SUMMING CURIOUS, SLOWLY CONVERGENT, HARMONIC SUBSERIES

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1. Abstract

The harmonic series diverges. But if we delete from it all terms whose denominators contain any string of digits such as "9", "42", or "314159", then the sum of the remaining terms converges. These series converge far too slowly to compute their sums directly. We describe an algorithm to compute these and related sums to high precision. For example, the sum of the series whose denominators contain no "314159" is approximately 2302582.33386. We explain why this sum is so close to $10^6 \log 10$ by developing asymptotic estimates for sums that omit strings of length n, as n approaches infinity.

2. Introduction and History

The harmonic series

$$\sum_{s=1}^{\infty} \frac{1}{s}$$

diverges. Suppose we delete from the series all terms whose denominators, in base 10, contain the digit 9 (that is, $1/9$, $1/19$, $1/29, \ldots$). Kempner [12] proved in 1914 that the sum of the remaining terms converges. At first glance, this is counter-intuitive because it appears that we are removing only every tenth term from the harmonic series. If that were the case, then the sum of the remaining terms would indeed diverge.

This series converges because in the long run, we in fact delete almost everything from it. We begin by deleting $1/9$, $1/19$, $1/29, \ldots$. But when we reach $1/89$, we delete 11 terms in a row: $1/89$, then $1/90$ through $1/99$. When we reach $1/889$, we delete 111 terms in a row: $1/889$, then $1/890$ through $1/899$, and finally $1/900$ through $1/999$.

Moreover, the vast majority of integers of, say, 100 digits contain at least one "9" somewhere within them. Therefore, when we apply our thinning process to 100-digit denominators, we will delete most terms. Only $8 \times 9^{99}/(9 \times 10^{99}) \approx 0.003\%$ of terms with 100-digit denominators will survive our thinning process. Schumer [13] argues that *the*

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problem is that we tend to live among the set of puny integers and generally ignore the vast infinitude of larger ones. How trite and limiting our view!

To be more precise: The fractions p_i of i-digit numbers not containing "9" or an alternative single digit are a geometric decaying sequence. In addition the sum of reciprocals of all i-digit numbers is smaller than "9" (see also Equation (3)). The sum of reciprocals of the survivors amongst them is bounded by $9\ p_i$. Summing over all i results in a convergent sequence that serves as an upper bound for the sum of the harmonic series missing "9". The proof for harmonic series missing any given n-digit integer $b \neq 0$ carries over. Here it is enough to construct an upper bound for the fractions not containing b by dividing the i-digit integers into substrings of length n and use the same argument. For example the fraction p_6 of 6-digit numbers not containing "15" is bounded by $89/90 \times (99/100)^2$ as we have divided the 6-digit numbers in 3 substrings of length 2.

Once a series is known to converge, the natural question is, "What is its sum?" Unfortunately, these series converge far too slowly to compute their sums directly [11, 12].

The problem has attracted wide interest through the years in books such as [1, p. 384], [4, pp. 81–83], [8, pp. 120–121] and [9, pp. 31–34]. The computation of these sums is discussed in [2], [6] and [17]. Fischer computed 100 decimals of the sum with "9" missing from the denominators, but his method does not readily generalize to other digits. His remarkable result is that the sum is

$$\beta_0 \ln 10 - \sum_{n=2}^{\infty} 10^{-n} \beta_{n-1} \zeta(n)$$

where $\beta_0 = 10$ and the remaining β_n values are given recursively by

$$\sum_{k=1}^{n} \binom{n}{k} (10^{n-k+1} - 10^k + 1) \beta_{n-k} = 10(11^n - 10^n).$$

Trott [16, pp. 1281–1282] has implemented Fischer's algorithm using MATHEMATICA.

In 1979, Baillie [2] published a method for computing the ten sums that arise when we delete terms containing each of the digits "0" through "9". The sum with "9" deleted is about 22.92067. But the sum of all terms with denominators up to 10^{27} still differs from the final sum by more than 1.

In order to compute sums whose denominators omit strings of two or more digits, we must generalize the algorithm of [2]. We do that here. We will show how to compute sums of $1/s$ where s contains no odd digits, no even digits, or strings like "42" or "314159" or even combinations of those constraints.

Recently the problem has attracted some interest again. The computation of the sum of $1/s$ where s does not contain "42" is a problem suggested by Bornemann et al. [5, p. 281].

Subsequently the problems related to those sums have been discussed in a German-speaking online mathematics forum[1]. In 2005 Bornemann presented his solution for the "42"-problem to Trefethen's problem solving squad at Oxford. His idea is very similar to the original approach of Baillie and is covered by our analysis.

Bold print indicates vectors, matrices or tensors. Sets are in calligraphic print.

