, we may set

$$N_{\pm} = \frac{\partial}{\partial t} \pm \frac{\sqrt{1-f^2}}{Y'} \frac{\partial}{\partial \chi},$$

since $\partial/\partial t$ is future-directed and $g(\partial/\partial t, N_{\pm}) = 1$. Since the area of the two-sphere at (t, χ) is $4\pi Y^2$, we can then easily compute

$$\theta_{\pm} = N_{\pm}[\log(4\pi Y^2)] = \frac{2f^3}{(1-\cos\eta)m} \left(\cot\frac{\eta}{2} \pm \frac{\sqrt{1-f^2}}{f} \right).$$

Note that in order for the two-sphere at (t, χ) to be marginally trapped, we are forced to have $\theta_+ = 0$ and $\theta_- < 0$ (since $\theta_+ > \theta_-$).

Let us consider first Tipler's definition. Choose $\mathcal{T}$ to be a constant-t hypersurface. An example of a vector pointing in the direction of $\mathcal{T}$ is $\nu = -(\theta_+)' (\partial/\partial \chi)$ (here primes are as in the join conditions above), since θ_+ will decrease to become negative on $\mathcal{T}$, where the two-spheres will be trapped surfaces. Thus, a sufficient condition for cosmological trapped surfaces in a neighbourhood of a marginally trapped surface at (t, χ) is

$$0 < g(\nu, N_-) = (\theta_+)' \frac{Y'}{\sqrt{1-f^2}}.$$

Expanding this a bit, letting $Q = m/f^3$, ignoring positive multipliers and using the fact that $\theta_+ = 0$, we obtain the condition

$$\frac{Y'}{\sqrt{1-f^2}} \left[\frac{t_0' + (\eta - \sin\eta)Q'}{4f^4 Q} - \frac{f'}{f^2\sqrt{1-f^2}} \right] > 0.$$

We now consider how Hayward's definition applies. We are taking the derivative in the N_- direction of θ_+, and determining its sign. To that end, we will have a Hayward black hole type marginally trapped surface if

$$0 > \mathcal{L}_- \theta_+ =$$

$$\frac{1}{Y'} \left\{ \frac{1}{2f^3 Q} \left(\frac{f'}{f} - \frac{Q'}{Q} \right) - \frac{\sqrt{1-f^2}}{2f^6 Q^2} [t_0' + (\eta - \sin\eta)Q'] \right\}.$$

6.5 Black holes in a joined universe

Now let us suppose that we are looking for a black hole in a pressureless dust universe which can be joined to the *N–Z* universe defined above. We first consider the case of the differentiable join. Leaving *m* and *f* free as in our derivation of the join conditions, we compute Tipler's criterion to be

$$\frac{Y'}{\sqrt{1-f^2}} \left[\frac{(T + \eta - \sin\eta)Q'}{4f^4 Q} - \frac{f'}{f^2\sqrt{1-f^2}} \right] > 0,$$

and Hayward's to be

$$\frac{1}{2Y'f^3 Q} \left[\frac{f'}{f} - \frac{Q'}{Q} - \frac{Q'(T + \eta - \sin\eta)\sqrt{1-f^2}}{Qf^3} \right] < 0,$$
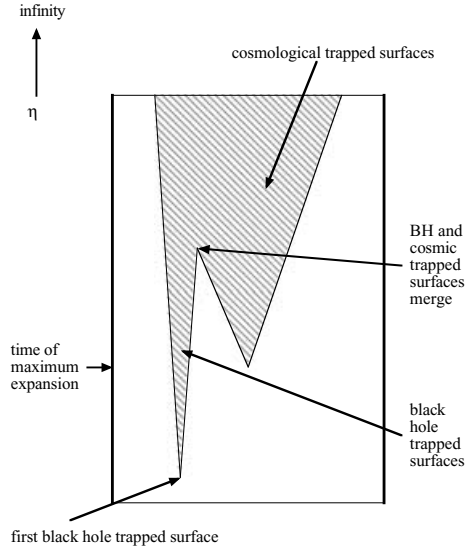
Figure 3. Black hole and cosmological trapped surfaces in a three-sphere Tolman–Bondi dust universe which is joined to an Omega Point universe. The points in the cross-hatched region represent two-spheres which are trapped surfaces. The black hole trapped surfaces are centred at the origin of coordinates, and they form long before the global collapse of the universe. The two types of trapped surface merge as the collapse proceeds, and become indistinguishable.

