Misunderstanding that the Effective Action is Convex under Broken Symmetry, Studied via Ising Model

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The widespread belief that the effective action is convex and has a flat bottom under broken symmetry is shown to be wrong. To draw concrete results we center on the 2-dimensional Ising model, and there it is shown that the exact transition point is not at $|M| = M_{\rm sp}$ but slightly shifted, and even though the extensive variable M, the magnetization, is used, discontinuity appears. A simple model of phase coexistence is found to obey the Born rule.

Much has been said about the effective action $\Gamma[\langle\varphi(x)\rangle]$ under broken symmetry where the field has the vacuum expectation value $\langle\varphi\rangle = \pm\varphi_{\rm sp} \neq 0$. For simplicity, we here use the language of a field theory with one scalar field φ. At its core lies the belief: "Because of (A) its downward convexity, (B) the effective action has a flat bottom for $|\langle\varphi\rangle| < \varphi_{\rm sp}$. (C) States in the flat bottom are realized as a linear superposition of the vacua $|\pm\Omega\rangle$, where $\langle\pm\Omega|\varphi(x)|\pm\Omega\rangle = \pm\varphi_{\rm sp}$."[1] But as we will see, all the points (A), (B) and (C) are wrong. Being unaware of it, numerous confused reports have been done.[2]

Instead of quantum field theories, we switch to statistical mechanics for a while. Unless otherwise stated, we use the 2-dimensional square Ising ferromagnetic model with nearest-neighbor interaction of which volume is $V = L^2$. Even though it has always been the best studied model, we will see its understanding is inadequate. We fix the temperature $T = 1/\beta$ lower than the critical temperature T_c, so that the model exhibits spontaneous magnetization $m = \pm m_{\rm sp}(T)$ under zero external field. Our model is, together with relevant symbols, defined as:

$$\begin{aligned} Z(\beta,h) &:= \sum_{\{\sigma_i=\pm1\}} \exp(-\beta H(h)), \\ H(h) &:= -J\sum_{n.n.}\sigma_i\sigma_j - hM, \\ M &:= \sum_i \sigma_i, \\ m &:= M/V, \\ \beta F(\beta,h) &:= -\log Z, \\ \beta D(\beta,m) &:= \beta(F+hM) \\ &= -\log \sum_{\{\sum\sigma_i=M\}} e^{-\beta(H+hM)}. \end{aligned}$$

$D(\beta,m)$, the Legendre transform of the free energy $F(\beta,h)$, plays a role similar to the effective potential $V(\langle\varphi\rangle)$, which is the effective action with spatially uniform vacuum configuration. It is the free energy when (β,m) are the variables that specify the state of the system, and $F(h=\pm0) = D(m = \pm m_{\rm sp})$. We will often present statements and equations which are exactly correct and/or meaningful only in thermodynamic limit, but readers will have no difficulty understanding them.

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The case where $|m| < m_{\rm sp}$, i.e. the region singular with respect to the external field (RSEF hereafter), is of our interest. To write down D, we notice $Z(h=0) = \sum_m \exp(-\beta D(m))$, so $P(m') := \exp(-\beta D(m'))/Z$ is the probability that m takes the value m' when (T,h) are specified. Fortunately the analytic formula of P is obtained for the 2d Ising model for free and periodic boundaries in thermodynamic limit, at least for low enough temperature.[3] It can be rewritten for D as:

$$\begin{aligned} D^+(m) &:= D(m) - D(m_{\rm sp}) \\ &= \begin{cases} aLT\sqrt{m_{\rm sp}-|m|} & \text{for } m_{\rm f} < |m| < m_{\rm sp}, \\ aLT\sqrt{m_{\rm sp}-m_{\rm f}} = \text{const.} & \text{for } |m| < m_{\rm f}, \end{cases} \end{aligned} \tag{1}$$

where $m_{\rm f} > 0$ and $a > 0$ are finite constants dependent on β and the boundary condition. It is now obvious that the claims (A), the convexity of D, and (B), the "flat bottom" are illusions. D is *not* always downward convex.

