Black Holes and the Second Law (*).

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Black-hole physics seems to provide at least two ways in which the second law of thermodynamics may be transcended or violated:

a) Let an observer drop or lower a package of entropy into a black hole; the entropy of the exterior world decreases. Furthermore, from an exterior observer's point of view a black hole in equilibrium has only three degrees of freedom: mass, charge and angular momentum [1]. Thus, once the black hole has settled down to equilibrium, there is no way for the observer to determine its interior entropy. Therefore, he cannot exclude the possibility that the total entropy of the universe may have decreased in the process. It is in this sense that the second law appears to be transcended [2].

b) A method for violating the second law has been proposed by GEROCH [3]: By means of a string one slowly lowers a body of rest mass m and nonzero temperature toward a Schwarzschild black hole of mass M. By the time the body nears the horizon, its energy as measured from infinity, $E = m(1 - 2M/r)^{\frac{1}{2}}$, is nearly zero; the body has already done work m on the agent which lowers the string. At this point the body is allowed to radiate into the black hole until its rest mass is $m - \Delta m$. Finally, by expending work $m - \Delta m$, one hauls the body back up.

The net result: a quantity of heat Δm has been completely converted into work. Furthermore, since the addition of the radiation to the black hole takes place at a point where $(1 - 2M/r)^{\frac{1}{2}} \approx 0$, the mass of the black hole is unchanged. Thus the black hole appears to be unchanged after the process. This implies a violation of the second law: « One may not transform heat entirely into work without compensating changes taking place in the surroundings. »

In this note we formulate the second law in a form suitable for black-hole physics which resolves the transcendence problem. We also indicate why Geroch's gedanken experiment does not, in fact, violate the second law.

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[1] C. W. MISNER, K. S. THORNE and J. A. WHEELER: *Gravitation* (San Francisco, 1972).
[2] For transcendence of the law of baryon conservation see J. D. BEKENSTEIN: *Phys. Rev. Lett.*, 28, 452 (1972); *Phys. Rev. D*, 5, 1239 (1972) and to be published.
[3] R. GEROCH: *Colloquium at Princeton University* (Dec. 1971).

We state the second law as follows: «Common entropy plus black-hole entropy never decreases.» By common entropy we mean entropy in the black-hole's exterior. By black-hole entropy $S_{\text{b.h.}}$ we mean

$$(1) \qquad\qquad S_{\text{b.h.}} = \eta k L_{\text{P}}^{-2} A \,,$$

where A is the area of the black hole [4] in question, L_{P} is the Planck length $(\hbar G/c^3)^{\frac{1}{2}} = 1.6 \cdot 10^{-33}$ cm, k is Boltzmann's constant $1.38 \cdot 10^{-16}$ erg/°K and η is a constant number of order unity.

The introduction of a black-hole entropy is necessitated by our first example. Without it the second law is definitely transcended. With black-hole entropy the second law becomes a well-defined statement susceptible to verification by an exterior observer. The choice of the area of a black hole as a measure of its entropy is motivated by the results of CHRISTODOULOU [5] and of HAWKING [6] that the area of a black hole never decreases, and that it increases for all but a very special class of black-hole transformations [5]. The area of a black hole appears to be the only one of its properties having this entropylike behavior which is so essential if the second law as we have stated it is to hold when entropy goes down a black hole.

By choosing $S_{\text{b.h.}}$ proportional to A rather than to some monotonically increasing function of A, we ensure that the total black-hole entropy of a system of black holes (the sum of individual $S_{\text{b.h.}}$) depends only on the total horizon area—also a nondecreasing quantity which is of fundamental importance in Hawking's work [6].

The introduction of k and of a length squared in (1) is necessitated by dimensional considerations. We choose Planck's length because it is the only truly universal constant with units of length. (We shall show later on that no larger scale of length will do.) Although a black hole is a thoroughly classical entity, black-hole entropy contains $\hbar$ because it relates to the interaction of a black hole with material systems. It is well known that the expression for entropy of any material system always contains $\hbar$; thus, the appearance of $\hbar$ in $S_{\text{b.h.}}$ is understandable. (In what follows we use units with $G = c = 1$; thus $L_{\text{P}}^2 = \hbar$.)

Let us illustrate the operation of the second law with a simple example. Consider the case of a narrow beam of black-body radiation of temperature T which is directed into a black hole of mass M. We are clearly assuming that geometrical optics is applicable. Thus, the characteristic wavelengths of the radiation must be much smaller than M. This implies that

$$(2) \qquad\qquad T \gg \hbar/kM \,.$$

We note also that, to a given energy E in the beam, there corresponds entropy [7] S, where

$$(3) \qquad\qquad S = E/T \,.$$

The increase in area of the black hole [4] caused by the infall of radiation with energy E may be calculated from the conservation of energy and angular momentum.

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[4] The area of a (Kerr-Newman) black hole of mass M, charge Q and angular momentum L is $A = 4\pi\{[M + (M^2 - Q^2 - L^2/M^2)^{\frac{1}{2}}]^2 + L^2/M^2\}$. An extreme Kerr black hole has $L = M^2$ and $Q = 0$.