3. Recurrence matrices

Let X be a string of $n \geqslant 1$ digits. Let $\mathcal{S}$ be the set of positive integers that do not contain X in base 10. We denote the sum by Ψ, that is,

$$\Psi = \sum_{s \in \mathcal{S}} \frac{1}{s}. \tag{1}$$

If X is the single digit m, Baillie's method partitions $\mathcal{S}$ into subsets $\mathcal{S}_i$. The i^{th} subset consists of those elements of $\mathcal{S}$ that have exactly i digits. The following recurrence connects $\mathcal{S}_i$ to $\mathcal{S}_{i+1}$:

$$\mathcal{S}_{i+1} = \bigcup_{s \in \mathcal{S}_i} \{10s, 10s+1, 10s+2, ..., 10s+9\} \backslash \{10s+m\}$$

From this, a recurrence formula is derived that allows us to compute $\sum_{s \in \mathcal{S}_{i+1}} 1/s^k$ from the sums $\sum_{s \in \mathcal{S}_i} 1/s^k$. If $n > 1$, there is no simple recurrence relation between $\mathcal{S}_i$ and $\mathcal{S}_{i+1}$. However, we can further partition $\mathcal{S}_i$ into subsets for $j = 1, 2, \ldots, n$ in a way that yields a recurrence between $\mathcal{S}_{i+1}^j$ and the sets $\mathcal{S}_i^1, \ldots, \mathcal{S}_i^n$. Once we have done this, we have

$$\Psi = \sum_{j=1}^{n} \sum_{i=1}^{\infty} \sum_{s \in \mathcal{S}_i^j} \frac{1}{s}.$$

The subsets are chosen so that, when we append a digit d to an element of $\mathcal{S}_i^j$, we get an $(i+1)$-digit integer in the d_j^{th} subset of $\mathcal{S}_{i+1}$, that is, $\mathcal{S}_{i+1}^{d_j}$. It will also be convenient to let $\mathcal{S}^j$ be the union of $\mathcal{S}_i^j$, over all i. We will represent the partition and the corresponding recurrence with an $n \times 10$ matrix $\mathbf{T}$. The (j, d) entry of $\mathbf{T}$ tells us which set we end up in when we append the digit d to each element of $\mathcal{S}^j$. If the digit d cannot be appended because the resulting number would not be in $\mathcal{S}$, then we set $T(j, d)$ to 0.

Here is an example that shows how to compute the matrix $\mathbf{T}$ for a given string. Let $\mathcal{S}$ be the set of integers containing no "314". We partition $\mathcal{S}$ into three subsets: $\mathcal{S}^1$ consists of the elements of $\mathcal{S}$ not ending in 3 or 31. $\mathcal{S}^2$ is the set of elements of $\mathcal{S}$ ending in 3.

[1] At `http://matheplanet.com/matheplanet/nuke/html/viewtopic.php?topic=9875` a few participants discuss the computation of the sum of $1/n$ where n does not contain an even digit. It seems their solution combines a direct summation and Richardson extrapolation and is of limited accuracy.

$\mathcal{S}^3$ is the set of elements of $\mathcal{S}$ ending in 31. The following matrix $\mathbf{T}$ shows what happens when we append the digits 0 through 9 to $\mathcal{S}^1$, $\mathcal{S}^2$, and $\mathcal{S}^3$.

$$\mathbf{T} = \left[\begin{array}{c|cccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 3 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 1 & 2 & 0 & 1 & 1 & 1 & 1 & 1 \end{array}\right].$$

When we append a 3 to an element of $\mathcal{S}^1$, we get an element of $\mathcal{S}^2$, so we set $\mathrm{T}(1,3) = 2$. Appending any other digit to an element of $\mathcal{S}^1$ yields another element of $\mathcal{S}^1$, so all other $\mathrm{T}(1,\cdot) = 1$. Consider elements of $\mathcal{S}^2$. Appending a 1 yields an element of $\mathcal{S}^3$; appending a 3 yields another element of $\mathcal{S}^2$. Appending any other digit yields an element of $\mathcal{S}^1$. The only special feature of $\mathcal{S}^3$ is that if we append a 4, we get a number ending in 314, which is not in $\mathcal{S}$, so we set $\mathrm{T}(3,4) = 0$.

Let us emphasize that the matrix $\mathbf{T}$ does not have to be induced by a string X. Indeed our approach is more general. We can also solve a puzzle stated by Boas [3] asking[2] for an estimate of the sum of $1/s$ where s has no even digits. Here it is enough to work with one set $\mathcal{S}$ but to forbid that an even integer can be attached. Hence $\mathbf{T}$ is

$$\mathbf{T} = \left[\begin{array}{c|cccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right].$$

Many more interesting examples are discussed in Section 6.

The recursive relations between the sets $\mathcal{S}^1, \ldots, \mathcal{S}^n$ may also be illustrated by means of a directed graph. There is a directed edge from $\mathcal{S}^i$ to $\mathcal{S}^j$ if by appending an integer d to elements of $\mathcal{S}^i$ we end up in $\mathcal{S}^j$, see Fig. 1. For further analysis we assume only that the associated direct graph is strongly connected, that is, there are directed paths from $\mathcal{S}^i$ to $\mathcal{S}^j$ and $\mathcal{S}^j$ to $\mathcal{S}^i$ for any pair $i \neq j$. Graphs that are not strongly connected can be induced by more exotic constraints but are not discussed here.

In the next Section, we show how $\mathbf{T}$ is used to compute $\sum_{s\in\mathcal{S}_{i+1}} 1/s^k$ from the values of $\sum_{s\in\mathcal{S}_i} 1/s^k$.