To understand it better, we have to be careful about the normalization of variables. Even though M is discrete, we would like to define $\partial D/\partial M$ and to expect it to be $= h$. There is no problem by defining it as $(D(M+n) - D(M))/n$ $(n \in \mathbb{Z})$, since $\to \partial D/V\partial m$ as $L \to \infty$. Or $\partial d/\partial m = h$, where $d := D^+/V$. But d is $= 0$ throughout the RSEF. So even though d is differentiable and globally convex, it is a "boring" quantity, containing less information than D.

It is odd that non-convexity has already been known for long in statistical mechanics, but it has been misunderstood to be an artifact of finite systems, and that convexity (of d) is recovered in thermodynamic limit.[4] It is true that $d \to 0$ as $L \to \infty$ inside the RSEF irrespective of the dimensionality and the broken symmetry, (in general there are multiple external fields h_k and $M_k = \partial F/\partial h_k$, but that does not matter) because it is what Legendre transform should satisfy. But that ought not to be confused with the fact that D is not convex. Rather, D^+ is, and has to be > 0, and it dictates the physics in the RSEF. This fact has also been well known in the field of Monte Carlo simulations. After all, if D were flat-bottomed, how could symmetry break spontaneously? To summarize, $D^+(|m| > m_{\rm sp}), D, M = O(L^2), D^+(|m| < m_{\rm sp}) = o(L^2)$ and $m = O(L^0)$, where O and o are Landau's symbols.

We here clarify the flaw in the proof of D's convexity: F is convex in h *inside the intervals where F is regular in h*, and this fact has to be used to prove D's convexity. Thus F's convexity does not imply that D is convex in the RSEF, which corresponds to one, singular point $h = 0$.

An intuitive understanding of the D^+'s equation is desirable.[3] First take the free boundary. When $|m| < m_{\rm f}$ ("flat region" hereafter), the system consists of two domains, the plus-spin rich phase and minus rich one, separated by a straight wall, running parallel to the two system edges, with $D^+ = sL$, where s is the surface free energy per unit length. (Although 2-dimensional, we use the word domain *wall*.) As long as m stays in the flat region, the wall can shift but remains straight, and D does not change. When $m_{\rm f} < |m| < m_{\rm sp}$, the smaller domain favors to be a droplet[5] at a corner to lower the energy. If the boundary is periodic, the domains in the flat region are strips with two walls, and the droplet can form anywhere. We immediately recognize that (i) the RSEF is a phase coexistence region, (ii) $L\sqrt{m_{\rm sp} - |m|}$ is of the order of the wall length, and (iii) the flatness of D means the wall's thickness is finite, or the wall is "sharp", so does not interact with the boundaries, or with the other wall under periodic boundary. For general boundaries the shape of the domain is determined by "Wulff construction".[6]

The phase transition at $|m| = m_{\rm f}$ is sometimes called droplet-strip transition. It is obviously first-order; both domains discontinuously change their geometries, and both sides of the transition point have the metastable branch. It also lacks universality: The situation that D is constant in the flat region is specific to rectangular systems, and is not true for general geometries. (On the other hand, the sharpness of the wall is common to any case.) There may exist geometries without droplet-strip transition point altogether. At the same time, it is noteworthy that D for rectangular systems has a continuous symmetry, by being constant, which connects the states of different m's. So even though we control m by hand, there may exist something similar to Goldstone boson. This continuous symmetry is also marked by its sudden termination at $|m| = m_{\rm f}$.

