# Skill Flows: A Theory of Human Capital and Unemployment

Ross Doppelt*

June 21, 2012

**Abstract**

I present a theoretical macroeconomic model that captures the fact that temporary job losses lead to life-long earnings losses, the severity of which depends on aggregate labor-market conditions. Workers accumulate both general and match-specific human capital on the job, while suffering human capital depreciation during unemployment. The model features endogenous growth from aggregate human-capital accumulation, business cycles from stochastic productivity shocks, and a time-varying distribution of skills. Learning by doing also changes the wage bargain by making the worker's outside option less attractive. I solve for a competitive equilibrium and derive conditions under which it will be efficient.

*Department of Economics, New York University. Contact: ross.doppelt@nyu.edu. I am grateful to Gadi Barlevy, Jess Benhabib, Joe Briggs, Jeff Campbell, Fatih Guvenen, Alejandro Justiniano, Ezra Oberfield, Tom Sargent, and seminar participants at the Chicago Fed for valuable discussions and criticisms. Any remaining imperfections are my own. I thank the NSF for financial support.

# 1 Introduction

A well-documented fact of the empirical labor literature is that temporary job losses lead to large earnings losses that are highly persistent, if not permanent. Moreover, the severity of these losses depends on aggregate labor-market conditions. This gives rise to the question: How do aggregate labor-market conditions change in an environment where workers are subject to lifetime earnings losses from being unemployed? To provide an answer, I take a stochastic version of the Mortensen-Pissarides (1994) search model and add on-the-job skill formation. When employed, agents accumulate two types of human capital. Some of this human capital is general and will benefit the worker in future jobs, while some human capital is match-specific. Once a worker loses her job, she loses all of her match-specific human capital, and her general skills depreciate as she spends time in unemployment.

This innovation changes two important aspects of the economy, which drive model's predictions at both high and low frequencies. First, human-capital formation has important consequences for the wage bargaining process. The standard model predicts that the worker's share of output depends on aggregate labor-market conditions. When it's easy to find a job, the worker has a valuable outside option, which boosts wages. But when workers accumulate match-specific human capital, the worker's outside option becomes less relevant as she gains tenure. With general human capital, unemployment spells represent not only losses of income, but losses of potential work experience. Consequently, spells of joblessness are especially damaging, so workers have less leverage when bargaining. Skill accumulation will also influence the cyclical properties of the wage bargain. Imagine that firms pay workers for their time, but workers pay firms for the opportunity to gain experience. During a boom, the value of labor to the firm goes up, but so does the value of experience to the worker. Ultimately, wages respond less to aggregate labor-market conditions. In turn, skill accumulation enhances the effects of business-cycle shocks on firms' hiring decisions.

Second, adding human capital to the model also creates an endogenous component to labor productivity. In the long run, skill accumulation leads to endogenous growth, and the trend rate of output growth is negatively correlated with unemployment. Over the business cycle, the employed human capital stock is subject to composition effects; during booms, more aggressive hiring draws in unemployed workers, who have lower skills on average than employed workers. These composition effects are only temporary. A persistent recession can drag down labor productivity because lengthy downturns will degrade the aggregate stock of human capital. In addition, I can characterize the behavior of the entire distribution of human capital. The model suggests that high unemployment can contribute to an increase in skill dispersion.

Beyond the positive features of the model, I solve the planner's problem associated with this economy. In this environment, the planner is not concerned with job flows, per se, but with skill flows; labor-market conditions are only relevant insofar as they determine the evolution of human capital, which is the driving force of output and productivity growth. Nevertheless, it turns out that a version of the Hosios (1990) condition still holds: The planner's allocation will coincide with the market allocation when the elasticity of the matching function with respect to vacancies is exactly equal to the firm's bargaining power. This result is surprising because there is a human-capital externality, as well as a search externality. A worker's stock of general human capital depends upon her complete employment history, so the skills she gains on the job remain relevant long after her match is terminated. Firms do not take this into account when posting vacancies; employers only care about productivity gains made by their own employees on their current jobs. However, the Hosios condition breaks down if we modify the matching process to make the firm's free entry condition depend on the distribution of general human capital in the unemployed population.

I will now review the facts that motivate the model. There is a large literature that measures the earnings losses associated with unemployment.[1] This line of research has consistently found that the damage from unemployment persists long after a worker finds a new job. Figure A.1 reproduces results from Davis and von Wachter (2011); this figure shows the effect of job loss on earnings for men under age 50 with at least three years of job tenure who are separated in a mass layoff event. Year zero on the horizontal axis is the period in which the worker is displaced, and the vertical axis shows earnings losses as a fraction of

[1] Examples include Jacobson, LaLonde, and Sullivan (1993); Couch and Placzek (2010); and von Wachter, Song, and Manchester (2009). See Davis and von Wachter (2011) for a good review.

average pre-displacement earnings. The behavior of a worker's earnings immediately after losing her job is not surprising: Earnings drop precipitously and then rebound. What's striking is the fact that earnings never seem to recover fully from a one-time job loss. Moreover, the magnitude of lost earnings depends upon aggregate labor-market conditions. During recessions, when the average unemployment duration is long, the drop in earnings is more severe. This earnings profile is consistent with the notion that workers become more productive as they gain experience: An unemployment spell represents a loss of potential experience and a diminution of human capital. And, the longer the unemployment spell, the greater is the loss in human capital.[2]

At the macro level, the data display a negative correlation between unemployment and productivity growth at low frequencies. Figure A.2 plots Hodrick-Prescott trends for the unemployment rate and the growth rate of aggregate labor productivity. The correlation coefficient between these series is -.5895. Other authors have also presented cross-country evidence on the negative correlation between unemployment and trend growth; see, for example, Pissarides and Vallanti (2007). This correlation is often interpreted to mean that productivity gains drive down unemployment, but it's not clear that the direction of causality runs in only one direction. If the individual earnings dynamics in Figure A.1 were driven by on-the-job accumulation of human capital, then one would expect aggregate productivity growth to go up when unemployment goes down because the aggregate stock of human capital grows more quickly when more people are employed.

A final pattern to note is the upward trend in wage dispersion. Inequality has increased markedly in recent decades. Figure A.3 plots the log coefficient of variation in average hourly wages for both men and women. Not only does the coefficient of variation increase over time, it appears to exhibit geometric growth. This geometric growth is evident for the entire period 1961-2002 for men and after 1980 for women. At the same time, the distribution

[2]It's also possible that firms fire their least productive workers. However, Davis and von Wachter use worker fixed effects, which should capture such selection effects. Plus, if firms did select their least productive workers to be laid off, it's reasonable to think that the workers who lose their jobs in expansions are of even lower quality than those who lose their jobs during recessions, yet the workers who lose their jobs in expansions suffer less severe earnings losses. From a theoretical standpoint, a story about skill accumulation is not mutually exclusive with a story about worker selection; I study the former now and defer the latter to future research.

of skills, as measured by education and experience, has become more spread out. No cyclical pattern is evident in Figure A.3, but as other authors have documented, alternative measures of dispersion show that earnings inequality becomes more severe during recessions, when unemployment is high. See, for example, Krueger et al. (2010) and Heathcote et al. (2010). There is reason to think that the relationship between unemployment and wage dispersion is related to the earnings dynamics in Figure A.1: When workers become unemployed, they experience a drop in earnings power, so as displaced workers find new jobs, they cause the distribution of wages to fan out.

In addition to being consistent with the aforementioned facts, the model that I build makes contributions to several areas of the theoretical literature. First, numerous authors have debated whether rapid trend growth will increase or decrease steady-state unemployment. The two main theories in this debate are associated with Aghion and Howitt (1994) and Mortensen and Pissarides (1998).[3] Both pairs of authors posit that technology follows an exogenous trend and examine how this trend influences labor demand. In Aghion and Howitt's model, technology is embodied in a worker-firm match, so the productivity of a worker hired at date $t$ is frozen at the date-$t$ level of technology until the match is dissolved. Rapid growth generates a "creative destruction" effect, where firms dispense with workers after a shorter amount of time in order to take advantage of new technology. In Mortensen and Pissarides's model, technology is disembodied, so the productivity of all workers is the same and constantly growing. Rapid growth generates a "capitalization" effect, where firms are more willing to pay the up-front cost of hiring in order to reap the benefits of better technology in the future. More recently, Elsby and Shapiro (2011) pointed out that technological growth compounds the returns to experience that workers would experience in a static economy, so trend growth increases labor supply.

One unsatisfying feature of all three models is that growth is exogenous, so it is assumed that growth determines the rate of unemployment. However, going back to Lucas (1988, 1993), growth theory has tried to incorporate the aggregate implications of workers becoming more productive by accumulating human capital on the job. This premise suggests that

[3] See chapter three of Pissarides (2000) for a summary and additional references.

unemployment could influence growth. Laing et al. (1995) and Chen et al. (2011) have models of endogenous growth with frictional labor markets, schooling, and human-capital accumulation. But these, too, leave some important issues unresolved. Unemployment arises in Laing et al. only as workers leave school and search for their first jobs, which the workers hold for the rest of their lives. In Chen et al., human capital is a public good that is not embodied in individual workers. Therefore, neither model can capture the earnings dynamics of Figure A.1. In contrast, I model how the human capital of individual workers changes during spells of employment and unemployment. Then, by aggregating, I highlight the direct link between labor-market conditions and the (endogenous) rate of growth.

This model also contributes to the literature on the role of human capital in labor-market dynamics. One branch of this literature is focused on the distribution of wages in the steady state. Recent examples include Burdett and Coles (2010), Burdett et al. (2010), and Carrillo-Tudela (2010). These models feature more sophisticated wage-determination mechanisms and generate elegant predictions about the distribution of wages and skills. However, these models are not well suited for looking at business cycles. Wages in this paper will be determined by Nash bargaining, which will allow me to analyze how human capital interacts with aggregate shocks. In addition, I can look at the separate effects of general and match-specific human capital and how they change over the cycle. Another branch of literature is focused on the determination of worker flows; examples include Ljungqvist and Sargent (1998, 2008), Pissarides (1992), and Esteban-Pretel (2007), amongst others. These papers generate their results by using numerical techniques (Esteban-Pretel), abstracting from the process that drives aggregate shocks (Pissarides), or both (Ljungqvist and Sargent). I complement this line of research by constructing a tractable environment that lends itself to pen-and-paper solutions but is still rich enough to feature endogenous growth, business cycles, and an evolving skill distribution.

I will proceed as follows. Section 2 contains the model. Section 3 defines an equilibrium and proves the equilibrium's existence and uniqueness. Section 4 analyzes labor-market dynamics, in particular wage determination and vacancy creation. Section 5 analyzes productivity and growth dynamics by looking at the behavior of aggregate human-capital flows.

Section 6 examines the distributional implications of the model. Section 7 discusses the welfare properties of the model. Section 8 discusses possibilities for future research and concludes. Proofs are in Appendix B.

# 2 The Model

## 2.1 Technology

There is a continuum of workers indexed by $i \in [0,1]$. Agent $i$ has two types of human capital: general and match-specific. Let $x^i$ be the stock of general human capital of agent $i$, and let $y^i$ be the stock of match-specific human capital for agent $i$.[4] Almost everywhere, an individual's general human-capital stock evolves according to:

$$\frac{\dot{x}^i}{x^i} = e^i \alpha - \left(1 - e^i\right)\delta, \tag{2.1}$$

where $e^i \in \{0,1\}$ is an indicator variable for whether agent $i$ is employed:

$$e^i = \begin{cases} 1 & \text{if agent } i \text{ is employed} \\ 0 & \text{if agent } i \text{ is unemployed.} \end{cases} \tag{2.2}$$

In other words, when an agent is employed, her stock of general human capital grows at a constant rate $\alpha$; when unemployed, her stock of human capital decays at rate $\delta$. This geometric growth of human capital is like the process used by Burdett et al. (2010), except with skill depreciation during unemployment. In addition, when an agent loses her job, she instantly loses a fraction $\zeta$ of her general human capital stock; this instant depreciation of human capital is like the "microeconomic turbulence" at work in Ljungqvist and Sargent (1998). I will maintain the assumption that $\alpha - \lambda\zeta \geq 0$; this assures that, in expectation, a worker does not lose general human capital by accepting a job. All unemployed agents have match-specific productivity $y^i = 1$; then, once a worker actually matches with a firm, her

[4] Because all variables will change over time, I will tend to omit time subscripts, except where they are needed for clarity.

match-specific human capital evolves according to:

$$\frac{\dot{y}^i}{y^i} = e^i \rho. \tag{2.3}$$

When a worker loses her job, $y^i$ resets to one. Define a worker's total stock of human capital as:

$$k^i = x^i y^i. \tag{2.4}$$

As I'll show in Section 4, an agent's earnings will be linear in $x^i$ and $k^i$. Consequently, this specification for human capital accumulation is consistent with the earnings behavior in Figure A.1. I will not take a stand on the relative importance of each component of the skill-accumulation process. If we just wanted to create earnings losses from job loss, it would suffice to model either $x^i$ or $y^i$, not both. However, I will argue that these different forms of skill play different roles in determining wages and shaping the distribution of earnings. Similarly, a permanent earnings loss could be captured by either instant depreciation ($\zeta$) or gradual decay ($\delta$) of general human capital.[5] I include both because they have different implications for aggregate labor-market dynamics, and including both allows for comparability with other authors.

There is an aggregate productivity variable that follows a two-state Markov switching process.[6] Denote the exogenous state variable by $s \in \{0, 1\}$. At Poisson rate $\beta$, a business-cycle shock arrives, and $s$ switches values. Aggregate productivity is given by $z_s$, where $z_0 < z_1$. There is one worker per firm, which produces goods with linear technology. That is, the flow output of a firm matched with worker $i$ in state $s$ is $z_s k^i$.

Matching is standard, following Mortensen and Pissarides (1994). Let $\theta$ be the vacancy-unemployment ratio; let $q(\theta)$ be the rate at which a firm finds a worker; let $\theta q(\theta)$ be the rate at which a worker finds a firm; let $\lambda$ be the constant and exogenous rate of job termination.

[5] In fact, we could generate permanent earnings losses without any decay of human capital by setting $\delta = 0$, so skills would simply stagnate during unemployment. We could even set $0 > \delta > -\alpha$, so that agents gained skill during unemployment, as long as they gained skills even more rapidly during employment. This model abstracts from human capital accumulated in school, but it's well known that school enrollment expands during recessions when unemployment is high. Setting $0 > \delta$ might capture some of this learning that takes place off the job.

[6] The model can accomodate a richer Markov process for the aggregate shocks, but specializing to two states provides for a cleaner exposition.

I will maintain the assumption that $\lambda > \rho$. Standard regularity conditions on $q(\cdot)$ apply: I will assume that $q(\cdot)$ is continuous, decreasing, convex, and $\theta q(\theta)$ is strictly increasing. Search is undirected: An agent's stock of human capital affects neither her probability of being matched with an employer when unemployed, nor her chance of being separated when employed. Aggregate employment is given by $e \equiv \int e^i di$. Given a level of market tightness $\theta$, the law of motion for $e$ is:

$$\dot{e} = \theta q(\theta)(1-e) - \lambda e. \tag{2.5}$$

Define the aggregate stocks of human capital for the employed and unemployed populations as $x^e \equiv \int x^i e^i di$, $x^u \equiv \int x^i \left(1 - e^i\right) di$, $y \equiv \int y^i e^i di$, and $k \equiv \int k^i e^i di$. Notice that total output will be given by $z_s k$. Likewise, we can define the average stocks of human capital for the employed and unemployed as $\bar{x}^e \equiv \frac{x^e}{e}$, $\bar{x}^u \equiv \frac{x^u}{1-e}$, $\bar{y} \equiv \frac{y}{e}$, and $\bar{k} \equiv \frac{k}{e}$. Given a level of market tightness $\theta$, the laws of motion for $x^e$, $x^u$, $k$, and $y$ are:

$$\begin{aligned}
\dot{x}^e &= \alpha x^e - \lambda x^e + \theta q(\theta) x^u & (2.6)\\
\dot{x}^u &= -\delta x^u - \theta q(\theta) x^u + \lambda(1-\zeta) x^e & (2.7)\\
\dot{k} &= (\alpha + \rho) k - \lambda k + \theta q(\theta) x^u & (2.8)\\
\dot{y} &= \rho y - \lambda y + \theta q(\theta)(1-e). & (2.9)
\end{aligned}$$

Equation (2.6) shows that changes in $x^e$ come from two sources: (1) Individual employees gaining skills and (2) the movement of workers into and out of unemployment. In a given instant, the stock of employed general human capital expands by $\alpha x^e$ from on-the-job accumulation of human capital. Simultaneously, employed workers lose their jobs at rate $\lambda$, and their human capital becomes unemployed. Hence, job destruction causes the stock of employed general human capital to decline by $\lambda x^e$. Meanwhile, unemployed workers find jobs at rate $\theta q(\theta)$, so the stock of employed human capital is augmented by $\theta q(\theta) x^u$. Similar logic explains the components of equations (2.7), (2.8), and (2.9).

## 2.2 Workers

Following Burdett and Coles (2010) and Burdett et al. (2010), I will assume that an unemployed worker gets utility from leisure (or value from home production) of $bx^i$, where $b < z_0$.[7] Denote by $W_s\left(x^i, y^i\right)$ the flow earnings of an employed agent in state $s$ with human capital $\left(x^i, y^i\right)$. All agents have linear utility and discount future payoffs at constant rate $r$. Goods are non-storable, so there is neither borrowing nor saving. To ensure boundedness of payoffs, I will maintain the assumption that $r > \alpha$. A worker at time $t_0$ seeks to maximize:

$$\mathbb{E}_{t_0}\left[\int_{t=0}^{\infty} \exp\{-tr\}\left[e^i_{t_0+t} W_{s(t_0+t)}\left(x^i_{t_0+t}, y^i_{t_0+t}\right) + \left(1 - e^i_{t_0+t}\right) bx^i_{t_0+t}\right] dt\right], \tag{2.10}$$

where $s(t_0+t)$ is the exogenous state at time $t_0+t$. The only choice facing a worker is whether or not to work when matched with a firm; wages are determined by Nash bargaining, so in equilibrium, it will be the case that the employee always chooses to work. Let $U_s\left(x^i\right)$ denote the value that agent $i$ associates with being unemployed in state $s$ with human capital $x^i$. Let $H_s\left(x^i, y^i\right)$ denote the value that agent $i$ associates with being employed in state $s$ with human capital $\left(x^i, y^i\right)$. As will be made clear in Section 3, I will seek a recursive equilibrium in which market tightness and the wage function are constant within each exogenous productivity regime $s$. Agent $i$'s Bellman equation when unemployed in state $s$ is:

$$rU_s\left(x^i\right) = bx^i + \beta\left[U_{1-s}\left(x^i\right) - U_s\left(x^i\right)\right] + \theta_s q\left(\theta_s\right)\left[H_s\left(x^i, 1\right) - U_s\left(x^i\right)\right] + U_s'\left(x^i\right)\dot{x}^i. \tag{2.11}$$

[7] The flow value of leisure is scaled by $x^i$ for three reasons. First, the economy will be growing in the long run, so after enough time, $b$ would play no virtually role in the agent's problem if it were not normalized by something that is, on average, growing. Second, we might think of $b$ as standing in for unemployment benefits, which are typically designed to be an increasing function of the wages a worker would be making if employed. Third, this assumption makes the model solution much more tractable.

Agent $i$'s Bellman equation when employed in state $s$ is:

$$\begin{aligned} rH_s\left(x^i, y^i\right) &= W_s\left(x^i, y^i\right) + \beta\left[H_{1-s}\left(x^i, y^i\right) - H_s\left(x^i, y^i\right)\right] \\ &\quad + \lambda\left[U_s\left(x^i\left(1-\zeta\right)\right) - H_s\left(x^i, y^i\right)\right] \\ &\quad + \frac{\partial H_s\left(x^i, y^i\right)}{\partial x^i}\dot{x}^i + \frac{\partial H_s\left(x^i, y^i\right)}{\partial y^i}\dot{y}^i. \end{aligned} \tag{2.12}$$

## 2.3 Firms and Bargaining

To search for a worker to hire, a potential firm owner can post a vacancy at flow cost $\kappa\bar{x}^u$.[8] Let $G_s\left(x^i, y^i\right)$ denote the value that a firm owner associates with employing worker $i$ in state $s$, and let $V_s$ denote the the value that a potential firm owner associates with opening a vacancy in state $s$. The Bellman equation for a firm owner is:

$$\begin{aligned} rG_s\left(x^i, y^i\right) &= z_s x^i y^i - W_s\left(x^i, y^i\right) + \beta\left[G_{1-s}\left(x^i, y^i\right) - G_s\left(x^i, y^i\right)\right] \\ &\quad + \lambda\left[V_s - G_s\left(x^i, y^i\right)\right] + \frac{\partial G_s\left(x^i, y^i\right)}{\partial x^i}\dot{x}^i + \frac{\partial G_s\left(x^i, y^i\right)}{\partial y^i}\dot{y}^i. \end{aligned} \tag{2.13}$$

The Bellman equation for a potential firm owner posting a vacancy is:

$$rV_s = -\kappa\bar{x}^u + q\left(\theta_s\right)\left[\frac{\int G_s\left(x^i, 1\right)\left(1-e^i\right)di}{\int\left(1-e^i\right)di} - V_s\right] + \beta\left[V_{1-s} - V_s\right] + \dot{V}_s. \tag{2.14}$$

Free entry requires that firms have zero expected profit from posting a vacancy:

$$V_s = \dot{V}_s = 0, \ \forall s. \tag{2.15}$$

Workers and firms engage in Nash bargaining to determine $W_s\left(x^i, y^i\right)$; workers have bargaining power $\eta \in [0, 1]$. Axiomatic Nash bargaining yields the usual first-order condition:

[8] There are two reasons for scaling the vacancy cost by the average human capital stock of the unemployed. First, this is a balanced-growth assumption that appears in all models that have both growth and unemployment: With long-run growth, it's necessary to scale the vacancy cost by the overall sophistication of the economy. Otherwise, the vacancy cost would become trivial relative to output, vacancies would grow arbitrarily large, and unemployment would tend to zero. Second, this normalization ensures a one-to-one mapping from the exogenous state $s$ to market tightness $\theta_s$. In principle, this assumption could be relaxed, but the model could not be solved by hand. I will revisit this assumption in Section 7.

$$\eta G_s\left(x^i, y^i\right) = (1-\eta)\left[H_s\left(x^i, y^i\right) - U_s\left(x^i\right)\right]. \tag{2.16}$$

# 3 Equilibrium

The features of the model lend themselves to a tractable solution. I will seek an equilibrium that is block recursive. That is, market tightness $\theta$ will be constant within each exogenous productivity regime $s$, so the job-finding rate and the vacancy-filling rate will follow a Markov switching process. The market tightness function $\theta_s$ will not depend on the distribution of human capital across workers; however, once we characterize $\theta_s$, we can characterize the evolution of the aggregate endogenous state variables and the skill distribution.

