# Communication and Detection Theory

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## Practice Exam

http://www.isi.ee.ethz.ch/teaching/courses/cdt

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### Problem 1 — *Transforming an Energy-Limited Signal*

Let $\mathbf{v}$ be an energy-limited signal of self-similarity function $\mathsf{R}_{\mathbf{vv}}$, and let $\mathbf{p}$ be the signal

$$p(t) = \mathsf{A} v(\alpha t), \quad t \in \mathbb{R},$$

where $\mathsf{A}$ and $\alpha$ are real numbers (not necessarily positive) and $\alpha \neq 0$. Is $\mathbf{p}$ energy-limited? If so, relate its self-similarity function $\mathsf{R}_{\mathbf{pp}}$ to $\mathsf{R}_{\mathbf{vv}}$.

### Problem 2 — *Bandlimited Signals and Bandwidth*

For every $\lambda \in (0,1)$ let $\check{\mathbf{g}}_\lambda$ be the Inverse Fourier Transform of $\mathbf{g}_\lambda$, where

$$g_\lambda(f) = \begin{cases} 0 & \text{if } f < -\lambda W \text{ or } f > (1-\lambda)W, \\ \frac{1}{\lambda W}(f + \lambda W) & \text{if } -\lambda W \leq f \leq 0, \\ -\frac{1}{(1-\lambda)W}\bigl(f - (1-\lambda)W\bigr) & \text{if } 0 < f \leq (1-\lambda)W, \end{cases} \qquad f \in \mathbb{R},$$

and where $W$ is a fixed positive constant.

(i) Plot $\mathbf{g}_\lambda$ for a $\lambda$ of your choice.

(ii) For which values of $\lambda$, if any, is $\check{\mathbf{g}}_\lambda$ bandlimited to $W$ Hz?

(iii) For which values of $\lambda$, if any, is $\check{\mathbf{g}}_\lambda$ bandlimited to $W/4$ Hz?

(iv) What is the bandwidth of $\check{\mathbf{g}}_\lambda$? Express your answer in terms of $\lambda$ and $W$.

(v) Which $\lambda$ minimizes the bandwidth of $\check{\mathbf{g}}_\lambda$? What is the corresponding minimal bandwidth?

Consider $\mathbf{h}_\lambda$ and $\mathbf{u}_\lambda$ defined for $f_c > 0$ by

$$h_\lambda(f) = g_\lambda(f - f_c) + g_\lambda(f + f_c), \quad f \in \mathbb{R},$$

and

$$u_\lambda(f) = g_\lambda(|f| - f_c), \quad f \in \mathbb{R}.$$

Let $\check{\mathbf{h}}_\lambda$ and $\check{\mathbf{u}}_\lambda$ be the Inverse Fourier Transforms of $\mathbf{h}_\lambda$ and $\mathbf{u}_\lambda$.

(vi) For which positive values of $f_c$ is $\check{\mathbf{h}}_\lambda$ a passband signal around $f_c$? For which positive values of $f_c$ is $\check{\mathbf{u}}_\lambda$ a passband signal around $f_c$?

For the remainder assume that $\check{\mathbf{h}}_\lambda$ and $\check{\mathbf{u}}_\lambda$ are passband signals around $f_c$.

(vii) What is the bandwidth of $\check{\mathbf{h}}_\lambda$ around $f_c$? What is the bandwidth of $\check{\mathbf{u}}_\lambda$ around $f_c$?

(viii) For which values of $\lambda$, if any, is $\check{\mathbf{h}}_\lambda$ a *real* passband signal around $f_c$? For which values of $\lambda$, if any, is $\check{\mathbf{u}}_\lambda$ a *real* passband signal around $f_c$?

## Problem 3 — *Smoothing a PAM Signal*

Let $\big(X(t)\big)$ be the result of mapping the IID random bits $D_1, \ldots, D_\mathsf{K}$ to the real numbers $X_1, \ldots, X_\mathsf{N}$ using $\mathbf{enc}\colon \{0,1\}^\mathsf{K} \to \mathbb{R}^\mathsf{N}$ and then mapping these symbols to the waveform

$$X(t) = \mathsf{A} \sum_{\ell=1}^{\mathsf{N}} X_\ell \, g(t - \ell \mathsf{T_s}), \quad t \in \mathbb{R},$$

where $\mathsf{A} > 0$, where $\mathbf{g}$ is an energy-limited pulse shape, and where $\mathsf{T_s} > 0$ is the baud period. Define the stochastic process $\big(Y(t)\big)$ as

$$Y(t) = \frac{1}{17} \int_t^{t+17} X(\tau) \, \mathrm{d}\tau, \quad t \in \mathbb{R}.$$

Can $\big(Y(t)\big)$ be viewed as a PAM signal? If so, of what pulse shape?