4. A recurrence formula

It may seem a bit odd to introduce sums of s^{-k} although only the case $k = 1$ is desired, but they enable us to exploit the recurrence relations between the aforementioned sets $\mathcal{S}^1, \ldots, \mathcal{S}^n$. The idea has been successfully applied in [2]. Let

$$\Psi^j_{i,k} = \sum_{s\in\mathcal{S}^j_i} \frac{1}{s^k}. \tag{2}$$

[2] Actually Boas is pointing to problem 3555 published in *School Science and Mathematics*, April 1975.

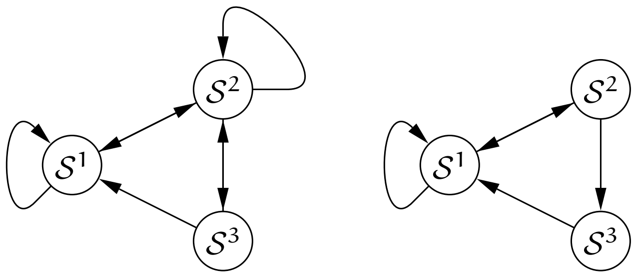

FIGURE 1. Directed, strongly connected graphs. Left: Graph for the partition induced by the string "314". Any other string with three distinct digits would have the same graph. Right: Graph of the partition induced by the string "333". Note that the strings "332" and "323" would induce two alternative graphs not shown here. The graph related to "233" would match the left graph.

Therefore the sum s^{-k} where s ranges over all i-digit integers is an upper bound for $\Psi^j_{i,k}$. There are at most $10^i - 10^{i-1}$ of these integers. Every such s is at least 10^{i-1}. Therefore,

$$\Psi^j_{i,k} \leqslant \left(10^i - 10^{i-1}\right) \frac{1}{10^{(i-1)k}} = \frac{9}{10^{(i-1)(k-1)}}. \tag{3}$$

Using the new notation the problem is to compute

$$\Psi = \sum_{j=1}^{n} \sum_{i=1}^{\infty} \Psi^j_{i,1}.$$

The recursive nature of the sets $\mathcal{S}^j_i$ is now used to derive recurrence relations for the sums $\Psi^j_{i,k}$. We introduce a tensor $\mathbf{f}$ of dimensions $n \times n \times 10$ with

$$f_{jlm} = \begin{cases} 1 & \text{if } T(l,m) = j \\ 0 & \text{else} \end{cases}. \tag{4}$$

This tells us in the sum below to either include a term ($f_{jlm} = 1$) or not include it ($f_{jlm} = 0$). Then

$$\Psi^j_{i,k} = \sum_{m=0}^{9} \sum_{l=1}^{n} f_{jlm} \sum_{s \in \mathcal{S}^l_{i-1}} (10s + m)^{-k}. \tag{5}$$

By construction of $\mathbf{f}$ the sum runs over all index pairs (l, m) such that $T(l, m) = j$, which indicates that

$$\left\{10s + m, s \in \mathcal{S}^l_{i-1}\right\} \subset \mathcal{S}^j_i.$$

Although Equation (5) is the crucial link to the recurrence matrices it is of no computational use. It is still a direct computation of $\Psi^j_{i,k}$. We should avoid summing over a range of integers in subsets $\mathcal{S}^l_{i-1}$. Using negative binomial series[3] we observe

$$(10s+m)^{-k} = (10s)^{-k} \sum_{w=0}^{\infty} (-1)^w \begin{pmatrix} k+w-1 \\ w \end{pmatrix} \left(\frac{m}{10s}\right)^w$$

where $0^0 = 1$ and define:

$$c(k,w) = (-1)^w \begin{pmatrix} k+w-1 \\ w \end{pmatrix}.$$

We replace the term $(10s+m)^{-k}$ in Equation (5) and get

$$\begin{aligned} \Psi^j_{i,k} &= \sum_{m=0}^{9} \sum_{l=1}^{n} f_{jlm} \sum_{s \in \mathcal{S}^l_{i-1}} (10s)^{-k} \sum_{w=0}^{\infty} c(k,w) \left(\frac{m}{10s}\right)^w \\ &= \sum_{m=0}^{9} \sum_{l=1}^{n} f_{jlm} \sum_{w=0}^{\infty} 10^{-k-w} c(k,w) m^w \sum_{s \in \mathcal{S}^l_{i-1}} s^{-k-w} \end{aligned}$$

by reordering the sum. To simplify the notation we introduce

$$a(k,w,m) = 10^{-k-w} c(k,w) m^w$$

and write therefore

$$\Psi^j_{i,k} = \sum_{m=0}^{9} \sum_{l=1}^{n} f_{jlm} \sum_{w=0}^{\infty} a(k,w,m) \Psi^l_{i-1,k+w}. \tag{6}$$

Again it may seem odd that we have replaced the finite summation (5) in s by an infinite sum in w. But the infinite sum decays so fast in w that truncation enables us to approximate (6) much faster than evaluating the sums of Equation (5).