The strategy will be to exploit the homogeneity of the dynamic programming problems, allowing us to replace the partial differential equations with ordinary differential equations. I conjecture that $W_s\left(x^i, y^i\right)/x^i$ does not depend on $x^i$. Then, there exists a function $w_s(\cdot)$ such that $W_s\left(x^i, y^i\right) = w_s\left(y^i\right)x^i$. Moreover, if $\theta_s$ is constant for a given $s$, then we can see that the Bellman equations (2.11), (2.12), and (2.13) have solutions that are homogeneous of degree one in $x^i$; i.e., there exist functions $u_s$, $h_s(\cdot)$, and $g_s(\cdot)$ such that:

$$U_s\left(x^i\right) = u_s x^i \tag{3.1}$$

$$H_s\left(x^i, y^i\right) = h_s\left(y^i\right)x^i \tag{3.2}$$

$$G_s\left(x^i, y^i\right) = g_s\left(y^i\right)x^i. \tag{3.3}$$

We can interpret $u_s$ as the marginal value of general human capital for an unemployed worker in state $s$; we can interpret $h_s\left(y^i\right)$ as the marginal value of general human capital for an employed worker in state $s$ with match-specific human capital $y^i$; and we can interpret $g_s\left(y^i\right)$ as the marginal value of general human capital for a firm in state $s$ paired with a worker with match-specific human capital $y^i$. We can replace (2.11), (2.12), (2.13), and

(2.14) with:

$$
\begin{aligned}
ru_s &= b+\beta\left[u_{1-s}-u_s\right]+\theta_s q\left(\theta_s\right)\left[h_s\left(1\right)-u_s\right]-\delta u_s && (3.4)\\
rh_s\left(y^i\right) &= w_s\left(y^i\right)+\beta\left[h_{1-s}\left(y^i\right)-h_s\left(y^i\right)\right] \\
&\quad +\lambda\left[\left(1-\zeta\right)u_s-h_s\left(y^i\right)\right]+\alpha h_s\left(y^i\right)+h_s'\left(y^i\right)\dot{y}^i && (3.5)\\
rg_s\left(y^i\right) &= z_s y_i-w_s\left(y^i\right)+\beta\left[g_{1-s}\left(y^i\right)-g_s\left(y^i\right)\right] \\
&\quad -\lambda g_s\left(y^i\right)+\alpha g_s\left(y^i\right)+g_s'\left(y^i\right)\dot{y}^i && (3.6)\\
\kappa &= q\left(\theta_s\right)g_s\left(1\right). && (3.7)
\end{aligned}
$$

The homogeneity of the Bellman equations implies that the Nash-bargaining condition becomes:

$$\eta g_s\left(y^i\right)=\left(1-\eta\right)\left[h_s\left(y^i\right)-u_s\right]. \tag{3.8}$$

We are now prepared to define a competitive equilibrium where $U_s\left(x^i\right)$, $G_s\left(x^i,y^i\right)$, $H_s\left(x^i,y^i\right)$, and $W_s\left(x^i,y^i\right)$ are homogeneous in $x^i$, and market tightness is a function only of the exogenous state $s$.

**Definition 1.** A recursive homogeneous equilibrium is:

1. A wage function $w_s\left(y^i\right)$
2. A market-tightness function $\theta_s$
3. Value functions $u_s$, $g_s\left(y^i\right)$, and $h_s\left(y^i\right)$
4. Random variables $e$, $x^e$, $x^u$, $y$, and $k$

such that:

1. The value functions satisfy (3.4), (3.5), and (3.6)
2. Free entry (3.7)
3. Nash bargaining (3.8)
4. $e$, $x^e$, $x^u$, $k$, and $y$ evolve according to (2.5), (2.6), (2.7), (2.8), and (2.9).

**Theorem 2.** *There exists a unique homogeneous equilibrium.*

*Proof.* See appendix. □

# 4 Labor-Market Dynamics

We can now characterize market tightness and the wage function in a recursive homogeneous equilibrium. The first thing to note is that market tightness is positively correlated with the aggregate productivity variable.

**Proposition 3.** *Market tightness will be procyclical; i.e., $\theta_1 > \theta_0$. Moreover, if $-\theta\frac{q'(\theta)}{q(\theta)}$ is weakly decreasing in $\theta$, then $\frac{\partial\theta}{\partial z}$ is an increasing function of $\alpha$, $\rho$, and $\delta$, whereas $\frac{\partial\theta}{\partial z}$ is a decreasing function of $\zeta$.*

*Proof.* See appendix. □

It is interesting to see how on-the-job accumulation of human capital relates to two prominent criticisms of the baseline Mortensen-Pissarides model. The first criticism is that the baseline model over-predicts the sensitivity of employee compensation to aggregate labor-market conditions. The second, related criticism is that the baseline model under-predicts the sensitivity of market tightness to productivity shocks. These arguments are due to Shimer (2005) and Hall (2005). Those authors claim that when a positive productivity shock hits the Mortensen-Pissarides economy, the value of being unemployed increases because it becomes easier to find a job. This improves the bargaining position of workers, so wages absorb most of the productivity gain. Consequently, profits go up only modestly, so firm owners do not have a strong incentive to post additional vacancies.

In the present model, hiring remains procyclical, but the degree of procyclicality depends on the parameters of the skill-acquisition process. Faster human-capital accumulation (higher $\alpha$ or $\rho$) will make market tightness more sensitive to changes in aggregate productivity. Interestingly, though, human-capital depreciation will increase the cyclical sensitivity of market tightness if human capital decays continuously during unemployment (higher $\delta$), but market tightness becomes less cyclically sensitive if human capital evaporates instantly with job loss (higher $\zeta$). It is easiest to interpret these results in the context of a social planner's problem.[9] An aggregate productivity shock makes all workers' human capital more productive. So, if workers accumulate human capital quickly, then there is more value in putting

[9]In Section 7, I will solve the planner's problem in detail.

people to work during a boom. Similarly, if human capital decays continuously during unemployment, it becomes more important to keep people out of unemployment when human capital is most productive. This is not the case when human capital depreciates instantly upon job loss. In that case, human capital is not lost from people being unemployed; rather, human capital is lost when people transition from employment to unemployment. Hence, if job loss leads to a sudden drop in human capital, it is better to have an uninterrupted unemployment spell than an unemployment spell of equal duration interrupted by a very brief employment spell.

Let's consider how human-capital accumulation affects the wage-determination process. We see that on-the-job learning creates a wedge in the wage equation.

**Proposition 4.** *The wage equation in a recursive homogeneous equilibrium is given by:*

$$w_s\left(y^i\right) = \eta z_s y^i + (1-\eta)\, b + \eta\kappa\theta_s - (1-\eta)\,(\alpha+\delta+\lambda\zeta)\, u_s. \tag{4.1}$$

*Proof.* See appendix. □

If we fix $y^i = 1$ and $\alpha = \delta = \zeta = \rho = 0$, then the above is just like the wage equation that appears in a standard Mortensen-Pissarides model. The static Nash outcome is a convex combination of the firm's reservation ($z_s$) and the worker's reservation ($b$); there is also a search wedge given by $\eta\kappa\theta_s$, representing the worker's outside option of continuing to search. For non-zero values of $\alpha$, $\delta$, $\zeta$, and $\rho$, there is a new wedge given by $-(1-\eta)(\alpha+\delta+\lambda\zeta)\, u_s$. This learning-by-doing wedge represents the fact that giving a worker employment not only gives her an instantaneous flow of wages, it also gives her more human capital, which will make her better off if she suddenly separates from her job. It makes sense, then, that the wedge is equal to an unemployed worker's marginal value of general human capital $u_s$, scaled by the firm's bargaining power $(1-\eta)$ and the rate of human-capital accumulation the worker enjoys plus the rate of human-capital depreciation the worker avoids $(\alpha+\delta+\lambda\zeta)$. It's almost as though firms pay employees for the value of their time, employees pay firms for the value of the experience they gain, and the difference is the wage we observe.

To contrast the present model with a textbook Mortensen-Pissarides model, we can think about general human capital as creating an impatience effect for unemployed workers. It is instructive to consider the case where $\zeta = \rho = 0$ and $y^i = 1$. Then, the Bellman equations (3.4), (3.5), and (3.6) become:

$$
\begin{aligned}
(r+\delta)\, u_s &= b + \beta\left[u_{1-s} - u_s\right] + \theta_s q\left(\theta_s\right)\left[h_s(1) - u_s\right] && (4.2)\\
(r-\alpha)\, h_s(1) &= w_s(1) + \beta\left[h_{1-s}(1) - h_s(1)\right] + \lambda\left[u_s - h_s(1)\right] && (4.3)\\
(r-\alpha)\, g_s(1) &= z_s - w_s(1) + \beta\left[g_{1-s}(1) - g_s(1)\right] - \lambda g_s(1)\,. && (4.4)
\end{aligned}
$$

The above equations, combined with (3.7) and (3.8), look just like the equations that characterize an equilibrium in a textbook Mortensen-Pissarides model, except with one difference: It appears as though employed workers and firm owners discount the future at rate $r-\alpha$, whereas unemployed workers discount the future more heavily at rate $r+\delta$. In other words, the unemployed behave as though they are more impatient than their employed selves and their prospective employers. Moreover, the difference in the effective rate of time preference equals the difference in human-capital growth rates $\alpha+\delta$. I emphasize the differing rates of effective time preference because bargaining theory suggests that impatient agents are at a disadvantage.[10] Because being unemployed carries the cost of foregone human-capital accumulation, an unemployed worker is more eager to arrive at a bargain over wages, and the magnitude of this impatience effect is governed by the potential for on-the-job skill formation.

The model also shows us how match-specific human capital also makes the worker's outside option less relevant in determining wages over the life of a job. Recall that $w_s\left(y^i\right)$ is the worker's earnings per unit of general human capital, so total earnings will be equal to

[10] Rubinstein (1982) showed that the more impatient agent in a non-cooperative bargaining game will receive a lower payout. Binmore, Rubinstein, and Wolinsky (1986) showed how Rubinstein's alternating-offers game produces outcomes equivalent to axiomatic Nash bargaining, which determines wages in the present model. In addition, a higher discount rate is equivalent to having a lower degree of bargaining power, captured by $\eta$.

$w_s\left(y^i\right)x^i$. Thus, the worker's share of output with tenure $t$ will be given by:

$$\text{Worker's share in state } s \text{ with tenure } t = \eta + \frac{(1-\eta)\,b + \eta\kappa\theta_s - (1-\eta)\,(\alpha+\delta+\lambda\zeta)\,u_s}{z_s \exp\{\rho t\}}. \tag{4.5}$$

The cyclically-varying objects in the above expression are $\theta_s$, $z_s$, and $u_s$, but as tenure grows long, these factors receive less weight in determining how much of the produce from the match goes to the worker. In fact, as a worker's tenure goes to infinity, the worker's share converges to $\eta$, the static Nash outcome. After workers build up a lot of match-specific human capital, the search wedge vanishes from the worker's share because looking for employment elsewhere would amount to starting over in a less productive job. The learning-by-doing wedge also vanishes, because a growing fraction of the worker's skills are exclusively valuable to her current employer.

Finally, let's look at how general human-capital accumulation affects the cyclical properties of wages. Notice that the magnitude of the learning-by-doing wedge in equation (4.1) is procyclical: It's better to be unemployed in a boom than in a recession, so $u_1 > u_0$.[11] This suggests that the learning-by-doing wedge blunts the cyclicality of $w_s\left(y^i\right)$. When a positive productivity shock hits the economy, $z_s$ and $\theta_s$ increase, which puts upward pressure on wages. However, $u_s$ also increases, which puts downward pressure on wages. To be more precise, we can decompose cyclical movements in wages into two components: cyclical changes in labor productivity and cyclical changes in market tightness.

**Proposition 5.** *The cyclical change in wages can be decomposed as:*

$$w_1\left(y^i\right) - w_0\left(y^i\right) = \eta y^i\left(z_1 - z_0\right) + \eta\kappa\left[1 - \frac{(\alpha+\delta+\lambda\zeta)\,(r+\delta)}{r+\delta+2\beta}\right](\theta_1-\theta_0)\,. \tag{4.6}$$

*Proof.* See appendix. □

We see that human capital dampens the contribution of aggregate labor-market conditions to the cyclical change in wages. The contribution of aggregate labor-market conditions will be determined by the quantity multiplying $(\theta_1 - \theta_0)$ in equation (4.6). If workers gain

[11]This can be seen formally from the characterization of $u_s$ given in the Appendix.

skills quickly on the job (i.e. if $\alpha$ is large), or if job loss leads to a large loss of human capital (i.e. if $\zeta$ or $\delta$ is large), then the quantity multiplying $(\theta_1 - \theta_0)$ decreases. In the standard model without human-capital accumulation, workers are in a better bargaining position during booms because it would be easier for them to find another job if they chose to quit; this fact allows them to extract higher wage payments. However, the value of the experience workers gain on the job will also rise during a boom, which offsets the importance of the worker's outside option.

## 5 Productivity and Growth Dynamics

The dynamics of productivity and growth will be determined by the dynamics of aggregate human capital. Now that we've characterized the behavior of market tightness, we can see how output $z_s k$ moves immediately after a shock. Suppose $s$ switches from the low state to the high state. This has two immediate impacts on $z_s k$. First, the increase in $z_s$ causes output to jump up discontinuously. Secondly, the increase in $z_s$ causes the hiring rate $\theta_s q(\theta_s)$ to increase; as can be seen from equation (2.8), the stock of employed human capital $k$ accelerates as unemployed workers find jobs at a faster rate. In the short run, both of these effects would be present in a model without skill accumulation: If the stock of human capital in the economy were just equal to the number of bodies in the labor force, we would still expect a positive productivity shock to drive down the unemployment rate, thereby driving up the level of output. In this model, however, the elevated hiring rate leads to the accumulation of new human capital. Thus, the dynamics of human capital accumulation play an important role over longer horizons.

Let's begin by looking at general human capital, because the accumulation of these skills drives long-run growth. Recall that $x^e$ and $x^u$ evolve according to equations (2.6) and (2.7). Defining $\mathbf{x} \equiv (x^e, x^u)'$, we can write this concisely as a linear system of ordinary differential equations:

$$\dot{\mathbf{x}} = \mathbf{Q}_s \mathbf{x} \tag{5.1}$$

$$\mathbf{Q}_s \equiv \begin{bmatrix} \alpha - \lambda & \theta_s q(\theta_s) \\ (1-\zeta)\lambda & -[\delta + \theta_s q(\theta_s)] \end{bmatrix}. \tag{5.2}$$

The nice thing about the coefficient matrix $\mathbf{Q}_s$ is that the coefficients are constant within each particular productivity regime. So, almost everywhere, we can characterize aggregate human capital flows using the typical tools for linear differential equations.

**Proposition 6.** *Let $t_n$ be the time of the $n^{th}$ switch in the exogenous state $s$. For $t \leq t_{n+1} - t_n$, the path of human capital is given by:*

$$\mathbf{x}_{t_n+t} = \Omega_s \text{diag}\left(\exp\{\gamma_s t\}\right) \Omega_s^{-1} \mathbf{x}_{t_n}, \tag{5.3}$$

*where $\gamma_s$ is a vector containing the eigenvalues of $\mathbf{Q}_s$, and $\mathbf{\Omega}_s$ is an orthonormal matrix, the columns of which are the corresponding eigenvectors.*

*Proof.* See appendix. □

**Definition 7.** The trend growth rate in state $s$, denoted $\tau_s$, is the maximal eigenvalue of $\mathbf{Q}_s$:

$$\tau_s \equiv \max_i \{\gamma_{i,s}\}. \tag{5.4}$$

Proposition 6 delivers an analytic solution for the path of general human capital. We see that $x^e$ and $x^u$ can be written as a linear combination of two geometrically growing variables, and the rates of geometric growth will be given by the eigenvalues of $\mathbf{Q}_s$. This fact motivates the definition of trend growth: If the economy remains in state $s$ for a sufficiently long time, the growth rate of general human capital will converge to $\tau_s$, the maximal eigenvalue of $\mathbf{Q}_s$. The law of motion for $k$ (2.8) makes it clear that the growth rate of output will converge to $\tau_s$ as well.[12]

[12] It's straightforward to compute the exact path of $k$ by writing $\left(\dot{x}^e, \dot{x}^u, \dot{k}\right)$ jointly as a linear system of

**Proposition 8.** *Trend growth $\tau_s$ is well defined and an increasing function of $s$. If $\alpha > \lambda + \delta + \theta_s q(\theta_s)$, then $\tau_s > 0$. Otherwise, the sign of trend growth in state $s$ is given by:*

$$\tau_s \gtreqless 0 \iff \theta_s q(\theta_s) \gtreqless \left(\frac{\lambda - \alpha}{\alpha - \zeta\lambda}\right)\delta. \tag{5.5}$$

*Proof.* See appendix. □

Proposition 8 illustrates a direct link between labor-market conditions and the economy's capacity for growth. When a shock induces a change in labor-market tightness, it affects the average length of an unemployment spell, which is given by $[\theta_s q(\theta_s)]^{-1}$. In turn, the average unemployment duration determines how much human capital a worker can expect to lose when she is separated from her job. Furthermore, aggregate human capital will be non-stationary: Switches in the *level* of $z_s$ induce changes in the *growth rates* of $x^e$, $x^u$, and $k$. Besides propagating business-cycle shocks, human-capital accumulation implies a negative relationship between steady-state unemployment and long-run growth: An economy with a higher job-finding rate will have both lower unemployment and a higher growth rate of human capital. Proposition 8 gives the exact conditions under which human capital accumulation can generate endogenous growth. Suppose that $\lambda > \alpha$.[13] Then, then the hiring rate needs to exceed a certain threshold $\left(\frac{\lambda-\alpha}{\alpha-\zeta\lambda}\right)\delta$ in order for trend growth to be positive. This threshold is lowered by faster individual skill accumulation (higher $\alpha$), and the threshold is raised by shorter employment spells (higher $\lambda$) and greater skill depreciation (higher $\delta$ or $\zeta$). If the hiring rate is too low, then the labor market is too weak to afford workers the opportunity to accumulate skills, so the economy cannot sustain positive growth. In principle, it's possible to have $\theta_1 q(\theta_1) > \left(\frac{\lambda-\alpha}{\alpha-\zeta\lambda}\right)\delta > \theta_0 q(\theta_0)$. In that case, the economy alternates between states of growth and depression.

Human capital also adds an endogenous component to labor productivity. Output per worker in this economy is given by $z_s\bar{k}$. Combining the employment law of motion (2.5)

differential equations. Likewise, $(\dot{y}, \dot{e})$ jointly form an affine system of differential equations, which also has an analytical solution; whereas $(x^e, x^u, k)$ can be explosive, $(y, e)$ is stable.

