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Rotating cylinders and the possibility of global causality violation *

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In 1936 van Stockum solved the Einstein equations $G_{\mu\nu} = -8\pi T_{\mu\nu}$ for the gravitational field of a rapidly rotating infinite cylinder. It is shown that such a field violates causality, in the sense that it allows a closed timelike line to connect any two events in spacetime. This suggests that a finite rotating cylinder would also act as a time machine.

Since the work of Hawking and Penrose,[1] it has become accepted that classical general relativity predicts some sort of pathological behavior. However, the exact nature of the pathology is under intense debate at present, primarily because solutions to the field equations can be found which exhibit virtually any type of bizarre behavior.[2,3] It is thus of utmost importance to know what types of pathologies might be expected to occur in actual physical situations. One of these pathologies is causality violation, and in this paper I shall argue that if we make the assumptions concerning the behavior of matter and manifold usual in general relativity, then it should be possible in principle to set up an experiment in which this particular pathology could be observed.

Because general relativity is a local theory with no *a priori* restrictions on the global topology, causality violation can be introduced into solutions quite easily by injudicious choices of topology; for example, we could assume that the timelike coordinate in the Minkowski metric is periodic, or we could make wormhole identifications in Reissner-Nordström space.[4] In both of these cases the causality violation takes the form of closed timelike lines (CTL) which are not homotopic to zero, and these need cause no worries since they can be removed by reinterpreting the metric in a covering space (following Carter,[5] CTL removable by such means will be called trivial—others will be called nontrivial).

In 1949, however, Gödel[6] discovered a solution to the field equations with nonzero cosmological constant that contained nontrivial CTL. Still, it could be argued that the Gödel solution is without physical significance, since it corresponds to a rotating, stationary cosmology, whereas the actual universe is expanding and apparently nonrotating.

The low-angular-momentum Kerr field, on the other hand, cannot be claimed to be without physical relevance: It appears to be the unique final state of gravitational collapse,[7] and so Kerr black holes probably exist somewhere, possibly in the center of our galaxy.[8] This field also contains nontrivial CTL, though the region of causality violation is confined within an event horizon; causality violation from this source could never be observed by terrestrial physicists.[9] In addition, since the CTL must thread their way through a region near the singularity, it is quite possible that matter of a collapsing star will replace this region, as matter replaces the past horizon in the case of spherical collapse.[10] The final Kerr field with collapsed star could be causally well behaved, so the CTL pathology might still be eliminated from general relativity's physical solutions.

I doubt this, because nontrivial causality violation also occurs in the field generated by a rapidly rotating infinite cylinder.

The field of such a cylinder in which the centrifugal forces are balanced by gravitational attraction was discovered by van Stockum in 1936.[11] The metric is expressed in Weyl-Papapetrou form:

$$ds^2 = H(dr^2 + dz^2) + L d\varphi^2 + 2M d\varphi dt - F dt^2 , \quad (1)$$

where z measures distance along the cylinder axis, r is the radial distance from the axis, φ is the angle coordinate, and t is required to be timelike at $r=0$. ($-\infty < z < \infty$, $0 < r < \infty$, $0 \leqslant \varphi \leqslant 2\pi$, $-\infty < t < \infty$.) The metric tensor is a function of r alone, and the coordinate condition $FL + M^2 = r^2$ has been imposed (units $G = c = 1$).

It is clear that since $g = \det g_{\mu\nu} = -r^2 H^2$ is negative, the metric signature is $(+++-)$ for all $r > 0$, provided $H \neq 0$. van Stockum assumes the Einstein equations

$$G^\mu{}_\nu = -8\pi T^\mu{}_\nu$$
$$= -8\pi\rho \frac{dx^\mu}{ds}\frac{dx_\nu}{ds} ,$$

where ρ is the particle mass density. Also

$$\frac{dr}{ds} = \frac{dz}{ds} = 0 ,$$

$$\frac{d\varphi}{ds} \bigg/ \frac{dt}{ds} = \text{constant} ,$$

$$T = T^\mu{}_\mu = -\rho$$

(particle paths required to be timelike).