5. Truncation and Extrapolation

We step into the numerical analysis of the problem. So far we have only reformulated the summation by introducing the partial sums $\Psi^j_{i,k}$. The ultimate goal is the efficient computation of $\Psi = \sum_{i,j} \Psi^j_{i,1}$. We use the following scheme:

For $i \leqslant 2$ the sums $\Psi^j_{i,k}$ ($k > 1$ is needed in the next step) are computed directly as suggested in Equation (2).

For $i > 2$ a recursive evaluation of (6) is used. The indices i and w both run from 3 (resp. 0) to infinity. For w we use a simple truncation by neglecting all terms $\Psi^l_{i-1,k+w}$ smaller than an a priori given bound ε, or to be precise: to neglect all terms $\Psi^l_{i-1,k+w}$

[3] http://mathworld.wolfram.com/NegativeBinomialSeries.html

where the estimate (3) is smaller than ε, that is, $k+w$ is sufficiently large. In practice we decrease ε until the first d digits of the result stop changing where d is a desired number of correct digits.

For the same reason we can neglect for large i all contributions from terms $\Psi^{l}_{i-1,k+w}$ with $k + w > 1$. Once the algorithm has achieved that stage it is possible to apply extrapolation. Equation (6) with $w = 0$ and $k = 1$ reads in matrix form

$$\begin{pmatrix} \Psi^{1}_{i,1} \\ \vdots \\ \Psi^{n}_{i,1} \end{pmatrix} \approx \underbrace{\sum_{m=0}^{9} 10^{-1} \begin{pmatrix} f_{11m} & \cdots & f_{1nm} \\ \vdots & & \vdots \\ f_{n1m} & \cdots & f_{nnm} \end{pmatrix}}_{\mathbf{A}_n} \begin{pmatrix} \Psi^{1}_{i-1,1} \\ \vdots \\ \Psi^{n}_{i-1,1} \end{pmatrix} \tag{7}$$

since $a(1, 0, m) = 10^{-1}$ for all m.

At this point the skeptical reader may not believe that the nonnegative matrix $\mathbf{A}_n$ is a contraction. The associated digraph of $\mathbf{A}_n$ with vertices $1, \ldots, n$ representing the sets $\mathcal{S}^j$, $j = 1, \ldots, n$ and arc_{ij} if and only if $A_n(i, j) \neq 0$ is exactly the graph illustrating the recurrence relations between the sets $\mathcal{S}^j$ as introduced above, see Fig. 1. We have assumed that this graph is strongly connected and hence $\mathbf{A}_n$ is irreducible[4] and therefore the Perron-Frobenius Theorem[10, Theorem 8.4.4] applies:

Theorem 1 (Perron-Frobenius). *Let* $\mathbf{A}$ *be a nonnegative*[5] *and irreducible matrix. Then*

- *there is an eigenvalue* λ_d *that is real and positive, with positive left and right eigenvectors,*
- *any other eigenvalue* λ *satisfies* $|\lambda| < \lambda_d$*,*
- *the eigenvalue* λ_d *is simple.*

The eigenvalue λ_d *is called the dominant eigenvalue of* $\mathbf{A}$*.*

It remains to show that the dominant eigenvalue λ_d of $\mathbf{A}_n$ is smaller than 1. Consider the l^{th} column of the matrix

$$\begin{pmatrix} f_{11m} & \cdots & f_{1nm} \\ \vdots & & \vdots \\ f_{n1m} & \cdots & f_{nnm} \end{pmatrix}.$$

By definition of $\mathbf{f}$ (4) there is a 1 in row $T(l, m)$ if $T(l, m) > 0$. All other entries in the column are zero. Therefore $\|\mathbf{a}_i\|_1 \leqslant 1$ if we denote by $\mathbf{a}_1, \ldots, \mathbf{a}_n$ the columns of $\mathbf{A}_n$.

The existence of columns which have no nonzero entry implies that there are columns of $\mathbf{A}_n$ with $\|\mathbf{a}_i\|_1 < 1$. Let $\mathbf{x}$ be the right eigenvector of $\mathbf{A}_n$ corresponding to λ_d. Applying

[4] A matrix is reducible if and only if its associated digraph is not strongly connected [7, p. 163 ff.].

[5] A matrix is nonnegative if and only if every entry is nonnegative.

the triangle inequality

$$\lambda_d \|\mathbf{x}\|_1 = \|\mathbf{A}_n\mathbf{x}\|_1 \leqslant \sum_{i=1}^{n} |x_i| \, \|\mathbf{a}_i\|_1 < \sum_{i=1}^{n} |x_i| = \|\mathbf{x}\|_1$$

we conclude that $\lambda_d < 1$. Note that we have used the fact that the elements x_i of the positive vector $\mathbf{x}$ can not vanish for any index i. If all columns of the nonnegative matrix $\mathbf{A}_n$ satisfy $\|\mathbf{a}_i\|_1 = 1$, the matrix $\mathbf{A}_n$ is called *stochastic* and $(1, \ldots, 1)$ is a left eigenvector with eigenvalue 1. In this case $\mathbf{A}_n$ is no longer a contraction. This situation occurs once $\mathbf{T}$ contains no zero and hence no integers are deleted at all. This may serve as an unusual explanation for the divergence of the harmonic series.