[13] This is a very reasonable supposition. For instance, when time is measured in years, an average job spell of ten years would correspond to $\lambda = .1$, so assuming that $\lambda > \alpha$ would mean that general skills of employed workers grow at less than 10% per year.

with the law of motion for employed human capital (2.8) yields:

$$\frac{\dot{\bar{k}}}{\bar{k}} = \alpha + \rho - \theta_s q\left(\theta_s\right)\left(\frac{1-e}{e}\right)\left(\frac{\bar{k}-\bar{x}^u}{\bar{k}}\right) \tag{5.6}$$

In the short run, we see how endogenous composition effects influence productivity dynamics immediately after an exogenous shock. We would typically expect that there is a higher average skill level amongst the employed, relative to the unemployed; i.e. $\bar{x}^u < \bar{k}$.[14] Suppose that this is the case, and there is a positive productivity shock. Then, equation (5.6) shows that an increase in the job-finding rate $\theta_s q\left(\theta_s\right)$ will induce the average human capital of the employed population $\bar{k}$ to decelerate. So, if exogenous productivity shifts from $z_0$ to $z_1$, then labor productivity will jump up discontinuously, but after this initial jump, productivity will slow down. The reason is that firms begin hiring more aggressively, which pulls less skilled workers into the labor force. One can see the same composition effects when looking at $\dot{\bar{y}}/\bar{y}$ or at $\dot{\bar{x}}^e/\bar{x}^e$. Consequently, given the wage equation (5.6), there is a composition effect for average wages as well. In the long run, though, the growth rate of labor productivity will be equal to $\tau_s$.

# 6 Characterizing the Skill Distribution

Because the wage received by agent $i$ is a function of $x^i$ and $y^i$, tracking the distribution of human capital, conditional on employment status, tells us about the distribution of wages. Although I will provide laws of motion to characterize how these distributions change at any point in time, I will focus primarily on the limiting behavior of the skill distributions as the amount of time spent in an exogenous state $s$ grows large. This will allow me to analyze the effects of persistent business cycles on the skill distribution.

[14] One can show that if the exogenous state $s$ is constant for a sufficiently long time, then the ratio $\bar{x}^u/\bar{k}$ will converge to a value less than one.

## 6.1 Match-Specific Human Capital

I will begin by looking at the distribution of match-specific human capital. Define the cumulative distribution function of $y^i$, conditional on being employed:

$$F_t^y(y) \equiv \frac{\int \mathbb{I}\left[y_t^i \leq y\right] e^i di}{\int e_t^i di}, \tag{6.1}$$

where $\mathbb{I}[\cdot]$ is the indicator function. Likewise, define the density as $f_t^y(y) \equiv \frac{\partial}{\partial y} F_t^y(y)$.[15]

**Theorem 9.** *Suppose that the initial distribution $F_0^y(y)$ has support $[1, \infty)$ and is everywhere continuously differentiable. Then, the distribution of $y^i$ is characterized by the following partial differential equation:*

$$\frac{\partial}{\partial t} F_t^y(y) = -\rho y f_t^y(y) + \theta_s q(\theta_s) \left(\frac{1-e_t}{e_t}\right) \left[1 - F_t^y(y)\right]. \tag{6.2}$$

*Proof.* See appendix. □

To gain some intuition for this result, we can write the above equation as:

$$\frac{\partial}{\partial t} \left[F_t^y(y) e_t\right] = \underbrace{-\rho y f_t^y(y) e_t}_{\text{Learning by doing}} \underbrace{-\lambda F_t^y(y) e_t}_{\text{Job Loss}} + \underbrace{\theta_s q(\theta_s)(1-e_t)}_{\text{Finding Employment}}. \tag{6.3}$$

Notice that $F_t^y(y) e_t$ is the measure of employed workers with match-specific human capital less than or equal to $y$. A measure $f_t^y(y) e_t$ of workers gain match-specific skills at rate $\rho y$, thereby removing themselves from the set of workers with $y^i \leq y$. At rate $\lambda$, employed workers with $y^i \leq y$ lose their jobs, and at rate $\theta_s q(\theta_s)$, unemployed workers find employment and start their jobs with $y^i = 1$. Suppose that the economy remains in state $s$ for a long time. Then, from the employment law of motion (2.5), we see that $\theta_s q(\theta_s)\left(\frac{1-e_t}{e_t}\right)$ will converge to $\lambda$. Hence, a time-invariant distribution, corresponding to $\frac{\partial}{\partial t} F_t^y(y) = 0$, satisfies:

$$\rho y f_t^y(y) = \lambda \left[1 - F_t^y(y)\right]. \tag{6.4}$$

[15] I will assume that the density is defined for the initial distribution, and the distribution will be well defined subsequently.

The above differential equation is solved by the cumulative distribution function of a Pareto distribution with tail index $\frac{\lambda}{\rho}$:

$$F_t^y(y) = 1 - y^{-\frac{\lambda}{\rho}}. \tag{6.5}$$

This implies that a low job-separation rate or a high growth rate of match-specific human will contribute to dispersion of $y^i$. Notice that the limiting distribution of $y^i$ does not depend on the exogenous state $s$. This is because the amount of match-specific human capital a worker accumulates in an employment spell is governed by how long she keeps her job, not how easily she found it. Data presented by Heathcote et al. (2010), however, suggest that wage dispersion increases during recessions. Even if we added exogenous shocks to the separation rate, they would make the distribution of $y^i$ become more equal during recessions.[16] In addition, the fact that the distribution of $y^i$ is stationary disallows an upward trend in wage dispersion, like the one in Figure A.3. This suggests that the behavior of wage dispersion is not well explained by the distribution of match-specific human capital. Hence, I turn to the distribution of general human capital.

## 6.2 General Human Capital

### 6.2.1 An Individual in the Steady State

Before looking at the entire distribution of general human capital and how it changes over time, let's take a moment to consider how human-capital growth for an individual agent behaves in a non-stochastic version of the model. With a fixed value of $z$, the job-finding rate $\theta q(\theta)$ will remain fixed, and the employment rate will converge monotonically to $e = \frac{\theta q(\theta)}{\lambda + \theta q(\theta)}$. Steady-state employment is an increasing function of the job finding rate, which in turn is an increasing function of $z$. In addition to being a fixed point for the mass of workers who are employed, the steady-state value of $e$ also characterizes a stationary distribution for idiosyncratic employment states of individuals. That is, if the economy is in the steady state, $e$ also represents the unconditional probability of a given individual being employed.

[16] It would not be difficult to add exogenous shocks to $\lambda$ to the model. I chose to omit shocks to $\lambda$ because of Shimer's (2012) conclusion that "fluctuations in the employment exit probability are quantitatively irrelevant during the last two decades."

From the technology for human-capital accumulation (2.1), the unconditional variance of agent $i$'s human-capital growth is a function of steady-state employment:

$$\mathbb{V}\left[\frac{\dot{x}^i}{x^i} \mid \text{steady state}\right] = (\alpha+\delta)^2 (1-e)\, e \tag{6.6}$$

So, the variance of individual human-capital growth is a concave function of $e$, and this variance is largest when the steady-state employment rate is one half (i.e. when $\theta q(\theta) = \lambda$).

If the stochastic economy remains in a given productivity regime for a long enough time, then the employment rate will converge to a constant. Then, we can apply the same reasoning as we did in the preceding paragraph. Suppose that the limiting employment rate is greater than one half for both exogenous productivity regimes. Then, agent $i$ can expect her stock of general human capital to fluctuate more when the economy has a prolonged recession than when the economy has a prolonged boom. The mechanism is employment volatility: If a worker is less consistently employed, then she will accumulate human capital at a more variable rate.

### 6.2.2 The Aggregate Distribution

Define the conditional cumulative distribution functions as:

$$F_t^e(x) \equiv \frac{\int \mathbb{I}\left[x_t^i \le x\right] e_t^i di}{\int e_t^i di} \tag{6.7}$$

$$F_t^u(x) \equiv \frac{\int \mathbb{I}\left[x_t^i \le x\right]\left(1-e_t^i\right) di}{\int \left(1-e_t^i\right) di}. \tag{6.8}$$

**Theorem 10.** *Suppose that the initial conditional distributions are continuously differentiable with support $\mathbb{R}_+$. Then, the evolution of the conditional cumulative distribution functions is characterized by the following system of partial differential equations:*

$$\frac{\partial}{\partial t} F_t^e(x) = -\alpha x f_t^e(x) + \theta_s q(\theta_s)\left(\frac{1-e_t}{e_t}\right)\left[F_t^u(x) - F_t^e(x)\right] \tag{6.9}$$

$$\frac{\partial}{\partial t} F_t^u(x) = \delta x f_t^u(x) + \lambda \frac{e_t}{1-e_t}\left[F_t^e\left(\frac{x}{1-\zeta}\right) - F_t^u(x)\right], \tag{6.10}$$

*where* $f_t^l(x) \equiv \frac{\partial F_t^l(x)}{\partial x}$, $l \in \{e, u\}$, *is the conditional density.*

*Proof.* See appendix. □

A more accessible way of writing Theorem 10 is to write:

$$\frac{\partial}{\partial t}\left[e_t F_t^e(x)\right] = \underbrace{-\alpha x f_t^e(x)\, e_t}_{\text{Learning by doing}} - \underbrace{\lambda e_t F_t^e(x)}_{\text{Job Loss}} + \underbrace{\theta q(\theta)(1-e_t)\, F_t^u(x)}_{\text{Finding Employment}}. \tag{6.11}$$

The above is the time derivative of the mass of the set of workers who are both employed and have human capital less than or equal to $x$. The most skilled workers in this set have mass $f_t^e(x)\, e_t$, and the rate at which they exit this set is the rate at which they accumulate human capital, which is equal to $\alpha x$. Hence, $\alpha x f_t^e(x)\, e_t$ workers per instant flow out of this set due to learning by doing. A fraction $\lambda$ of this set of workers exits the state of employment, so $\lambda e_t F_t^e(x)$ workers per instant flow out of this set by exiting employment. Finally, a mass $(1-e_t)\, F_t^u(x)$ of workers are unemployed and have human capital less than or equal to $x$, and these workers find jobs at rate $\theta q(\theta)$. Similar accounting explains the components of the law of motion for $F_t^u(x)$.

I will turn my attention to the coefficient of variation for general human capital.[17] Define the (squared) conditional coefficient of variation as the ratio of human-capital variance to mean squared, conditional on employment state:

$$c_t^l \equiv \frac{\int \left(x - \bar{x}_t^l\right)^2 f_t^l(x)\, dx}{\left(\bar{x}_t^l\right)^2}, \quad l \in \{e, u\}. \tag{6.12}$$

**Proposition 11.** *Suppose that the initial conditional distributions are continuously differentiable with finite second moments, and that* $f_0^e(0) = f_0^u(0) = 0$. *The coefficients of variation* $(c^e, c^u)$ *are jointly characterized by an affine system of differential equations with time-varying coefficients. If the economy remains in state* $s$ *for a sufficiently long time, then*

[17]Obviously, there are many other measures of wage dispersion, the time derivatives of which we could calculate by virtue of knowing Theorem 10. However, it becomes unwieldy to keep track of how the entire probability distribution function − an infinite-dimensional object − evolves over time; the time derivative of, say, the Gini coefficient is similarly unwieldy. As will become clear in a moment, the advantage of looking at the coefficient of variation is that we can characterize its evolution with a two-variable system of differential equations.

$\frac{\dot{c}^e}{c^e}$ *and* $\frac{\dot{c}^u}{c^u}$ *converge to a positve constant, denoted* $\iota_s$*. If the employment rate converges to a value sufficiently close to one, then* $0 < \iota_1 < \iota_0$.

*Proof.* See appendix. See equations (B.92) and (B.93) for the expressions for $\dot{c}^e$ and $\dot{c}^u$. $\square$

Proposition 11 says that skill inequality will be trending upward over time.[18] The conclusion implied by the model is that disparate employment experiences across workers causes the skill distribution to fan out over time, so unemployment can contribute to growing wage dispersion. For economies that are near full employment, a lower level of unemployment corresponds to a slower increase in skill dispersion. Consequently, a persistent boom will decelerate inequality. In addition, the model predicts that the coefficient of variation will grow geometrically in the long run, which is exactly what we see in Figure A.3. Undoubtedly, the trend in wage dispersion has to do with factors besides on-the-job accumulation of human capital, but there is reason to believe that the distribution of skills plays an important part. For example, Lemieux (2006) looks at wage data from the CPS and concludes that much of the growth in wage dispersion between 1973 and 2003 can be explained by composition effects linked to education and experience. Lemieux attributes the majority of these composition effects to education, with experience playing a supporting role. Although the distribution of educational attainment did change considerably, I suspect that Lemieux's approach underestimates the importance of skills gained on the job because he controls for potential experience rather than actual experience. The difference is important in light of the model presented here; a worker observed in the late 1980s who has spent ten years in the labor force is likely to have accumulated less human capital on the job than a worker observed in the late 1990s who has spent ten years in the labor force. Even if the magnitude is unclear, the source of wage dispersion at work in the model appears to be one, though certainly not the only, source of wage dispersion in the data.

[18] One might worry that this result comes from the fact that agents are infinitely lived, but this result is robust to the introduction of some demographics. If agents die at Poisson rate $\mu$, and new agents are born into unemployment with average skills, then a necessary condition for $\iota_s \leq 0$ is $\mu \geq \lambda$. In other words, for the coefficient of variation to converge to a constant, workers must die at a faster rate than they are separated from their jobs. Although it's mathematically possible for $c^e$ and $c^u$ to converge in the long run, this will not happen within the empirically reasonable portion of the parameter space.

# 7 Welfare

Consider the problem facing a social planner who chooses market tightness to maximize flow consumption, subject to the laws of motion for the aggregate state variables. Flow consumption is given by:

$$\begin{aligned}
\text{flow consumption} &= \text{production of employed} + \text{consumption of unemployed} \\
&\quad - \text{vacancy creation costs} \\
&= z_s k + b x^u - \kappa \bar{x}^u \times \text{vacancies} \\
&= z_s k + (b - \kappa\theta)\, x^u. \qquad (7.1)
\end{aligned}$$

Notice that the planner doesn't care about the level of employment, per se, because employment doesn't enter into the expression for flow consumption, nor does it enter into the laws of motion for $k$, $x^e$, nor $x^u$. The planner chooses market tightness as a means of controlling skill flows, not job flows. Given an initial condition and the planner's choice of market tightness, we can compute the implied path of employment; in this context, however, we can interpret employment as an optimal utilization rate of the economy's human capital stock. The planner's dynamic programming problem is given by:

$$\begin{aligned}
r v_s(k, x^e, x^u) &= \max_{\theta} \left\{ z_s k + (b - \kappa\theta)\, x^u + \frac{\partial v_s}{\partial k}\dot{k} + \frac{\partial v_s}{\partial x^e}\dot{x}^e + \frac{\partial v_s}{\partial x^u}\dot{x}^u \right\} \\
&\quad + \beta \left[ v_{1-s}(k, x^e, x^u) - v_s(k, x^e, x^u) \right] \qquad (7.2) \\
&\quad \text{s.t.:} \begin{cases} \dot{x}^e &= (\alpha - \lambda)\, x^e + \theta q(\theta)\, x^u \\ \dot{x}^u &= -\left[\delta + \theta q(\theta)\right] x^u + \lambda (1 - \zeta)\, x^e \\ \dot{k} &= (\alpha + \rho - \lambda)\, k + \theta q(\theta)\, x^u. \end{cases}
\end{aligned}$$

The first-order condition is:

$$\kappa = \left[ \frac{\partial v_s(k, x^e, x^u)}{\partial k} + \frac{\partial v_s(k, x^e, x^u)}{\partial x^e} - \frac{\partial v_s(k, x^e, x^u)}{\partial x^u} \right] \left[1 - \epsilon(\theta)\right] q(\theta), \qquad (7.3)$$

where $\epsilon(\theta) \equiv -\theta \frac{q'(\theta)}{q(\theta)}$ is the elasticity of the vacancy-filling rate with respect to market tightness. The left-hand side of equation (7.3) is the marginal cost of opening a vacancy. The right-hand side of equation (7.3) is the marginal benefit of opening a vacancy: Creating a new vacancy decreases the vacancy-filling rate from $q(\theta)$ to $[1 - \epsilon(\theta)] q(\theta)$, and when a vacancy is filled, the human capital of the newly matched worker augments $k$ and $x^e$ but is subtracted from $x^u$.

**Theorem 12.** *Suppose that $\epsilon(\theta)$ is a constant $\epsilon$. If $\epsilon = \eta$, then:*

$$v_s(k, x^e, x^u) = G_s\left(x^e, \frac{k}{x^e}\right) + H_s\left(x^e, \frac{k}{x^e}\right) + U_s(x^u), \tag{7.4}$$

*and the values of market tightness chosen by the planner coincide with the values of market tightness in the competitive equilibrium.*

*Proof.* See appendix. □

Theorem 12 is a generalization of a result originally derived by Hosios (1990): With frictional labor markets, the economy will be constrained efficient only when the elasticity of the matching function with respect to vacancies is exactly equal to the bargaining power of firms in the Nash problem.[19] The mechanism at work in Hosios's model is a congestion externality. When one firm posts a vacancy, it becomes easier for unemployed workers to find jobs, but it becomes harder for all the other firms posting vacancies to find workers. This congestion externality exists in the present model as well, but we also need to consider the role of skill accumulation. It's not too surprising that the Hosios condition is robust to the introduction of match-specific human capital. As a worker gets better at her job, her outside option does not change, nor does the outside option for the firm owner change. Moreover, the match-specific human capital that a worker accumulates at her current job has no bearing on her next job. General human capital, which is embodied in the worker, behaves differently. A worker's stock of general human capital depends on her complete employment history. The worker will benefit from the experience she gains on her current

[19] Hosios examined a non-stochastic environment; Shimer (2005) extended this result to an economy with business-cycle shocks. Theorem 12 would be true in a version of this model with a richer Markov process governing the aggregate shocks.

job even when she moves on to her next job, whereas firm owners only benefit from the productivity of current employees. However, recall that in the wage equation (4.1), we saw that the learning-by-doing wedge was scaled by the bargaining power of firms. Thus, the firm can extract some of the value of new human capital from the worker. When $\epsilon = \eta$, the firm's power to make the worker pay for her experience will cause the private value of posting a vacancy to coincide with the social value of posting a vacancy.

But there is potentially another externality from the accumulation of general human capital: When a firm hires a worker today, that worker's next employer will also benefit from the experience that the employee gains in the current job. Hence, if individual firms hire aggressively, they boost the average quality of the labor force, which makes it more profitable for more firms to post more vacancies. Likewise, weak hiring depletes the quality of the labor force, which reduces the incentives for vacancy creation. Pissarides (1992) calls this the "thin market" externality. Because of the assumed form of vacancy-creation costs, however, this thin-market externality plays no role in the present model. Recall that the cost of posting a vacancy is $\kappa\bar{x}^u$, and the expected benefit to posting a vacancy is $q(\theta_s)\, g_s(1)\, \bar{x}^u$. Suppose that there is a prolonged slump that results in a drop in $\bar{x}^u$. Both the costs and the benefits of posting a vacancy fall in equal proportion, so it makes no difference what the average quality of the pool of prospective hires is. In principle, we can reintroduce the thin-market externality by changing the specification for vacancy-creation costs. One strategy would be to make the vacancy-creation cost a function of past $\bar{x}^u$:

$$\kappa_t = \kappa(\nu + \tau) \int_{-\infty}^{t} \exp\{-\nu(t - j)\}\, \bar{x}_j^u dj, \tag{7.5}$$

where $\kappa$ and $\nu$ are scalar parameters, and $\tau$ is the trend growth rate associated with the non-stochastic economy. With this distributed-lag specification, there would exist a unique balanced growth path; along this balanced growth path, $\kappa_t/\bar{x}_t^u$ would be equal to $\kappa$, and the allocation would be identical to the one that would prevail under the original specification.[20]

[20] It would be possible to use different specifications for $\kappa_t$, but other choices could lead to multiple balanced growth path. Pissarides (1992) examines the thin-market externality in an overlapping-generations economy without long-run growth, and he finds multiple steady-state equilibria.

But off the balanced growth path, $\kappa_t$ would adjust more rigidly than $\bar{x}_t^u$. Then, if a recession diminished the quality of prospective hires, the cost of posting a vacancy would rise relative to the benefit. Unfortunately, this specification would make it impossible to solve the stochastic model by hand. Investigating the importance of the thin-market externality with a quantitative approach would be interesting, but that undertaking falls outside the scope of this paper.

# 8 Conclusions and Future Research

The model I have presented has numerous implications for labor-market dynamics, business cycles, and long-run growth. Workers must effectively compensate their employers for the skills that they gain and can take with them to their next job. Because skills are more valuable during booms, allowing workers to accumulate general human capital affects the cyclicality of wage determination. If workers accumulate match-specific human capital as well, then a worker's outside option becomes less and less relevant as her tenure grows long. Both general and match-specific human capital can make vacancy creation more sensitive to aggregate shocks. In the long run, the economy grows endogenously as agents accumulate human capital. With human capital coming from on-the-job learning, the model establishes a link between labor-market conditions and the economy's capacity for growth. As workers have differing employment experiences, wage inequality will also trend upward over time.

One direction in which to extend this theory would be a more comprehensive welfare analysis. As I mentioned, this model could be modified to incorporate the thin-market externality discussed in Pissarides (1992). Another interesting modification would be risk-averse preferences, which would make workers even more sensitive to the lifetime earnings losses associated with unemployment. In turn, a planner would like to curb the upward trend in wage dispersion that appears in the present model. Finally, one could use this framework to analyze public employment policies. If the government were to post vacancies for hiring civil servants, it could create distortions in the labor market; however, this public-sector hiring could stem the loss of human capital associated with recessions.