In a frame in which the matter is at rest, the equations give for the interior field

$$H = e^{-a^2r^2}, \quad L = r^2(1-a^2r^2), \quad \rho = 4a^2e^{a^2r^2}, \quad M = ar^2, \quad F = 1, \tag{2}$$

where a is the angular velocity of the cylinder. For $r > 1/a$, the lines $r=$constant, $t=$constant, $z=$constant are CTL (in fact, by a theorem due to Carter,[5] nontrivial CTL can be found which intersect any two events in the manifold), but one could hope that the causality violation could be eliminated by requiring the boundary of the cylinder to be at $r = R < 1/a$. Here the interior solution would be joined to an exterior solution which would be (hopefully) causally well behaved; indeed, the resulting upper bound to the "velocity" aR would equal 1, the speed of light in our units (though the orbits of the particles creating the field are timelike for all r).

van Stockum has developed a procedure which generates an exterior solution for all $aR > 0$. When $0 < aR < \frac{1}{2}$, the exterior solution is

$$\begin{aligned} &{}^{-a^2R^2}(r/R)^{-2a^2R^2} \\ &L = \frac{Rr\sinh(3\epsilon+\theta)}{2\sinh 2\epsilon\cosh\epsilon}, \\ &M = \frac{r\sinh(\epsilon+\theta)}{\sinh 2\epsilon}, \\ &F = \frac{r\sinh(\epsilon-\theta)}{R\sinh\epsilon}, \end{aligned} \tag{3a}$$

with

$$\theta = (1-4a^2R^2)^{1/2}\ln(r/R),$$
$$\epsilon = \tanh^{-1}(1-4a^2R^2)^{1/2}.$$

For $aR = \frac{1}{2}$,

$$\begin{aligned} &H = e^{-1/4}(r/R)^{-1/2}, \\ &L = \tfrac{1}{4}Rr[3+\ln(r/R)], \\ &M = \tfrac{1}{2}r[1+\ln(r/R)], \\ &F = (r/R)[1-\ln(r/R)]. \end{aligned} \tag{3b}$$

For $aR > \frac{1}{2}$,

$$\begin{aligned} &H = e^{-a^2R^2}(r/R)^{-2a^2R^2}, \\ &L = \frac{Rr\sin(3\beta+\gamma)}{2\sin 2\beta\cos\beta}, \\ &M = \frac{r\sin(\beta+\gamma)}{\sin 2\beta}, \\ &F = \frac{r\sin(\beta-\gamma)}{R\sin\beta}, \end{aligned} \tag{3c}$$

with

$$\gamma = (4a^2R^2-1)^{1/2}\ln(r/R),$$
$$\beta = \tan^{-1}(4a^2R^2-1)^{1/2}$$

[as in the interior solution, $FL + M^2 = r^2$, so the metric signature is $(+++-)$ for $R \le r < \infty$].

We see that causality violation is avoided for $aR \le \frac{1}{2}$, but Carter's theorem tells us that it is possible to connect any two events by nontrivial CTL when $aR > \frac{1}{2}$.

There are several objections to be met before this result can be interpreted physically. First of all, Eqs. (3), which van Stockum derived by assuming a special functional form for the $g_{\mu\nu}$, might not be the only candidates for the exterior field; it is known, for instance, that the gravitational field (3a) is *static*[12] in the sense that a "transformation" of the form

$$t' = At + B\varphi, \quad A, B, C, D \text{ constants} \qquad \varphi' = Ct + D\varphi \tag{4}$$

will eliminate the $g_{\varphi t}$ component. [Transformation is placed in quotes since t' is a periodic coordinate: $t' \equiv t' + B2\pi$. Interpreted globally, the new metric covers a manifold with topology

$$S^2 \times (\text{half plane}).$$

We can return to the original topology by taking a covering space, an operation which is *not* equivalent to changing a coordinate system.]

Fortunately, it is easy to prove that (3) are the only possible exterior fields for a rotating infinite cylinder. Levy and Robinson[13] have shown that in this case, the Weyl-Papapetrou metric can be written [modulo (4)] in the form

$$ds^2 = -e^{2u}(dt + ad\varphi)^2 + e^{2(k-u)}(dr^2 + dz^2) + r^2e^{-2u}d\varphi^2, \tag{5}$$

where u, a, k are functions of r only. A procedure developed by Davies and Caplan[14] and myself allows the equations $R_{\mu\nu} = 0$ to be integrated; the solutions are equivalent to (3). (Details of the uniqueness proof can be found in the Appendix.)