Next we consider the phase transition at $\mu := m_{\rm sp} - |m| = 0$. Let us compute the energy inside the RSEF:

$$E^+(m) := E(m) - E(m_{\rm sp}) = \frac{\partial \beta D^+}{\partial \beta}|_\mu = b\mu^{1/2}, \qquad (2)$$

where b depends on β but not on μ. Here, the derivative with β is taken by fixing μ, or equivalently by first fixing m and $m_{\rm sp}(\beta_0)$, and by putting $\beta_0 = \beta$ after the differential. (If m and $m_{\rm sp}(\beta)$ are fixed in the derivative, E^+ diverges as $\mu^{-1/2}$. This definition must be wrong.)

Because we are using the extensive variable m, the transition looks continuous and there is no latent heat. More precisely, it is first-order when approached from the the opposite limit $|m| \searrow m_{\rm sp}$, by not showing any sign of transition beforehand, and the transition happens suddenly at $\mu = 0$. After the system passes the transition point, it can go to the metastable branch, too. On the other hand, inside the RSEF there are relations $D^+, E^+ \propto \mu^{1/2}$ which look like scaling laws, just like continuous transitions. This is partly expected, because phase coexistence terminates at the point $\mu = 0$, or put differently cannot extend over that point, so there must be a singularity, and by definition, $D^+, E^+ = 0$ at $\mu = 0$.

In reality this transition is *not* continuous and accompanies discontinuity, and as we will see, it is inevitable. For $m > m_{\rm sp}$ D is regular, so

$$D(m) = D(m_{\rm sp}) + L^2(c_2\mu^2 + c_3\mu^3 + ...), \qquad (3)$$

where $c_2 = \partial h/\partial m|_{h=0} = O(1)$. This equation should also describe the metastable branch inside the RSEF. By comparing this expression with Eq. (1), inside the RSEF $aLT\mu^{1/2} > c_2L^2\mu^2$ for small enough $\mu > 0$, and the true transition point μ_t is found slightly inside the RSEF, at $\mu_t = aTc_2^{-1}L^{-2/3}$, or $M_{\rm sp} - |M_t| \propto L^{4/3}$. This was rigorously proven in Ref. 9, but we were able to obtain it easily by Eq. (1), giving another physical interpretation. Although $\mu_t \to 0$ in thermodynamic limit, this shift is observable since $M_{\rm sp} - |M_t| \to \infty$. Latent heat is there, too. (If not, special symmetry is hidden.) In words, even if μ is put to 0 from the inside of the RSEF, the big one droplet disappears all of a sudden, and this discontinuity is necessary even if the *extensive* variable M is utilized. Also notice $D^+ \propto \mu^{1/2}$ represents a metastable branch for $0 < \mu < \mu_t$.

This is not an exceptional, isolated case. Consider a general system with the dimension k. When the symmetry is continuous, domain walls cannot form,[11] but anyway as we stated D^+ should be $\propto L^{k-\epsilon}$ for some $\epsilon > 0$ inside the RSEF. So one likely scenario is that $D^+ \propto (M_{\rm sp} - |M|)^{(k-\epsilon)/k} = (L^k\mu)^{1-\epsilon/k}$. For the cases this assumption holds, D^+ is *upward* convex at least for $1 \gg \mu$, and this D^+ is bigger than the regular branch as $\mu \to 0$. The transition point then lies always inside the RSEF at $\mu_t \propto L^{-k\epsilon/(k+\epsilon)} \stackrel{L\to\infty}{\to} 0$ or $M_{\rm sp} - |M_t| \propto L^{k^2/(k+\epsilon)} \stackrel{L\to\infty}{\to} \infty$.

We move on to the issue of bulk order parameters. In Ising models the total magnetization M or its per spin value m is almost always employed, but it loses the meaning as an order parameter in the RSEF because it is not what the system shows in response, but at our disposal. However, impose free boundary (we postpone the periodic boundary case) and define

$$r := \frac{1}{V}\sum_i \langle\sigma_i\rangle^2,$$

brackets meaning the thermal average. Then r does not depend on m (remember that the wall's thickness is finite), thus $r = m_{\rm sp}^2$. So, even inside the RSEF, there still is an order parameter which measures the magnitude of spontaneous magnetization, and depends only on T. The r's independence of m must be the consequence of the symmetry that all states in the RSEF correspond to $h = 0$, but we fail to derive r in that way.