My focus in this paper has been entirely theoretical, but it's straightforward to adapt the model to discrete time, which would facilitate a quantitative exploration. Work by other authors suggests that the other forces I have discussed will be important in empirical applications. In a calibrated real business cycle model, Esteban-Pretel (2007) considers two classes of agents with different levels of skill in a Mortensen-Pissarides model, and he finds more volatility of hiring and unemployment, addressing the criticisms of Shimer (2005) and Hall (2005). In an estimated DSGE model with a representative worker, Chang et al. (2002) show that learning by doing can propagate non-technology shocks as well. As discussed in Section 7, a quantitative version of my model in discrete time could incorporate the thin-market externality and quantify its relevance. In addition to capturing these effects, an estimated version of my model would have the novelty of incorporating the role of human capital accumulation at low frequencies. There is good reason to think that this innovation will be quantitatively important. Cogley (2001) demonstrates that trend misspecification is an especially damaging source of bias in estimated rational-expectations models. When agents are maximizing present discounted values of payoffs, their responses to shocks are sensitive to the long-run structure of the economy. Thus, incorrect assumptions about trend growth inject misspecification into both the low- and high-frequency components of the model. Canova (2008) argues that DSGE models should account for growth using a flexible statistical model that is auxiliary to the economic model used to explain business cycles. He writes: "I share with Cogley the point of view that economic theory has not much to say about non-cyclical fluctuations."[21] This attitude, I think, is misguided. A better approach would be to use an economic model, such as the one I have constructed here, to explain both the trend and the cycle. In related research, Comin and Gertler (2006) document "medium-term" fluctuations in the U.S. economy, linking slower-moving components of output to several factors often included in endogenous growth models; however, one factor not considered by Comin and Gertler is human capital. An estimated version of my model seems like a promising way of quantifying the relative importance of skill accumulation in explaining both growth and business cycles.

[21] Canova (2008), p. 19. Cogley does not actually assert this point of view; however, this quotation fairly represents Canova's argument.

# 9 References


1. Aghion, Philippe and Peter Howitt. 1994. "Growth and Unemployment." *The Review of Economic Studies*, vol. 61, no. 3, pp. 477-494.

2. Binmore, Ken, Ariel Rubinstein, and Asher Wolinsky. 1986. "The Nash Bargaining Solution in Economic Modelling." *The RAND Journal of Economics*, vol. 17, no. 2, pp. 176-188.

3. Burdett, Kenneth and Melvyn G. Coles. 2010. "Tenure and Experience Effects on Wages: A Theory." Working Paper.

4. Burdett, Kenneth, Carlos Carrillo-Tudela, and Melvyn G. Coles. 2011. "Human Capital Accumulation And Labor Market Equilibrium." *International Economic Review*, vol. 52, no. 3, pp. 657-677.

5. Canova, Fabio. 2008. "Bridging Cyclical DSGE Models and the Raw Data." Working Paper.

6. Chang, Yongsung, Joao F. Gomes, and Frank Schorfheide. 2002. "Learning-by-Doing as a Propagation Mechanism." *American Economic Review*, vol. 92, no. 5, pp. 1498-1520.

7. Carrillo-Tudela, Carlos. 2010. "Job Search, Human Capital and Wage Inequality." Working Paper.

8. Chen, Been-Lon, Hung-Ju Chen, and Ping Wang. 2011. "Labor-Market Frictions, Human Capital Accumulation, and Long-Run Growth: Positive Analysis and Policy Implications." *International Economic Review*, vol. 52, no. 1, pp. 131-160.

9. Cogley, Timothy. 2001. "Estimating and Testing Rational Expectations Models when the Trend Specification is Uncertain." *Journal of Economic Dynamics and Control*, vol. 25, no. 10, pp. 1485-1525.

10. Comin, Diego and Mark Gertler. 2006. "Medium-Term Business Cycles." *American Economic Review*, vol. 96, no. 3, pp. 523-555.

11. Couch, Kenneth A. and Dana W. Placzek. 2010. "Earnings Losses of Displaced Workers Revisited." *American Economic Review*,, vol. 100, no. 1, pp. 572-589.

12. Davis, Steven and Till von Wachter. 2011. "Recessions and the Cost of Job Loss." *Brookings Papers on Economic Activity*, Fall 2011.

13. Eckstein, Zvi and Eva Nagypal. 2004. "The Evolution of U.S. Earnings Inequality: 1961-2002." *Federal Reserve Bank of Minneapolis Quarterly Review*, vol. 28, no. 2, pp. 10-29.

14. Elsby, Michael and Matthew Shapiro. 2011. "Why Does Trend Growth Affect Equilibrium Employment? A New Explanation of an Old Puzzle." *American Economic Review*, forthcoming.

15. Esteban-Pretel, Julen. 2007. "The Effects of the Loss of Skill on Unemployment Fluctuations." Working Paper.

16. Hall, Robert E. 2005. "Employment Fluctuations with Equilibrium Wage Stickiness." *American Economic Review*, vol. 95, no. 1, pp. 50-65.

17. Heathcote, Jonathan, Fabrizio Perri, and Giovanni L. Violante. 2010. "Unequal We Stand: An Empirical Analysis of Economic Inequality in the United States: 1967-2006." *Review of Economic Dynamics*, vol. 13, no. 1, pp. 15-51.

18. Hosios, Arthur J. 1990. "On the Efficiency of Matching and Related Models of Search and Unemployment." *Review of Economic Studies*, vol. 57, no. 2, pp. 279-298.

19. Jacobson, Louis S., Robert J. LaLonde, and Daniel G. Sullivan. 1993. "Earnings Losses of Displaced Workers." *American Economic Review*, vol. 83, no. 4, pp. 685-709.

20. Krueger, Dirk, Fabrizio Perri, Luigi Pistaferri, Giovanni L. Violante. 2010. "Cross Sectional Facts for Macroeconomists." *Review of Economic Dynamics*, vol. 13, no. 1, pp. 1-14.

21. Lemieux, Thomas. 2006. “Increasing Residual Wage Inequality: Composition Effects, Noisy Data, or Rising Demand for Skill?” *American Economic Review*, vol. 96, no. 3, pp. 461-498.

22. Laing, Derek, Theodore Palivos, and Ping Wang, 1995. “Learning, Matching and Growth.” *Review of Economic Studies*, vol. 62, no. 1, pp. 115-129.

23. Ljungqvist, Lars and Thomas J. Sargent. 1998. “The European Unemployment Dilemma.” *Journal of Political Economy*, vol. 106, no. 3, pp. 514-550.

24. Ljungqvist, Lars and Thomas J. Sargent. 2008. “Two Questions about European Unemployment.” *Econometrica*, vol. 76, no. 1, pp. 1-29.

25. Lucas, Jr., Robert E. 1988. “On the mechanics of economic development.” *Journal of Monetary Economics*, vol. 22, no. 1, pp. 3-42.

26. Lucas, Jr., Robert E. 1993. “Making a Miracle.” *Econometrica*, vol. 61, no. 2, pp. 251-272.

27. Mortensen, Dale T. and Christopher A. Pissarides. 1994. “Job Creation and Job Destruction in the Theory of Unemployment.” *Review of Economic Studies*, vol. 61, no. 3, pp. 397-415.

28. Mortensen, Dale T. and Christopher A. Pissarides. 1998. “Technological Progress, Job Creation and Job Destruction.” *Review of Economic Dynamics*, vol. 1, no. 4, pp. 733-753.

29. Pissarides, Christopher A. 1992. “Loss of Skill during Unemployment and the Persistence of Employment Shocks.” *The Quarterly Journal of Economics*, vol. 107, no. 4, pp. 1371-1391.

30. Pissarides, Christopher A. 2000. *Equilibrium Unemployment Theory*, 2nd Edition. MIT Press Books, The MIT Press.

31. Pissarides, Christopher A. and Giovanna Vallanti. 2007. "The Impact Of TFP Growth On Steady-State Unemployment." *International Economic Review*, vol. 48, no. 2, pp. 607-640.

32. Rubinstein, Ariel. 1982. "Perfect Equilibrium in a Bargaining Model." *Econometrica*, vol. 50, no. 1, pp. 97-109.

33. Shimer, Robert. 2005. "The Cyclical Behavior of Equilibrium Unemployment and Vacancies." *American Economic Review*, vol. 95, no. 1, pp. 25-49.

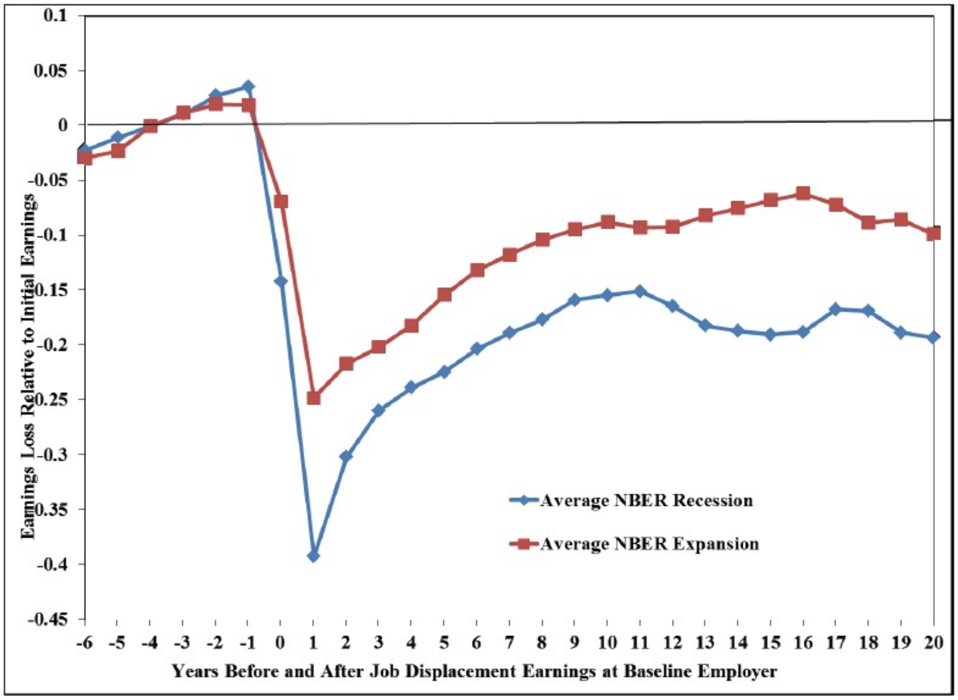
34. Shimer, Robert. 2012. "Reassessing the Ins and Outs of Unemployment." *Review of Economic Dynamics*, vol. 15, no. 2, pp. 127-148.

35. Uhlig, Harald. 1996. "A law of large numbers for large economies." *Economic Theory*, vol. 8, no. 1, pp 41-50.

36. von Wachter, Till, Jae Song, and Joyce Manchester. 2009. "Long-Term Earnings Losses due to Mass Layoffs During the 1982 Recession: An Analysis Using U.S. Administrative Data from 1974 to 2004." Working Paper.

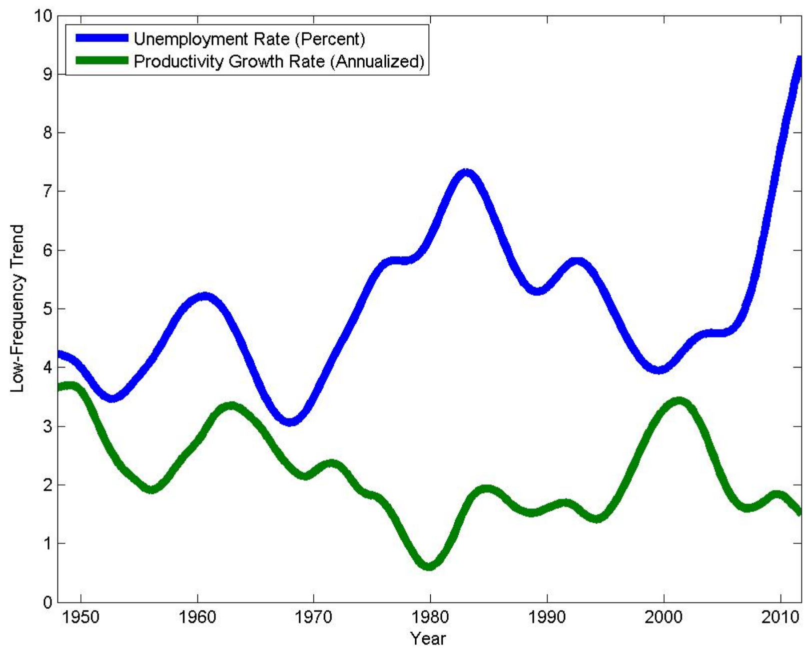
# A Figures

Figure A.1: Permanent Earnings Loss, Estimated by Davis and von Wachter (2011)



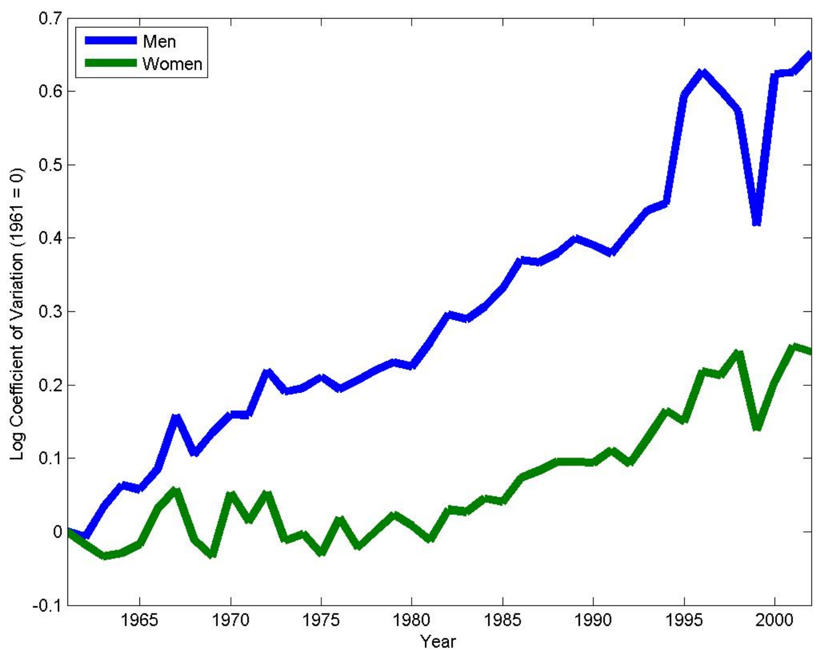

The image shown here is Figure 5A from the working-paper version of Davis and von Wachter; it appears as Figure 4 in the published version. The earnings losses computed by Davis and von Wachter are measured relative to a counterfactual earnings trend. Those authors control for worker effects, calendar-year effects, age, and interaction terms between calendar-year fixed effects and individual average earnings in the five years preceding displacement. Davis and von Wachter use the administrative data on W2 earnings used in von Wachter, Song, and Manchester (2009). See these these two papers for additional details and discussion.

Figure A.2: Productivity Growth and Unemployment at Low Frequencies




The unemployment rate is the quarterly average of monthly unemployment for workers ages 20 and above. The growth rate of labor productivity is 400 times the log difference in quarterly output per manhour in the non-farm business sector. Both series are treated with a Hodrick-Prescott filter using a smoothing parameter of 1600.

Figure A.3: Trends in Wage Dispersion




This figure is constructed using data from Eckstein and Nagypal (2004). Those authors use data from the March CPS for full-time (35+ hours per week) full-year (40+ weeks per year) employees between the ages of 22 and 65. See the appendix of Eckstein and Nagypal for details; the raw data is available from <http://faculty.wcas.northwestern.edu/~een461/QRproject/>.

# B Proof Appendix: For Online Publication

## B.1 Theorem 2

*Claim.* There exists a unique homogeneous equilibrium.

*Proof.* I will first show that an equilibrium can be characterized by a single, implicit function of $\theta \equiv (\theta_0, \theta_1)'$. To do so, I will show that the value functions are affine in $y^i$. Then, I will show that the function that characterizes $\theta$ has a unique solution.

Observe that:

$$\begin{aligned}
(r+\delta+\beta)\,u_s &= b + \beta u_{1-s} + \theta_s q(\theta_s)\left[h_s(1) - u_s\right] \\
&= b + \beta u_{1-s} + \theta_s q(\theta_s)\frac{\eta}{1-\eta} g_s(1) \\
&= b + \beta u_{1-s} + \frac{\eta\kappa}{1-\eta}\theta_s, \qquad \text{(B.1)}
\end{aligned}$$

where the first equality comes from (3.8), and the second equality comes from (3.7). Define:

$$\mathbf{u} \equiv \begin{bmatrix} u_0 & u_1 \end{bmatrix}' \qquad \text{(B.2)}$$

$$\Pi \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \qquad \text{(B.3)}$$

We can write:

$$\mathbf{u} = \left[(r+\delta+\beta)\,\mathbf{I} - \beta\Pi\right]^{-1}\left[b\mathbf{1}_{2\times 1} + \frac{\eta\kappa}{1-\eta}\theta\right]. \qquad \text{(B.4)}$$

I will now demonstrate the form that $g_s\left(y^i\right)$ must take. In Proposition 4, I show that $w_s\left(y^i\right)$ is an affine function of $y^i$. So, we can express the wage equation as:

$$w_s\left(y^i\right) = w_s^0 + w_s^1 y^i, \qquad \text{(B.5)}$$

for scalars $\left\{w_s^0, w_s^1\right\}_{s\in\{0,1\}}$. In light of (3.6), we can write $g_s\left(y^i\right)$ as:

$$[(r-\alpha+\beta+\lambda)\,\mathbf{I}-\beta\Pi]\begin{bmatrix} g_0\left(y^i\right) \\ g_1\left(y^i\right) \end{bmatrix} = -\begin{bmatrix} w_0^0 \\ w_1^0 \end{bmatrix} + \begin{bmatrix} z_0 - w_0^1 \\ z_1 - w_1^1 \end{bmatrix} y^i + \rho \begin{bmatrix} g_0'\left(y^i\right) \\ g_1'\left(y^i\right) \end{bmatrix} y^i. \tag{B.6}$$

In more concise vector notation:

$$[(r-\alpha+\beta+\lambda)\,\mathbf{I}-\beta\Pi]\,\mathbf{g}\left(y^i\right) = -\mathbf{w}^0 + \left(\mathbf{z}-\mathbf{w}^1\right) y^i + \rho\mathbf{g}'\left(y^i\right) y^i. \tag{B.7}$$

Thus, we see that $g_s\left(y^i\right)$ assumes the form:

$$g_s\left(y^i\right) = g_s^0 + g_s^1 y^i + g_s^2 \cdot \left(y^i\right)^\xi, \tag{B.8}$$

where $\left\{g_s^0, g_s^1, g_s^2\right\}_{s\in\{0,1\}}$ and $\xi$ are coefficients to be determined. Because $h_s\left(y^i\right) = \frac{\eta}{1-\eta} g_s\left(y^i\right) + u_s$, it must also be the case that:

$$h_s\left(y^i\right) = h_s^0 + h_s^1 y^i + h_s^2 \cdot \left(y^i\right)^\xi, \tag{B.9}$$

where $\left\{h_s^0, h_s^1, h_s^2\right\}_{s\in\{0,1\}}$ are coefficients to be determined. Define:

$$N_s\left(y^i\right) \equiv h_s\left(y^i\right) + g_s\left(y^i\right). \tag{B.10}$$

In turn, we can write $N_s\left(y^i\right) = N_s^0 + N_s^1 y^i + N_s^2 \cdot \left(y^i\right)^\xi$, where $\left\{N_s^0, N_s^1, N_s^2\right\}_{s\in\{0,1\}}$ are coefficients to be determined. Summing equations (3.5) and (3.6), we see that this form implies:

$$\begin{aligned}
&\left[N_s^0 + N_s^1 y^i + N_s^2 \cdot \left(y^i\right)^{\xi}\right] \\
&\qquad \times (r-\alpha+\beta) &=& \quad (r-\alpha+\beta)\left[h_s\left(y^i\right) + g_s\left(y^i\right)\right] \\
&&=& \quad w_s\left(y^i\right) + \lambda\left[u_s - h_s\left(y^i\right)\right] - \lambda\zeta u_s + \beta h_{1-s}\left(y^i\right) + h_s'\left(y^i\right)\dot{y}^i \\
&&& \quad + z_s y^i - w_s\left(y^i\right) - \lambda g_s\left(y^i\right) + \beta g_{1-s}\left(y^i\right) + g_s'\left(y^i\right)\dot{y}^i \\
&&=& \quad z_s y^i + \lambda(1-\zeta)u_s - \lambda\left[h_s\left(y^i\right) + g_s\left(y^i\right)\right] \\
&&& \quad + \beta\left[h_{1-s}\left(y^i\right) + g_{1-s}\left(y^i\right)\right] + \left[h_s'\left(y^i\right) + g_s'\left(y^i\right)\right]\dot{y}^i \\
&&=& \quad z_s y^i + \lambda(1-\zeta)u_s - \lambda N_s\left(y^i\right) + \beta N_{1-s}\left(y^i\right) + N_s'\left(y^i\right)\dot{y}^i \\
&&=& \quad z_s y^i + \lambda(1-\zeta)u_s - \lambda\left[N_s^0 + N_s^1 y^i + N_s^2 \cdot \left(y^i\right)^{\xi}\right] \\
&&& \quad + \beta\left[N_{1-s}^0 + N_{1-s}^1 y^i + N_{1-s}^2 \cdot \left(y^i\right)^{\xi}\right] \\
&&& \quad + \left[N_s^1 + \xi N_s^2 \cdot \left(y^i\right)^{\xi-1}\right]\rho y^i \\
&&=& \quad \lambda(1-\zeta)u_s - \lambda N_s^0 + \beta N_{1-s}^0 \\
&&& \quad + \left[z_s - \lambda N_s^1 + \rho N_s^1 + \beta N_{1-s}^1\right] y^i \\
&&& \quad + \left[\rho\xi N_s^2 - \lambda N_s^2 + \beta N_{1-s}^2\right]\left(y^i\right)^{\xi}. \qquad \text{(B.11)}
\end{aligned}$$