Since the causality problems come from the sinusoid factors of (3c), we might hope to avoid these factors by "transforming" (3a) via (4) and then attempting to join the interior field to the "new" (topologically distinct) field. But the "transformation" (4) will not change the exponents of r, which for $aR > \frac{1}{2}$ become imaginary—in fact, for $aR > \frac{1}{2}$, (3a) *is* (3c) with the substitutions $\epsilon = i\beta$ and $\theta = i\gamma$.

Thus we expect causality violation to occur in the matter-free space surrounding a rapidly rotating infinite cylinder. As Thorne[15] has emphasized, however, it is risky to claim that the properties of such a cylinder also hold for realistic cylinders.

In addition to the already mentioned static nature of the field, there is the fact that it is not even asymptotically Minkowskian (especially when $aR > \frac{1}{2}$). Still, the gravitational potential of the cylinder's Newtonian analog also diverges at radial infinity, yet this potential is a good approximation near the surface in the middle of a long but finite cylinder, and if we shrink the rotating cylinder down to a "ring" singularity, we end up with the Kerr field, which also has CTL. These facts suggest that there is a region near the surface of a *finite* cylinder where $g_{\varphi\varphi}$ becomes negative, implying causality violation.

Since $H \neq 0$ for $r \neq 0$, there are no event horizons around the infinite cylinder. By analogy with the static case,[16] I expect this to be true for a finite cylinder; if so, then a timelike line from any event in the universe could enter the region where $g_{\varphi\varphi}$ is negative and return to any other event.[17]

In short, general relativity suggests that if we construct a sufficiently large rotating cylinder, we create a time machine.

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APPENDIX: PROOF THAT VAN STOCKUM'S EXTERIOR SOLUTIONS (3) ARE THE ONLY POSSIBLE EXTERIOR FIELDS FOR AN INFINITE ROTATING CYLINDER

Davies and Caplan have shown[14] that the field equations $R_{\mu\nu}=0$ for the Levy-Robinson metric [Eq. (5)] reduce to

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}+\frac{e^{4u}}{2r^2}\left(\frac{da}{dr}\right)^2=0\,, \tag{A1}$$

$$\frac{d^2a}{dr^2}-\frac{1}{r}\frac{da}{dr}+4\frac{da}{dr}\frac{du}{dr}=0\,, \tag{A2}$$

$$\frac{2}{r}\frac{dk}{dr}-2\left(\frac{du}{dr}\right)^2+\frac{1}{2r^2}e^{4u}\left(\frac{da}{dr}\right)^2=0\,. \tag{A3}$$

We have three coupled equations for three functions: second order in u, second order in a, first order in k. Thus we expect five arbitrary constants. A general physical solution to the above system will be defined to be a set of functions a, u, k in which the five constants are allowed to assume all real values from $-\infty$ to ∞. I will show that this general solution is given by Eqs. (3a)–(3c).

Equation (A2) can be written

$$r\frac{d}{dr}\left(\frac{1}{r}e^{4u}\frac{da}{dr}\right)=0\,.$$

Thus $(1/r)e^{4u}\,da/dr=2\omega$ (where ω is a constant). Substituting this into (A1), we obtain

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}+2\omega^2e^{-4u}=0\,.$$

Suppose first that $\omega=0$. Then a little manipulation yields

$$u=A(\ln r)+B\,,\quad k=A^2(\ln r)+C\,,\quad a=D\,,$$

where A, B, C, D are constants.

By the transformation $t=t'-a\varphi$, $\varphi=\varphi'$, $z=z'$, $r=r'$, we discover that except for global topology this solution is just the Weyl solution (3a).

Suppose now that $\omega\neq 0$. It is at this point that Davies and Caplan err; their "general" solution in fact places implicit restrictions on the value of their constant A. The complete general solution is obtained via the following procedure. Let $v=e^{-4u}$, $p=(\omega r)^2$, so that $u=-\frac{1}{4}\ln(v)$, and $d/dr=2\omega^2 r(d/dp)$, which gives

$$\frac{du}{dr}=-\frac{1}{4v}\frac{dv}{dp}2\omega^2 r\,.$$

Equation (A4) becomes

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right)+2\omega^2e^{-4u}$$
$$=\frac{2\omega^2 r}{r}\frac{d}{dp}\left(\frac{2\omega^2r^2}{4v}\frac{dv}{dp}\right)+2\omega^2v=0$$