where $T(\chi)$ is defined as

$$T = \frac{3\sin\eta_{\mathcal{J}}}{2 + \cos\eta_{\mathcal{J}}} - \eta_{\mathcal{J}}.$$

It is clear that we can choose m, f and t_f in the C^1 join in such a way that the resulting joined universe contains black holes in either the Hayward or the Tipler sense, and satisfies the energy conditions. A black hole whose centre is at the origin of coordinates in such a joined universe is pictured in Fig. 3. Note that we show only the spherically symmetric trapped surfaces. There may exist non-spherically symmetric marginally trapped surfaces extending outside the cross-hatched region, though we believe these do not exist.

6.6 Wheeler's black hole definition

Following Wheeler & Qadir (1985), we will consider black holes in the spherically symmetric dust universe to be regions in which the universe is collapsing much faster than elsewhere. An intuitive measure of the rate of collapse of the universe in any region is a measurement of the elapsed time between the initial and final singularities.

We can use physical models to approximate the elapsed time between singularities in black hole regions. For simplicity, we will consider a one solar mass black hole. Misner et al. (1973) give the time for a particle to fall from radius R_i to the singularity in a standard Schwarzschild black hole as $\pi(R_i^3/8GM)^{1/2}$. Taking R_i to be the Schwarzschild event horizon $R_i = 2GM/c^2$, we have that the elapsed time from the penetration of the event horizon to the final singularity is approximately $t_{horizon} = 5 \times 10^{-6}$ s $(M/M_{\odot})$. A dust cloud generating such a black hole would have a total lifetime of twice this, giving us the elapsed time from initial to final singularity

as

$$t_{\text{total}} = 10^{-5} \text{ s} \left(\frac{M}{\text{M}_\odot} \right) \tag{6.1}$$

in the vicinity of a black hole.

Misner et al. (1973) cite the *illustrative* time that might be expected to elapse from the beginning to the end of a typical closed FRW universe (without black hole regions) as

$$t_{\text{total}} = 60 \times 10^9 \text{ yr} = 1.8 \times 10^{17} \text{ s}.$$

Combining this result with the above calculation, we conclude that the elapsed time between initial and final singularities in the vicinity of a 1-M$_\odot$ black hole will be of the order of 10^{22} times smaller than that in non-black hole regions of the universe. Since an upper bound to the mass of a black hole in the current epoch of universe history is believed to be 10^{10} M$_\odot$, we would expect that the elapsed time between the initial and final singularities inside the largest black hole in existence today would be of the order of 10^{12} times smaller than that in non-black hole regions of the universe, since by equation (6.1), a black hole lifetime scales linearly with its mass.

The elapsed time from the beginning to the end of the universe along any time-like curves of constant χ can be obtained from equation (5.2) by computing $t(2\pi, \chi) - t(0, \chi)$. Thus, the total time elapsed from initial to final singularity along a time-like curve of constant χ is simply $2\pi \, m(\chi)/f(\chi)^3$. Therefore, we can assert that if χ_a corresponds to a black hole region of some hypersurface and χ_b corresponds to the cosmological region, we should obtain that

$$\frac{t_{\text{total}}(\chi_b)}{t_{\text{total}}(\chi_a)} = \frac{m(\chi_b)/f(\chi_b)^3}{m(\chi_a)/f(\chi_a)^3} > 10^{13}. \tag{6.2}$$

To apply this notion of black holes in the dust universe to our joined metric, we consider any two-sphere with coordinate radius χ_a to be inside a black hole if equation (6.2) is satisfied when χ_b is the coordinate radius of a two-sphere whose size evolves like the two-spheres of spherical symmetry in an FRW universe. It will be sufficient that there exist a pair of two-spheres with coordinate radii χ_a and χ_b such that equation (6.2) is satisfied. Alternatively, we could simply restrict attention to 1-M$_\odot$ black holes and fix χ_b. An elementary calculation shows that a solar mass black hole is a two-sphere with radius corresponding to the radial value $\chi = 10^{-23}$. A third way of picking the pair of two-spheres is to require that the χ_a two-sphere is the 'largest' two-sphere in the universe 'today'. This would mean that the χ_a two-sphere is an extremal – maximal – two-sphere embedded in the three-sphere corresponding to 'today'. We have to require the two-sphere to be extremal since one can construct a non-extremal two-sphere of arbitrary size embedded in a three-sphere.