Evidently, the coefficients are characterized by:

$$(r-\alpha+\beta) N_s^0 = \lambda(1-\zeta)u_s - \lambda N_s^0 + \beta N_{1-s}^0 \qquad \text{(B.12)}$$

$$(r-\alpha+\beta) N_s^1 = z_s - \lambda N_s^1 + \rho N_s^1 + \beta N_{1-s}^1 \qquad \text{(B.13)}$$

$$(r-\alpha+\beta) N_s^2 = \rho\xi N_s^2 - \lambda N_s^2 + \beta N_{1-s}^2. \qquad \text{(B.14)}$$

First, I will show that $N_s^2 = 0\ \forall s$. We see that $\mathbf{N}^2 \equiv \left(N_0^2, N_1^2\right)'$ must satisfy:

$$\left[(r-\alpha+\beta+\lambda-\rho\xi)\mathbf{I} - \beta\Pi\right]\mathbf{N}^2 = \mathbf{0}_{2\times 1}. \qquad \text{(B.15)}$$

The above implies that $\mathbf{N^2}$ is zero, or the matrix premultiplying $\mathbf{N}^2$ has reduced rank. However, we see that this matrix cannot have reduced rank, because the determinant is strictly positive:

$$
\begin{aligned}
\det\left[\left(r-\alpha+\beta+\lambda-\rho\xi\right)\mathbf{I}-\beta\Pi\right] &= \det\begin{bmatrix} \left(r-\alpha+\beta+\lambda-\rho\xi\right) & -\beta \\ -\beta & \left(r-\alpha+\beta+\lambda-\rho\xi\right) \end{bmatrix} \\
&= \left(r-\alpha+\beta+\lambda-\rho\xi\right)^2+\beta^2 \\
&> 0. \qquad \text{(B.16)}
\end{aligned}
$$

Hence, $\mathbf{N}^2 = \mathbf{0}_{2\times 1}$, and we can conclude that $N_s\left(y^i\right)$ is affine. It remains to determine the coefficients $\left\{N_s^0, N_s^1\right\}_{s\in\{0,1\}}$. In vector notation, we can write (B.12) and (B.13) as:

$$
\begin{aligned}
\mathbf{N}^0 &\equiv \begin{bmatrix} N_0^0 & N_1^0 \end{bmatrix}' \\
&= \lambda\left(1-\zeta\right)\left[\left(r-\alpha+\beta+\lambda\right)\mathbf{I}-\beta\Pi\right]^{-1}\mathbf{u} \qquad \text{(B.17)} \\
\mathbf{N}^1 &\equiv \begin{bmatrix} N_0^1 & N_1^1 \end{bmatrix}' \\
&= \left[\left(r-\alpha+\beta+\lambda-\rho\right)\mathbf{I}-\beta\Pi\right]^{-1}\mathbf{z}. \qquad \text{(B.18)}
\end{aligned}
$$

Note that adding $\left(1-\eta\right)g_s\left(y^i\right)$ to both sides of (3.8) yields:

$$
g_s\left(y^i\right) = \left(1-\eta\right)\left[N_s\left(y^i\right)-u_s\right]. \qquad \text{(B.19)}
$$

Hence, the free-entry condition (3.7) becomes:

$$
\kappa = q\left(\theta_s\right)\left(1-\eta\right)\left[N_s\left(1\right)-u_s\right]. \qquad \text{(B.20)}
$$

Define:

$$
\tilde{\mathbf{q}}\left(\theta\right) \equiv \begin{bmatrix} \frac{1}{q(\theta_0)} & \frac{1}{q(\theta_1)} \end{bmatrix}'. \qquad \text{(B.21)}
$$

Combining the above results, we see that $\theta$ is characterized by the following implicit function:

$$\begin{aligned}
\mathbf{0}_{2\times 1} &= \frac{\kappa}{1-\eta}\tilde{\mathbf{q}}(\theta)+\mathbf{u}-\mathbf{N}^{0}-\mathbf{N}^{1} \\
&= \frac{\kappa}{1-\eta}\tilde{\mathbf{q}}(\theta)-\left[(r-\alpha+\beta+\lambda-\rho)\mathbf{I}-\beta\Pi\right]^{-1}\mathbf{z} \\
&\quad +\left[\mathbf{I}-\lambda(1-\zeta)\left[(r-\alpha+\beta+\lambda)\mathbf{I}-\beta\Pi\right]^{-1}\right] \\
&\quad \times\left[(r+\delta+\beta)\mathbf{I}-\beta\Pi\right]^{-1}\left[b\mathbf{1}_{2\times 1}+\frac{\eta\kappa}{1-\eta}\theta\right].
\end{aligned} \tag{B.22}$$

To establish that this implicit function has a unique solution, I will show that the Jacobian is positive definite. The Jacobian of the above expression is:

$$\mathbf{J} = \frac{\kappa}{1-\eta}\begin{bmatrix} -\frac{q'(\theta_0)}{[q(\theta_0)]^2} & 0 \\ 0 & -\frac{q'(\theta_1)}{[q(\theta_1)]^2} \end{bmatrix}+\mathbf{J}_0\mathbf{J}_1. \tag{B.23}$$

$$\mathbf{J}_0 \equiv \frac{\eta\kappa}{1-\eta}\left[\mathbf{I}-\lambda(1-\zeta)\left[(r-\alpha+\beta+\lambda)\mathbf{I}-\beta\Pi\right]^{-1}\right] \tag{B.24}$$

$$\mathbf{J}_1 \equiv \left[(r+\delta+\beta)\mathbf{I}-\beta\Pi\right]^{-1}. \tag{B.25}$$

Note that the first term in $\mathbf{J}$ is a diagonal matrix, and since $q(\cdot)$ is decreasing, all of the elements along the main diagonal are strictly positive. Hence, the first term in the above expression is a positive definite matrix. It remains to show that $\mathbf{J}_0\mathbf{J}_1$ is positive definite. Because both $\mathbf{J}_0$ and $\mathbf{J}_1$ are symmetric $2\times 2$ matrices, if each of these matrices is positive definite, then their product will be positive definite as well. Sufficient conditions for a symmetric $2\times 2$ matrix to be positive definite are that the upper-left element is strictly positive and greater in absolute value than the upper-right element. Observe that:

$$\begin{aligned}
\mathbf{J}_1 &= \begin{bmatrix} r+\delta+\beta & -\beta \\ -\beta & r+\delta+\beta \end{bmatrix}^{-1} \\
&= \frac{1}{(r+\delta+\beta)^2-\beta^2}\begin{bmatrix} r+\delta+\beta & \beta \\ \beta & r+\delta+\beta \end{bmatrix}.
\end{aligned} \tag{B.26}$$

It is clear that the above matrix satisfies the conditions for being positive definite. Also, observe that:

$$
\begin{aligned}
\frac{1-\eta}{\eta\kappa}\mathbf{J}_0 &= \mathbf{I}-\frac{\lambda(1-\zeta)}{(r-\alpha+\beta+\lambda)^2-\beta^2}\begin{bmatrix} r-\alpha+\beta+\lambda & \beta \\ \beta & r-\alpha+\beta+\lambda \end{bmatrix} \\
&= \begin{bmatrix} 1-\frac{\lambda(1-\zeta)(r-\alpha+\beta+\lambda)}{(r-\alpha+\beta+\lambda)^2-\beta^2} & -\frac{\lambda(1-\zeta)\beta}{(r-\alpha+\beta+\lambda)^2-\beta^2} \\ -\frac{\lambda(1-\zeta)\beta}{(r-\alpha+\beta+\lambda)^2-\beta^2} & 1-\frac{\lambda(1-\zeta)(r-\alpha+\beta+\lambda)}{(r-\alpha+\beta+\lambda)^2-\beta^2} \end{bmatrix}. \qquad \text{(B.27)}
\end{aligned}
$$

Note that:

$$
\begin{aligned}
1-\frac{\lambda(1-\zeta)(r-\alpha+\beta+\lambda)}{(r-\alpha+\beta+\lambda)^2-\beta^2} &> 0 \\
\Longleftrightarrow (r-\alpha+\lambda)^2+2\beta(r-\alpha+\lambda) &> \lambda(1-\zeta)(r-\alpha+\beta+\lambda) \\
\Longleftrightarrow (r-\alpha+2\beta+\lambda\zeta)(r-\alpha+\lambda) &> \lambda(1-\zeta)\beta \\
\Longleftrightarrow (r-\alpha+\beta+\lambda\zeta)(r-\alpha+\lambda)+(r-\alpha)\beta &> -\lambda\zeta\beta,
\end{aligned}
$$

which must hold, since $r>\alpha$ and all parameters above are positive. Also, note that:

$$
\begin{aligned}
1-\frac{\lambda(1-\zeta)(r-\alpha+\beta+\lambda)}{(r-\alpha+\beta+\lambda)^2-\beta^2} &> \frac{\lambda(1-\zeta)\beta}{(r-\alpha+\beta+\lambda)^2-\beta^2} \\
\Longleftrightarrow 1-\frac{\lambda(1-\zeta)(r-\alpha+2\beta+\lambda)}{(r-\alpha+\beta+\lambda)^2-\beta^2} &> 0 \\
\Longleftrightarrow (r-\alpha+\beta+\lambda)^2-\beta^2 &> \lambda(1-\zeta)(r-\alpha+2\beta+\lambda) \\
\Longleftrightarrow (r-\alpha+\lambda)^2+2\beta(r-\alpha+\lambda) &> \lambda(1-\zeta)(r-\alpha+2\beta+\lambda).
\end{aligned}
$$

For the above to hold, it is sufficient to show that:

$$
\begin{aligned}
(r-\alpha+\lambda)^2+2\beta(r-\alpha+\lambda) &> \lambda(r-\alpha+2\beta+\lambda) \\
\Longleftrightarrow (r-\alpha+\lambda)^2+2\beta(r-\alpha) &> \lambda(r-\alpha+\lambda) \\
\Longleftrightarrow (r-\alpha)(r-\alpha+\lambda)+2\beta(r-\alpha) &> 0,
\end{aligned}
$$

which again must be true because $r>\alpha$. Thus, $\mathbf{J}_0$ is also positive definite. $\square$

## B.2 Proposition 3

*Claim.* Market tightness will be procyclical; i.e., $\theta_1 > \theta_0$. Moreover, if $-\theta \frac{q'(\theta)}{q(\theta)}$ is weakly decreasing in $\theta$, then $\frac{\partial \theta}{\partial z}$ is an increasing function of $\alpha$, $\rho$, and $\delta$, whereas $\frac{\partial \theta}{\partial z}$ is a decreasing function of $\zeta$.

*Proof.* Recall equation (B.22) from the proof of Theorem 2, which provides an implicit function that characterizes $\theta$:

$$\begin{aligned}
\mathbf{0}_{2\times 1} \quad &= \quad \tilde{\mathbf{q}}(\theta) - \left[(r-\alpha+\beta+\lambda-\rho)\mathbf{I} - \beta\Pi\right]^{-1}\mathbf{z} \\
&\quad + \left[\mathbf{I} - \lambda(1-\zeta)\left[(r-\alpha+\beta+\lambda)\mathbf{I} - \beta\Pi\right]^{-1}\right] \\
&\quad \times \left[(r+\delta+\beta)\mathbf{I} - \beta\Pi\right]^{-1}\left[b\mathbf{1}_{2\times 1} + \frac{\eta\kappa}{1-\eta}\theta\right]. \qquad \text{(B.28)}
\end{aligned}$$

Premultiplying both sides of the above by $\begin{bmatrix} 1 & -1 \end{bmatrix}$ yields:

$$\begin{aligned}
&\begin{bmatrix} 1 & -1 \end{bmatrix} \\
&\times \left[(r-\alpha+\beta+\lambda-\rho)\mathbf{I} - \beta\Pi\right]^{-1}\mathbf{z} \quad = \quad \frac{\kappa}{1-\eta}\left[\frac{1}{q(\theta_0)} - \frac{1}{q(\theta_1)}\right] \\
&\qquad + \begin{bmatrix} 1 & -1 \end{bmatrix} \\
&\qquad \times \left[\mathbf{I} - \lambda(1-\zeta)\left[(r-\alpha+\beta+\lambda)\mathbf{I} - \beta\Pi\right]^{-1}\right] \\
&\qquad \times \left[(r+\delta+\beta)\mathbf{I} - \beta\Pi\right]^{-1}\frac{\eta\kappa}{1-\eta}\theta. \qquad \text{(B.29)}
\end{aligned}$$

Note that:

$$\begin{aligned}
\begin{bmatrix} 1 & -1 \end{bmatrix} & \\
\times \left[(r-\alpha+\beta+\lambda-\rho)\,\mathbf{I}-\beta\Pi\right]^{-1} &= \frac{1}{(r-\alpha+\beta+\lambda-\rho)^2-\beta^2} \\
&\quad \times \begin{bmatrix} 1 & -1 \end{bmatrix} \\
&\quad \times \begin{bmatrix} (r-\alpha+\beta+\lambda-\rho) & \beta \\ \beta & (r-\alpha+\beta+\lambda-\rho) \end{bmatrix} \\
&= \frac{(r-\alpha+\lambda-\rho)}{(r-\alpha+\beta+\lambda-\rho)^2-\beta^2} \begin{bmatrix} 1 & -1 \end{bmatrix}. \qquad \text{(B.30)}
\end{aligned}$$

Also, note that:

$$\begin{aligned}
\begin{bmatrix} 1 & -1 \end{bmatrix} & \\
\times \left[\mathbf{I}-\lambda(1-\zeta)\left[(r-\alpha+\beta+\lambda)\,\mathbf{I}-\beta\Pi\right]^{-1}\right] &= \begin{bmatrix} 1 & -1 \end{bmatrix} \\
&\quad \times \begin{bmatrix} 1-\frac{\lambda(1-\zeta)(r-\alpha+\beta+\lambda)}{(r-\alpha+\beta+\lambda)^2-\beta^2} & -\frac{\lambda(1-\zeta)\beta}{(r-\alpha+\beta+\lambda)^2-\beta^2} \\ -\frac{\lambda(1-\zeta)\beta}{(r-\alpha+\beta+\lambda)^2-\beta^2} & 1-\frac{\lambda(1-\zeta)(r-\alpha+\beta+\lambda)}{(r-\alpha+\beta+\lambda)^2-\beta^2} \end{bmatrix} \\
&= \left[1-\frac{\lambda(1-\zeta)(r-\alpha+\lambda)}{(r-\alpha+\beta+\lambda)^2-\beta^2}\right] \begin{bmatrix} 1 & -1 \end{bmatrix}. \qquad \text{(B.31)}
\end{aligned}$$

Also, note that:

$$\begin{aligned}
\begin{bmatrix} 1 & -1 \end{bmatrix} \left[(r+\delta+\beta)\,\mathbf{I}-\beta\Pi\right]^{-1} &= \begin{bmatrix} 1 & -1 \end{bmatrix} \frac{1}{(r+\delta+\beta)^2-\beta^2} \\
&\quad \times \begin{bmatrix} (r+\delta+\beta) & \beta \\ \beta & (r+\delta+\beta) \end{bmatrix} \\
&= \frac{(r+\delta)}{(r+\delta+\beta)^2-\beta^2} \begin{bmatrix} 1 & -1 \end{bmatrix}. \qquad \text{(B.32)}
\end{aligned}$$

Thus:

$$\begin{aligned}\frac{(r-\alpha+\lambda-\rho)}{(r-\alpha+\beta+\lambda-\rho)^2-\beta^2}(z_0-z_1) &= \frac{\kappa}{1-\eta}\left[\frac{1}{q(\theta_0)}-\frac{1}{q(\theta_1)}\right] \\ &\quad+\frac{\eta\kappa}{1-\eta}\left[1-\frac{\lambda(1-\zeta)(r-\alpha+\lambda)}{(r-\alpha+\beta+\lambda)^2-\beta^2}\right] \\ &\quad\times\frac{(r+\delta)}{(r+\delta+\beta)^2-\beta^2}(\theta_0-\theta_1). \qquad \text{(B.33)}\end{aligned}$$

The left-hand side of the above expression is negative. Because $[q(\theta)]^{-1}$ is an increasing function of $\theta$, it must be the case that $[q(\theta_0)]^{-1}-[q(\theta_1)]^{-1}$ has the same sign as $\theta_0-\theta_1$. Also, the expressions multiplying $[q(\theta_0)]^{-1}-[q(\theta_1)]^{-1}$ and $\theta_0-\theta_1$ are positive, so the sign of the right-hand side (which, of course, must be the sign of the left-hand side) must be the sign of $\theta_0-\theta_1$. Thus, we see that $\theta_1>\theta_0$. Dividing both sides of the above expression by $(z_1-z_0)$ yields:

$$\begin{aligned}\frac{(r-\alpha+\lambda-\rho)}{(r-\alpha+\beta+\lambda-\rho)^2-\beta^2} &= \left[\frac{q(\theta_1)^{-1}-q(\theta_0)^{-1}}{\theta_1-\theta_0}\right]\frac{\eta\kappa}{1-\eta}\left(\frac{\theta_1-\theta_0}{z_1-z_0}\right) \\ &\quad+\left[1-\frac{\lambda(1-\zeta)(r-\alpha+\lambda)}{(r-\alpha+\beta+\lambda)^2-\beta^2}\right] \\ &\quad\times\frac{(r+\delta)}{(r+\delta+\beta)^2-\beta^2}\frac{\eta\kappa}{1-\eta}\left(\frac{\theta_1-\theta_0}{z_1-z_0}\right). \qquad \text{(B.34)}\end{aligned}$$

Taking the limit as $z_1-z_0\to 0$, we get:

$$\begin{aligned}\frac{(r-\alpha+\lambda-\rho)}{(r-\alpha+\beta+\lambda-\rho)^2-\beta^2} &= \frac{\eta\kappa}{1-\eta}\frac{\epsilon(\theta)}{\theta q(\theta)}\frac{\partial\theta}{\partial z} \\ &\quad+\frac{\eta\kappa}{1-\eta}\left[1-\frac{\lambda(1-\zeta)(r-\alpha+\lambda)}{(r-\alpha+\beta+\lambda)^2-\beta^2}\right] \\ &\quad\times\frac{(r+\delta)}{(r+\delta+\beta)^2-\beta^2}\frac{\partial\theta}{\partial z}, \qquad \text{(B.35)}\end{aligned}$$

where $\epsilon(\theta) \equiv -\theta\frac{q'(\theta)}{q(\theta)}$. Thus:

$$\frac{\partial\theta}{\partial z} = \frac{\frac{(r-\alpha+\lambda-\rho)}{(r-\alpha+\beta+\lambda-\rho)^2-\beta^2}\frac{1-\eta}{\eta\kappa}}{\frac{\epsilon(\theta)}{\theta q(\theta)} + \left[1-\frac{\lambda(1-\zeta)(r-\alpha+\lambda)}{(r-\alpha+\beta+\lambda)^2-\beta^2}\right]\frac{(r+\delta)}{(r+\delta+\beta)^2-\beta^2}}. \tag{B.36}$$

Because computation of $\frac{\partial\theta}{\partial z}$ involves taking the limit as $z_1 - z_0$ goes to zero, equation (B.22) implies that the value of $\theta$ that appears on the right-hand side of (B.36) is characterized by:

$$0 = \frac{\kappa}{1-\eta}\frac{1}{q(\theta)} - \frac{z}{(r-\alpha+\lambda-\rho)} + \left[1-\frac{\lambda(1-\zeta)}{r-\alpha+\lambda}\right]\frac{1}{r+\delta}\left[b+\frac{\eta\kappa}{1-\eta}\theta\right]. \tag{B.37}$$