or

$$\frac{d}{dp}\left(\frac{p}{v}\frac{dv}{dp}\right)-2v=0\,. \tag{A5}$$

Let $w=pv$, giving $dv/dp\equiv v'=w'/p-w/p^2$. (A5) becomes

$$\frac{d}{dp}\left(\frac{pw'}{w}\right)-\frac{2w}{p}=0\,. \tag{A6}$$

Let $t=\ln(p)$, $d/dp=(1/p)d/dt$. (A6) becomes

$$\frac{d}{dt}\left(\frac{\dot w}{w}\right)-2w=0\,. \tag{A7}$$

Let $Q=\dot w\equiv dw/dt$, $d/dt=(Q)d/dw$. (A7) becomes

$$Q\frac{d}{dw}\left(\frac{Q}{w}\right)-2w=0$$

or

$$\frac{d}{dw}\left(\frac{Q}{w}\right)-2\frac{w}{Q}=0\,.$$

Thus

$$\frac{Q}{w}d\left(\frac{Q}{w}\right)=2dw$$

or

$$(Q/w)^2=4w\pm A^2\,,$$

which can be written

$$\dot{w} = w(4w \pm A^2)^{1/2}. \tag{A8}$$

The next integral depends on the sign choice in (A8). First choose the + sign. Then performing the integration, we obtain

$$\frac{1}{A}\ln\left[\frac{(4w+A^2)^{1/2}-A}{(4w+A^2)^{1/2}+A}\right] = \ln p + \frac{1}{A}\ln B.$$

[The constant of integration $(1/A)\ln B$ has values from $-\infty$ to ∞, though $0<B<\infty$.] This can be inverted (after the appropriate substitutions are made) to give

$$u = \frac{1}{4}\ln\left[\frac{(1-\omega^{2A}r^{2A}B)^2}{A^2(\omega r)^{2A-2}B}\right],$$

which is identical to Eq. (2.3) of Davies and Caplan (in Ref. 14). The computation proceeds as they outline to obtain k and a. Frehland[14] has shown that this solution is the same as the Weyl solution (3a).

Suppose now that $A=0$. We get

$$\begin{aligned}
g_{zz} &= g_{rr} = Fr^{-1/2},\\
g_{\varphi t} &= -r(1 - E\omega \ln\omega^2 r^2 D^2),\\
g_{tt} &= \omega r \ln(\omega^2 r^2 D^2),\\
g_{\varphi\varphi} &= -rE(2 + E\omega\ln\omega^2 r^2 D^2),
\end{aligned} \tag{A9}$$

where E, D, F are constants. These solutions are identical to (3b), with a suitable choice of constants. Suppose now that the integration constant is $-A^2$. We obtain

$$\begin{aligned}
g_{tt} &= \frac{2\omega r}{A}\cos[\ln(\omega^A r^A)+C],\\
g_{zz} &= g_{rr} = Fr^{-(1+A^2)/2},\\
g_{\varphi t} &= r\{\sin[\ln(\omega^A r^A)+C]\\
&\qquad + D\cos[\ln(\omega^A r^A)+C]\}.
\end{aligned} \tag{A10}$$

$g_{\varphi\varphi}$ is determined by the relation $FL + M^2 = r^2$, where A, C, D, F are constants.

Thus the general exterior field is given by (3).

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[5]B. Carter, Phys. Rev. 174, 1559 (1968). Carter's causality theorem can be stated as follows: A necessary and sufficient condition for nontrivial causality violation in a connected, time-oriented spacetime with a timewise orthogonally transitive Abelian isometry group is the nonexistence of a covariant vector in the Lie algebra such that the corresponding differential form in the surface of transitivity is everywhere well behaved and everywhere timelike. If the above criterion is satisfied, then there exist both future- and past-directed timelike lines between any two points of the spacetime. For the van Stockum metrics (2) and (3a)–(3b), the group generated by the Killing vectors $(\partial/\partial z, \partial/\partial t, \partial/\partial\varphi)$ is timewise orthogonally transitive and Abelian. It is easily checked that for $r>1/a$ in (2) and $aR>\frac{1}{2}$ in (3), there is no linear combination $\psi = At + B\varphi + Cz$ (where A, B, C are constants) such that the form $d\psi$ is everywhere timelike.

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