Having shown that the spectrum of $\mathbf{A}_n$ lies within the unit disk, we can simplify

$$\sum_{i=1}^{\infty} \begin{pmatrix} \Psi^1_{i+K,1} \\ \vdots \\ \Psi^n_{i+K,1} \end{pmatrix} \approx \sum_{i=1}^{\infty} \mathbf{A}_n^i \begin{pmatrix} \Psi^1_{K,1} \\ \vdots \\ \Psi^n_{K,1} \end{pmatrix} \tag{8}$$

by a Neumann series

$$\sum_{k=1}^{\infty} \mathbf{A}_n^k = (\mathbf{I} - \mathbf{A}_n)^{-1} - \mathbf{I} =: \mathbf{B}_n^{\infty}$$

where $\mathbf{I}$ is an identity matrix of appropriate dimension. Hence

$$\Psi \approx \sum_{j=1}^{n} \sum_{i=1}^{K} \Psi^j_{i,1} + \left\| \mathbf{B}_n^{\infty} \begin{pmatrix} \Psi^1_{K,1} \\ \vdots \\ \Psi^n_{K,1} \end{pmatrix} \right\|_1 . \tag{9}$$

Using the same idea we can also estimate the result of truncating the series after, say, $M + K$ digits using

$$\sum_{k=1}^{M} \mathbf{A}_n^k = (\mathbf{I} - \mathbf{A}_n^{M+1})(\mathbf{I} - \mathbf{A}_n)^{-1} - \mathbf{I} =: \mathbf{B}_n^{M}. \tag{10}$$

In practice, one would use an eigendecomposition of $\mathbf{A}_n$ in order to determine $\mathbf{A}_n^{M+1}$.

6. Examples

We compute a few sums to a precision of 100 decimals, although if desired, more could easily be obtained.

First, let us compute the sum originally considered by Kempner [12], namely, where the digit "9" is missing from the denominators. Here, there is only one set $\mathcal{S} = \mathcal{S}^1$, namely, the set of integers that do not contain a "9". When we append a "9" to an element of

$\mathcal{S}$, we get a number not in $\mathcal{S}$, so $\mathrm{T}(1,9) = 0$. When we append any other digit, we get another element of $\mathcal{S}$, so all other $\mathrm{T}(1,\cdot) = 1$:

$$\mathbf{T} = \left[\begin{array}{c|cccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{array}\right].$$

To 100 decimals, the sum is

$$\begin{aligned} &22.92067\ 66192\ 64150\ 34816\ 36570\ 94375\ 93191\ 49447\ 62436\ 99848 \\ &\qquad 15685\ 41998\ 35657\ 21563\ 38189\ 91112\ 94456\ 26037\ 44820\ 18989 \ldots . \end{aligned}$$

In Section 3 we mentioned the sum of $1/s$ where s has no even digits. The sum is

$$\begin{aligned} &3.17176\ 54734\ 15904\ 95722\ 87097\ 08750\ 61165\ 67970\ 50708\ 39628 \\ &\qquad 57241\ 64186\ 89843\ 71376\ 88585\ 61926\ 68852\ 31080\ 74715\ 60454 \ldots . \end{aligned}$$

Similarly, the sum over denominators with no odd digits can be found using the matrix

$$\mathbf{T} = \left[\begin{array}{c|cccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{array}\right].$$

In this case the sum is

$$\begin{aligned} &1.96260\ 84129\ 94616\ 98515\ 91542\ 64737\ 29435\ 67128\ 30665\ 51443 \\ &\qquad 53546\ 71522\ 23586\ 65760\ 95274\ 32927\ 13468\ 24171\ 73826\ 12704 \ldots . \end{aligned}$$

In Section 3, we gave the matrix $\mathbf{T}$ that corresponds to the sum of $1/s$ where s has no string "314". The sum is:

$$\begin{aligned} &2299.82978\ 27675\ 18338\ 45358\ 63536\ 11974\ 36784\ 61556\ 88394\ 19837 \\ &\qquad 51645\ 98202\ 17625\ 43309\ 41712\ 63285\ 37992\ 24266\ 07454\ 90945 \ldots . \end{aligned}$$

We can also compute the sum of $1/s$ where s has no string "314159". Then the sum is:

$$\begin{aligned} &2302582.33386\ 37826\ 07892\ 02375\ 60364\ 84435\ 61276\ 86862\ 90972\ 08627 \\ &\qquad 80786\ 90557\ 30669\ 81792\ 73645\ 44979\ 47969\ 47311\ 14619\ 12012 \ldots . \end{aligned}$$

This sum demonstrates the power of the technique presented here. Using (10) we calculate that the partial sum of all terms whose denominators have 100000 or fewer digits is about $219121.34825\ldots$. This is still only $1/10^{\text{th}}$ as large as the final sum, and illustrates the futility of direct summation. Notice that this sum is close to $10^6 \log 10$. But there is nothing special about "314159"; we observe similar results for other strings of six digits. We say more about this in the next section.