Here the fact that symmetry of inversion along the x- and y-axes is broken in equilibrium is used. This can be proved by adding for example an x-symmetry breaking term $u\Upsilon$ to the Hamiltonian, where $\Upsilon := \sum_i x_i\sigma_i$, x_i is the x-coordinate of the site i, and u is an external field. We need the case of $u \to \pm 0$. Do Legendre transform from u to the conjugate extensive variable Υ and plot the free energy. By an argument similar to that of D vs M, the states of the extrema of Υ are favored, since the second domain wall accompanies intermediate states.

There is extra intricacy for periodic boundary, namely translational invariance. For Ising models with free m, in other words where (T, h) are the independent variables, it is known for $d \geq 3$ equilibrium translational invariance is broken under various boundary conditions.[6] In rigorous proofs, typical situations are such that $+/-$ spin is favored at $x = 0$ or $= L$, or antiperiodic in x-direction, and the rest boundaries are free or periodic. In $d = 2$, translational invariance is preserved. In general, translational invariance breakdown is more likely for fixed m than free m because wall's deviation is more restricted, and for our case where m is the input, we give two simple (handwaving) arguments to suggest translational invariance breakdown. Let us consider a $2L \times L$ system so that the wall runs along y-direction. The first is based on the linearity of correlation functions: For any translationally invariant equilibrium of a 2d system with free m, there exists α such that any correlation function $\langle\sigma_{i_1}...\sigma_{i_n}\rangle$ can be expressed as $\alpha\langle\sigma_{i_1}...\sigma_{i_n}\rangle_+ + (1-\alpha)\langle\sigma_{i_1}...\sigma_{i_n}\rangle_-$, where $\langle\sigma_i\rangle_\pm = \pm m_{\rm sp}$.[8] But when m is fixed to 0, $\langle\sigma(0,y)\sigma(L,y)\rangle = -m_{\rm sp}^2$ cannot be a linear sum of $\langle\sigma(0,y)\sigma(L,y)\rangle_\pm = m_{\rm sp}^2$. (Strictly speaking, in the argument of free m the spins of correlation functions are fixed, while for fixed m we considered spins' separation which depends on L.) In the second, we borrow notations from field theories, and again take the case $m = 0$. Let $|s\rangle$ symbolically mean the state where $\langle s|\varphi(x)|s\rangle = \varphi_{\rm sp}$ for $s < x < s+L$, and $= -\varphi_{\rm sp}$ otherwise. The problem is, roughly speaking, if the equilibrium is unique, being $= \int dx|x\rangle$, or each of $|x\rangle$ is one of the equilibria. Add to Hamiltonian a translational invariance breaking term $-\varepsilon\int_t^{t+L} dx\varphi(x)$ for any t, and see if the limit $\varepsilon \to 0$ depends on t or not. For any finite $\varepsilon > 0$, however small, the state $|x\rangle$ costs energy $\propto |x-t|L$ but the entropy gain of translational invariance preservation is $\propto \log L$, so $|t\rangle$ is the only equilibrium.

These are not proofs, but if translational invariance is broken under periodic boundary, then r is again $= m_{\rm sp}^2$ for any m, so it really is a good order parameter, and unlike d which is also constantly $= 0$, r is not trivial. We also ask this question: We specify m, so there is no more spontaneous breakdown of the symmetry of m being $= \pm m_{\rm sp}$. Has it changed its guise to the breakdown of translational invariance?