[5] D. CHRISTODOULOU: Phys. Rev. Lett., 25, 1596 (1970); D. CHRISTODOULOU and R. RUFFINI: Phys. Rev. D, 4, 3552 (1971).

[6] S. W. HAWKING: Phys. Rev. Lett., 26, 1344 (1971); Comm. Math. Phys., 25, 152 (1972).

[7] A black-body cavity emits entropy E/T as it emits energy E; the propagation of the beam is a reversible process.

(One neglects gravitational radiation losses since they are of $O(E^2/M)$.) This increase is minimized when the black hole is extreme Kerr, and when the beam of radiation travels in the equatorial plane, and adds to the black hole as much angular momentum as is consistent with the requirement that it be captured by the black hole [8], The increase is [8]

$$(4) \qquad\qquad (\Delta A)_{\min} = 8\pi(2 - 3^{\frac{1}{2}}) ME .$$

The corresponding minimum possible increase in $S_{\mathrm{b.h.}}$ is

$$(5) \qquad\qquad (\Delta S_{\mathrm{b.h.}})_{\min} = 8\pi(2 - 3^{\frac{1}{2}}) \eta k ME/\hbar .$$

On the other hand, according to (2) and (3) the corresponding decrease of common entropy is

$$(6) \qquad\qquad |\Delta S| \ll kME/\hbar .$$

Thus, provided η is of-order unity, the overall entropy (common entropy plus $S_{\mathrm{b.h.}}$) always increases for all M. It should be clear that the length L_{P} appearing in (1) cannot be chosen much larger than $\hbar^{\frac{1}{2}}$ if the second law is to hold in all cases.

At first sight it would appear that the second law will break down if $T \lesssim \hbar/kM$. However, in this regime one cannot continue treating the radiation by geometrical optics, but must treat it by wave theory. If this is done, one finds [8] that statistical fluctuations in the radiation are all important, and thus the second law is irrelevant in the first place. Thus, for this gedanken experiment the second law as here formulated holds whenever fluctuations are unimportant.

CHRISTOULOU [5] has described a process by means of which a particle can be added to a black hole without there resulting an increase in the black hole's area. One could imagine that the particle in question carries entropy. It would then appear that this entropy goes down the black hole without there being a compensating increase in $S_{\mathrm{b.h.}}$ as demanded by the second law.

In Cristodoulou's process the particle in question must be created at the black hole's horizon by the fission of a primary particle (the other fission product escaping to infinity). It follows that the primary particle is required to fall from far away to the horizon. Recent results [9] indicate that in such a fall the particle will radiate into the black hole gravitational waves with energy of the order of its rest mass. One can show [8,10] that this radiation will always cause a nonnegligible increase in the black hole's area corresponding to the increase in $S_{\mathrm{b.h.}}$ demanded by the second law. In this manner one reconciles Christodoulou's process with the second law.

Let us now try to resolve the difficulty raised by Geroch's gedanken experiment. The key assumption made which leads to conflict with the second law is that one can calculate the work done by (on) the body by taking differences of its « potential energy » $m(1 - 2M/r)^{\frac{1}{2}}$ at two different points. Such a procedure assumes the process of lowering (raising) the body to be of a conservative or reversible character. However, the tendency of a black hole's area to increase irreversibly at the slightest provocation [6] warns us that there may be an element of irreversibility in the problem.

[8] J. D. BEKENSTEIN: to be published.
[9] M. DAVIS, R. RUFFINI and J. TIOMNO: Phys. Rev. D, to be published (1972).
[10] J. D. BEKENSTEIN: Ph. D. Thesis, Princeton University (1972) (unpublished).

Indeed, some calculations based on the metric for a Schwarzschild black hole placed in an exterior quadrupole gravitational field [11], coupled with simple physical arguments, suggest that the black hole in question is deformed, and its area increases all along as the body is lowered by the string [8,10]. This increase in area is irreversible. Thus, when the body is hauled back up, the black hole returns to an undeformed Schwarzschild configuration of larger area (and hence larger mass) than the initial one. Therefore, the work expended to raise the body must be larger than the one calculated from the potential method; the work done by the body is less than Δm. This means that the heat Δm is, at best, only partially converted into work. No violation of the second law is entailed here. One should also notice that the « surroundings », i.e. the black hole, are changed in the overall process.

Further details on the above examples, other examples and implications of the second law for black-hole physics will be reported elsewhere.

We conclude on a speculative note. One sees from (1) that the natural unit of area of a black hole is the Planck length squared. One cannot help wondering about the possible connection between this feature and the expected quantum structure of space-time at a scale of the order of the Planck length [12].

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[11] YA. B. ZEL'DOVICH and I. D. NOVIKOV: Stars and Relativity (Chicago, 1972), p. 138.
[12] J. A. WHEELER: a chapter in Relativity, Groups and Topology, edited by B. DEWITT and C. DEWITT (New York, 1965).