Wheeler & Qadir (1985) point out that the natural meaning of 'today' – the choice of a space-like hypersurface though the Earth – is the constant mean curvature hypersurface through the Earth today. Tipler (1994, p. 440) has shown that if the strong energy condition holds, and if the universe began close to homogeneity and isotropy, an Omega Point space–time can be uniquely foliated by constant mean curvature hypersurfaces, so Wheeler's proposal does indeed define a unique 'today' over the entire universe. (Tipler also shows that a constant mean curvature hypersurface probably coincides with the rest frame of the CBR (cosmic background radiation) at any event, so Wheeler's 'today across the entire universe' is even easy to locate experimentally.) Putting all of these criteria together yields

Definition 6. (Wheeler) *a* **black hole** *is the set of all space–time points p such that* $J^+(J^+(p) \cap J^-(\mathcal{S})) \subset J^-(\mathcal{S})$, *where* $\mathcal{S}$ *is any two-sphere with coordinate radius* χ_a *for which equation (6.2) holds when* χ_b *is the coordinate radius of a maximal two-sphere in the constant mean curvature hypersurface which includes the two-sphere with coordinate radius* χ_a.

It is clear that since equation (6.2) does not depend on the function t_0, just on the functions m and f, it is possible to construct even in the case of the C^1 join a universe which is essentially a closed dust FRW everywhere outside a small black hole by Wheeler's definition, a black hole which is centred at the origin of coordinates $\chi = 0$.

7 QUINTESSENCE AND RECOLLAPSE

The 'no event horizon solution' to the black hole information problem requires that the universe recollapse to a final singularity before black holes have time to evaporate. However, the best observations (Bahcall et al. 1999), independently confirmed by a number of groups, indicate that the universe is currently accelerating. Furthermore, the observed structure is best explained (given a Hubble constant of 65 ± 5 km s^{-1}-Mpc and spatial flatness) via a Λ cold dark matter model (Turner & White 1997). If this acceleration were to continue – as it would if it were due to a positive cosmological constant – then the universe would expand forever, and our proposed solution to the black hole unitarity problem would be incorrect: unitarity would be violated.

However, Barrow (1987), see also Burd & Barrow (1988) was the first to point out that an accelerating universe today need not preclude a recollapse in the far future of a closed universe. Since the acceleration of the scalefactor R is given by equation (2.3), namely $\ddot{R} = -(4\pi G/3)R(\rho + 3p) = -(4\pi G/3)R(1 + 3w)\rho$, acceleration today implies that $w < -1/3$ today (the data give $w < -5/9$ today at the 95 per cent confidence level (Turner 1999), but if eventually $w > -1/3$ for all time greater than some far future value t_{future}, then the recollapse theorems of Barrow et al. (1986) will apply, and recollapse will occur.

Thus, unitarity implies that the observed acceleration 'today' (meaning most of past *proper* time) cannot be due to a positive cosmological constant, but must instead be due to quintessence. This is of course the general expectation of cosmologists, since the only plausible non-zero values of the cosmological constant are near the Planck density of $(10^{19} \text{ GeV})^4$, or near the density of the SM Higgs field at its minimum $\sim (200 \text{ GeV})^4$, whereas the observed density of the material causing the acceleration is of the order of the closure density, $(10^{-3} \text{ eV})^4$ (Weinberg 1989; Carroll, Press & Turner 1992).