Let $\mathcal{S}$ be the set of integers containing no "42". Compute Ψ in this case is the challenge recently posed by Bornemann et al. [5]. In this case

$$\mathbf{T} = \left[\begin{array}{c|cccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 0 & 1 & 2 & 1 & 1 & 1 & 1 & 1 \end{array}\right]$$

and the sum is given by

$$228.44630\,41592\,30813\,25414\,80861\,26250\,58957\,81629\,27539\,83036$$
$$11859\,13460\,00045\,28607\,68650\,21430\,70480\,46117\,41443\,21741\ldots.$$

The next example combines various constraints and illustrates the flexibility of our approach. Let $\mathcal{S}$ be the set of integers containing no even digits, no "55" and no "13579". Then we define the following partition of $\mathcal{S}$. $\mathcal{S}^2$ is the subset of numbers ending in 5 (but not ending in 135), $\mathcal{S}^3$ is the set of numbers ending in 1, $\mathcal{S}^4$ is the set of numbers ending in 13, $\mathcal{S}^5$ is the subset of numbers ending in 135 and the elements of $\mathcal{S}^6$ end in 1357. All remaining elements of $\mathcal{S}$ are in $\mathcal{S}^1$. Following those rules the matrix $\mathbf{T}$ is given by

$$\mathbf{T} = \left[\begin{array}{c|cccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline 1 & 0 & 3 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 1 \\ 2 & 0 & 3 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 3 & 0 & 3 & 0 & 4 & 0 & 2 & 0 & 1 & 0 & 1 \\ 4 & 0 & 3 & 0 & 1 & 0 & 5 & 0 & 1 & 0 & 1 \\ 5 & 0 & 3 & 0 & 1 & 0 & 0 & 0 & 6 & 0 & 1 \\ 6 & 0 & 3 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 0 \end{array}\right].$$

Here the sum is

$$3.09084\,91496\,53806\,46825\,46563\,73157\,80175\,63888\,91119\,39765$$
$$22149\,64013\,36906\,53946\,19395\,87929\,18235\,63131\,88124\,97325\ldots.$$

Our algorithm can be easily generalized for other bases than 10. The sum of $1/s$ where s has no "0" in base 100 is

$$460.52520\,26385\,12471\,14293\,67535\,66415\,29497\,12569\,09908\,47934$$
$$06016\,95672\,87250\,06818\,86421\,46967\,22875\,07176\,27582\,54794\ldots.$$

All sorts of other experiments are possible. The interested reader might experiment by removing from the harmonic series their personal favorite number, such as their birthday (as a single string, or as a set of three strings), their phone number, etc.

7. Asymptotic behavior

We now discuss the asymptotic behavior of the sums that arise when we remove from the harmonic series terms whose denominators contain a string X_n of length n digits.

Data for several random strings of $n = 20$ digits are given in Table 1. It is striking that, for each random string X_n, the "normalized" sum, $\Psi_{X_n}/10^n$, is very close to $\log 10 = 2.30258\,50929\,94045\,68401\,79914\ldots$.

n	String X_n	$\Psi_{X_n}/10^n$
20	21794968139645396791	2.30258 50929 94045 68397 52162
20	31850115459210380210	2.30258 50929 94045 68399 08824
20	67914499976105176602	2.30258 50929 94045 68401 09579
20	98297963712691768117	2.30258 50929 94045 68401 77079

Table 1. Sums for several random 20-digit strings

Table 2 shows what happens for strings that consist of repeated patterns of substrings. If a string consists of shorter substrings repeated two or more times, we define the *period* of the string to be the length of that shortest substring. So, "11111" has period 1, while "535353" has period 2.

Here, the normalized sums appear to approach different limits. The limits do not depend on which digits comprise the strings, but instead depend on the periods. When all digits are identical (period 1), the limit of the normalized sum seems to be $(10/9)\log 10 = 2.55842\ldots$. When the period is 2, the limit seems to be $(100/99)\log 10 = 2.32584\ldots$. For period 5, it's $(100000/99999)\log 10 = 2.30260\ldots$.

We emphasize that we observe these same limits for other strings of digits. In the limit, the particular digits in a string do not matter. What matters is the structure of the strings.

Equation (9) is based on a truncation and is correct in the limit for $K \to \infty$. Nevertheless we fix K to $n-1$ and introduce an error term $\mathbf{e}_n$. In particular n is the only free parameter left. The error term implies

$$\frac{\Psi_{X_n}}{10^n} = \frac{1}{10^n}\left(\sum_{j=1}^{n}\sum_{i=1}^{n-1}\Psi_{i,1}^{j} + \left\|\mathbf{B}_n^{\infty}\begin{pmatrix}\Psi_{n-1,1}^{1}\\ \vdots \\ \Psi_{n-1,1}^{n}\end{pmatrix}\right\|_1 + \mathbf{e}_n\right).$$

We observe that for increasing n the first term on the right-hand side converges to zero due to the slow growth of the double sum. The error term scaled by 10^{-n} representing the truncated sums $\Psi_{i,1+w}^{l}$ with $l = 1, \ldots, n$, $i \geqslant n-1$ and $w > 0$ meets the same fate.