Differentiating the above expression with respect to parameter values yields:

$$0 < \frac{\partial\theta}{\partial\alpha} \tag{B.38}$$

$$0 < \frac{\partial\theta}{\partial\rho} \tag{B.39}$$

$$0 < \frac{\partial\theta}{\partial\delta} \tag{B.40}$$

$$0 > \frac{\partial\theta}{\partial\zeta}. \tag{B.41}$$

Suppose that $\epsilon(\theta)$ is weakly decreasing in $\theta$. Then, $\frac{\epsilon(\theta)}{\theta q(\theta)}$ is a strictly decreasing function of $\theta$. This implies, along with the signs of the above partial derivatives, that $\frac{\epsilon(\theta)}{\theta q(\theta)}$ is decreasing in $\alpha$, $\rho$, and $\zeta$, and $\frac{\epsilon(\theta)}{\theta q(\theta)}$ is increasing in $\zeta$. Now, observe that:

$$0 < \frac{\partial}{\partial\alpha}\left[\frac{(r-\alpha+\lambda-\rho)}{(r-\alpha+\beta+\lambda-\rho)^2-\beta^2}\frac{1-\eta}{\eta\kappa}\right] \tag{B.42}$$

$$0 < \frac{\partial}{\partial\rho}\left[\frac{(r-\alpha+\lambda-\rho)}{(r-\alpha+\beta+\lambda-\rho)^2-\beta^2}\frac{1-\eta}{\eta\kappa}\right] \tag{B.43}$$

$$0 < \frac{\partial}{\partial\zeta}\left[\left[1-\frac{\lambda(1-\zeta)(r-\alpha+\lambda)}{(r-\alpha+\beta+\lambda)^2-\beta^2}\right]\frac{(r+\delta)}{(r+\delta+\beta)^2-\beta^2}\right] \tag{B.44}$$

$$0 > \frac{\partial}{\partial\alpha}\left[\left[1-\frac{\lambda(1-\zeta)(r-\alpha+\lambda)}{(r-\alpha+\beta+\lambda)^2-\beta^2}\right]\frac{(r+\delta)}{(r+\delta+\beta)^2-\beta^2}\right] \tag{B.45}$$

$$0 > \frac{\partial}{\partial\delta}\left[\left[1-\frac{\lambda(1-\zeta)(r-\alpha+\lambda)}{(r-\alpha+\beta+\lambda)^2-\beta^2}\right]\frac{(r+\delta)}{(r+\delta+\beta)^2-\beta^2}\right]. \tag{B.46}$$

It follows that an increase in $\alpha$ or $\rho$ increases the numerator of equation (B.36) while decreasing the denominator; an increase in $\delta$ decreases the denominator of (B.36); and an increase in $\zeta$ increases the denominator of equation (B.36). Thus, $\frac{\partial \theta}{\partial z}$ is increasing in $\alpha$, $\rho$, and $\delta$, but decreasing in $\zeta$. $\square$

## B.3 Proposition 4

*Claim.* The wage equation in a recursive homogeneous equilibrium is given by:

$$w_s\left(y^i\right) = \eta z_s y^i + (1-\eta)\, b + \eta\kappa\theta_s - (1-\eta)\left(\alpha+\delta+\lambda\zeta\right) u_s. \tag{B.47}$$

*Proof.* Note that:

$$\begin{aligned}
(r-\alpha+\beta)\, u_s &= b+\theta q(\theta)\left[h_s(1)-u_s\right]+\beta u_{1-s}-(\alpha+\delta)\, u_s && \text{(B.48)}\\
(r-\alpha+\beta)\, h_s\left(y^i\right) &= w_s\left(y^i\right)+\lambda\left[u_s-h_s\left(y^i\right)\right]-\lambda\zeta u_s \\
&\quad +\beta h_{1-s}\left(y^i\right)+h_s'\left(y^i\right)\dot{y}^i && \text{(B.49)}\\
(r-\alpha+\beta)\, g_s\left(y^i\right) &= z_s y^i - w_s\left(y^i\right)-\lambda g_s\left(y^i\right)+\beta g_{1-s}\left(y^i\right)+g_s'\left(y^i\right)\dot{y}^i. && \text{(B.50)}
\end{aligned}$$

Hence:

$$\begin{aligned}
(r-\alpha+\beta)\left[h_s\left(y^i\right)-u_s\right] &= w_s\left(y^i\right)+\lambda\left[u_s-h_s\left(y^i\right)\right]-\lambda\zeta u_s+\beta h_{1-s}\left(y^i\right)+h_s'\left(y^i\right)\dot{y}^i\\
&\quad -\left[b+\theta q(\theta)\left[h_s(1)-u_s\right]+\beta u_{1-s}-(\alpha+\delta)\, u_s\right]\\
&= w_s\left(y^i\right)-b-\lambda\left[h_s\left(y^i\right)-u_s\right]-\theta q(\theta)\left[h_s(1)-u_s\right]\\
&\quad +(\alpha+\delta+\lambda\zeta)\, u_s+\beta\left[h_{1-s}\left(y^i\right)-u_{1-s}\right]+h_s'\left(y^i\right)\dot{y}^i\\
&= w_s\left(y^i\right)-b-\lambda\frac{\eta}{1-\eta}g_s\left(y^i\right)-\theta q(\theta)\frac{\eta}{1-\eta}g_s(1)\\
&\quad +(\alpha+\delta+\lambda\zeta)\, u_s+\beta\frac{\eta}{1-\eta}g_{1-s}\left(y^i\right)+\frac{\eta}{1-\eta}g_s'\left(y^i\right)\dot{y}^i\\
&= w_s\left(y^i\right)-b-\lambda\frac{\eta}{1-\eta}g_s\left(y^i\right)-\theta\frac{\eta}{1-\eta}\kappa+(\alpha+\delta+\lambda\zeta)\, u_s\\
&\quad +\beta\frac{\eta}{1-\eta}g_{1-s}\left(y^i\right)+\frac{\eta}{1-\eta}g_s'\left(y^i\right)\dot{y}^i. && \text{(B.51)}
\end{aligned}$$

Multiplying the above by $1-\eta$, equating it with $\eta\left(r-\alpha+\beta\right)g_s\left(y^i\right)$, and canceling redundant terms yields:

$$w_s\left(y^i\right)=\eta z_s y^i+\left(1-\eta\right)b+\eta\kappa\theta-\left(1-\eta\right)\left(\alpha+\delta+\lambda\zeta\right)u_s. \tag{B.52}$$

$\square$

## B.4 Proposition 5

*Claim.* The cyclical change in wages can be decomposed as:

$$w_1\left(y^i\right)-w_0\left(y^i\right)=\eta y^i\left(z_1-z_0\right)+\eta\kappa\left[1-\frac{\left(\alpha+\delta+\lambda\zeta\right)\left(r+\delta\right)}{r+\delta+2\beta}\right]\left(\theta_1-\theta_0\right). \tag{B.53}$$

*Proof.* Recalling equation (B.4), notice that:

$$\begin{aligned}
\left(1-\eta\right)\left(u_1-u_0\right) &= \left(1-\eta\right)\begin{bmatrix}-1 & 1\end{bmatrix}\mathbf{u}\\
&= \begin{bmatrix}-1 & 1\end{bmatrix}\left[\left(r+\delta+\beta\right)\mathbf{I}-\beta\Pi\right]^{-1}\left[\left(1-\eta\right)b\mathbf{1}_{2\times1}+\eta\kappa\theta\right]\\
&= \frac{\left(r+\delta\right)\eta\kappa}{\left(r+\delta+\beta\right)^2-\beta^2}\begin{bmatrix}-1 & 1\end{bmatrix}\theta\\
&= \frac{\left(r+\delta\right)\eta\kappa}{r+\delta+2\beta}\left(\theta_1-\theta_0\right).
\end{aligned} \tag{B.54}$$

Taking the difference between the wage equation (4.1) evaluated at $s=1$ and $s=0$ yields:

$$\begin{aligned}
w_1\left(y^i\right)-w_0\left(y^i\right) &= \eta y^i\left(z_1-z_0\right)+\eta\kappa\left(\theta_1-\theta_0\right)-\left(\alpha+\delta+\lambda\zeta\right)\left(1-\eta\right)\left(u_1-u_0\right)\\
&= \eta y^i\left(z_1-z_0\right)+\eta\kappa\left(\theta_1-\theta_0\right)-\left(\alpha+\delta+\lambda\zeta\right)\frac{\left(r+\delta\right)\eta\kappa}{r+\delta+2\beta}\left(\theta_1-\theta_0\right)\\
&= \eta y^i\left(z_1-z_0\right)+\eta\kappa\left[1-\frac{\left(\alpha+\delta+\lambda\zeta\right)\left(r+\delta\right)}{r+\delta+2\beta}\right]\left(\theta_1-\theta_0\right)
\end{aligned} \tag{B.55}$$

$\square$

## B.5 Proposition 6

*Claim.* Let $t_n$ be the time of the $n^{th}$ switch in the exogenous state $s$. For $t \leq t_{n+1} - t_n$, the path of human capital is given by:

$$\mathbf{x}_{t_n+t} = \Omega_s \text{diag}\left(\exp\left\{\gamma_s t\right\}\right) \Omega_s^{-1} \mathbf{x}_{t_n}, \tag{B.56}$$

where $\gamma_s$ is a vector containing the eigenvalues of $\mathbf{Q}_s$, and $\mathbf{\Omega}_s$ is an orthonormal matrix, the columns of which are the corresponding eigenvectors.

*Proof.* An eigendecomposition of $\mathbf{Q}_s$ allows us to write:

$$\mathbf{Q}_s = \mathbf{\Omega}_s \text{diag}\left(\gamma_s\right) \mathbf{\Omega}_s^{-1}, \tag{B.57}$$

$\gamma_s$ is a vector containing the eigenvalues of $\mathbf{Q}_s$, and $\mathbf{\Omega}_s$ is an orthonormal matrix, the $i^{th}$ column of which is the $i^{th}$ eigenvector of $\mathbf{Q}_s$. Define $\mathbf{a}_{t_n+t} \equiv \mathbf{\Omega}_s^{-1} \mathbf{x}_{t_n+t}$; it follows that $\dot{\mathbf{a}}_{t_n+t} \equiv \mathbf{\Omega}_s^{-1} \dot{\mathbf{x}}_{t_n+t}$. Note that:

$$\dot{\mathbf{x}}_{t_n+t} = \mathbf{Q}_s \mathbf{x}_{t_n+t} = \mathbf{\Omega}_s \text{diag}\left(\gamma_s\right) \mathbf{\Omega}_s^{-1} \mathbf{x}_{t_b+t} = \mathbf{\Omega}_s \text{diag}\left(\gamma_s\right) \mathbf{a}_{t_n+t}. \tag{B.58}$$

Premultiplying both sides of the above by $\Omega_s^{-1}$ yields $\dot{\mathbf{a}}_{t_n+t} = \text{diag}\left(\gamma_s\right) \mathbf{a}_{t_n+t}$. Hence, $\mathbf{a}_{i,t_n+t} = \exp\left\{\gamma_{i,s} t\right\} \mathbf{a}_{i,t_n}$; in vector notation:

$$\mathbf{\Omega}_s^{-1} \mathbf{x}_{t_n+t} = \mathbf{a}_{t_n+t} = \text{diag}\left(\exp\left\{\gamma_s t\right\}\right) \mathbf{a}_{t_n} = \text{diag}\left(\exp\left\{\gamma_s t\right\}\right) \Omega_s^{-1} \mathbf{x}_{t_n}. \tag{B.59}$$

Premultiplying both sides of the above by $\Omega_s$ completes the proof. $\square$

## B.6 Proposition 8

*Claim.* Trend growth $\tau_s$ is well defined and an increasing function of $s$. If $\alpha > \lambda+\delta+\theta_s q(\theta_s)$, then $\tau_s > 0$. Otherwise, the sign of trend growth in state $s$ is given by:

$$\tau_s \gtreqless 0 \iff \theta_s q(\theta_s) \gtreqless \left(\frac{\lambda - \alpha}{\alpha - \zeta\lambda}\right)\delta. \tag{B.60}$$

*Proof.* Observe that the eigenvalues of $\mathbf{Q}_s$ are given by:

$$\frac{\alpha - \lambda - \delta - \theta_s q(\theta_s) \pm \sqrt{4\theta_s q(\theta_s)(1-\zeta)\lambda + [\alpha - \lambda + \delta + \theta_s q(\theta_s)]^2}}{2} \tag{B.61}$$

The first thing to notice is that the expression under the radical above is positive, so both eigenvalues of $\mathbf{Q}_s$ are real. Hence, taking the maximum over the eigenvalues is a well-defined operation. Also, it is clear that $\tau_s$ will be the eigenvalue associated with the plus sign. If $\alpha - \lambda - \delta - \theta_s q(\theta_s) > 0$, then $\tau_s$ will be positive. Suppose $\alpha - \lambda - \delta - \theta_s q(\theta_s) \leq 0$. Then, $\tau_s \gtreqless 0$ if, and only if:

$$\begin{aligned}
\frac{\sqrt{4\theta_s q(\theta_s)(1-\zeta)\lambda + [\alpha - \lambda + \delta + \theta_s q(\theta_s)]^2}}{2} & \\
+\frac{\alpha - \lambda - \delta - \theta_s q(\theta_s)}{2} &\gtreqless 0 \\
\iff \sqrt{4\theta_s q(\theta_s)(1-\zeta)\lambda + [\alpha - \lambda + \delta + \theta_s q(\theta_s)]^2} &\gtreqless -\alpha + \lambda + \delta + \theta_s q(\theta_s) \\
\iff 4\theta_s q(\theta_s)(1-\zeta)\lambda + [\alpha - \lambda + \delta + \theta_s q(\theta_s)]^2 &\gtreqless [-\alpha + \lambda + \delta + \theta_s q(\theta_s)]^2 \\
\iff 4\theta_s q(\theta_s)(1-\zeta)\lambda + ((\alpha - \lambda) + \delta)^2 + 4(\alpha - \lambda)\theta_s q(\theta_s) &\gtreqless (\delta - (\alpha - \lambda))^2 \\
\iff \theta_s q(\theta_s) &\gtreqless \left(\frac{\lambda - \alpha}{\alpha - \zeta\lambda}\right)\delta.
\end{aligned} \tag{B.62}$$

To see that $\tau_s$ is increasing in $s$, it is sufficient to show that $\tau_s$ is increasing in the job-finding rate $\theta_s q(\theta_s)$, since the job-finding rate is increasing in $\theta_s$, and I've already shown that $\theta_s$ is increasing in $s$. Note that:

$$\begin{aligned}
\frac{\partial \tau_s}{\partial \theta_s q(\theta_s)} &= -\frac{1}{2}+\frac{1}{4}\frac{4(1-\zeta)\lambda+2[\alpha-\lambda+\delta+\theta_s q(\theta_s)]}{\sqrt{4\theta_s q(\theta_s)(1-\zeta)\lambda+[\alpha-\lambda+\delta+\theta_s q(\theta_s)]^2}} \\
&= \frac{2(1-\zeta)\lambda+\alpha-\lambda+\delta+\theta_s q(\theta_s)}{2\sqrt{4\theta_s q(\theta_s)(1-\zeta)\lambda+[\alpha-\lambda+\delta+\theta_s q(\theta_s)]^2}} \\
&\quad -\frac{\sqrt{4\theta_s q(\theta_s)(1-\zeta)\lambda+[\alpha-\lambda+\delta+\theta_s q(\theta_s)]^2}}{2\sqrt{4\theta_s q(\theta_s)(1-\zeta)\lambda+[\alpha-\lambda+\delta+\theta_s q(\theta_s)]^2}}. \qquad \text{(B.63)}
\end{aligned}$$

Hence, $\frac{\partial \tau_s}{\partial \theta_s q(\theta_s)} \gtreqless 0$ if, and only if:

$$\begin{aligned}
2(1-\zeta)\lambda+\alpha-\lambda+\delta+\theta_s q(\theta_s) &\gtreqless \sqrt{4\theta_s q(\theta_s)(1-\zeta)\lambda+[\alpha-\lambda+\delta+\theta_s q(\theta_s)]^2} \\
\iff [2(1-\zeta)\lambda+\alpha-\lambda+\delta+\theta_s q(\theta_s)]^2 &\gtreqless 4\theta_s q(\theta_s)(1-\zeta)\lambda+[\alpha-\lambda+\delta+\theta_s q(\theta_s)]^2 \\
\iff [2(1-\zeta)\lambda+(\alpha-\lambda+\delta)]^2 &\gtreqless (\alpha-\lambda+\delta)^2 \\
\iff \alpha-\zeta\lambda+\delta &\gtreqless 0. \qquad \text{(B.64)}
\end{aligned}$$

which must hold under the maintained assumption that $a-\zeta\lambda \geq 0$. □

## B.7 Theorem 9

*Claim.* Suppose that the initial distribution $F_0^y(y)$ has support $[1,\infty)$ and is everywhere continuously differentiable. Then:

$$\frac{\partial}{\partial t}F_t^y(y) = -\rho y\frac{\partial}{\partial y}[F_t^y(y)]+\theta_s q(\theta_s)\left(\frac{1-e_t}{e_t}\right)[1-F_t^y(y)]. \qquad \text{(B.65)}$$

*Proof.* This proof (and the proof of Theorem 10) makes extensive use of the law of large numbers on a continuum of random variables. That is, if $\{X(i) \mid i\in[0,1]\}$ is a continuum of pairwise uncorrelated random variables, each of which with expected value $\bar{X}$, then:

$$\int_0^1 X(i)\,di = \bar{X}, \qquad \text{(B.66)}$$

with probability one. A formal justification of this can be found in Uhlig (1996). Note that for any $y \geq 1$:

$$\begin{aligned}
F^y_{t+\Delta}(y)\,e_{t+\Delta} &= \int \mathbb{I}\left[y^i_{t+\Delta} \leq y\right] e^i_{t+\Delta} di \\
&= \int \mathbb{I}\left[y^i_{t+\Delta} \leq y\right] e^i_{t+\Delta} e^i_t di + \int \mathbb{I}\left[y^i_{t+\Delta} \leq y\right] e^i_{t+\Delta}\left(1-e^i_t\right) di \\
&= (1-\Delta\lambda) \int \mathbb{I}\left[y^i_{t+\Delta} \leq y\right] e^i_t di \\
&\quad +\Delta\theta_s q(\theta_s) \int \mathbb{I}\left[y^i_{t+\Delta} \leq y\right]\left(1-e^i_t\right) di + o(\Delta) \\
&= (1-\Delta\lambda) \int \mathbb{I}\left[y^i_t + \Delta\dot{y}^i_t \leq y\right] e^i_t di \\
&\quad +\Delta\theta_s q(\theta_s) \int \mathbb{I}\left[y^i_t + \Delta\dot{y}^i_t \leq y\right]\left(1-e^i_t\right) di + o(\Delta) \\
&= (1-\Delta\lambda) \int \mathbb{I}\left[y^i_t \leq \frac{y}{1+\Delta\rho}\right] e^i_t di \\
&\quad +\Delta\theta_s q(\theta_s) \int \mathbb{I}\left[y^i_t \leq y\right]\left(1-e^i_t\right) di + o(\Delta) \\
&= (1-\Delta\lambda) F^y_t\left(\frac{y}{1+\Delta\rho}\right) e_t + \Delta\theta_s q(\theta_s)(1-e_t) + o(\Delta). \qquad \text{(B.67)}
\end{aligned}$$

Subtracting $F^y_t(y)\,e_t$ from both sides of the above and dividing by $\Delta$ yields:

$$\begin{aligned}
\frac{F^y_{t+\Delta}(y)\,e_{t+\Delta} - F^y_t(y)\,e_t}{\Delta} &= -\frac{e_t}{\Delta}\left[F^y_t(y) - F^y_t\left(\frac{y}{1+\Delta\rho}\right)\right] - \lambda F^y_t\left(\frac{y}{1+\Delta\rho}\right) e_t \\
&\quad +\theta_s q(\theta_s)(1-e_t) + \frac{o(\Delta)}{\Delta} \\
&= -e_t\rho y\left[\frac{F^y_t(y) - F^y_t(y-\Delta\rho y)}{\Delta\rho y}\right] \\
&\quad -\lambda F^y_t\left(\frac{y}{1+\Delta\rho}\right) e_t + \theta_s q(\theta_s)(1-e_t) + \frac{o(\Delta)}{\Delta}. \qquad \text{(B.68)}
\end{aligned}$$

Taking the limit of the above expression as $\Delta \to 0$ yields:

$$\frac{\partial}{\partial t}\left[F^y_t(y)\,e_t\right] = -e_t\rho y \frac{\partial}{\partial y}\left[F^y_t(y)\right] - \lambda F^y_t(y)\,e_t + \theta_s q(\theta_s)(1-e_t). \qquad \text{(B.69)}$$

Applying the product rule to the left-hand side of the above and rearranging terms yields:

$$\frac{\partial}{\partial t} F_t^y\left(y\right) = -\rho y \frac{\partial}{\partial y}\left[F_t^y\left(y\right)\right] + \theta_s q\left(\theta_s\right)\left(\frac{1-e_t}{e_t}\right)\left[1 - F_t^y\left(y\right)\right]. \tag{B.70}$$

□

## B.8 Theorem 10

*Claim.* Suppose that the initial conditional distributions are continuously differentiable with support $(0, +\infty)$. Then, the evolution of the conditional cumulative distribution functions is characterized by the following system of partial differential equations:

$$\frac{\partial}{\partial t} F_t^e\left(x\right) = -\alpha x f_t^e\left(x\right) + \theta_s q\left(\theta_s\right)\left(\frac{1-e_t}{e_t}\right)\left[F_t^u\left(x\right) - F_t^e\left(x\right)\right] \tag{B.71}$$

$$\frac{\partial}{\partial t} F_t^u\left(x\right) = \delta x f_t^u\left(x\right) + \lambda \frac{e_t}{1-e_t}\left[F_t^e\left(\frac{x}{1-\zeta}\right) - F_t^u\left(x\right)\right], \tag{B.72}$$

where $f_t^l\left(x\right) \equiv \frac{\partial F_t^l(x)}{\partial x}$, $l \in \{e, u\}$, is the conditional density.