Finally we would like to come back to field theory, i.e. the continuum limit. First remember how to get φ^4-field theory from the Ising model:[12] Convert the spin variables $\{\sigma_i\}$ to real-valued variables $\{\varphi_i\}$ by Hubbard-Stratonovich transform and take some more steps to clean up. Notice the condition $\sum\sigma_i = M$ is mapped to $\sum\varphi_i = M$, since $T\partial\log Z/\partial h_i = \langle\sigma_i\rangle = \langle\varphi_i\rangle$ (with appropriate normalization), where h_i is a per-site external field, and since Z is the generating functional. Then Fourier-expand φ_i and retain only low-frequency modes.

We have to deal with the obstacle that m was specified in the lattice system. We continue to use L to mean the system's linear extent. We look at four options in turn: (i) Keep L finite. In this case, M is finite, but there is no phase transition, even if the "mass", i.e. the quadratic term in Lagrangian is negative. Although D has a maximum at $m = 0$, this case is not interesting. (ii) This is not a field theory, but what if L is finite and render the lattice spacing to 0? Then the wall is a singular line, and $\langle\varphi(x)\varphi(y)\rangle = 0$ for $x \neq y$. This theory is unusable. (iii) Let $L \to \infty$, and keep the wall near the origin. In this limit, the domain wall gets stretched into a straight line, with the boundary conditions, say $\langle\varphi\rangle \to \pm\varphi_{\rm sp}$ as $x^1 \to \pm\infty$, and the original condition of m has vanished. (iv) Still $L \to \infty$, and also send the domain wall infinitely far. Then we get the theory $\langle\varphi\rangle = \pm\varphi_{\rm sp}$, even though we started from the lattice theory inside the RSEF. This is not intriguing. In cases (iii) and (iv), we assumed that we can neglect the constraint $\int dx\varphi(x) =: \tilde{M}$, which is nonlocal. This must be no problem insofar as there are only finite excitations. Variant models in which the system edge is there, so e.g. y is limited to ≥ 0, are also possible.[13]

We elaborate the case (iii) above. To be general, let the spational dimensionality be d and the Hamiltonian H have the potential V:

$$H = \int d^d x \left(\frac{1}{2}\overrightarrow{\frac{d^2}{dx^2}}\varphi + V(\varphi) \right),$$

where $V(\varphi)$ takes the absolute minimum 0 at $\pm\varphi_{\rm sp}$. We impose the boundary condition $\varphi(x) \to \pm\varphi_{\rm sp}$ as $x^1 \to \pm\infty$. At the classical level, H has the minimum value, or the vacuum energy,

$$E_{\rm vac} := L^{d-1}\int_{-\varphi_{\rm sp}}^{\varphi_{\rm sp}} d\xi\,\sqrt{2V(\xi)} > 0,$$

with the field which satisfies

$$\frac{\partial}{\partial x^1}\varphi = \sqrt{2V(\varphi)},$$

and uniform in time and other spatial directions.[14] For uniform vacuum, $\Gamma = TE_{\rm vac}$ where $T := \int dt$ and $E_{\rm vac} = 0$.[15] Assuming that this expression of Γ is valid when a wall is present, $\Gamma \propto TL^{d-1} > 0$ — this assumption is plausible, since it conforms to the result of the lattice statistical mechanics we saw. Then at least in these field theories, but probably generally, Γ does not have the flat bottom.

The vacuum with a wall is usually considered to be a topological condition. In the present discussion it was naturally introduced as the normalization condition which was originally there in the lattice theory, but has become invisible in the continuum theory.

We have skipped the discussion of the breakdown of translational invariance on quantization, but anyway it is clear that the statement (C), the realization of the flat bottom by linear superposition of the vacua $|\pm\Omega\rangle$ that give $\langle\pm\Omega|\varphi|\pm\Omega\rangle = \pm\varphi_{\rm sp}$, is wrong. Even if translational invariance is preserved, the vacuum will be a superposition of states with a wall, and the vacuum energy is higher by the order L^{d-1}. It must also be mentioned that the two uniform vacua $|\pm\Omega\rangle$ cannot be added, because they belong to two different Lagrangians $\mathcal{L}[J = \pm 0]$, where J is the source coupling to φ.

