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THE FINITE ACTION PRINCIPLE OR, SINGULARITIES WITHOUT SINGULARITIES

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ABSTRACT

Physical theories have their most fundamental expression as action integrals. This implies that the total action of the universe should be the most fundamental physical quantity, hence this universal action must be finite. We investigate the consequences of a finite universal action. We show that if a finite action universe obeys the Copernican Cosmological Principle, it must be spatially and temporally closed. Thus, the classical curvature singularites in cosmology are seen to be essential if an even more fundamental singularity, a singularity or rather an infinity in the universal action, is to be avoided. Finally, we show that finite universal action places constraints on the types of matter that can dominate the dynamics of the early universe.

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AN ESSAY FOR THE GRAVITY RESEARCH FOUNDATION

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In most discussions, the initial cosmological singularity predicted by classical general relativity is regarded as an indication of the breakdown of the theory rather than a prediction of an extant singularity on the boundary of space-time. This view of the initial singularity is a manifestation of the widespread opinion that the occurrence of singularities in a physical theory indicates, not the actual occurrence of singularities in Nature, but rather the breakdown of the theory. There is considerable evidence for this opinion. For example, waves in perfect fluids often develop into shocks having discontinuities in the pressure. But discontinuities do not arise in real shock fronts, because the perfect fluid model does not accurately describe the behaviour of fluids in shock fronts. Another oft-quoted example is the atomic instability predicted by classical electromagnetic theory: the electron is predicted to spiral rapidly into the proton of the hydrogen atom, which therefore collapses in upon itself with the emission of an infinite pulse of radiation. This annihilation of hydrogen atoms is not seen because quantum mechanics, not classical mechanics, applies at the atomic level. The classically predicted singularity in atoms merely signals that classical theory has broken down. By analogy, it is believed that the prediction of cosmological singularities simply means that classical relativity has broken down.

But this analogy is incomplete in at least two ways. First, we know the atomic singularity does not occur because atoms in fact exist; therefore there must be a mechanism preventing the collapse. Furthermore, the existence of atoms means that the perfect fluid model cannnot be valid at shock fronts. In contrast, the initial singularity, if it actually occurred, did so only at the beginning of time. There is no singularity occurring now, and if the strong cosmic censorship hypothesis is valid, no singularity can ever be produced in the laboratory. Thus we have no observations showing that an initial cosmological singularity did not occur, whereas direct observations do rule out atomic singularities and shock discontinuities. Second, in the cosmological case -- in contrast to the atomic and perfect fluid cases -- an infinity in a local physical quantity can be avoided only at the price of an infinity in an even more fundamental global physical quantity, the total action of the universe.

Max Planck[1] was perhaps the first in the twentieth century (see[2] for a discussion of earlier work) to argue strongly for regarding the action as the most basic quantity in physics. He, and later Dirac[3], justified this on the ground of the manifest Lorentz (and $GL(4,\mathbf{R})$) invariance of the action integral. In contemporary physics, the action is used as the starting point of fundamental theories because of its manifest invariance under gauge transformations of Yang-Mills and supersymmetric fields, and also because the central importance of the path-integral method of quantization, which uses the action in a central way[4,5].

It is well-known that the requirement of finite action has important physical significance for actions defined in Minkowski and Euclidean space. Recall that a finite action solution to the Yang-Mills-Higgs equations for the fields (A,Φ) in Minkowski space is called a *soliton*[6], and if the space is Euclidean, a finite action solution to pure Yang-Mills ($\Phi = 0$) is termed an *instanton* [6,7].

In Euclidean space, the finite action requirement imposes a large number of strong restrictions on the solutions to the Yang-Mills-Higgs equations; for example: (1) the fields must die off sufficiently rapidly at Euclidean infinity; (2) the minima of the pure Yang-Mills action must be instantons[6]; (3) instantons must be self-dual or anti-self-dual[6]; (4) if $d > 4$, where d is the dimension of the Euclidean space, then there are no nontrivial solutions to the Yang-Mills-Higgs equations[6]; and (5) if $d < 4$, there are no nontrivial solutions to the pure Yang-Mills equations[6].