n	String X_n	$\Psi_{X_n}/10^n$
5	00000	2.55840 22969
10	0000000000	2.55842 78808 48652
15	000000000000000	2.55842 78811 04492 64603
20	00000000000000000000	2.55842 78811 04495 20443 88506
20	11111111111111111111	2.55842 78811 04495 20435 05433
20	44444444444444444444	2.55842 78811 04495 20442 19551
20	99999999999999999999	2.55842 78811 04495 20443 88506
4	4242	2.32542 92748
10	4242424242	2.32584 35278 62555
16	4242424242424242	2.32584 35282 76813 40798 19419
20	42424242424242424242	2.32584 35282 76813 82219 89695
20	09090909090909090909	2.32584 35282 76813 82221 85405
5	12345	2.30250 59575
10	1234512345	2.30260 81180 53596
15	123451234512345	2.30260 81190 75226 21998
20	12345123451234512345	2.30260 81190 75236 43628 01912

TABLE 2. Sums for strings of periods $1, 2$, and 5

We need a more practical notation for the vector we take the 1-norm of and introduce

$$\boldsymbol{\psi}_{n-1}^{n} = \begin{pmatrix} \Psi_{n-1,1}^{1} \\ \vdots \\ \Psi_{n-1,1}^{n} \end{pmatrix}.$$

We observe that $\|\boldsymbol{\psi}_{n-1}^{n}\|_1$ is exactly the sum over all positive integers with exactly $n-1$ digits. Not a single integer has been deleted at this stage yet. Hence $\|\boldsymbol{\psi}_{n-1}^{n}\|_1$ converges to $\log 10$ because

$$\|\boldsymbol{\psi}_{n-1}^{n}\|_1 \xrightarrow[n\to\infty]{} \int_{10^{n-1}}^{10^n} \frac{1}{t}\, dt = \log 10.$$

The same is true

$$\|\boldsymbol{\psi}_{k+n}^{n}\|_1 \xrightarrow[n\to\infty]{} \log 10$$

for any fixed $k \geqslant 0$. Here we delete amongst integers of length $k+n$ those which contain a certain string of length n. Hence the number of integers we delete is bounded no matter how large n is. But the contribution from these deleted integers converges to zero for increasing n.

Equation (8) reveals what in numerical linear algebra is called a power iteration [15]. Note that any vector with nonnegative entries and positive norm can never be orthogonal to the dominant eigenvector of $\mathbf{A}_n$. Hence the vectors $\boldsymbol{\psi}^n_{k+n}$ quickly line up with the dominant eigenvector of $\mathbf{A}_n$ for increasing k. Contributions in other directions are damped more quickly. The factor of amplification of the "asymptotic eigenvector" $\boldsymbol{\psi}$ is therefore given by the dominant eigenvalue of $\mathbf{B}^\infty_n$. Hence

$$\begin{aligned} 0 &= \lim_{k\to\infty}\lim_{n\to\infty}\left(\Psi_{X_n}/10^n - \frac{1}{10^n}\|\mathbf{B}^\infty_n\boldsymbol{\psi}\|_1\right) \\ &= \lim_{n\to\infty}\left(\Psi_{X_n}/10^n - \log 10\frac{\lambda_{\mathbf{B}^\infty_n}}{10^n}\right). \end{aligned}$$

If $\lambda_{\mathbf{A}_n} < 1$ is the dominant eigenvalue of $\mathbf{A}_n$, then $\lambda_{\mathbf{B}^\infty_n} = 1/(1-\lambda_{\mathbf{A}_n}) - 1$ is the largest eigenvalue of $\mathbf{B}^\infty_n$. Both matrices share their eigenvectors and therefore

$$\begin{aligned} 0 &= \lim_{n\to\infty}\left(\Psi_{X_n}/10^n - \log 10\frac{1/(1-\lambda_{\mathbf{A}_n})-1}{10^n}\right) \\ &= \lim_{n\to\infty}\left(\Psi_{X_n}/10^n - \log 10\frac{1/(1-\lambda_{\mathbf{A}_n})}{10^n}\right). \end{aligned}$$

Note that does not imply that $\Psi_{X_n}/10^n$ converges. We have proved:

Theorem 2. *Let Ψ_{X_n} be the sum of $1/s$ where s does not contain the substring X_n. Let $\mathbf{A}_n$ be the matrix defined in (7). Then*

$$\lim_{n\to\infty}\left(\Psi_{X_n}/10^n - \frac{1/(1-\lambda_{\mathbf{A}_n})}{10^n}\right) = 0$$

where $\lambda_{\mathbf{A}_n}$ is the dominant eigenvalue of $\mathbf{A}_n$.

It all boils down to locating this dominant eigenvalue or getting some good estimates for it. At least for special cases this is simple:

Lemma 1. *Let X_n be any string of n digits that are all the same. Let Ψ_{X_n} be the sum of $1/s$ where s does not contain the substring X_n. Then*

$$\lim_{n\to\infty}\Psi_{X_n}/10^n = \frac{10}{9}\log 10.$$

Proof. If all digits in X_n are the same then the $n \times n$ matrix $\mathbf{A}_n$ has the form

$$\mathbf{A}_n = \frac{1}{10}\begin{pmatrix} 9 & 9 & \cdots & \cdots & 9 \\ 1 & 0 & & & 0 \\ 0 & 1 & & & \vdots \\ & & \ddots & & \vdots \\ & & & 1 & 0 \end{pmatrix},$$

assuming that $\mathcal{S}^1$ is the set of integers not ending in the integer d. $\mathcal{S}^2$ is ending in d, $\mathcal{S}^3$ is ending in dd and so on.