A model of domain wall motion under random external field—We model the fluctuation of a rigid domain wall, initially in the flat region and then exposed to a random external field. Consider a one dimensional lattice whose points are denoted by $x \in \{-M_{\rm f}, -M_{\rm f}+1, ..., M_{\rm f}\}, M_{\rm f} \in \mathbb{N}$. The "wall" randomly walks on this lattice. Call the regions "left" and "right" to the wall as "spin-plus phase" and "-minus" one, respectively. At each time step the wall moves to one of the neighbor points with equal probabilities 1/2. When the wall reaches an end point $x = \pm M_{\rm f}$ so getting out of the flat region, then "spontaneous symmetry breaking" happens "irreversibly," and the system ends up in a uniform state. This simplest model is unique in that if the wall is initially at x, or equivalently the plus region shares $v := (x + M_{\rm f})/2M_{\rm f}$ part of the system's whole volume $2M_{\rm f}$, then the probability that the final state is the plus phase is v, which is easy to prove.

This immediately allows to be made more abstract and general: Suppose each spin can take q states, and there can be arbitrary number of walls, where if any of two walls meet they are annihilated. The boundary can be periodic, but it does not matter. Again, the probability that i-th state domain is the only survivor is proportional to its volume. (To prove it, notice that the state can be expressed as a point on the lattice on a surface of q-simplex, where $x_i \in \{0, 1, ..., N\}$ and $\sum x_i = N$.) This model satisfies two important conditions of quantum measurement, the collapse of the state and the Born rule, and we *imagined* that (i) the time development is deterministic, (ii) randomness is brought about by noise, and (iii) large degrees of freedom (thermodynamic limit) is necessary.

Discussion and outlook—Textbook explanations of Landau's theory of phase transition are also confused regarding convexity. It is now an easy task to rewrite them.

As for the transition at $|m| = m_{\rm sp}$ some questions are in order: First we don't know the order of magnitude of its energy gap in L. And by using energy (or entropy) as the independent variable instead of T, the transition can be observed continuously. Usually in such cases phase coexistence occur, but it is unlikely in the current case. What exactly happens? We also note that in Ref. 9 it is shown (not rigorous but in a physically plausible way) that in gas-liquid transitions in general, there appears a model independent function that is related to the above mentioned shift of the transition point into the RSEF. In our recent paper[16] we discovered a new universality that the energy difference and the latent heat *along* the gas-liquid coexistence curve are well described by power laws $\Delta E \propto (T_c - T)^a, T\Delta S \propto (T_c - T)^b$, where a and b are constants independent of fluids. This transition is a classical one among other first-order transitions, but awaits more exploration.

Probably a pure mathematical model which obeys the Born rule and is equivalent to ours has already been reported, but the discussion here motivates to take the Copenhagen interpretation more seriously. Admittedly we cannot hint at anything on the model's relation to quantum mechanics, and its relevance to Ising/Potts models is weak; reasonable will be the criticism that we normalized "volume" arbitrarily, by using M_f, not $M_{\rm sp}$. However, when discussing quantum measurement the stability of thermodynamic limit vacuum has scarcely, if not never, been considered in explicit conjunction with a model which yields the Born rule.[17] The Copenhagen interpretation is unpopular because linearity is thought to be never violated. In our opinion it is not obvious at all; in fact, we refuted the RSEF as the superposition of $\langle\varphi\rangle = \pm\varphi_{\rm sp}$ vacua, which has not been questioned before. Discussions based on finite systems should be invalid, for the same reason that phase transitions can only happen in thermodynamic limit. The importance of the correct understanding of vacuum can never be underestimated.

We had to limit ourselves to the 2d Ising model for concrete results, but thorough investigation of the nature of the RSEF of other lattice models is called for. The case of the 2d Ising model we considered is the most basic of all, and yet our brief and simple tour already opens a way to deep insight. Of course, theories with continuous symmetry are by no means less interesting, but more challenging. Our result raises interest in many other directions.