*Proof.* Note that:

$$\begin{aligned}
e_{t+\Delta}F^{e}_{t+\Delta}(x) &= \int \mathbb{I}\left[x^{i}_{t+\Delta}\le x\right] e^{i}_{t+\Delta}di \\
&= \int \mathbb{I}\left[x^{i}_{t+\Delta}\le x\right] e^{i}_{t+\Delta}e^{i}_{t}di + \int \mathbb{I}\left[x^{i}_{t+\Delta}\le x\right] e^{i}_{t+\Delta}\left(1-e^{i}_{t}\right)di \\
&= (1-\Delta\lambda)\int \mathbb{I}\left[x^{i}_{t+\Delta}\le x\right] e^{i}_{t}di \\
&\quad +\Delta\theta q(\theta)\int \mathbb{I}\left[x^{i}_{t+\Delta}\le x\right]\left(1-e^{i}_{t}\right)di + o(\Delta) \\
&= (1-\Delta\lambda)\int \mathbb{I}\left[x^{i}_{t}+\Delta\dot{x}^{i}_{t}+o(\Delta)\le x\right] e^{i}_{t}di \\
&\quad +\Delta\theta q(\theta)\int \mathbb{I}\left[x^{i}_{t}+\Delta\dot{x}^{i}_{t}+o(\Delta)\le x\right]\left(1-e^{i}_{t}\right)di + o(\Delta) \\
&= (1-\Delta\lambda)\int \mathbb{I}\left[(1+\Delta\alpha)x^{i}_{t}+o(\Delta)\le x\right] e^{i}_{t}di \\
&\quad +\Delta\theta q(\theta)\int \mathbb{I}\left[(1-\Delta\delta)x^{i}_{t}+o(\Delta)\le x\right]\left(1-e^{i}_{t}\right)di + o(\Delta) \\
&= (1-\Delta\lambda)\int \mathbb{I}\left[x^{i}_{t}\le \frac{x-o(\Delta)}{1+\Delta\alpha}\right] e^{i}_{t}di \\
&\quad +\Delta\theta q(\theta)\int \mathbb{I}\left[x^{i}_{t}\le \frac{x-o(\Delta)}{1-\Delta\delta}\right]\left(1-e^{i}_{t}\right)di + o(\Delta) \\
&= (1-\Delta\lambda)e_{t}F^{e}_{t}\left(\frac{x-o(\Delta)}{1+\Delta\alpha}\right) \\
&\quad +\Delta\theta q(\theta)(1-e_{t})F^{u}_{t}\left(\frac{x-o(\Delta)}{1-\Delta\delta}\right) + o(\Delta) \qquad \text{(B.73)}
\end{aligned}$$

Subtracting $e_{t}F^{e}_{t}(x)$ from both sides of the above and dividing by $\Delta$ yields:

$$\begin{aligned}
\frac{e_{t+\Delta}F^{e}_{t+\Delta}(x)-e_{t}F^{e}_{t}(x)}{\Delta} &= \frac{e_{t}}{\Delta}\left[F^{e}_{t}\left(\frac{x-o(\Delta)}{1+\Delta\alpha}\right)-F^{e}_{t}(x)\right]-\lambda e_{t}F^{e}_{t}\left(\frac{x-o(\Delta)}{1+\Delta\alpha}\right) \\
&\quad +\theta q(\theta)(1-e_{t})F^{u}_{t}\left(\frac{x-o(\Delta)}{1-\Delta\delta}\right)+\frac{o(\Delta)}{\Delta} \\
&= -e_{t}\left[\frac{\alpha}{1+\Delta\alpha}x+\frac{o(\Delta)}{\Delta}\frac{1}{1+\Delta\alpha}\right] \\
&\quad \times\left[\frac{F^{e}_{t}(x)-F^{e}_{t}\left(x-\left[\frac{\Delta\alpha}{1+\Delta\alpha}x+\frac{o(\Delta)}{1+\Delta\alpha}\right]\right)}{\frac{\Delta\alpha}{1+\Delta\alpha}x+\frac{o(\Delta)}{1+\Delta\alpha}}\right] \\
&\quad -\lambda e_{t}F^{e}_{t}\left(\frac{x-o(\Delta)}{1+\Delta\alpha}\right) \\
&\quad +\theta q(\theta)(1-e_{t})F^{u}_{t}\left(\frac{x-o(\Delta)}{1-\Delta\delta}\right)+\frac{o(\Delta)}{\Delta}. \qquad \text{(B.74)}
\end{aligned}$$

Taking the limit as $\Delta \to 0$ yields:

$$\frac{\partial}{\partial t}\left[e_t F_t^e(x)\right] = -e_t \alpha x \frac{\partial}{\partial x}\left[F_t^e(x)\right] - \lambda e_t F_t^e(x) + \theta q(\theta)(1-e_t) F_t^u(x). \tag{B.75}$$

Applying the product rule to the left-hand side of the above expression and rearranging terms yields:

$$\frac{\partial}{\partial t} F_t^e(x) = -\alpha x f_t^e(x) + \theta_s q(\theta_s)\left(\frac{1-e_t}{e_t}\right)\left[F_t^u(x) - F_t^e(x)\right]. \tag{B.76}$$

Similarly, note that:

$$\begin{aligned}
(1-e_{t+\Delta}) F_{t+\Delta}^u(x) &= \int \mathbb{I}\left[x_{t+\Delta}^i \leq x\right]\left(1-e_{t+\Delta}^i\right) di \\
&= \int \mathbb{I}\left[x_{t+\Delta}^i \leq x\right]\left(1-e_{t+\Delta}^i\right)\left(1-e_t^i\right) di \\
&\quad + \int \mathbb{I}\left[x_{t+\Delta}^i \leq x\right]\left(1-e_{t+\Delta}^i\right) e_t^i di \\
&= \int \mathbb{I}\left[x^i + \Delta \dot{x}^i \leq x\right]\left(1-e_{t+\Delta}^i\right)\left(1-e_t^i\right) di \\
&\quad + \int \mathbb{I}\left[(1-\zeta)\left(x_t^i + \Delta \dot{x}^i\right) \leq x\right]\left(1-e_{t+\Delta}^i\right) e_t^i di + o(\Delta) \\
&= \left[1-\Delta\theta q(\theta)\right] \int \mathbb{I}\left[x^i + \Delta \dot{x}^i \leq x\right]\left(1-e_t^i\right) di \\
&\quad + \Delta\lambda \int \mathbb{I}\left[(1-\zeta)\left(x_t^i + \Delta \dot{x}^i\right) \leq x\right] e_t^i di + o(\Delta) \\
&= \left[1-\Delta\theta q(\theta)\right] \int \mathbb{I}\left[x^i \leq \frac{x}{1-\Delta\delta}\right]\left(1-e_t^i\right) di \\
&\quad + \Delta\lambda \int \mathbb{I}\left[x_t^i \leq \frac{x}{(1-\zeta)(1+\Delta\alpha)}\right] e_t^i di + o(\Delta) \\
&= \left[1-\Delta\theta q(\theta)\right](1-e_t) F_t^u\left(\frac{x}{1-\Delta\delta}\right) \\
&\quad + \Delta\lambda e_t F_t^e\left(\frac{x}{(1-\zeta)(1+\Delta\alpha)}\right) + o(\Delta).
\end{aligned} \tag{B.77}$$

Subtracting $(1-e_t)\,F_t^u(x)$ from both sides of the above and dividing by $\Delta$ yields:

$$\begin{aligned}
\frac{(1-e_{t+\Delta})\,F_{t+\Delta}^u(x)-(1-e_t)\,F_t^u(x)}{\Delta} &= \left(\frac{1-e_t}{\Delta}\right)\left[F_t^u\left(\frac{x}{1-\Delta\delta}\right)-F_t^u(x)\right]+\frac{o(\Delta)}{\Delta} \\
&\quad -\theta q(\theta)\,(1-e_t)\,F_t^u\left(\frac{x}{1-\Delta\delta}\right) \\
&\quad +\lambda e_t F_t^e\left(\frac{x}{(1-\zeta)\,(1+\Delta\alpha)}\right) \\
&= (1-e_t)\,\frac{\delta x}{(1-\Delta\delta)}\left[\frac{F_t^u\left(x+\frac{\Delta\delta x}{(1-\Delta\delta)}\right)-F_t^u(x)}{\frac{\Delta\delta x}{(1-\Delta\delta)}}\right] \\
&\quad -\theta q(\theta)\,(1-e_t)\,F_t^u\left(\frac{x}{1-\Delta\delta}\right)+\frac{o(\Delta)}{\Delta} \\
&\quad +\lambda e_t F_t^e\left(\frac{x}{(1-\zeta)\,(1+\Delta\alpha)}\right).
\end{aligned} \tag{B.78}$$

Taking the limit as $\Delta\to 0$ yields:

$$\frac{\partial}{\partial t}\left[(1-e_t)\,F_t^u(x)\right]=(1-e_t)\,\delta x\frac{\partial}{\partial x}\left[F_t^u(x)\right]-\theta q(\theta)\,(1-e_t)\,F_t^u(x)+\lambda e_t F_t^e\left(\frac{x}{1-\zeta}\right). \tag{B.79}$$

Applying the product rule to the left-hand side of the above expression and rearranging terms yields:

$$\frac{\partial}{\partial t}F_t^u(x)=\delta x f_t^u(x)+\lambda\frac{e_t}{1-e_t}\left[F_t^e\left(\frac{x}{1-\zeta}\right)-F_t^u(x)\right]. \tag{B.80}$$

□

## B.9 Proposition 11

*Claim.* Suppose that the initial conditional distributions are continuously differentiable with finite second moments, and that $f_0^e(0)=f_0^u(0)=0$. The coefficients of variation $(c^e,c^u)$ are jointly characterized by an affine system of differential equations with time-varying coefficients. If the economy remains in state $s$ for a sufficiently long time, then $\frac{\dot{c}^e}{c^e}$ and $\frac{\dot{c}^u}{c^u}$ converge to a positve constant, denoted $\iota_s$. If the employment rate converges to a value sufficiently close to one, then $\iota_1<\iota_0$.

*Proof.* I will begin by deriving the laws of motion for $c^e$ and $c^u$. Observe that for $l \in \{e, u\}$:

$$
\begin{aligned}
\frac{\dot{c}_t^l}{c_t^l} &= \frac{\frac{d}{dt}\int\left(x-\bar{x}_t^l\right)^2 f_t^l(x)\,dx}{\int\left(x-\bar{x}_t^l\right)^2 f_t^l(x)\,dx} - 2\frac{\frac{d}{dt}\bar{x}_t^l}{\bar{x}_t^l} \\
&= \frac{\int\left[-2\left(x-\bar{x}_t^l\right) f_t^l(x)\frac{d}{dt}\bar{x}_t^l + \left(x-\bar{x}_t^l\right)^2\frac{\partial}{\partial t}\left[f_t^l(x)\right]\right]dx}{\int\left(x-\bar{x}_t^l\right)^2 f_t^l(x)\,dx} - 2\frac{\frac{d}{dt}\bar{x}_t^l}{\bar{x}_t^l} \\
&= \frac{2\int\left(x-\bar{x}_t^l\right) f_t^l(x)\,dx\frac{d}{dt}\bar{x}_t^l + \int\left(x-\bar{x}_t^l\right)^2\frac{\partial}{\partial t}\frac{\partial}{\partial x}\left[F_t^l(x)\right]dx}{\int\left(x-\bar{x}_t^l\right)^2 f_t^l(x)\,dx} - 2\frac{\frac{d}{dt}\bar{x}_t^l}{\bar{x}_t^l} \\
&= \frac{\int\left(x-\bar{x}_t^l\right)^2\frac{\partial}{\partial x}\frac{\partial}{\partial t}\left[F_t^l(x)\right]dx}{\int\left(x-\bar{x}_t^l\right)^2 f_t^l(x)\,dx} - 2\frac{\frac{d}{dt}\bar{x}_t^l}{\bar{x}_t^l}. \qquad \text{(B.81)}
\end{aligned}
$$

I will now compute all of the pieces of the above expression. From the laws of motion for aggregate human capital and employment, we have:

$$
\begin{aligned}
\frac{\frac{d}{dt}\bar{x}_t^e}{\bar{x}_t^e} &= \frac{\dot{x}_t^e}{\bar{x}_t^e} - \frac{\dot{e}_t}{e_t} \\
&= \alpha + \theta q(\theta)\frac{x^u}{x^e} - \theta q(\theta)\left(\frac{1-e}{e}\right) \\
&= \alpha + \theta q(\theta)\left(\frac{1-e}{e}\right)\left(\frac{\bar{x}^u-\bar{x}^e}{\bar{x}^e}\right) \qquad \text{(B.82)} \\
\frac{\frac{d}{dt}\bar{x}_t^u}{\bar{x}_t^u} &= \frac{\dot{x}_t^u}{\bar{x}_t^u} + \frac{\dot{e}_t}{1-e_t} \\
&= \lambda(1-\zeta)\frac{x^e}{x^u} - \delta - \left(\frac{e}{1-e}\right)\lambda \\
&= \lambda\left(\frac{e}{1-e}\right)\left[\frac{(1-\zeta)\bar{x}^e-\bar{x}^u}{\bar{x}^u}\right] - \delta. \qquad \text{(B.83)}
\end{aligned}
$$

Assuming the second moments of all these distributions exist, integration by parts gives us for $l \in \{e, u\}$:

$$
\begin{aligned}
\int\left(x-\bar{x}_t^l\right)^2\frac{\partial}{\partial x}\left[f_t^l(x)\right]dx &= \left[\left(x-\bar{x}_t^l\right)^2 f_t^l(x)\right]_{x=0}^{\infty} - 2\int\left(x-\bar{x}_t^l\right) f_t^l(x)\,dx \\
&= \left(\bar{x}_t^l\right)^2 f_t^l(0). \qquad \text{(B.84)}
\end{aligned}
$$

Assuming that $f_t^l(0) = 0$ means that the above is zero. Note that:

$$\begin{aligned}
\int (x - \bar{x}_t^e)^2 f_t^u(x)\,dx &= \int \left[(x - \bar{x}_t^u) + (\bar{x}_t^u - \bar{x}_t^e)\right]^2 f_t^u(x)\,dx \\
&= \int \left[(x - \bar{x}_t^u) + (\bar{x}_t^u - \bar{x}_t^e)\right]^2 f_t^u(x)\,dx \\
&= \int (x - \bar{x}_t^u)^2 f_t^u(x)\,dx + 2(\bar{x}_t^u - \bar{x}_t^e)\int (x - \bar{x}_t^u) f_t^u(x)\,dx \\
&\quad + (\bar{x}_t^u - \bar{x}_t^e)^2 \int f_t^u(x)\,dx \\
&= \int (x - \bar{x}_t^u)^2 f_t^u(x)\,dx + (\bar{x}_t^u - \bar{x}_t^e)^2. \qquad \text{(B.85)}
\end{aligned}$$

Likewise:

$$\begin{aligned}
\int (x - \bar{x}_t^u)^2 f_t^e\left(\frac{x}{1-\zeta}\right) dx &= \int \left(\left[\frac{x}{1-\zeta} - \frac{\bar{x}^e}{1-\zeta}\right] - \left[\frac{\bar{x}_t^u}{1-\zeta} - \frac{\bar{x}^e}{1-\zeta}\right]\right)^2 f_t^e\left(\frac{x}{1-\zeta}\right) dx \\
&\quad \times (1-\zeta)^2 \\
&= (1-\zeta)^2 \int \left[\frac{x}{1-\zeta} - \frac{\bar{x}^e}{1-\zeta}\right]^2 f_t^e\left(\frac{x}{1-\zeta}\right) dx \\
&\quad + (1-\zeta)^2 \left[\frac{\bar{x}_t^u}{1-\zeta} - \frac{\bar{x}^e}{1-\zeta}\right]^2 \\
&= (1-\zeta)^2 \int \left[\frac{x}{1-\zeta} - \frac{\bar{x}^e}{1-\zeta}\right]^2 f_t^e\left(\frac{x}{1-\zeta}\right) dx + (\bar{x}_t^u - \bar{x}^e)^2 \\
&= \int (x - \bar{x}^e)^2 f_t^e\left(\frac{x}{1-\zeta}\right) dx + (\bar{x}_t^u - \bar{x}^e)^2. \qquad \text{(B.86)}
\end{aligned}$$

Note that:

$$\begin{aligned}
\int \left(x - \bar{x}_t^l\right)^2 x \frac{\partial}{\partial x} f_t^l(x)\,dx &= \left[\left(x - \bar{x}_t^l\right)^2 x f_t^l(x)\right]_{x=0}^{\infty} - \int \left[2\left(x - \bar{x}_t^l\right) x + \left(x - \bar{x}_t^l\right)^2\right] f_t^l(x)\,dx \\
&= -3 \int \left(x - \bar{x}_t^l\right)^2 f_t^l(x)\,dx. \qquad \text{(B.87)}
\end{aligned}$$

Hence:

$$
\begin{aligned}
\int \left(x-\bar{x}_t^e\right)^2 \frac{\partial}{\partial x}\frac{\partial}{\partial t}\left[F_t^e(x)\right]dx &= \int \left(x-\bar{x}_t^e\right)^2 \frac{\partial}{\partial x}\left[-\alpha x f_t^e(x)+\theta q(\theta)\left(\frac{1-e}{e}\right)\left[F_t^u(x)-F_t^e(x)\right]\right]dx \\
&= -\alpha\left[\int \left(x-\bar{x}_t^e\right)^2 f_t^e(x)\,dx+\int \left(x-\bar{x}_t^e\right)^2 x\frac{\partial}{\partial x}f_t^e(x)\,dx\right] \\
&\quad +\theta q(\theta)\left(\frac{1-e}{e}\right)\left[\int \left(x-\bar{x}_t^e\right)^2 f_t^u(x)\,dx-\int \left(x-\bar{x}_t^e\right)^2 f_t^e(x)\,dx\right] \\
&= 2\alpha\int \left(x-\bar{x}_t^e\right)^2 f_t^e(x)\,dx \\
&\quad +\theta q(\theta)\left(\frac{1-e}{e}\right)\left[\int \left(x-\bar{x}_t^e\right)^2 f_t^u(x)\,dx-\int \left(x-\bar{x}_t^e\right)^2 f_t^e(x)\,dx\right] \\
&= \theta q(\theta)\left(\frac{1-e}{e}\right) \\
&\quad \times\left[\int \left(x-\bar{x}_t^u\right)^2 f_t^u(x)\,dx-\int \left(x-\bar{x}_t^e\right)^2 f_t^e(x)\,dx+\left(\bar{x}_t^u-\bar{x}_t^e\right)^2\right] \\
&\quad +2\alpha\int \left(x-\bar{x}_t^e\right)^2 f_t^e(x)\,dx. \qquad \text{(B.88)}
\end{aligned}
$$

It follows that:

$$
\begin{aligned}
\frac{\dot{c}_t^e}{c_t^e} &= \frac{\int \left(x-\bar{x}_t^e\right)^2 \frac{\partial}{\partial x}\frac{\partial}{\partial t}\left[F_t^e(x)\right]dx}{\int \left(x-\bar{x}_t^e\right)^2 f_t^e(x)\,dx}-2\frac{\frac{d}{dt}\bar{x}_t^e}{\bar{x}_t^e} \\
&= \frac{2\alpha\int \left(x-\bar{x}_t^e\right)^2 f_t^e(x)\,dx+\theta q(\theta)\left(\frac{1-e}{e}\right)\left[\int \left(x-\bar{x}_t^u\right)^2 f_t^u(x)\,dx-\int \left(x-\bar{x}_t^e\right)^2 f_t^e(x)\,dx+\left(\bar{x}_t^u-\bar{x}_t^e\right)^2\right]}{\int \left(x-\bar{x}_t^e\right)^2 f_t^e(x)\,dx} \\
&\quad -2\left[\alpha+\theta q(\theta)\left(\frac{1-e}{e}\right)\left(\frac{\bar{x}^u-\bar{x}^e}{\bar{x}^e}\right)\right] \\
&= \theta q(\theta)\left(\frac{1-e}{e}\right)\left[\frac{c^u}{c^e}\left(\frac{\bar{x}^u}{\bar{x}^e}\right)^2+1-2\frac{\bar{x}^u}{\bar{x}^e}+\frac{1}{c^e}\left(\frac{\bar{x}_t^u-\bar{x}_t^e}{\bar{x}_t^e}\right)^2\right]. \qquad \text{(B.89)}
\end{aligned}
$$