The reason for considering $d = 4$ instantons -- Euclidean actions -- is that the path integral is usually defined by Wick-rotating the Minkowski space action S to a $d = 4$ Euclidean action, so that the path integral weight-factor exp[-S] defines a Wiener measure. The classical Euclidean solutions with finite action provide a starting point for a semi-classical approximation to the Euclidean path integral, which can be analytically continued back into Minkowski space to obtain the physical path integral.

Of fundamental importance is the fact that instantons are useful only in the semi-classical approximation; in Euclidean space, the paths with finite action are a set of measure zero[6,8] in the function space of paths, in spite of the weight exp[-S] being zero for infinite action! As is clear from the proof of this fact for the simple harmonic oscillator[8], the infinite action paths will have an overwhelming importance in any natural measure on Euclidean space; Euclidean infinity will dominate the path integral, and make its presence felt in any local experiment. Thus if the finite action principle is to apply in quantum as well as classical mechanics, the function space upon which the path integral of the entire Universe is defined must be such as to give dominant weight to the finite action paths, or (what is physically equivalent), the Universal action operator and possibly other observables (self-adjoint operators) must be bounded operators.

We now consider the implications of finite action in cosmology. The general action[9-11] for Einstein's gravitational theory is

$$S = (1/16\pi G)\int_M([R + 2\Lambda]\sqrt{-g})d^4x + \int_M(L_m\sqrt{-g})d^4x + (1/8\pi G)\int_{\partial M}(\mathrm{tr}K + C)(\sqrt{\pm h})d^3x \qquad (1)$$

where R is the 4-D Ricci scalar , Λ is the cosmological constant, and L_m is the matter Lagrangian. The integration is over an appropriate 4-manifold M with space-time metric g_{ab}, whose determinant is g. The boundary of M is ∂M, where ∂M has induced metric $h_{\mu\nu}$ and extrinsic curvature K_{ab}. The plus (minus) sign in $(\pm h)^{1/2}$ is chosen if the boundary is spacelike (timelike) respectively. C is a function of $h_{\mu\nu}$ alone, chosen so that the metric g satisfies the Einstein equations when S is an extremum under all variations of g which vanish on the boundary. If S is to be the universal action, the appropriate 4-manifold over which the integration is performed must be the whole of space-time, so the boundary terms either are zero or infinite. Therefore, we shall henceforth drop the boundary terms from (1). The fact that the ugly boundary terms can be removed from (1) only if the universal action is finite is itself an argument for the principle of finite action. The universal action will be singularity-free -- that is, finite -- only if each term in (1) is finite. As a general rule, the finiteness of one term in (1) will imply the finiteness of the others because of the field equations, so it will generally (but not always) be sufficient to establish the finiteness of one term in (1) to establish the finiteness of all terms.

If the Universe is roughly homogeneous and isotropic; i.e., if the Copernican Cosmological Principle holds, then it must be spatially closed, else the integration over space by itself would cause all terms in (1) to diverge (assuming that the Lagrangians are not identically zero). In particular, the steady-state theory, both in the original version developed by Bondi, Gold, Hoyle and Narlikar[12], and also in the reincarnation now defended by Gott[13], Guth[14], and Linde[15], has singular universal action, because the steady-state theory in either form is spatially infinite.

But spatial closure is not sufficient for finite action. The action of the static Einstein universe is also infinite because $L_m\sqrt{-g}$, $R\sqrt{-g}$, and $\Lambda\sqrt{-g}$ are independent of time, and so the integrals in (1) diverge when the time integration is carried out. In both the Einstein static universe and the steady-state universe, the absence of singularities in curvature invariants is purchased at the price of a singularity in the universal action.