## Problem 4 — *Hypothesis Testing*

Let the binary random variable $H$ take on the values 0 and 1 equiprobably. Let $\mathbf{W}$ be a standard Gaussian 3-vector that is independent of $H$. Consider the problem of guessing $H$ based on the observation $\mathbf{Y}$, where conditional on $H = 0$,

$$\mathbf{Y} = \mathsf{A}\mathbf{W},$$

and conditional on $H = 1$,

$$\mathbf{Y} = \mathsf{B}\mathbf{W},$$

where $\mathsf{A}$ and $\mathsf{B}$ are the deterministic matrices

$$\mathsf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix} \quad \text{and} \quad \mathsf{B} = \begin{pmatrix} 0 & 0 & \beta \\ 0 & \alpha & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

and where $\alpha$ and $\beta$ are positive real numbers and $\beta \neq 1$.

(i) If you must guess $H$ based on exactly two out of the three components of

$$\mathbf{Y} = (Y^{(1)}, Y^{(2)}, Y^{(3)})^\mathsf{T},$$

which two components would you choose in order to minimize the probability of error? Why?

(ii) Determine the conditional densities $f_{\mathbf{Y}|H=0}$ and $f_{\mathbf{Y}|H=1}$.

(iii) Find a one-dimensional sufficient statistic for guessing $H$ based on $\mathbf{Y}$.

(iv) Describe an optimal decision rule for guessing $H$ based on $\mathbf{Y}$.

(v) Compute the Bhattacharyya Bound on the optimal probability of error $p^\star(\text{error})$.

(vi) Compute $\lim_{\beta\to\infty} p^\star(\text{error})$.

## Problem 5 — *A Guessing Rule*

Let the received waveform $\bigl(Y(t)\bigr)$ be

$$Y(t) = X\,s(t) + N(t), \quad t \in \mathbb{R},$$

where $\mathbf{s}$ is a real, deterministic, integrable signal that is bandlimited to $W$ Hz, $X$ is a RV taking on the values $\pm 1$ equiprobably, and $\bigl(N(t)\bigr)$ is noise.

(i) Consider a decoder that guesses "$X = +1$" if $(\mathbf{Y} \star \mathbf{h})(0)$ is positive and guesses "$X = -1$" otherwise. Here $\mathbf{h}$ is some real, deterministic, integrable signal that is bandlimited to $W_c$ Hz. What is the probability of error of this decoder under the assumption that $\bigl(N(t)\bigr)$ is white Gaussian noise of PSD $N_0/2$ with respect to $\max\{W, W_c\}$? Express your answer using the $\mathcal{Q}$-function, $\mathbf{s}$, $\mathbf{h}$, and $N_0$.

(ii) Find an $\mathbf{h}$ (of whichever bandwidth you like) that minimizes the probability of error.

(iii) Evaluate the probability of error when $s(t) = A\operatorname{sinc}^2(Wt)$ for all $t \in \mathbb{R}$ and the frequency response of $\mathbf{h}$ closely resembles that of an ideal unit-gain LPF of cutoff frequency $W_c$. Which choice of $W_c$ minimizes the probability of error?

(iv) Suppose now that $X$ takes on the values $\pm 1, \pm 3$ equiprobably, and consider the guessing rule

$$\text{Guess} = \begin{cases} +3 & \text{if } (\mathbf{Y} \star \mathbf{h})(0) > \alpha, \\ +1 & \text{if } \alpha \geq (\mathbf{Y} \star \mathbf{h})(0) > 0, \\ -1 & \text{if } 0 \geq (\mathbf{Y} \star \mathbf{h})(0) > -\alpha, \\ -3 & \text{if } -\alpha \geq (\mathbf{Y} \star \mathbf{h})(0), \end{cases}$$

where $\alpha$ is some real number. Assume that $(\mathbf{s} \star \mathbf{h})(0) > 0$. Which choice of $\alpha$ minimizes the probability of error?