An analysis of the characteristic polynomial of $\mathbf{A}_n$ shows that any eigenvalue of λ is a solution of the equation

$$\lambda^n(1-\lambda) = \frac{9}{10^{n+1}}. \tag{11}$$

The term on the right hand side might be interpreted as a perturbation of the homogeneous equation with a root at $\lambda = 1$. This root is moving on the real line towards the origin such that the factor $(1-\lambda)$ is still larger than $9/10^{n+1}$ balancing the fast decay of λ^n. Hence

$$\lim_{n\to\infty} \frac{\lambda_{\mathbf{B}_n^\infty}}{10^n} = \lim_{n\to\infty} \frac{1}{(1-\lambda_{\mathbf{A}_n})10^n} = \frac{10}{9}\lim_{n\to\infty} \lambda_{\mathbf{A}_n}^n \tag{12}$$

by utilizing (11).

It remains to show that although $\lambda_{\mathbf{A}_n} < 1$ we get $\lim_{n\to\infty} \lambda_{\mathbf{A}_n}^n = 1$. If $\lambda > \frac{n}{n+1}$ the graph of $\lambda^n(1-\lambda)$ is monotonic decreasing. But for $\lambda = \frac{10^n-1}{10^n}$, $\lambda^n(1-\lambda)$ is still larger than $9/10^{n+1}$ and hence $\lambda_{\mathbf{A}_n} > \lambda$. Yet $\lim_{n\to\infty} \lambda^n = 1$ and we get desired result. Therefore

$$\lim_{n\to\infty} \Psi/10^n = \log 10 \lim_{n\to\infty} \lambda_{\mathbf{B}_n^\infty}/10^n = \frac{10}{9}\log 10.$$

□

We now consider the more general case of periodic strings of period $p \geqslant 1$.

Lemma 2. *Let X_n be any string of n digits having period p. Let Ψ_{X_n} be the sum of $1/s$ where s does not contain the substring X_n. Then*

$$\lim_{n\to\infty} \Psi_{X_n}/10^n = \frac{10^p}{10^p-1}\log 10.$$

Proof. The matrix $\mathbf{A}_n$ mildly depends on the structure of the digits in the periodic substring of length p but in all cases the characteristic polynomial is given by

$$p_n(\lambda) = g(\lambda) - \frac{1}{\lambda}\left(g(\lambda)-1\right)$$

where

$$g(\lambda) = \sum_{m=0}^{\nu} 10^{mp}\,\lambda^{mp} = \frac{1 - 10^{(\nu+1)p}\lambda^{(\nu+1)p}}{1 - 10^{p}\lambda^{p}}.$$

A complete proof based on the block structure of $\mathbf{A}_n$ (the pattern of $\mathbf{A}_n$ can be solely described by the leading $p \times p$ block) is omitted here. Hence for any eigenvalue

$$1/(1-\lambda) = g(\lambda).$$

In the spirit of (12)

$$\lim_{n\to\infty} \frac{\lambda_{\mathbf{B}_n^\infty}}{10^n} = \lim_{n\to\infty} \frac{g(\lambda_{\mathbf{A}_n})}{10^n} = \frac{10^p}{10^p - 1} \lim_{n\to\infty} \lambda_{\mathbf{A}_n}^n.$$

For $\lambda = \frac{10^n-1}{10^n}$ we have $p_n(\lambda) < 0$, but $p_n(\lambda) \to \infty$ for $\lambda \to \infty$. Hence there is a root larger than λ. Therefore $\lim_{n\to\infty} \lambda_{\mathbf{A}_n}^n = 1$ which finishes the proof. □

Lemma 3. *Let X_n be any string of n digits with no periodic pattern. Let Ψ_{X_n} be the sum of $1/s$ where s does not contain the substring X_n. Then*

$$\lim_{n\to\infty} \Psi_{X_n}/10^n = \log 10.$$

Proof. This can be interpreted as the limit of the result of Lemma 2 for $p \to \infty$. □

8. Conclusions

We have derived an algorithm able to cope with various constraints on the set of integers deleted from the harmonic series. The algorithm relies on truncation and extrapolation and avoids direct summation for large integers with more than 2 digits. Embedding the problem in the language of linear algebra we draw on nothing beyond eigenvalues and eigenvectors to give an asymptotic analysis of the problem.

Our MATHEMATICA implementation can be downloaded from the webpage of the first author[6]. It fits on one page and runs in less than 5 seconds to produce 10 digits. This is a model for good scientific computing that has recently been put forward by Trefethen [14].

This paper may provide some opportunities for further exploration. It might be good fun to derive an algorithm for the inverse problems. What set of simple constraints might one apply in order to make the sum as close as possible to a given number? The N^{th} partial sum of the harmonic series is never an integer [9, p. 24]. What about partial sums of the series we consider here? And are these sums rational, irrational, algebraic, or transcendental?

[6] http://web.comlab.ox.ac.uk/oucl/people/thomas.schmelzer.html

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