Consider specifically, an evolving, closed ($k = +1$) Friedman universe with $\Lambda = 0$ and containing a perfect fluid satisfying $p = (\gamma - 1)\mu$, where μ is the density and p is the pressure. In such a spacetime, $R = -8\pi G T^a{}_a \propto \mu - 3p = (\gamma - 4/3)\mu$. In the Friedman models, $\mu \propto a^{-3\gamma}$, where $a(t)$ is the Friedman scale factor. Near the singularity the behaviour of the scale factor a will be asymptotically the same as it is in the $k = 0$ Friedman universe; i.e., $a \propto t^{2/3\gamma}$, where t is the proper time (assuming $\gamma > 2/3$). Thus near each singularity, each term in the action will be proportional to

$\int R\sqrt{-g}d^4x \propto (\gamma - 4/3)\int a^{3(1-\gamma)}dt \propto (\gamma - 4/3)\int t^{2(1-\gamma)/\gamma}dt \propto (\gamma - 4/3)t^{(2-\gamma)/\gamma}$ if $\gamma \neq 2$, and $\int R\sqrt{-g}d^4x \propto \ln[t]$ if $\gamma = 2$. Hence, when $\gamma = 2$, the action diverges at the $t = 0$ limit of integration. If $\gamma \leq 2/3$, the universe expands forever, but if $\gamma > 2/3$, perfect fluid closed Friedman universes begin and end in singularities[16]: that is, the time integration is over a finite range of time. Thus the action is finite if and only if $2/3 < \gamma < 2$.

The important point here is that the finiteness of the action results from the fact that time integration is restricted to a finite range. This finite range in turn is a manifestation of the fact that the closed Friedman universe begins and ends in singularities. In other words, the finiteness of the action depends on the existence of curvature singularities a finite temporal distance in the past and future. Without these curvature singularities, the time evolution would proceed without limit, and this infinity would manifest itself in infinite action. Thus in general, there is a trade-off between space-time singularities: a singularity in the action is avoided only at the price of a singularity in curvature invariants, and vice versa. In cosmology, some sort of singularity seems inevitable. Even an ultimate "Theory of Everything" (TOE) would in general eliminate the classical singularity only at the price of a singular universal action, for if we ignore the possibility of cyclic time because of the second law of thermodynamics, the elimination of the classical singularity in general would mean an infinite 4-volume in which the classical action (1) is an excellent approximation to the ultimate quantum TOE action[17].

The finite action principle in cosmology restricts the allowed forms of matter just as it does in Euclidean Yang-Mills theory. We indicated above that dominant perfect fluids with $\gamma \geq 2$ or $\gamma < 2/3$ near the singularity are ruled out. Scalar fields in closed universes are also ruled out, as one would expect since scalar fields behave like $\gamma = 2$ perfect fluids in the $a \to 0$ limit. For the massless scalar field, $\Phi(t)$, in the Friedman universe ($L_m \propto (d\Phi/dt)^2$), we have near each singularity $d\Phi/dt \propto t^{-1}$ and $a(t) \propto t^{1/3}$ (this is exact in flat models), so that near each singularity the matter term in the action goes as $S_m \propto \int (d\Phi/dt)^2 a^3 dt \propto \ln[t]$, which diverges as $t \to 0$. Massive scalar fields also give a divergent S_m, since the mass term is either negligible near the singularity, or causes a bounce[18].

The divergence of the scalar field action near singularities suggests that scalar fields are inappropriate matter sources to test the strong cosmic censorship hypothesis [19], as has recently be done by Christodoulou[20]. The divergence of the action integral is as bad as the shell-crossing singularities generated by collapse of shells of dust[21]. Just as shell-crossing singularities can be removed by a slight change in the equation of state, so we would predict that the violation of cosmic censorship caused by the divergence of scalar fields outside event horizons can be removed by modifying the equation of state to give an action integral that is bounded near a singularity. Further evidence for our prediction is the fact that static, spherically symmetric space-times containing massless scalar fields possess no event horizons, but instead have naked singularities.