For the unemployed, we have:

$$
\begin{aligned}
\int \left(x-\bar{x}_t^u\right)^2 \frac{\partial}{\partial x}\frac{\partial}{\partial t}\left[F_t^u(x)\right] dx &= \int \left(x-\bar{x}_t^u\right)^2 \frac{\partial}{\partial x}\left[\delta x f_t^u(x) + \lambda \frac{e}{1-e}\left[F_t^e\left(\frac{x}{1-\zeta}\right) - F_t^u(x)\right]\right] dx \\
&= \delta \int \left(x-\bar{x}_t^u\right)^2 \left[f_t^u(x) + x\frac{\partial}{\partial x} f_t^u(x)\right] dx \\
&\quad +\lambda \frac{e}{1-e}\int \left(x-\bar{x}_t^u\right)^2 \left[f_t^e\left(\frac{x}{1-\zeta}\right) - f_t^u(x)\right] dx \\
&= -2\delta \left[\int \left(x-\bar{x}_t^u\right)^2 f_t^u(x)\, dx + \int \left(x-\bar{x}_t^u\right)^2 x \frac{\partial}{\partial x} f_t^u(x)\, dx\right] \\
&\quad +\lambda \frac{e}{1-e}(1-\zeta)^2 \int \left(\frac{x}{1-\zeta} - \frac{\bar{x}_t^u}{1-\zeta}\right)^2 f_t^e\left(\frac{x}{1-\zeta}\right) dx \\
&\quad -\lambda \frac{e}{1-e}\int \left(x-\bar{x}_t^u\right)^2 f_t^u(x)\, dx \\
&= -2\delta \int \left(x-\bar{x}_t^u\right)^2 f_t^u(x)\, dx \\
&\quad +\lambda \frac{e}{1-e}\int \left(x-\bar{x}_t^e\right)^2 f_t^e(x)\, dx + \left(\bar{x}_t^u - \bar{x}_t^e\right)^2 \\
&\quad -\lambda \frac{e}{1-e}\int \left(x-\bar{x}_t^u\right)^2 f_t^u(x)\, dx
\end{aligned} \tag{B.90}
$$

Hence:

$$
\begin{aligned}
\frac{\dot{c}_t^u}{c_t^u} &= \frac{\frac{d}{dt}\int \left(x-\bar{x}_t^u\right)^2 f_t^u(x)\, dx}{\int \left(x-\bar{x}_t^u\right)^2 f_t^u(x)\, dx} - 2\frac{\frac{d}{dt}\bar{x}_t^u}{\bar{x}_t^u} \\
&= -2\delta + \lambda \frac{e}{1-e}\left[\frac{\int \left(x-\bar{x}_t^e\right)^2 f_t^e(x)\, dx}{\int \left(x-\bar{x}_t^u\right)^2 f_t^u(x)\, dx} + \frac{1}{\int \left(x-\bar{x}_t^u\right)^2 f_t^u(x)\, dx}\left(\bar{x}_t^u - \bar{x}_t^e\right)^2 - 1\right] \\
&\quad -2\left[\lambda\left(\frac{e}{1-e}\right)\left(\frac{\bar{x}^e - \bar{x}^u}{\bar{x}^u}\right) - \delta\right] \\
&= -2\delta + \lambda \frac{e}{1-e}\left[\frac{c^e}{c^u}\left(\frac{\bar{x}^e}{\bar{x}^u}\right)^2 + \frac{1}{c^u}\left(\frac{\bar{x}_t^u - \bar{x}_t^e}{\bar{x}^u}\right)^2 - 1\right] \\
&\quad -2\left[\lambda\left(\frac{e}{1-e}\right)\left(\frac{\bar{x}^e - \bar{x}^u}{\bar{x}^u}\right) - \delta\right] \\
&= \lambda \frac{e}{1-e}\left[\frac{c^e}{c^u}\left(\frac{\bar{x}^e}{\bar{x}^u}\right)^2 + \frac{1}{c^u}\left(\frac{\bar{x}_t^u - \bar{x}_t^e}{\bar{x}^u}\right)^2 - \frac{\bar{x}^e}{\bar{x}^u}\right].
\end{aligned} \tag{B.91}
$$

Thus, we get the following affine system of differential equations:

$$\begin{aligned}
\dot{c}^e &= \theta q(\theta)\left(\frac{1-e}{e}\right)\left(\frac{\bar{x}^u}{\bar{x}^e}\right)(1+c^u)\left(1+\frac{\bar{x}^u-\bar{x}^e}{\bar{x}^e}\right) \\
&\quad -\theta q(\theta)\left(\frac{1-e}{e}\right)\left(\frac{\bar{x}^u}{\bar{x}^e}\right)(1+c^e)\left(1+\frac{\bar{x}^u-\bar{x}^e}{\bar{x}^u}\right) \qquad \text{(B.92)} \\
\dot{c}^u &= \lambda\frac{e}{1-e}\left(\frac{\bar{x}^e}{\bar{x}^u}\right)\left[(1+c^e)\left(\frac{\bar{x}^e}{\bar{x}^u}\right)+\left(\frac{\bar{x}^u-\bar{x}^e}{\bar{x}^e}\right)-(1+c^u)\right]. \qquad \text{(B.93)}
\end{aligned}$$

Concisely, we can write the above as:

$$\begin{aligned}
\dot{\mathbf{c}}_t &= \mathbf{P}_t(\mathbf{c}_t-\bar{\mathbf{c}}_t) \qquad \text{(B.94)} \\
\mathbf{c}_t &\equiv \begin{bmatrix} c_t^e & c_t^u \end{bmatrix}', \qquad \text{(B.95)}
\end{aligned}$$

where the elements of $\mathbf{P}_t$ and $\bar{\mathbf{c}}_t$ depend on model parameters and $\theta q(\theta)$, $e$, and $\frac{\bar{x}^u}{\bar{x}^e}$. Within state $s$, the job-finding rate $\theta_s q(\theta_s)$ is constant. From the employment law of motion (2.5), we see that $e$ converges to $\frac{\theta_s q(\theta_s)}{\theta_s q(\theta_s)+\lambda}$. It's straigtforward to show from the laws of motion for $x^e$ (2.6) and $x^u$ (2.7) that $\frac{x^u}{x^e}$ also converges to a constant as the amount of time in state $s$ grows long; it follows that $\frac{\bar{x}^u}{\bar{x}^e}$ converges to a constant as well. Thus, as the amount of time spent in state $s$ grows long, $\mathbf{P}_t$ and $\bar{\mathbf{c}}_t$ converge to constants $\mathbf{P}_s$ and $\bar{\mathbf{c}}_s$. If $\dot{\mathbf{c}}_t=\mathbf{P}_s(\mathbf{c}_t-\bar{\mathbf{c}}_s)$, then the growth rate of $c^e$ and $c^u$ converges to the maximal eigenvalue of $\mathbf{P}_s$.[22] Define $\iota_s$ to be the maximal eigenvalue of $\mathbf{P}_s$.

I will now show that $\iota_s$ is positive. If system (B.94) did have a steady state and system (B.94) were not explosive, then $c^e$ and $c^u$ would converge to that steady state. But since $c^e$ and $c^u$ are coefficients of variation, we know that the must be positive. Hence, to show that system (B.94) is explosive, it is sufficient to show that it does not have a steady state with $c^e>0$ and $c^u>0$. Define $\chi_s$ to be the limiting value of $\frac{\bar{x}^u}{\bar{x}^e}$ in state $s$. The $\dot{c}^e=0$ locus and

[22] To see this, let $\Psi_s$ be a vector of the eigenvalues of $\mathbf{P}_s$ on the main diagonal, and let $\Upsilon_s$ be a matrix containing the corresponding eigenvectors. Let $\hat{\mathbf{c}}_t\equiv(\mathbf{c}_t-\bar{\mathbf{c}}_s)$. Since $\frac{d}{dt}\hat{\mathbf{c}}_t=\mathbf{P}_s\hat{\mathbf{c}}_t$, we know that $\hat{\mathbf{c}}_{t_n+t}=\Upsilon_s\mathrm{diag}\left(\exp\{\Psi_s t\}\right)\Upsilon_s^{-1}\hat{\mathbf{c}}_{t_n+t}$, using the same reasoning as in the proof of Proposition 6. This shows that the growth rate of $\hat{\mathbf{c}}_t$ converges to the maximal element of $\Psi_s$, so the growth rate of $\mathbf{c}_t$ must also converge to the maximal value of $\Psi_s$.

the $\dot{c}^u = 0$ locus are, respectively:

$$c^e = \frac{-(\chi_s - 1)^2 - \chi_s^2 c^u}{1 - 2\chi} \tag{B.96}$$

$$c^e = \chi_s c^u - (1 - \chi_s)^2 . \tag{B.97}$$

Consider the case where $1 - 2\chi_s > 0$. Then, the $\dot{c}^e = 0$ locus is a downward-sloping line with negative intercept in $c^u - c^e$ space. So, the $\dot{c}^e = 0$ locus does not pass through the first quadrant, meaning that it cannot intersect the $\dot{c}^u = 0$ locus at a point where $c^e > 0$ and $c^u > 0$. Now, suppose that $1 - 2\chi_s < 0$. Then, the $\dot{c}^e = 0$ locus is an upward-sloping line with positive intercept in $c^u - c^e$ space; the $\dot{c}^u = 0$ locus is an upward-sloping line with negative intercept in $c^u - c^e$ space. So, for these lines to intersect with $c^e > 0$ and $c^u > 0$, it must be the case that the $\dot{c}^u = 0$ locus is steeper than the $\dot{c}^e = 0$ locus. That is, a positive steady state requires:

$$\chi_s > -\frac{\chi_s^2}{1 - 2\chi_s}. \tag{B.98}$$

But this holds if, and only if, $\chi_s > 1$; in other words, the unemployed must have a higher average quality than the employed. I will show that this will not be the case. In the limit, as the amount of time in state $s$ grows long, $\frac{d}{dt}\bar{x}^e / \bar{x}^e = \frac{d}{dt}\bar{x}^u / \bar{x}^u$. The laws of motion for $\bar{x}^e$ and $\bar{x}^u$ imply that $\chi_s$ is characterized by:

$$\alpha + \delta + \lambda \left( \frac{e_s}{1 - e_s} - 1 \right) + \lambda \chi_s = \lambda \left( \frac{e_s}{1 - e_s} \right) (1 - \zeta) \frac{1}{\chi_s}, \tag{B.99}$$

where $e_s \equiv \frac{\theta_S q(\theta_s)}{\theta_S q(\theta_s) + \lambda}$ is the limiting value of the employment rate. Hence:

$$0 = \lambda \chi_s^2 + \left[ \alpha + \delta + \lambda \left( \frac{e_s}{1 - e_s} - 1 \right) \right] \chi_s - \lambda \left( \frac{e_s}{1 - e_s} \right) (1 - \zeta) . \tag{B.100}$$

The above quadratic has a unique positive root given by:

$$\chi_s = \frac{-\left[\alpha+\delta+\lambda\left(\frac{e_s}{1-e_s}-1\right)\right]+\sqrt{\left[\alpha+\delta+\lambda\left(\frac{e_s}{1-e_s}-1\right)\right]^2+4\lambda^2\left(\frac{e_s}{1-e_s}\right)(1-\zeta)}}{2\lambda}. \tag{B.101}$$

We see that $\chi_s \gtreqless 1$ if, and only if:

$$\begin{aligned}
\sqrt{\left[\alpha+\delta+\lambda\left(\frac{e_s}{1-e_s}-1\right)\right]^2+4\lambda^2\left(\frac{e_s}{1-e_s}\right)(1-\zeta)} &\gtreqless \lambda+\alpha+\delta+\lambda\frac{e_s}{1-e_s} \\
\iff \sqrt{\left[\alpha+\delta+\lambda\left(\frac{e_s}{1-e_s}-1\right)\right]^2+4\lambda^2\left(\frac{e_s}{1-e_s}\right)(1-\zeta)} &\gtreqless \alpha+\delta+\lambda\left(\frac{e_s}{1-e_s}+1\right) \\
\iff \left[\alpha+\delta+\lambda\left(\frac{e_s}{1-e_s}-1\right)\right]^2+4\lambda^2\left(\frac{e_s}{1-e_s}\right)(1-\zeta) &\gtreqless \left[\alpha+\delta+\lambda\left(\frac{e_s}{1-e_s}+1\right)\right]^2 \\
\iff \lambda^2\left(\frac{e_s}{1-e_s}-1\right)^2+4\lambda^2\left(\frac{e_s}{1-e_s}\right)(1-\zeta) &\gtreqless 4(\alpha+\delta)\lambda \\
&\quad +\lambda^2\left(\frac{e_s}{1-e_s}+1\right)^2 \\
\iff 0 &\gtreqless \alpha+\delta \\
&\quad +\lambda\left(\frac{e_s}{1-e_s}\right)\zeta. \qquad \text{(B.102)}
\end{aligned}$$

Thus, $\chi_s < 1$, so $\iota_s > 0$.

It remains to show that $\iota_1 < \iota_0$ if $e_0$ is sufficiently close to one. This will follow from a continuity argument. We know that in an economy with full employment ($e_s = 1$), all agents accumulate general human capital at the same constant rate, so $c^e$ must be constant, implying that $\iota_s = 0$. We also know that for any non-stochastic economy with some unemployment, $\iota_s > 0$. Also, $\iota_s$ is a continuous function of $\theta_s q(\theta_s)$, $e_s$, and $\chi_s$. Without loss of generality, we can write $\iota_s$ as a continuous function of $e_s$, since we can write both $\chi_s$ and $\theta_s q(\theta_s)$ as a continuous function of $e_s$. Thus, local to $e_s = 1$, $\iota_s$ must be decreasing in $e_s$. This implies that if $e_0$ is sufficiently close to one, then $\iota_1 < \iota_0$ because $e_0 < e_1$. $\square$

## B.10 Theorem 12

*Claim.* Suppose that $\epsilon(\theta)$ is a constant $\epsilon$. If $\epsilon = \eta$, then:

$$v_s(k, x^e, x^u) = G_s\left(x^e, \frac{k}{x^e}\right) + H_s\left(x^e, \frac{k}{x^e}\right) + U_s(x^u), \tag{B.103}$$

and the values of market tightness chosen by the planner coincide with the values of market tightness in the competitive equilibrium.

*Proof.* I conjecture that:

$$v_s(k, x^e, x^u) = \omega_s^k k + \omega_s^e x^e + \omega_s^u x^u. \tag{B.104}$$

Under this conjecture, the first-order condition (7.3) becomes:

$$\kappa = \left(\omega_s^k + \omega_s^e - \omega_s^u\right)(1-\epsilon)\, q(\theta). \tag{B.105}$$

This implies that the planner makes $\theta$ constant within each productivity regime $s$. Denote this maximizing value $\theta_s$. Plugging the conjecture into the Bellman equation, evaluated at the maximum, yields:

$$\begin{aligned}
r\left(\omega_s^k k + \omega_s^e x^e + \omega_s^u x^u\right) \quad = \quad & z_s k + (b - \kappa\theta_s)\, x^u + \omega_s^k\left[(\alpha + \rho - \lambda)\, k + \theta_s q(\theta_s)\, x^u\right] \\
& + \omega_s^e\left[(\alpha - \lambda)\, x^e + \theta_s q(\theta_s)\, x^u\right] \\
& + \omega_s^u\left[\lambda(1-\zeta)\, x^e - \left[\delta + \theta_s q(\theta_s)\right] x^u\right] \\
& + \beta\left[\left(\omega_{1-s}^k - \omega_s^k\right) k + \left(\omega_{1-s}^e - \omega_s^e\right) x^e + \left(\omega_{1-s}^u - \omega_s^u\right) x^u\right] \\
= \quad & \left[z_s + \omega_s^k(\alpha + \rho - \lambda) + \beta\left(\omega_{1-s}^k - \omega_s^k\right)\right] k \\
& + \left[b - \kappa\theta_s + \theta_s q(\theta_s)\left(\omega_s^k + \omega_s^e - \omega_s^u\right) - \omega_s^u\delta + \beta\left(\omega_{1-s}^u - \omega_s^u\right)\right] x^u \\
& + \left[\omega_s^e(\alpha - \lambda) + \omega_s^u\lambda(1-\zeta) + \beta\left(\omega_{1-s}^e - \omega_s^e\right)\right] x^e
\end{aligned} \tag{B.106}$$

Evidently:

$$r\omega_s^k = z_s + \omega_s^k (\alpha + \rho - \lambda) + \beta \left(\omega_{1-s}^k - \omega_s^k\right) \tag{B.107}$$

$$r\omega_s^e = \omega_s^e (\alpha - \lambda) + \omega_s^u \lambda (1 - \zeta) + \beta \left(\omega_{1-s}^e - \omega_s^e\right) \tag{B.108}$$

$$r\omega_s^u = b - \kappa\theta_s + \theta_s q(\theta_s) \left(\omega_s^k + \omega_s^e - \omega_s^u\right) - \omega_s^u \delta + \beta \left(\omega_{1-s}^u - \omega_s^u\right). \tag{B.109}$$

Notice that we can use the first-order condition to simplify the last of the above equations:

$$\begin{aligned}
r\omega_s^u &= b - \frac{\kappa\theta_s}{1-\epsilon}(1-\epsilon) + \frac{\theta_s}{1-\epsilon}(1-\epsilon) q(\theta_s) \left(\omega_s^k + \omega_s^e - \omega_s^u\right) - \delta\omega_s^u + \beta \left(\omega_{1-s}^u - \omega_s^u\right) \\
&= b + \frac{\epsilon\kappa}{1-\epsilon}\theta_s + \frac{\theta_s}{1-\epsilon}\left[(1-\epsilon) q(\theta_s) \left(\omega_s^k + \omega_s^e - \omega_s^u\right) - \kappa\right] - \delta\omega_s^u + \beta \left(\omega_{1-s}^u - \omega_s^u\right) \\
&= b + \frac{\epsilon\kappa}{1-\epsilon}\theta_s - \delta\omega_s^u + \beta \left(\omega_{1-s}^u - \omega_s^u\right).
\end{aligned} \tag{B.110}$$

Switching to vector notation, we can solve for the coefficients in terms of market tightness and primitives:

$$(r+\beta)\Omega^k = \mathbf{z} + (\alpha + \rho - \lambda)\Omega^k + \beta\Pi\Omega^k \tag{B.111}$$

$$(r+\beta)\Omega^e = (\alpha - \lambda)\Omega^e + \lambda(1-\zeta)\Omega^u + \beta\Pi\Omega^e \tag{B.112}$$

$$(r+\beta)\Omega^u = b\mathbf{1}_{S\times 1} + \frac{\epsilon\kappa}{1-\epsilon}\theta - \delta\Omega^u + \beta\Pi\Omega^u, \tag{B.113}$$

where

$$\Pi \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{B.114}$$

$$\Omega^l \equiv \begin{bmatrix} \omega_0^l & \omega_1^l \end{bmatrix}', \ l \in \{k, e, u\}. \tag{B.115}$$

Thus:

$$\Omega^k = [(r - \alpha + \beta + \lambda - \rho)\,\mathbf{I} - \beta\Pi]^{-1}\,\mathbf{z} \tag{B.116}$$

$$\Omega^e = \lambda(1-\zeta)\,[(r - \alpha + \beta + \lambda)\,\mathbf{I} - \beta\Pi]^{-1}\,\Omega^u \tag{B.117}$$

$$\Omega^u = [(r + \delta + \beta)\,\mathbf{I} - \beta\Pi]^{-1}\left[b\mathbf{1}_{S\times 1} + \frac{\epsilon\kappa}{1-\epsilon}\theta\right]. \tag{B.118}$$

The above uniquely determines $\Omega^k$, but we still need to determine $\theta$ in order to determine $\Omega^e$ and $\Omega^u$. Recall from the proof of Proposition 2:

$$\mathbf{N}^1 = [(r - \alpha + \beta + \lambda - \rho)\,\mathbf{I} - \beta\Pi]^{-1}\,\mathbf{z} \tag{B.119}$$

$$\mathbf{N}^0 = \lambda(1-\zeta)\,[(r - \alpha + \beta + \lambda)\,\mathbf{I} - \beta\Pi]^{-1}\,\mathbf{u} \tag{B.120}$$

$$\mathbf{u} = [(r + \delta + \beta)\,\mathbf{I} - \beta\Pi]^{-1}\left[b\mathbf{1}_{2\times 1} + \frac{\eta\kappa}{1-\eta}\theta\right]. \tag{B.121}$$

In other words, when $\eta = \epsilon$, we have $\Omega^k = \mathbf{N}^1$, $\Omega^e = \mathbf{N}^0$, and $\Omega^u = \mathbf{u}$. Thus, we can write the planner's first-order condition as:

$$\begin{aligned} \kappa\tilde{\mathbf{q}}(\theta) &= (1-\epsilon)\left[\Omega^k + \Omega^e - \Omega^u\right] \\ &= (1-\epsilon)\left[\mathbf{N}^0 + \mathbf{N}^1 - \mathbf{u}\right], \end{aligned} \tag{B.122}$$

which is identical to the implicit function that determines market tightness in a competitive equilibrium. Theorem 2 establishes that the solution to this equation exists and is unique. Finally, note that:

$$\begin{aligned} v_s(k, x^e, x^u) &= N_s^1 k + N_s^0 x^e + u_s x^u \\ &= \left(g_s^1 + h_s^1\right) k + \left(g_s^0 + h_s^0\right) x^e + u_s x^u \\ &= \left[\left(g_s^0 + g_s^1 \frac{k}{x^e}\right) + \left(h_s^0 + h_s^1 \frac{k}{x^e}\right)\right] x^e + u_s x^u \\ &= G_s\left(x^e, \frac{k}{x^e}\right) + H_s\left(x^e, \frac{k}{x^e}\right) + U_s(x^u). \end{aligned} \tag{B.123}$$

□