The 'standard model' of quintessence (Carroll 1998; Turner 1999) is a scalar field ϕ with a very shallow potential $V(\phi)$ in the present epoch, resulting in scalar field excitations of very small mass, $m_\phi \equiv \sqrt{V''(\phi)/2} \leqslant H_0 \sim 10^{-33}$ eV. Since we know very little more than this about $V(\phi)$, the potential could have a minimum around which the field will oscillate in the far future. In such a case, in the far future the leading term in the expansion of the potential about the minimum would be $\frac{1}{2}m_\phi^2\phi^2$, yielding (Turner 1983) an oscillation frequency $\omega = m_\phi$ and $w \to 0$ in the far future where $m_\phi \gg H(t)$. With such a potential for the quintessence, recollapse would occur, since the curvature term in the Friedmann equation decreases as R^{-2} whereas the quintessence term would eventually decrease like the matter, R^{-3}.

These potentials yield the most popular current models of quintessence, since such potentials are suggested by supersymmetry. There are many exponential potentials which allow recollapse, as established by Barrow (1987). For example, if the potential dies

off sufficiently fast with ϕ, then in the far future, the density will drop off as R^{-6} as does a massless scalar field, the universe will become matter and then curvature dominated in the far future, and recollapse will result.

In summary, there are many quintessence models consistent with all current observations which allow recollapse in the far future. Thus, the scenario of horizon elimination proposed in this paper is consistent with all current astronomical observations.

8 CONCLUSION

The 'holographic principle' (Birmingham 1999; Easther & Lowe 1999; Kraus & Balasubramanian 1999) claims that all physics on a manifold – especially quantum gravity – can be completely described by a theory defined only on the boundary of that manifold. This is a completely reasonable principle in the case that the boundary of the manifold is a Cauchy surface for the manifold, because in this situation the data on the boundary uniquely determine the manifold and the properties of all physical fields defined on the manifold. For a classical black hole which forms by collapse in an asymptotically flat space–time and then settles down to a Schwarzschild exterior in the far future, the black hole event horizon is indeed a Cauchy surface for the interior. More generally, if the space–time is globally hyperbolic, we would expect the event horizon to still be a Cauchy surface for the black hole interior. If we include the c-boundary points in anti-deSitter space, then the Cauchy horizons surrounding a region plus the points on the c-boundary where the horizon generators terminate form a Cauchy surface for the interior space–time region enclosed by the Cauchy horizons; once again we would expect the holographic principle to be valid.

However, there are problems with the holographic principle in the case of black holes which evaporate to completion. Since the entire space–time is no longer globally hyperbolic, it is not clear that the event horizon is a Cauchy surface for the interior. There are problems with the c-boundary completion: looked at from inside a black hole, the c-boundary inside a spherically symmetric black hole is a two-sphere [the TIPs (Terminal Indecompressible Pasts) define a two-sphere], whereas looked at from the future after the evaporation is complete, the c-boundary is a single point [the TIFs (Terminal Indecompressible Futures) define a point], that is, the causal completion does not define the boundary of the interior manifold uniquely. Even if the event horizon were actually a Cauchy surface for the black hole interior, the information can never leave the horizon to the exterior space–time, since the event horizon generators must terminate at the singularity which ends the black hole evaporation.

This problem is obviated in an Omega Point space–time. The null generators of the black hole apparent horizon will actually be a Cauchy horizon for the entire space–time, for it can be shown that $\partial I^+(p)$ is a Cauchy surface for the entire space–time for any point p in the space–time (lemma 1 in Tipler 1994, p. 436). Thus, the holographic principle is true for all manifolds which are future sets (sets for which $I^+(S) \subset S$). In particular, for all points p_i we wish to include in 'black holes' (by any of the definitions given above), the boundary $\partial I^+(\cup\, p_i)$ will be a Cauchy surface for the space–time, and so the holographic principle will hold for the surfaces of black holes in Omega Point space–times.