To summarize, we studied the Ising model and quantum scalar field theory under broken symmetry. In the lattice theory M, the total magnetization, is not restricted, and we can specify the state of the system with it. The transition point is not located at $|M| = M_{\rm sp}$, but slightly smaller, and the use of the extensive variable M can not avoid discontinuity. M corresponds to $\int dx \langle\varphi\rangle$ in field theory, but spatially uniform vacua with $|\langle\varphi\rangle| < \varphi_{\rm sp}$ are prohibited, and the spatial configuration of the vacuum needs to be specified explicitly. (Stable) vacua with a domain wall are allowed. The effective action of field theory, and its counterpart in statistical mechanics, are not convex in the region of phase coexistence when appropriately normalized.

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1) Almost ubiquitously K. Symanzik, Comm. Math. Phys. **16**, 48 (1970) is cited, presumably intending the attribution of point (A). However, Symanzik explicitly excludes the "flat-bottom" region from discussion. See its appendix.

2) For the recent situation, see for example E. N. Argyres, M. T. M. Kessel, and R. H. P. Kleiss, The European Physical Journal C **65**, 303 (2009), and B. Delamotte, in *Renormalization Group and Effective Field Theory Approaches to Many-Body Systems,* eds. J. Polonyi and A. Schwenk (Springer, Berlin, 2012), Sect. 2.2.5, and references therein.

3) S. B. Shlosman, Comm. Math. Phys. **125**, 81 (1989).

4) D. Ioffe, J. Stat. Phys. **74**, 411 (1994).

5) The word "droplet" is a standard nomenclature in the context of Wulff construction, where the droplet is sole and macroscopic. Do not confuse it with nucleation in usual first-order phase transitions, where there are many droplets, and the growth of initially microscopic droplets to macroscopic size is studied.

6) For reviews see for example S. Miracle-Sole, Scholarpedia **8**, 31226 (2013), and D.B. Abraham in Ref. 7, Chap. 1.

7) C. Domb and J. L. Lebowitz (eds), *Phase Transitions and Critical Phenomena* (Academic Press, London, 1986), Vol. 10.

8) G. Gallavotti and S. Miracle-Solé, Phys. Rev. B **5**, 2555 (1972).

9) M. Biskup, L. Chayes, and R. Kotecký, Europhys. Lett. **60**, 21 (2002); M. Biskup, L. Chayes, and R. Kotecký, Comm. Math. Phys. **242**, 137 (2003). Their results are confirmed by a Monte Carlo simulation.[10]

10) A. Nußbaumer, E. Bittner, T. Neuhaus, and W. Janke, Physics Procedia **7**, 52 (2010).

11) M. A. Shifman: *Advanced Topics in Quantum Field Theory : A Lecture Course* (Cambridge University Press, Cambridge, 2012), Chap. 2.

12) See for example A. Altland and B. Simons: *Condensed Matter Field Theory* (Cambridge University Press, 2010) 2nd ed., Chap 4.

13) In this case, renormalization at the system boundary is necessary. For reviews on surface/interface renormalization see e.g. Ref. 7, Chaps. 2 and 3.

14) E. B. Bogomolny, Sov. J. Nucl. Phys. **24**, 449 (1976) [Yad. Fiz. **24**, 861 (1976)]. For an introduction, see S. Weinberg, *The Quantum Theory of Fields* (Cambridge University Press, Cambridge, 2005), Vol. 2, Sect. 23.1.

15) K. Symanzik in Ref. 1, see appendix. For an introduction, see S. Weinberg in Ref. 14, Sect. 16.3.

16) Asanuma N.-H., submitted to J. Phys. Soc. Jpn.

17) A. Bassi, K. Lochan, S. Satin, T. P. Singh, and H. Ulbricht, Rev. Mod. Phys. **85**, 471 (2013).