Hitherto we have considered only Friedman models, but the addition of anisotropy does not seem to significantly change the above results. For example, for perfect fluids in the Kasner-like (Bianchi Type I) models, we have $a^3 \propto t$ and $\mu \propto a^{-3\gamma} \propto t^{-\gamma}$, so $S_m \propto (\gamma - 4/3)\int \mu t dt \propto t^{2-\gamma}$ if $\gamma \neq 2$, and $\propto \ln t$ if $\gamma = 2$. An S^3 spatial topology Taub-like (Bianchi Type IX) exact solution for $\gamma = 2$ is given in[22]; S_m can be shown to diverge near either singularity. On the other hand, the Kantowski-Sachs $S^2 \times S^1$ homogeneous dust-filled closed universe[23], has finite action.

Some types of matter can only exist in anisotropic models, for example electromagnetic fields. In this case we have $S_m \propto \int (1/2) F_{ab} F^{ab} (-g)^{1/2} dt d^3x \propto \int (B^2 - E^2)(-g)^{1/2} dt d^3x$. For electromagnetic radiation we have $B^2 = E^2$, so $S_m = 0$. The Brill electro-vac universe[24] -- basically a Taub universe (Bianchi Type IX) with electromagnetic fields -- is a model wherein $(B^2 - E^2) \neq 0$, and S_m is finite. Thus in general we would expect electromagnetic cosmological actions to be finite, and similarly for Yang-Mills actions in cosmology, since generally they correspond closely to electromagnetic actions[25].

As with anisotropy, inhomogeneity does not seem to change significantly the conclusions reached in the Friedman models. The dust-filled Tolman[26] (S^3 spatial topology) models, and the Szekeres[27] (S^3 and $S^2 \times S^1$ spatial topology) models both have finite action, whilst the closed $p = \mu$ and massless scalar field gravitational solitons of Belinskii[28] have infinite actions.

The finite action principle is most powerful when confronted with higher order corrections to the Einstein gravitational Lagrangian (1). In fact, it can be used to eliminate most known cosmological models with quadratic gravitational Lagrangians[29], in which $L_g = R$ is replaced by $L_{quad} = R + \alpha R^2 + \beta R_{ab}R^{ab} + \delta R_{abcd}R^{abcd}$. In four-dimensional space-time we may set $\delta = 0$ without loss of generality and in the Friedman models we may also set $\beta = 0$. The resulting theory is well-defined for $\alpha > 0$ (when $\alpha < 0$, standard general relativity is not obtained in the $\alpha \to 0$ limit, and in any case S is infinite[29]). When $\alpha \neq 0$, there exists a class of closed Friedman universe solutions containing a perfect fluid with $\gamma > 4/3$, and having $a(t) \propto t^{4/3\gamma}$ as $t \to 0$, so S_m diverges. There also exists a class of vacuum closed Friedman universes with $a(t) \propto t^{1/2}$ as $t \to 0$. These have finite action. The zero action solution of general relativity ($\alpha = 0$), obtained for $\gamma = 4/3$, is also a special exact solution of the quadratic theory, but this solution is unstable as $t \to 0$. The general behaviour of the R^2 Friedman universes is not presently known.

The fact that the universal action will be finite when the Copernican Cosmological Principle holds only if the Universe is closed in both space and time imposes strong constraints on the spatial topology. If the standard energy conditions hold and if the initial and final singularities are sufficiently strong, then a maximal hypersurface -- a time of maximum expansion -- must occur in any universe with a compact Cauchy surface[2,16,30,31]. The topology of a non-flat maximal hypersurface in a closed universe satisfying the weak energy condition is strongly restricted[2,30,31]; in fact, the only simple topologies allowed are S^3 and $S^2 \times S^1$.

In summary, the principle of finite action is as useful in cosmology as it is in classical Yang-Mills theory: it provides a powerful constraint on admissible matter fields, and if the Copernican Cosmological Principle holds, it requires the universe to be spatially closed, have special spatial topology, and be bounded in time by singularities. Conversely, a universe with nonsingular curvature invariants must be be singular in its most fundamental invariant, the action.

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