Another area of general relativity that is naturally complemented by the no-event-horizon resolution of the black hole information problem is the computation of gravitational radiation from colliding black holes. Matzner et al. (1996) have noted that the computer simulation of a black hole collision is much simpler if characteristic

evolution is used in the black hole exterior, because in asymptotically flat space–times, the characteristic formulation can be compressed. In an Omega Point space–time, the characteristic formulation is *automatically* compressed: the null boundary $\partial I^+(p)$ of any point p in an Omega Point space–time has been shown by Tipler to be a compact Cauchy surface (Tipler 1994, p. 436), as we pointed out above. We conjecture that the calculation would be even easier done in an Omega Point background space, such as the spherically symmetric Omega Point space–times exhibited in Sections 2 and 4. In an Omega Point space–time, it is not necessary to add the c-boundary points to compress characteristic null surfaces like $\partial I^+(p)$. This is important, because as York has recently emphasized, in general coordinate systems, the initial value problem cannot be well posed in general relativity. However, Tipler has shown (Tipler 1994, p. 440) that Omega Point space–times which satisfy the strong energy condition and begin in a 'crushing' singularity (all FRW singularities are of this type, as are all 'stable' singularities) possess a unique foliation by constant mean curvature hypersurfaces, and York has shown that the initial data problem *is* well posed on such a hypersurface. [If the universe is currently accelerating, the strong energy condition will not hold everywhere, but nevertheless a constant mean curvature foliation will still exist, (Tipler 1994, p. 439). However, the foliation may only be unique in the very early universe and in the very late universe where the strong energy condition will hold.] As we have emphasized repeatedly, we define black holes operationally in terms of trapped surfaces, just as is done by the groups trying to compute the amount of radiation emitted from colliding black holes. Locally, their calculations of the black hole surfaces would be the same in an asymptotically flat space and in an Omega Point space–time. No quantum effects would affect the location or the size or the existence of trapped surfaces evolved in the black hole collision calculations.

Finally, we point out that many of the well-known difficulties associated with doing quantum field theory in curved space–times disappear in Omega Point space–times. As we mentioned in Section 1, Omega Point space–times necessarily are foliated by compact Cauchy surfaces, and in space–times with compact Cauchy surfaces – that is, in closed universes – there exists a natural unitary equivalence class of quantum field theory constructions, specifically, those constructed from all the Hadamard vacuum states (Wald 1994, p. 96). [Roughly speaking, a 'Hadamard state' is one in which the short-distance singularity structure of the two point function in curved space is the same as it is in Minkowski space (Wald 1994, pp. 92–95).] In space–times without a compact Cauchy surface, there are no unitarily equivalent representations of the quantum field algebra, and it was this fact which led many relativists to give up the postulate of unitarity. In an Omega Point space–time, it is not necessary to give up unitarity.

It is not even necessary to give up the notion of 'particle' or 'vacuum state' in a curved Omega Point space–time, as many relativists have previously believed (e.g. Wald 1994, p. 59 and p. 96). The method of Hamiltonian diagonalization (Wald 1994, p. 65) will define a unique vacuum and Fock space with respect to any given Cauchy surface, and we pointed out above that a unique foliation of the space–time by a constant mean curvature exists in a physically realistic Omega Point space–time (unique except *possibly* in the periods where the universe is accelerating). These constant mean curvature hypersurfaces are the natural 'rest frame' of the universe, and are the natural corresponding frames to the global Lorentz frames in a Minkowski space. In FRW universes, the constant mean curvature hypersurfaces are the 'rest frames' of the CBR – observers on worldlines normal to these hypersurfaces would measure isotropic

CBR temperature. With respect to such a unique global foliation, the notion of 'particle' and 'vacuum state' is defined and is unique in a curved space–time, and such notions must be defined if the quantum field theory in a curved space–time is to be a legitimate low energy limit of the (still unknown) quantum theory of gravity.

Weinberg (1995, p. 2) has pointed out that quantum field theories are regarded today as 'mere effective field theories', just low-energy approximations to a more fundamental theory. Quantum field theories are not themselves fundamental, but we use them only because any relativistic quantum theory will closely approximate a quantum field theory when applied to particles at a low enough energy. If this is true, then quantum particles are more fundamental than quantum fields, and thus a semiclassical theory like the quantum field theory in a classical curved space background must contain a natural definition of the more fundamental entity, the particle.

In short, assuming the universe to end in a c-boundary which is a single point – assuming the universe to be an Omega Point space–time – solves the black hole information problem, allows the standard concepts of relativistic quantum mechanics to be carried over into curved space–times, simplifies the characteristic initial value problem, and is consistent with all astronomical observations. The actual universe may indeed be an Omega Point space–time.

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